# A FINITE ELEMENT METHOD FOR THE STOKES PROBLEM ON QUADRILATERAL GRIDS YIELDING DIVERGENCE FREE APPROXIMATIONS

by

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**A FINITE ELEMENT METHOD FOR THE STOKES PROBLEM ON QUADRILATERAL GRIDS YIELDING DIVERGENCE FREE APPROXIMATIONS**

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In this thesis project, a pair of conforming, stable and divergence free finite elements for the Stokes problem on two dimensional rectangular grids with no-slip boundary conditions is constructed. Pointwise continuous $Q_{3,2} \times Q_{2,3}$ polynomials that are partially $C^1$ at the vertices and $Q_{2,2}$ polynomials that are continuous at the vertices are used as the functions forming the velocity and pressure spaces, respectively. In the construction of these finite element spaces, a Stokes complex is formed to verify the incompressibility of the velocity approximation.

With the definition of appropriate norms and the use of the Piola transform, the *inf-sup* stability condition is satisfied on each rectangular element and then in the entire domain. Furthermore, by the application of Nitsche's method to the problem, the existence and the uniqueness of the solution to the Stokes problem are justified and error estimates are obtained.

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# 1.0 INTRODUCTION

In this thesis project, the goal is to construct a pair of conforming and stable finite elements for the two dimensional Stokes problem with no-slip boundary conditions given by:

$$-\gamma\Delta u + \nabla p = f \qquad in \quad \Omega, \tag{1.1}$$

$$div(u) = 0 \qquad in \quad \Omega, \tag{1.2}$$

$$u = 0 \qquad on \quad \partial\Omega. \tag{1.3}$$

Here $\Omega$ is assumed to be an open, bounded, simply-connected polyhedral domain, $\gamma > 0$ is the viscosity, $f \in L^2(\Omega)$ is an external force applied to the fluid, $u$ and $p$ are the velocity and the pressure of the fluid, respectively. For simplicity, we assume that $\gamma$ is constant.

The conformity of the finite element discretization implies that $V_h \subset H^1(\Omega)$ and $Q_h \subset L^2(\Omega)$. The *inf-sup* stability condition, which is also known as the Ladyzenskaja-Babuska-Brezzi(LBB) condition, is given by:

$$\alpha \|q\|_{L^2(\Omega)} \leq \sup_{v \in V_h \setminus \{0\}} \frac{\int_\Omega div(v) q dx}{\|v\|_{H^1(\Omega)}} \qquad \forall q \in Q_h, \tag{1.4}$$

where $\alpha > 0$ is a constant independent of $h$. The *inf-sup* condition implies that the spurious pressure modes are eliminated and a unique finite element solution to the problem is guaranteed. Furthermore, it can be deduced that the optimal convergence [8],

$$\|u - u_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq c \inf_{v \in V_h, q \in Q_h} (\|u - v\|_{H^1(\Omega)} + \|p - q\|_{L^2(\Omega)}) \tag{1.5}$$

where $c > 0$ is a constant that depends on $\gamma$ and $\alpha$, but not on $h$, is obtained.

Over the past years, many mixed finite element methods for the Stokes problem on triangular meshes have been developed. Although conforming and stable approximations have been made by most of these methods, the incompressibility condition, i.e., $div(u) = 0$, has been only weakly satisfied. Taylor-Hood elements, the MINI element [1], the Crouzeix-Raviart elements [10] and the $P_2 - P_0$ pair in [5] are among the methods that belong to this class. On the other hand, in recent years, Scott-Vogelius[18], Neilan-Guzman [15] and Neilan-Falk[12] have constructed stable and conforming finite element pairs which are divergence free on triangular meshes. Arnold and Qin showed that $P_k - P_{k-1}$ pairs which consist of $P_k$ continuous velocity and $P_{k-1}$ discontinuous pressure fields satisfy the incompressibility condition on certain types of uniform triangulations[2].

It is known that the finite element spaces that only satisfy the incompressibility condition weakly may lead to instabilities in nonlinear problems. Additionally, since the discrete divergence free condition can be interpreted as the conservation of mass, the inexact satisfaction of the incompressibility condition also leads to the lack of mass conservation.

The discrete velocity solution is divergence free if and only if the image of the divergence of the discrete velocity finite element space is a subset of the discrete pressure finite element space. In cases where the image of the discrete divergence operator is smaller than the discrete pressure finite element space, the kernel of the discrete gradient operator, $grad_h$, that maps the discrete pressure space with zero mean to the discrete velocity space is non-trivial. As a result, spurious pressure modes occur and the presence of the spurious pressure modes in the finite element methods for the incompressible fluid flow problems invalidates both the uniqueness of the discrete pressure solution and the *inf-sup* stability condition. For instance, the $Q_1 - P_0$ element does not satisfy the *inf-sup* stabiity condition. This element depends highly on the mesh and global spurious modes, which can't be eliminated easily, are observed on some regular meshes. Another example is the $P_1 - P_0$ element. In this case, the dimension of the kernel of the discrete gradient operator, which defines the spurious modes, grows as the mesh size tends to zero [5].

The first conforming, divergence free element on a rectangular mesh was proposed by Austin, Manteuffel and McCormick [3]. The finite element space they introduced is a continuous space that is based on the Raviart-Thomas finite element space, which is a discontinuous finite element space that has a discrete Helmholtz decomposition. The authors constructed a $Q_{3,2} \times Q_{2,3}$ finite element space as a direct sum of two $L_2$ orthogonal spaces and they proved the optimal convergence in the energy norm for tensor product grids.

Another conforming, divergence free element on rectangular grids was proposed by Zhang [21]. Here, it is shown that the $Q_{k+1,k} \times Q_{k,k+1} - Q_k^-$ mixed finite element, where $Q_k^-$ denotes the discontinuous polynomials of separated degree $k$ or less with spurious modes filtered, are stable and yield an optimal order of approximation for the Stokes problem for all $k \geq 2$. Furthermore, it's shown by Stenberg and Suri [19] that the finite element with different polynomial degrees in different directions, $Q_{k+1,k} \times Q_{k,k+1} - Q_{k-1}$, give the same approximation results as the element, $Q_{k+1} \times Q_{k-1}$.

In 2011, Huang and Zhang introduced a stable, conforming and divergence free mixed finite element, $Q_{2,1} \times Q_{1,2} - Q_1^-$, for the Stokes problem on rectangular grids [23]. The finite elements in [23] are obtained by taking $k = 1$ for the elements defined in [21].

In this project, we construct a pair of conforming and stable finite elements for the Stokes problem on two dimensional rectangular grids with no-slip boundary conditions. We use pointwise continuous $Q_{3,2} \times Q_{2,3}$ polynomials that are partially $C^1$ at the vertices of the rectangular elements and $Q_{2,2}$ polynomials that are continuous at the vertices of the rectangular elements as the functions forming the velocity and pressure spaces respectively. Defining appropriate norms and using the Piola transform, we verify the *inf-sup* stability condition on each rectangular element and then in the entire domain. In chapter 2, we define the local finite element spaces, an affine transformation, $F(\cdot)$, mapping a rectangular element to a reference element, which is assumed to be the unit square throughout this project, and a scaled Piola transform, $P(\cdot)$, to verify that the *inf-sup* stability condition holds. In chapter 3, we define the global finite element spaces with and without homogeneous boundary conditions and justify the conformity and the *inf-sup* stability. Then applying Nitsche's method to the Stokes problem with homogeneous boundary conditions, we define a bilinear form, $A(\cdot,\cdot)$, and prove the coercivity and the continuity of $A$. We show that the problem at hand has a unique solution and the convergence is obtained in a $H^1$ type norm.

## 2.0 THE LOCAL FINITE ELEMENT SPACES

We assume that $\Omega$ is an open, simply connected, bounded polyhedral domain with edges parallel to the coordinate axes. Let $\mathcal{Q}_h$ be a quasi-uniform rectangular mesh of $\Omega$ discretized by $h$.

In this chapter, we construct the local finite element spaces for the Stokes problem. We denote the vertices of a rectangular element in $\mathcal{Q}_h$, by $\{a_i\}_{i=1}^4$, and let $L_1$, $L_2$, $L_3$ and $L_4$ denote the edges of a rectangular element in $\mathcal{Q}_h$ such that:

$$\begin{aligned} L_1 &= \{(x, y_0) : x_0 \le x \le x_1\}, \\ L_2 &= \{(x, y_1) : x_0 \le x \le x_1\}, \\ L_3 &= \{(x_1, y) : y_0 \le y \le y_1\}, \\ L_4 &= \{(x_0, y) : y_0 \le y \le y_1\}, \end{aligned}$$

where $a_1 = (x_0, y_0)$, $a_2 = (x_1, y_0)$. $a_3 = (x_1, y_1)$ and $a_4 = (x_0, y_1)$. The space of polynomials of degree $m$ in $x$ and $n$ in $y$ is denoted by $Q_{m,n}$.

Our first goal is to construct an exact subcomplex of the Stokes complex defined as [12]:

$$\mathbb{R} \overset{\subseteq}{\longrightarrow} H^2(\Omega) \overset{curl}{\longrightarrow} H^1(\Omega) \overset{div}{\longrightarrow} L^2(\Omega) \rightarrow 0, \tag{2.1}$$

where the *curl* operator is defined as: $curl(z) = (-\frac{\partial z}{\partial y}, \frac{\partial z}{\partial x})^t$ for $z \in H^2(\Omega)$. The complex (2.1) is exact provided $\Omega$ is simply connected, i.e., the range of each map in the complex is the null space of the succeeding map. Thus, we wish to construct finite element spaces such that for $\forall q \in Q_h \subseteq L^2(\Omega)$, there exists $v \in V_h \subseteq H^1(\Omega)$ satisfying $div(v) = q$, and if $v \in V_h$ with $div(v) = 0$, then $v = curl(z)$ for some $z \in \Sigma_h \subseteq H^2(\Omega)$. Therefore, our goal is to define finite element spaces $(\Sigma_h, V_h, Q_h)$ such that the following is an exact subcomplex of the complex given in (2.1):

$$\mathbb{R} \overset{\subseteq}{\longrightarrow} \Sigma_h \overset{curl}{\longrightarrow} V_h \overset{div}{\longrightarrow} Q_h \rightarrow 0. \tag{2.2}$$

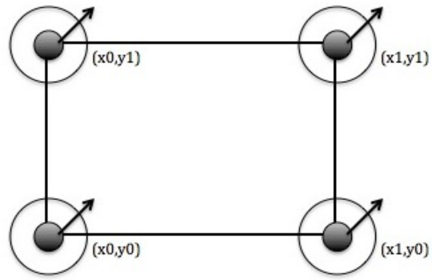


Figure 2.1: The degrees of freedom of $\Sigma_h(\mathcal{Q})$. The solid circles represent the function evaluations, larger circles represent the gradient evaluations, the arrows represent the second order mixed derivatives.

## 2.1 THE $C^1$ BOGNER-FOX-SCHMIDT FINITE ELEMENT SPACE, $\Sigma_h(\mathcal{Q})$

For $\mathcal{Q} \in \mathcal{Q}_h$, the local Bogner-Fox-Schmidt space is given by $\Sigma_h(\mathcal{Q}) = Q_{3,3}$ [12]. The degrees of freedom of $\Sigma_h(\mathcal{Q})$ illustrated in Figure 2.1 are as follows:

$$S_1 = \{z(a_i),\, \nabla z(a_i),\, \frac{\partial z}{\partial x \partial y}(a_i) :\; i = 1, 2, 3, 4\},$$

where $z \in \Sigma_h(\mathcal{Q})$. Note that the cardinality of $S_1$ is $8+4+4+16$ =dim$(Q_{3,3})$. Therefore, it suffices to prove that if $z$ nullifies $\Sigma_h(\mathcal{Q})$, then $z = 0$ to conclude the degrees of freedom form a unisolvent set. If $z \in \Sigma_h(\mathcal{Q})$, then we can write $z(x, y) = s_1(x)s_2(y)$ where $s_1$ and $s_2$ are cubic polynomials in one variable. Then, we have

$$\begin{aligned}
z(x_0, y_0) = s_1(x_0)s_2(y_0) &= z(x_1, y_0) = s_1(x_1)s_2(y_0) = z(x_0, y_1) \\
&= s_1(x_0)s_2(y_1) = z(x_1, y_1) = s_1(x_1)s_2(y_1) = 0, \\
\frac{\partial z}{\partial x}(x_0, y_0) = s_1'(x_0)s_2(y_0) &= \frac{\partial z}{\partial x}(x_1, y_0) = s_1'(x_1)s_2(y_0) = \frac{\partial z}{\partial x}(x_0, y_1) \\
&= s_1'(x_0)s_2(y_1) = \frac{\partial z}{\partial x}(x_1, y_1) = s_1'(x_1)s_2(y_1) = 0, \\
\frac{\partial z}{\partial x \partial y}(x_0, y_0) = s_1'(x_0)s_2'(y_0) &= \frac{\partial z}{\partial x \partial y}(x_0, y_1) = s_1'(x_0)s_2'(y_1) = \frac{\partial z}{\partial x \partial y}(x_1, y_0) \\
&= s_1'(x_1)s_2'(y_0) = \frac{\partial z}{\partial x \partial y}(x_1, y_1) = s_1'(x_1)s_2'(y_1) = 0.
\end{aligned}$$

If $s_1'(x_0) \neq 0$, then $s_2'(y_0) = s_2'(y_1) = s_2(y_0) = s_2(y_1) = 0$. As a result, $s_2 = 0$ and therefore, $z|_{L_4} = 0$. Similar computations show $z|_{\partial\mathcal{Q}} = 0$. Thus, $z = \alpha b$ where $b = L_1(x)L_2(x)L_3(y)L_2(y) \in Q_{2,2}$ is a bubble function and $\alpha \in Q_{1,1}$. Suppose $a_i = a_1$.

$$\begin{aligned}\frac{\partial z}{\partial x\partial y}(a_1) &= \frac{\partial \alpha}{\partial x\partial y}(a_1)\cdot b(a_1) + \frac{\partial \alpha}{\partial x}(a_1)\cdot\frac{\partial b}{\partial y}(a_1) + \frac{\partial \alpha}{\partial y}(a_1)\cdot\frac{\partial b}{\partial x}(a_1) + \alpha(a_1)\cdot\frac{\partial b}{\partial x\partial y}(a_1)\\ &= 0 + \frac{\partial \alpha}{\partial x}(a_1)\cdot(L_1(x_0)L_2(x_0)L_3'(y_0)L_4(y_0) + L_1(x_0)L_2(x_0)L_3(y_0)L_4'(y_0))\\ &+ \frac{\partial \alpha}{\partial y}(a_1)\cdot(L_1'(x_0)L_2(x_0)L_3(y_0)L_4(y_0) + L_1(x_0)L_2'(x_0)L_3(y_0)L_4(y_0)) + \alpha(a_1)\cdot\frac{\partial b}{\partial x\partial y}(a_1)\\ &= \alpha(a_1)\cdot\frac{\partial b}{\partial x\partial y}(a_1)\end{aligned}$$

since $b(a_1) = L_1(x_0) = L_4(y_0) = 0$. Furthermore, since $L_1'(x_0) \neq 0$ and $L_4'(y_0) \neq 0$, $L_2(x_0) \neq 0$ and $L_3(y_0) \neq 0$, $\frac{\partial z}{\partial x\partial y}(a_1) = 0$ implies $\alpha = 0$, i.e. $z = 0$. Hence, the degrees of freedom form a unisolvent set.

## 2.2 THE VELOCITY SPACE, $V_h(\mathcal{Q})$

The local velocity space is given by $V_h(\mathcal{Q}) = Q_{3,2} \times Q_{2,3}$ and the degrees of freedom $V_h(\mathcal{Q})$ shown in Figure 2.2, are given by:

$$S_2 = \{v(a_i), \frac{\partial v_1}{\partial x}(a_i), \frac{\partial v_2}{\partial y}(a_i), \int_{\mathcal{Q}} vp\,dx, , \int_{L_3} v_1\,ds, \int_{L_4} v_1\,ds, \int_{L_1} v_2\,ds, \int_{L_2} v_2\,ds;\, p \in Q_{1,1},\, i = 1,2,3,4\},$$

where $v = (v_1, v_2)$.

**Lemma 1.** *The degrees of freedom stated by $S_2$ are unisolvent on $V_h(\mathcal{Q})$.*

*Proof.* It suffices to show that if $v$ nullifies the degrees of freedom of $V_h$, then $v = 0$, since $\dim(V_h(Q)) = 24$ which equals the number of degrees of freedom [6].

We write $v_1(x, y) = s_1(x)s_2(y)$, where $s_1$ and $s_2$ are cubic and quadratic polynomials, respectively. Then, we have

$$\begin{aligned} v_1(x_0, y_0) &= s_1(x_0)s_2(y_0) = 0,\\ v_1(x_1, y_0) &= s_1(x_1)s_2(y_0) = 0,\\ \frac{\partial v_1}{\partial x}(x_0, y_0) &= s_1'(x_0)s_2(y_0) = 0,\\ \frac{\partial v_1}{\partial x}(x_1, y_0) &= s_1'(x_1)s_2(y_0) = 0.\end{aligned}$$

If $s_2(y_0) \neq 0$, then $s_1(x_0) = s_1(x_1) = s_1'(x_0) = s_1'(x_1) = 0$. This implies $s_1 = 0$. Thus, $v_1|_{L_1} = 0$. Similar arguments show that $v_1|_{L_2} = 0$. Moreover, $\int_{L_3} v_1 ds = \int_{L_4} v_1 ds = 0$ implies $s_2(y^*) = 0$ for some $y^* \in (y_0, y_1)$. Thus, $s_2(y)$ is a quadratic polynomial with three zeros, and therefore $s_2 = 0$ on $L_3$ and $L_4$, i.e., $v_1|_{L_3} = v_1|_{L_4} = 0$. Hence, $v_1 \in H_0^1(\mathcal{Q}) \cap Q_{2,3}$ and therefore we may write $v_1 = q_1 b_{\mathcal{Q}}$, where $b_{\mathcal{Q}} = L_1(x)L_2(x)L_3(y)L_4(y) \in Q_{2,2}$ is a bubble function and $q_1 \in Q_{1,0}$. Similar computations show $v_2 = q_2 b_{\mathcal{Q}}$ where $q_2 \in Q_{0,1}$. Thus, we have shown that $v = 0$ on the boundary.

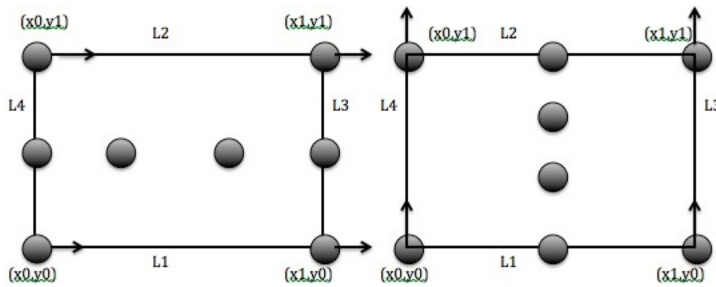


Figure 2.2: The degrees of freedom of $V_h(\mathcal{Q})$. The arrows represent the partial derivatives

In order to show $v = 0$ in $\mathcal{Q}$, we need to show $q_1 = q_2 = 0$. We have $\int_{\mathcal{Q}} v_1\, p\, dx = 0$ for $\forall p \in Q_{1,0}$ and $\int_{\mathcal{Q}} v_2\, p\, dx = 0$ for $\forall p \in Q_{0,1}$. Consider $\int_{\mathcal{Q}} v_1 p\, dx = 0$ for $\forall p \in Q_{1,0}$. Letting $p = q_1$, we have

$$\int_{\mathcal{Q}} v_1 q_1\, dx = \int_{\mathcal{Q}} q_1^2 b_{\mathcal{Q}}\, dx = 0.$$

Since $b_{\mathcal{Q}} > 0$ in $\mathcal{Q}$, $q_1^2 = 0$, i.e., $q_1 = 0$. Thus, $v_1 = 0$ in $\mathcal{Q}$. Similar arguments show that $q_2 = 0$, i.e., $v_2 = 0$ in $\mathcal{Q}$. Therefore, $v = 0$ in $\mathcal{Q}$. □

## 2.3 THE PRESSURE SPACE, $W_h(\mathcal{Q})$

The local pressure space denoted by $W_h(\mathcal{Q})$ consists of $Q_{2,2}$ polynomials. The degrees of freedom of $W_h(\mathcal{Q})$ illustrated in Figure 2.3 are as follows:

$$S_3 = \{q(a_i), \int_{\mathcal{Q}} q\, r\, dx,\ where\ r \in Q_{2,2}\ such\ that\ r(a_i) = 0\ for\ i = 1, 2, 3, 4\}.$$

Note that the cardinality of $S_3$ is $4 + (2+1)^2 - 4 = 9$, i.e., dim $(Q_{2,2})$=card$(S_3)$. Since $q(a_i) = 0$ for $\forall i = 1, 2, 3, 4$ and $\int_{\mathcal{Q}} q\, r\, dx = 0$ for $\forall r \in Q_{2,2}$ such that $r(a_i) = 0$ for $\forall i = 1, 2, 3, 4$, we can let $r = q$. This yields $\int_{\mathcal{Q}} q^2\, dx = 0$, therefore, $q = 0$. As a result, the given set of degrees of freedom determines $W_h(\mathcal{Q})$.

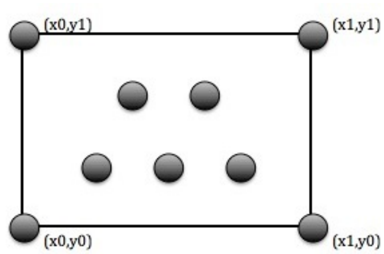


Figure 2.3: The degrees of freedom of $W_h(\mathcal{Q})$

### 2.4 A LOCAL CHARACTERIZATION OF THE DIVERGENCE OPERATOR

We now define local finite element spaces with imposed boundary conditions and prove the stability properties at the local level. To this end, we define for each $\mathcal{Q} \in \mathcal{Q}_h$,

- $\Sigma_{h,0}(\mathcal{Q}) = H_0^2(\mathcal{Q}) \cap Q_{3,3}(\mathcal{Q})$.
- $V_{h,0}(\mathcal{Q}) = (H_0^1(\mathcal{Q}))^2 \cap (Q_{3,2}(\mathcal{Q}) \times Q_{2,3}(\mathcal{Q}))$.
- $W_{h,0}(\mathcal{Q}) = \{q \in Q_{2,2}(\mathcal{Q}) : \int_{\mathcal{Q}} q\, dx = 0,\ q(a_i) = 0,\ i = 1, 2, 3, 4\}$.

**Lemma 2.** *The space $\Sigma_{h,0}(\mathcal{Q})$ is the trivial set, i.e.,*

$$\Sigma_{h,0}(\mathcal{Q}) = \{0\}.$$

*Proof.* If $z \in \Sigma_{h,0}(\mathcal{Q})$, then $z = b_{\mathcal{Q}}^2\, w$ where $b_{\mathcal{Q}}^2 \in H_0^2(\mathcal{Q})$ is a biquadratic bubble function. Since $b_{\mathcal{Q}}^2 \in Q_{4,4}(\mathcal{Q})$ and $z \in Q_{3,3}(\mathcal{Q})$, we conclude that $w = 0$, i.e., $z = 0$. □

**Lemma 3.** *There holds*

$$div(V_{h,0}(\mathcal{Q})) \subseteq W_{h,0}(\mathcal{Q}).$$

*Proof.* Let $r \in div(V_{h,0}(\mathcal{Q}))$. Then $\exists v \in V_{h,0}(\mathcal{Q})$ such that $div(v) = r$. By the divergence theorem and since $v \in (H_0^1(Q))^2$,

$$\int_{\mathcal{Q}} r\, dx = \int_{\mathcal{Q}} div(v)\, dx = \int_{\partial\mathcal{Q}} v \cdot n\, ds = 0,$$

where $n$ denotes the unit outward normal vector of $\partial\mathcal{Q}$. Moreover, at each vertex $a_i$,

$$r(a_i) = div(v(a_i)) = \frac{\partial v_1}{\partial x}(a_i) + \frac{\partial v_2}{\partial y}(a_i) = 0$$

since $\frac{\partial v}{\partial x}(a_i) = \frac{\partial v}{\partial y}(a_i) = \vec{0}$ as a result of $v|_{\partial\mathcal{Q}} = 0$. Thus, $r \in W_{h,0}(\mathcal{Q})$.

□

**Lemma 4.** *The kernel of the divergence operator acting on $V_{h,0}(Q)$ is given by*

$$Ker(div(V_{h,0}(\mathcal{Q}))) = curl(\Sigma_{h,0}(\mathcal{Q})).$$

*Proof.* Clearly,

$$curl(\Sigma_{h,0}) \subseteq Ker(div(V_{h,0})).$$

since $\Sigma_{h,0} = \{0\}$ and $div(0) = 0$.

We need to show

$$Ker(div(V_{h,0})) \subseteq curl(\Sigma_{h,0}),$$

to complete the proof.

Let $v \in Ker(div(V_{h,0}))$ such that $div(v) = 0$. This implies $v = curl(z)$ for some $z \in H_0^2(\mathcal{Q})$ [13]. Since $v \in Q_{3,2} \times Q_{2,3}$, we must have $z \in Q_{3,3}$ and therefore $z \in \Sigma_{h,0}(\mathcal{Q})$. Thus, $Ker(div(V_{h,0})) = curl(\Sigma_{h,0})$. □

**Theorem 1.** *The mapping $div : V_{h,0}(\mathcal{Q}) \to W_{h,0}$ is bijective, i.e.,*

$$div(V_{h,0}(\mathcal{Q})) = W_{h,0}(\mathcal{Q}) \text{ and } Ker(div(V_{h,0}(\mathcal{Q}))) = \{0\}.$$

*Proof.* By Lemmas $2-4$, it suffices to show that $dim(div(V_{h,0}(\mathcal{Q}))) = dim(W_{h,0})$ to conclude that the divergence map is bijective.
Since there are 5 linearly independent constraints imposed on the space $W_{h,0}(\mathcal{Q})$, we have $dim(W_{h,0}) = 3^2 - 5 = 4$. Furthermore, by Lemmas 2-4,

$$\begin{aligned} dim(div(V_{h,0}(\mathcal{Q})) &= dim(V_{h,0}(\mathcal{Q})) - dim(Ker(div(V_{h,0}(\mathcal{Q}))) \\ &= dim(V_{h,0}(\mathcal{Q})) - dim(curl(\Sigma_{h,0}(\mathcal{Q}))) \\ &= dim(V_{h,0}(\mathcal{Q})) - 0 = dim(Q_{1,0} \times Q_{0,1}) - 0 = 4. \end{aligned}$$

Hence, $div(V_{h,0}(\mathcal{Q})) = W_{h,0}(\mathcal{Q})$, i.e., for $\forall q \in W_{h,0}(\mathcal{Q})$, $\exists v \in V_{h,0}(\mathcal{Q})$ such that $div(v) = q$ and since $Ker(div(V_{h,0}(\mathcal{Q}))) = \{0\}$, this $v$ is unique. Therefore, $div : V_{h,0}(\mathcal{Q}) \to W_{h,0}$ is bijective. ☐

Consider the affine transformation $F : \hat{\mathcal{Q}} \to \mathcal{Q}$ such that $F(\hat{x}) = B\hat{x} + b$, where $\hat{\mathcal{Q}}$ is the unit square, $B$ is an $(2 \times 2)$ matrix and $b$ is a two dimensional vector. Then it is easy to show that $DF = B$. Furthermore, since $F$ maps edges to edges, we can define $F$ as follows:

$$F(\hat{x}, \hat{y}) = (x_0 + \hat{x}(x_1 - x_0), y_0 + \hat{y}(y_1 - y_0)) = (x, y),$$

Let $h_x = x_1 - x_0$ and $h_y = y_1 - y_0$. Then,

$$B = \begin{pmatrix} h_x & 0 \\ 0 & h_y \end{pmatrix}, \qquad b = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Note that $B$ is diagonal. Therefore, if $v \in V_{h,0}(\mathcal{Q})$, then $v(F(\hat{x})) \in V_{h,0}(\hat{\mathcal{Q}})$.

**Theorem 2.** *Let $w \in V_{h,0}(\mathcal{Q})$ and $q \in W_{h,0}(\mathcal{Q})$ such that $div(w) = q$. Then,*

$$\|w\|_{H^1(\mathcal{Q})} \leq c\|q\|_{L^2(\mathcal{Q})}, \tag{2.5}$$

*where $c$ is $h$-independent. Thus, the local* inf-sup *condition is satisfied:*

$$\sup_{w \in V_{h,0}(\mathcal{Q}) \setminus \{0\}} \frac{\int_{\mathcal{Q}} q\, div(w) dx}{\|q\|_{L^2(\mathcal{Q})} \|w\|_{H^1(\mathcal{Q})}} \geq \alpha, \qquad \forall q \in W_{h,0}(\mathcal{Q}). \tag{2.6}$$

*Proof.* Firstly, we consider the Piola transformation of $v$ given by the following formula:

$$P(\hat{v})(\hat{x}) = v(x) := B\hat{v}(\hat{x}). \tag{2.7}$$

We then have,

$$Dv(x) = \frac{BD\hat{v}(F^{-1}(x))}{det(B)} = \frac{B\hat{D}\hat{v}(\hat{x})DF^{-1}(x)}{det(B)} = \frac{B\hat{D}\hat{v}(\hat{x})B^{-1}}{det(B)}.$$

Since $div(v) = tr(Dv)$ and $tr(B\hat{D}\hat{v}B^{-1}) = tr(\hat{D}\hat{v})$ as the trace is similarity invariant, we get $div(v) = \hat{div}(\hat{v})$.

Then, we define $|||\hat{v}||| = \|\hat{div}(\hat{v})\|_{L^2(\hat{\mathcal{Q}})}$ where $P(\hat{v}) = v$. By the following argument, we can see that $|||\hat{v}|||$ defines a norm on $V_{h,0}(\hat{\mathcal{Q}})$.

1. *(Positivity)* Trivially, $|||\hat{v}||| \geq 0$. Suppose $|||\hat{v}||| = 0$, then $\hat{div}(\hat{v}) = 0$ i.e. $div(v) = 0$. Due to the construction of the finite element spaces, this implies $v = curl(z)$ for some $z \in \Sigma_{h,0}(\mathcal{Q})$. Since $\Sigma_{h,0}(\mathcal{Q}) = \{0\}$, $v = 0$, and therefore, $\hat{v} = 0$. Thus, $|||\hat{v}||| = 0$ iff $\hat{v} = 0$.

2. *(Scalar multiplication)* $|||\widehat{cv}||| = \|\hat{div}(\widehat{cv})\|_{L^2(\hat{\mathcal{Q}})} = c\|\hat{div}(\hat{v})\|_{L^2(\hat{Q})} = c|||\hat{v}|||$.

3. *(Triangle Inequality)* $|||\hat{v} + \hat{w}||| = \|\hat{div}(\widehat{v+w})\|_{L^2(\hat{Q})} \leq \|\hat{div}(\hat{v})\|_{L^2(\hat{Q})} + \|\hat{div}(\hat{w})\|_{L^2(\hat{Q})} = |||\hat{v}||| + |||\hat{w}|||$.

By the equivalence of the norms in finite dimension, there exist a constant $c$ such that

$$\|\hat{w}\|_{H^1(\hat{\mathcal{Q}})} \leq c|||\hat{w}||| = c\|\hat{div}(\hat{w})\|_{L^2(\hat{\mathcal{Q}})}, \tag{2.8}$$

where $\hat{w}$ is the Piola transformation of $w$.

Let $\hat{q}$ be defined by the relation $q(x) = \hat{q}(\hat{x})$ with $x = F(\hat{x})$. It can be shown that $\hat{q} \in W_{h,0}(\hat{\mathcal{Q}})$ by the following argument.

Since $q \in Q_{2,2}(\mathcal{Q})$ and $\hat{q}(\hat{x}) = q(x)$, there holds $\hat{q} \in Q_{2,2}(\hat{\mathcal{Q}})$ as a consequence of $B$'s diagonality. By a change of variables,

$$0 = \int_{\mathcal{Q}} q(x)dx = \int_{\hat{\mathcal{Q}}} \hat{q}(\hat{x})|JF^{-1}|d\hat{x}.$$

Since $F$ is an invertible affine mapping and $|JF^{-1}| > 0$,

$$\int_{\hat{\mathcal{Q}}} \hat{q}(\hat{x})\, d\hat{x} = 0.$$

Moreover, $q(a_i) = 0$ for $\forall i \in \{1,2,3,4\}$ implies $\hat{q}(\hat{a_i}) = 0$ for $\forall i \in \{1,2,3,4\}$, since affine transformations maps vertices to vertices. Hence, $\hat{q} \in W_{h,0}(\hat{\mathcal{Q}})$.

By a scaled Piola transform, we can write

$$\widehat{div}(\hat{w}(\hat{x})) = div(w(x)) = q(x) = \hat{q}(\hat{x}).$$

Therefore, by (2.8),

$$\|\hat{w}\|_{H^1(\hat{Q})} \leq c\|\hat{q}\|_{L^2(\hat{Q})} \tag{2.9}$$

Recall the scaling estimates [9]:

$$|\hat{w}|_{H^m(\hat{\mathcal{Q}})} \leq c\|B\|^m(det(B))^{-1/2}|w|_{H^m(\mathcal{Q})}, \tag{2.10}$$

$$|w|_{H^m(\mathcal{Q})} \leq c\|B^{-1}\|^m(det(B))^{1/2}|\hat{w}|_{H^m(\hat{\mathcal{Q}})}. \tag{2.11}$$

for $w(x)=\hat{w}(\hat{x})$ and $x=F(\hat{x})$.
Since we have $w(x)=B\hat{w}(\hat{x})$,

$$|w|_{H^m(\mathcal{Q})}\leq c\|B^{-1}\|^m(det(B))^{1/2}|\hat{w}|_{H^m(\hat{\mathcal{Q}})}\leq ch^{-m}h|\hat{w}|_{H^m(\hat{\mathcal{Q}})}=ch^{-m+1}|\hat{w}|_{H^m(\hat{\mathcal{Q}})}.$$

Here we have used the following estimates: $\|B\|\leq h$ and $det(B)=\frac{|\mathcal{Q}|}{|\hat{\mathcal{Q}}|}\leq h^2$ [9]. Therefore, by (2.9), (2.10) and (2.11), we have:

$$\begin{aligned}\|w\|^2_{H^1(\mathcal{Q})}=\|w\|^2_{L^2(\mathcal{Q})}+\|\nabla w\|^2_{L^2(\mathcal{Q})} &\leq ch^2\|\hat{w}\|^2_{L^2(\hat{\mathcal{Q}})}+ch^2\|\hat{w}\|^2_{H^1(\hat{\mathcal{Q}})}\\ &\leq ch^2\|\hat{w}\|^2_{H^1(\hat{\mathcal{Q}})}\leq ch^2\|\hat{q}\|^2_{L^2(\hat{\mathcal{Q}})}\\ &\leq c\|q\|^2_{L^2(\mathcal{Q})}.\end{aligned}$$

since $\|\hat{w}\|_{H^1(\hat{\mathcal{Q}})}\leq c\|\hat{q}\|_{L^2(\hat{\mathcal{Q}})}$. Thus, we get

$$\|w\|_{H^1(\mathcal{Q})}\leq c\|q\|_{L^2(\mathcal{Q})}.$$

## 3.0 THE GLOBAL FINITE ELEMENT SPACES

In this chapter, we define the global finite element spaces. By the use of Scott-Zhang interpolant, inverse inequalities, a new norm defined on the global vector space $V_h$ and Nitsche's method, we prove the conformity and stability constraints on the domain $\Omega$ for the global problem with and without boundary conditions. The global finite element spaces without boundary conditions are defined as follows:

- $\Sigma_h$ ={ $z \in H^2(\Omega)$: $z|_{\mathcal{Q}} \in Q_{3,3}$, $\frac{\partial^2 z}{\partial x \partial y}$ is continuous at the vertices,}.
- $V_h$={ $v \in (H^1(\Omega))^2$: $v|_{\mathcal{Q}} \in Q_{3,2} \times Q_{2,3}$, $\frac{\partial v_1}{\partial x}$ and $\frac{\partial v_2}{\partial y}$ are continuous at the vertices }.
- $W_h$= { $q \in L^2(\Omega) : q|_{\mathcal{Q}} \in Q_{2,2}$: $q$ is continuous at the vertices}.

**Lemma 5.** *There holds*

$$Ker(div(V_h)) = curl(\Sigma_h). \tag{3.1}$$

*Proof.* Firstly, we show curl($\Sigma_h$) $\subseteq Ker(div(V_h))$.

Let $v \in curl(\Sigma_h)$. Then there exists $z \in \Sigma_h$ such that $curl(z) = v$. It is easy to see that $v \in V_h$. Since the divergence of the curl operator is zero, we have $div(v) = 0$, i.e., $v \in Ker(div(V_h))$. Therefore, we have $curl(\Sigma_h) \subseteq Ker(div(V_h))$.

Then, we need to show $Ker(div(V_h)) \subseteq curl(\Sigma_h)$.

Let $v \in Ker(div(V_h))$, then $div(v) = 0$. Then, there exists $z \in H_0^2(\Omega)$ such that $v = curl(z)$ [13]. Since $v|_{\mathcal{Q}} \in Q_{2,3} \times Q_{2,3}$, we have $z|_{\mathcal{Q}} \in Q_{3,3}$. Furthermore, since $\frac{\partial v_1}{\partial x}$ and $\frac{\partial v_2}{\partial y}$ are continuous at the vertices. Therefore, $z \in \Sigma_h$. This implies $Ker(div(V_h)) \subseteq curl(\Sigma_h)$.

□

**Remark 1.** *By the rank nullity theorem, we conclude:*

$$dim(div(V_h)) = dim(V_h) - dim(curl(\Sigma_h)).$$

**Theorem 3.** *The Stokes complex given in* (2.2) *is exact.*

*Proof.* By Lemma 5, it suffices to show $div : V_h \to Q_h$ is surjective. Since $div(V_h) \subseteq W_h$, it suffices to show dim($div(V_h)$) =dim($W_h$). By the rank-nullity theorem,

$$
\begin{aligned}
dim(div(V_h)) = dim(V_h) - dim(curl(\Sigma_h)) &= dim(V_h) - (dim(\Sigma_h) - 1) \\
&= (2(2\mathbb{T} + 2\mathbb{V}) + \mathbb{E}) - 4\mathbb{V} + 1 \\
&= 4\mathbb{T} + 4\mathbb{V} + \mathbb{E} - 4\mathbb{V} + 1 \\
&= 4\mathbb{T} + \mathbb{E} + 1.
\end{aligned}
$$

where $\mathbb{T}$, $\mathbb{V}$ and $\mathbb{E}$ denote the number of faces, vertices and edges of the rectangular elements in the mesh, respectively. Thus, $dim(W_h) - dim(div(V_h)) = (\mathbb{V} + 5\mathbb{T}) - 4\mathbb{T} - \mathbb{E} - 1 = \mathbb{V} + \mathbb{T} - \mathbb{E} - 1 = 0$ by Euler's formula, i.e., $div(V_h) = W_h$, and the result follows. □

**Lemma 6.** *For any $q \in W_h$, there exists $v^{(1)} \in V_h$ such that $(q - div(v^{(1)}))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$ for all $\mathcal{Q} \in \mathcal{Q}_h$.*

*Proof.* For $q \in W_h$, there exists $w \in H^1(\Omega)$ such that $div(w) = q$ and $\|w\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}$ [13].
Define $v^{(1)} \in V_h$ such that it satisfies the following conditions:

1. $v^{(1)}(a) = I_h w(a)$ at all vertices $a$ in $\mathcal{Q}_h$, where $I_h w$ is the Scott-Zhang interpolant of $w$ [17].
2. $\frac{\partial v_1^{(1)}(a)}{\partial x} = \frac{\partial v_2^{(1)}(a)}{\partial y} = \frac{q(a)}{2}$ at all vertices $a \in \mathcal{Q}_h$.
3. $\int_{\mathcal{Q}} v^{(1)} \cdot s \, dx = \int_{\mathcal{Q}} w \cdot s \, dx$ for $s \in Q_{1,0} \times Q_{0,1}$ and $\mathcal{Q} \in \mathcal{Q}_h$.
4. $\int_{L_3,L_4} v_2^{(1)} \, ds = \int_{L_3,L_4} w_2 \, ds$, $\int_{L_1,L_2} v_1^{(1)} \, ds = \int_{L_1,L_2} w_1 \, ds$.

Note that (4) yields $\int_{\partial\mathcal{Q}} v^{(1)} \, ds = \int_{\partial\mathcal{Q}} w \, ds$ for all $\mathcal{Q} \in \mathcal{Q}_h$. Also, from (1) we get $div(v^{(1)}(a)) = q(a)$ at all vertices $a$, i.e., $(q - div(v^{(1)}))(a) = 0$ at all vertices $a$.
We need to show that $\int_{\mathcal{Q}} (q - div(v^{(1)}))dx = 0$ for all $\mathcal{Q} \in \mathcal{Q}_h$ to complete the proof, since $(q - div(v^{(1)}))|_{\mathcal{Q}} \in Q_{2,2}$.
Applying (3) and (4) and the divergence theorem twice yields:

$$
\begin{aligned}
\int_{\mathcal{Q}} (q - div(v^{(1)})) \, dx &= \int_{\mathcal{Q}} q \, dx - \int_{\mathcal{Q}} div(v^{(1)}) \, dx \\
&= \int_{\mathcal{Q}} q \, dx - \int_{\partial\mathcal{Q}} v^{(1)} \cdot n \, dx \\
&= \int_{\mathcal{Q}} q \, dx - (\int_{L_3,L_4} v^{(1)} \cdot n \, dx + \int_{L_1,L_2} v^{(1)} \cdot n \, dx) \\
&= \int_{\mathcal{Q}} q \, dx - (\int_{L_4} v^{(1)} \, d(-x) + \int_{L_3} v^{(1)} \, dx + \int_{L_1} v^{(1)} \, d(-y) + \int_{L_2} v^{(1)} \, dy) \\
&= \int_{\mathcal{Q}} q \, dx - (\int_{L_4} w \, d(-x) + \int_{L_3} w \, dx + \int_{L_1} w \, d(-y) + \int_{L_2} w \, dy) \\
&= \int_{\mathcal{Q}} q \, dx - \int_{\partial Q} w \cdot n \, ds \\
&= \int_{\mathcal{Q}} (q - div(w)) \, dx = 0.
\end{aligned}
$$

Thus, $\int_{\mathcal{Q}} (q - div(v^{(1)})) \, dx = 0$, and therefore $(q - div(v^{(1)}))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$ .

□

By Lemma 6, for any $q \in W_h$, there exists $v^{(1)} \in V_h$ satisfying $(q - div(v^{(1)}))|_{\mathcal{Q}} \in W_{h,0}(\mathcal{Q})$. On the other hand, by Theorem 2, for each $\mathcal{Q} \in \mathcal{Q}_h$, there exists $v^{(2)}_{\mathcal{Q}} \in V_{h,0}(\mathcal{Q})$ such that $div(v^{(2)}_{\mathcal{Q}}) = (q - div(v^{(1)})|_{\mathcal{Q}}$ and

$$\|v^{(2)}_{\mathcal{Q}}\|_{H^1(\mathcal{Q})} \leq c\|q - div(v^{(1)})\|_{L^2(\mathcal{Q})} \leq c(\|q\|_{L^2(\mathcal{Q})} + \|v^{(1)}\|_{H^1(\mathcal{Q})}). \tag{3.2}$$

**Lemma 7.** *Let $v^{(2)}_{\mathcal{Q}}$ be defined as above and $v^{(2)}$ be such that $v^{(2)}|_Q := v^{(2)}_{\mathcal{Q}} \in V_{h,0}(\mathcal{Q})$. Then, $v^{(2)} \in V_h$ and $\|v^{(2)}\|_{H^1(\Omega)} \leq c(\|q\|_{L^2(\Omega)} + \|v^{(1)}\|_{H^1(\Omega)})$.*

*Proof.* If $v^{(2)}_{\mathcal{Q}} \in V_{h,0}(\mathcal{Q})$, then $v^{(2)}_{\mathcal{Q}} \in (H^1_0(\mathcal{Q}))^2 \cap (Q_{3,2}(\mathcal{Q}) \times Q_{2,3}(\mathcal{Q}))$ by definition. Since $\nabla v^{(2)}_{\mathcal{Q}}(a_i) = 0$, where $\mathcal{Q}$ is an arbitrary rectangle with vertices $a_i, \nabla v^{(2)}(a_i) = 0$. Therefore, $\nabla v^{(2)}$ is continuous at the vertices. Consequently, $v^{(2)} \in V_h$. Additionally, (3.2) yields $\|v^{(2)}\|_{H^1(\Omega)} \leq c(\|q\|_{L^2(\Omega)} + \|v^{(1)}\|_{H^1(\Omega)})$.

□

**Theorem 4.** *For any $q \in W_h$, there exists $v \in V_h$ such that $div(v) = q$ and*

$$\|v\|_{H^1(\Omega)} \leq c\|q\|_{L^2(\Omega)}.$$

*Proof.* For $q \in W_h$, let $v^{(1)}$ and $v^{(2)}$ be given by Lemma 6 and Lemma 7 respectively and let $v := v^{(1)} + v^{(2)}$. Then, we have,

$$\|v\|_{H^1(\mathcal{Q})} \leq c(\|v^{(1)}\|_{H^1(\mathcal{Q})} + \|v^{(2)}\|_{H^1(\mathcal{Q})}) \leq c(\|v^{(1)}\|_{H^1(\mathcal{Q})} + \|q\|_{L^2(\mathcal{Q})}) \tag{3.3}$$

By scaling,

$$\|v^{(1)}\|_{H^1(\mathcal{Q})} \leq c\|\hat{v}^{(1)}\|_{H^1(\hat{\mathcal{Q}})} \tag{3.4}$$

where $\hat{v}^{(1)}(\hat{x}) = v^{(1)}(x)$, $x = F(\hat{x})$ and $F$ is the affine transformation defined in Section 2.4.

Consider the norm:

$$|||\hat{v}||| := \Big(\sum_{j=1}^{4}\Big(|\hat{v}(\hat{a_j})|^2 + \sum_{i=1}^{2}\Big|\frac{\hat{\partial}\hat{v}_i(\hat{a}_j)}{\partial \hat{x}_i}\Big|^2\Big) + \sum_{i=1}^{2}\Big|\int_{x_i=const} \hat{v}_i \, d\hat{s}\Big|^2 + \sup_D\Big|\int_{\hat{\mathcal{Q}}} \hat{v}\, \hat{r}\, d\hat{s}\Big|^2\Big)^{1/2},$$

where $D = \{\hat{r} \in Q_{1,0} \times Q_{0,1} : \|\hat{r}\|_{L^2(\hat{\mathcal{Q}})} = 1\}$ and $x_1 = x$, $x_2 = y$. By Lemma 1, $|||\hat{v}|||$ is a norm on $Q_{3,2} \times Q_{2.3}$. Let $I_h w : (H^1(\Omega))^2 \to V_h$ be the Scott-Zhang interpolant of $w$ [17]. By the triangle inequality and equivalence of norms,

$$\|\hat{v}^{(1)}\|_{H^1(\hat{\mathcal{Q}})} \leq \|\hat{v}^{(1)} - \widehat{I_h w}\|_{H^1(\hat{\mathcal{Q}})} + \|\widehat{I_h w}\|_{H^1(\hat{\mathcal{Q}})} \leq c|||\hat{v}^{(1)} - \widehat{I_h w}||| + \|\widehat{I_h w}\|_{H^1(\hat{\mathcal{Q}})}. \tag{3.5}$$

Estimating the first term in the right-hand side expression,

$$|||\hat{v}^{(1)} - \widehat{I_h w}|||^2 = \sum_{j=1}^{4}\Big(|(\hat{v}^{(1)} - \widehat{I_h w})(\hat{a_j})|^2 + \sum_{i=1}^{2}\Big|\frac{\hat{\partial}(\hat{v}^{(1)} - \widehat{I_h w})^{(i)}(\hat{a}_j)}{\partial \hat{x}_i}\Big|^2\Big) + \sum_{i=1}^{2}\Big|\int_{\hat{x}_i=const} (\hat{v}^{(1)} - \widehat{I_h w})^{(i)} d\hat{s}\Big|^2$$

$$+ \sup_D\Big|\int_{\hat{\mathcal{Q}}} (\hat{v}^{(1)} - \widehat{I_h w})\hat{r} d\hat{s}\Big|^2. \tag{3.6}$$

Since $D$ is a finite dimensional space, it is closed. Therefore, $\hat{r}$ attains a supremum over $D$.
Let $\hat{r}^* = \sup_{\hat{\mathcal{Q}}} \hat{r}(\hat{x})$. Then,

$$\sup_D \Big| \int_{\hat{\mathcal{Q}}} (\hat{v}^{(1)} - \widehat{I_h w}) \, \hat{r} \, d\hat{s} \Big|^2 = \Big| \int_{\hat{\mathcal{Q}}} (\hat{v}^{(1)} - \widehat{I_h w}) \, \hat{r}^* \, d\hat{s} \Big|^2.$$

By using a change of variables, the inequality $|\mathcal{Q}| \leq h^2$, the Cauchy-Schwarz inequality and the property 3 of $v^{(1)}$ stated in Lemma 6,

$$\begin{aligned}
\Big| \int_{\hat{\mathcal{Q}}} (\hat{v}^{(1)} - \widehat{I_h w}) \, \hat{r}^* \, d\hat{s} \Big|^2 &\leq \frac{c}{|\mathcal{Q}|^2} \Big| \int_{\mathcal{Q}} (v^{(1)} - I_h w) \, r^* \, ds \Big|^2 \leq c h^{-4} \Big| \int_{\mathcal{Q}} (v^{(1)} - I_h w) \, r^* \, ds \Big|^2 \\
&\leq c h^{-4} \Big| \int_{\mathcal{Q}} v^{(1)} \, r^* \, ds - \int_{\mathcal{Q}} I_h w \, r^* \, ds \Big|^2 = c h^{-4} \Big| \int_{\mathcal{Q}} w \, r^* \, ds - \int_{\mathcal{Q}} I_h w \, r^* \, ds \Big|^2 \\
&\leq c h^{-4} \Big| \int_{\mathcal{Q}} (w - I_h w) \, r^* \, ds \Big|^2 \\
&\leq c h^{-4} \| w - I_h w \|^2_{L^2(\mathcal{Q})} \| r^* \|^2_{L^2(\mathcal{Q})}.
\end{aligned}$$

By definition, $\|\hat{r}^*\|_{L^2(\hat{\mathcal{Q}})} = 1$ and a change of variables yields:

$$\| r^* \|^2_{L^2(\mathcal{Q})} = \int_{F(\hat{\mathcal{Q}})} |r^*(x)|^2 dx = 2|\mathcal{Q}| \int_{\hat{\mathcal{Q}}} |\hat{r}^*|^2 (\hat{x}) d\hat{x} = 2|\mathcal{Q}|.$$

Thus,

$$\sup_D \Big| \int_{\hat{\mathcal{Q}}} (\hat{v}^{(1)} - \widehat{I_h w}) \, \hat{r} \, d\hat{s} \Big|^2 \leq c h^{-4} \| w - I_h w \|^2_{L^2(\mathcal{Q})} |\mathcal{Q}| \leq c h^{-2} \| w - I_h w \|^2_{L^2(\mathcal{Q})}. \tag{3.7}$$

Now consider the second part of the first term of (3.6), by a change of variables, equivalence of norms and the property 2 of $v^{(1)}$ stated in Lemma 6,

$$\begin{aligned}
\sum_{j=1}^{4} \Big( |(\hat{v}^{(1)} - \widehat{I_h w})(\hat{a}_j)|^2 + \sum_{i=1}^{2} \Big| \frac{\hat{\partial} (\hat{v}^{(1)} - \widehat{I_h w})_i}{\hat{\partial} \hat{x}_i} (\hat{a}_j) \Big|^2 \Big) &\leq \sum_{j=1}^{4} \sum_{i=1}^{2} \Big| \frac{\hat{\partial} (\hat{v}^{(1)} - \widehat{I_h w})_i}{\hat{\partial} \hat{x}_i} (\hat{a}_j) \Big|^2 \\
&\leq \sum_{j=1}^{4} \sum_{i=1}^{2} \Big( \Big| \frac{\hat{\partial} \hat{v}_i^{(1)}}{\hat{\partial} \hat{x}_i} (\hat{a}_j) \Big|^2 + \Big| \frac{\hat{\partial} \widehat{I_h w}}{\hat{\partial} \hat{x}_i} (\hat{a}_j) \Big|^2 \Big) \\
&\leq \sum_{j=1}^{4} \sum_{i=1}^{2} \Big| \frac{\hat{\partial} \hat{v}_i^{(1)}}{\hat{\partial} \hat{x}_i} (\hat{a}_j) \Big|^2 + c \, \| \widehat{I_h w} \|^2_{H^1(\hat{\mathcal{Q}})} \\
&\leq \sum_{j=1}^{4} c \, h^2 |\hat{q}(\hat{a}_j)|^2 + c \| I_h w \|^2_{H^1(\mathcal{Q})} \\
&\leq c ( h^2 \| \hat{q} \|^2_{L^2(\hat{\mathcal{Q}})} + \| I_h w \|^2_{H^1(\mathcal{Q})} ) \\
&\leq c ( \| q \|^2_{L^2(\mathcal{Q})} + \| I_h w \|^2_{H^1(\mathcal{Q})} )
\end{aligned} \tag{3.8}$$

Now we will evaluate the line integrals that appear as the third term in (3.6) on each of the lines in the boundary, $\partial \mathcal{Q}$, separately. Consider the lines where $x$ is constant, namely, $L_3$ and $L_4$. Since $B$ is diagonal, $F$ preserves paralelism, $\hat{x}_1 = \hat{x}$ is constant implies $x_1 = x$ is constant. By the Cauchy-Schwarz inequality, a change of variables and the property 4 of $v^{(1)}$ stated in Lemma 6,

$$
\begin{aligned}
|\int_{\hat{x}_i=const}(\hat{v}^{(1)}-\widehat{I_h w})_i d\hat{s}|^2 &\le |h^{-1}\int_{x_i=const}(v^{(1)}-I_h w)_i ds|^2 \\
&= |h^{-1}(\int_{x_i=const} v_i^{(1)}\, ds - \int_{x_i=const}(I_h w)_i\, ds)|^2 \\
&= |h^{-1}(\int_{x_i=const} w_i\, ds - \int_{x_i=const}(I_h w)_i\, ds)|^2 \\
&\le |h^{-1}\int_{x_i=const}(w-I_h w)_i\, ds|^2 \\
&\le h^{-2}|\int_{x_i=const} 1^2 ds|\cdot \int_{x_i=const}|(w-I_h w)_i|^2 ds \\
&\le ch^{-2}\cdot h\int_{x_i=const}|(w-I_h w)_i|^2\, ds \\
&= ch^{-1}\int_{x_i=const}|(w-I_h w)_i|^2\, ds.
\end{aligned}
$$

Thus,

$$
\sum_{i=1}^{2}|\int_{\hat{x}_i=const}(\hat{v}^{(1)}-\widehat{I_h w})_i d\hat{s}|^2 \le ch^{-1}\|w-I_h w\|^2_{L^2(\partial\mathcal{Q})}. \tag{3.9}
$$

Note that by the trace theorem,

$$
\begin{aligned}
ch^{-1}\|w-I_h w\|^2_{L^2(\partial\mathcal{Q})} &\le ch^{-1}(h^{-1}\|w-I_h w\|^2_{L^2(\mathcal{Q})}+h\|w-I_h w\|^2_{H^1(\mathcal{Q})}) \\
&= ch^{-2}\|w-I_h w\|^2_{L^2(\mathcal{Q})}+\|w-I_h w\|^2_{H^1(\mathcal{Q})}.
\end{aligned}
$$

As a result, (3.9) gives:

$$
\sum_{i=1}^{2}|\int_{\hat{x}_i=const}(\hat{v}^{(1)}-\widehat{I_h w})_i d\hat{s}|^2 \le ch^{-2}\|w-I_h w\|^2_{L^2(\mathcal{Q})}+\|w-I_h w\|^2_{H^1(\mathcal{Q})}. \tag{3.10}
$$

Hence, combining (3.4) – (3.10),

$$
\|v^{(1)}\|^2_{H^1(\mathcal{Q})} \le c(\|q\|^2_{L^2(\mathcal{Q})}+\|I_h w\|^2_{H^1(\mathcal{Q})}+h^{-2}\|w-I_h w\|^2_{L^2(\mathcal{Q})}+\|w-I_h w\|^2_{H^1(\mathcal{Q})}). \tag{3.11}
$$

Since $\sum_{\mathcal{Q}\in\mathcal{Q}_h} h^{-2}\|w-I_h w\|^2_{L^2(\mathcal{Q})} \le \|w\|^2_{H^1(\Omega)}$ and $\sum_{\mathcal{Q}\in\mathcal{Q}_h}\|I_h w\|^2_{H^1(\mathcal{Q})} \le \|w\|^2_{H^1(\Omega)}$ [17], by scaling and summing over the rectangles $\mathcal{Q}$ in $\mathcal{Q}_h$, from (3.11), we get:

$$
\sum_{\mathcal{Q}\in\mathcal{Q}_h}\|v^{(1)}\|^2_{H^1(\mathcal{Q})} \le c\|w\|^2_{H^1(\Omega)} \le c\|q\|^2_{L^2(\Omega)}.
$$

Consequently, by $(3,3)$ we have

$$
\|v\|_{H^1(\Omega)} \le c\|q\|_{L^2(\Omega)}.
$$

□

## 3.1 THE FINITE ELEMENT SPACES WITH HOMOGENEOUS BOUNDARY CONDITIONS

In this chapter, we impose homogeneous boundary conditions on the finite element spaces we have defined in Chapter 2.

Consider the following finite element spaces as candidates:

- $\Sigma_{h,0} = \Sigma_h \cap H_0^2(\Omega)$.
- $V_{h,0} = V_h \cap (H_0^1(\Omega))^2$.
- $W_{h,0} = W_h \cap L_0^2(\Omega) = \{q \in W_h;\, \int_\Omega q\, dx = 0\}$.

Note that $z \in \Sigma_h$ is in $\Sigma_{h,0}$ if and only if:

1. $z(a) = 0$, $\forall a \in \mathbb{V}_b$.
2. $\nabla z(a) = 0$, $\forall a \in \mathbb{V}_b$.
3. $\frac{\partial z}{\partial y \partial x}(a) = 0$, $\forall a \in \mathbb{V}_b$.

where $\mathbb{V}_b$ denotes the boundary vertices. The number of constraints imposed on $\Sigma_{h,0}$ is $4V_b$. Thus, we have,

$$dim(\Sigma_{h,0}) = dim(\Sigma_h) - 4|\mathbb{V}_b| = 4|\mathbb{V}| - 4|\mathbb{V}_b|.$$

Also, $v \in V_{h,0}$ if and only if:

1. $v(a) = 0$, $\forall a \in \mathbb{V}_b$,
2. $\frac{\partial v_1}{\partial x}(a) = 0$, $\forall e \in \mathbb{V}_{b,y} \cup \mathbb{V}_c$,
3. $\frac{\partial v_2}{\partial y}(a) = 0$, $\forall e \in \mathbb{V}_{b,x} \cup \mathbb{V}_c$,
4. $\int_e v_1 ds = 0$, $\forall e \in \mathbb{E}_x^b$,
5. $\int_e v_2 ds = 0$, $\forall e \in \mathbb{E}_y^b$,

where $\mathbb{V}_c$ denotes the corner vertices,$\mathbb{V}_{b,x}$ and $\mathbb{V}_{b,y}$ denote the vertices among $\mathbb{V}_b \setminus \mathbb{V}_c$, where $x$ is constant and where $y$ is constant, respectively, and $\mathbb{E}_x^b$ are the edges where $x$ is constant and $\mathbb{E}_y^b$ are the edges where $y$ is constant. Therefore, we can write $\mathbb{V}_b = \mathbb{V}_{b,x} \cup \mathbb{V}_{b,y} \cup \mathbb{V}_c$ and $\mathbb{E}^b = \mathbb{E}_x^b \cup \mathbb{E}_y^b$.

The number of constraints imposed on $V_{h,0}$ is $3|\mathbb{V}_b| + |\mathbb{V}_c| + |\mathbb{E}^b|$ , we then have:

$$dim(V_{h,0}) = dim(V_h) - 3|\mathbb{V}_b| - |\mathbb{V}_c| - |\mathbb{E}^b| = (4|\mathbb{T}| + 4|\mathbb{V}| + |\mathbb{E})|) - 3|\mathbb{V}_b| - |\mathbb{V}_c| - |\mathbb{E}^b|.$$

On the other hand.

$$dim(W_{h,0}) = dim(W_h) - 1 = (4|\mathbb{T}| + |\mathbb{E}| + 1) - 1 = 4|\mathbb{T}| + |\mathbb{E}|.$$

**Lemma 8.** *There holds*

$$Ker(div(V_{h,0})) = curl(\Sigma_{h,0}).$$

*Proof.* Firstly, we show that

$$curl(\Sigma_{h,0}) \subseteq Ker(div(V_{h,0})).$$

Let $v \in curl(\Sigma_{h,0})$. Then there exists $z \in \Sigma_{h,0}$ such that $curl(z) = v$. Since the divergence of the curl operator is zero, we have $div(v) = 0$, i.e., $v \in Ker(div(V_{h,0}))$. Therefore, we have $curl(\Sigma_{h,0}) \subseteq Ker(div(V_{h,0}))$.
Then we show

$$Ker(div(V_{h,0})) \subseteq curl(\Sigma_{h,0}).$$

Let $v \in Ker(div(V_{h,0}))$ so that $div(v) = 0$. Then, there exists $z \in H_0^2(\Omega)$ such that $v = curl(z)$. Since $v|_{\mathcal{Q}} \in Q_{2,3} \times Q_{2,3}$, we have $z|_{\mathcal{Q}} \in Q_{3,3}$. Furthermore, since $\frac{\partial v_1}{\partial x}$ and $\frac{\partial v_2}{\partial y}$ are continuous at the vertices, $\frac{\partial^2 z}{\partial x \partial y}$ is continuous at the vertices. Therefore, $z \in \Sigma_{h,0}$, and this implies $Ker(div(V_{h,0})) \subseteq curl(\Sigma_{h,0})$. Hence, $curl(\Sigma_{h,0}) = Ker(div(V_{h,0}))$. ☐

By Lemma 8 and the rank-nullity theorem,

$$\begin{aligned} dim(W_{h,0}) - dim(div(V_{h,0})) &= dim(W_{h,0}) - (dim(V_{h,0}) - dim(Ker(div(V_{h,0})))) \\ &= dim(W_{h,0}) - (dim(V_{h,0}) - dim(curl(\Sigma_{h,0}))) \\ &= dim(W_{h,0}) - (dim(V_{h,0}) - dim(\Sigma_{h,0})). \end{aligned} \tag{3.12}$$

From the above calculations, it is easy to see that:

$$\begin{aligned} dim(\Sigma_{h,0}) + dim(W_{h,0}) - dim(V_{h,0}) &= (4|\mathbb{V}| - 4|\mathbb{V}_b|) + (4|\mathbb{T}| + |\mathbb{E}|) - 4|\mathbb{T}| - 4|\mathbb{V}| - |\mathbb{E}| + 3|\mathbb{V}_b| + |\mathbb{V}_c| + |\mathbb{E}^b| \\ &= -|\mathbb{V}_b| + |\mathbb{V}_c| + |\mathbb{E}^b| = |\mathbb{V}_c| > 0 \end{aligned}$$

since $|\mathbb{E}^b| = |\mathbb{V}_b|$. As a result, $(\Sigma_{h,0}, V_{h,0}, W_{h,0})$ does not form an exact sequence.
Note that $dim(W_{h,0}) > dim(div(V_{h,0}))$, i.e., the pressure space is larger than desired.
Now, consider the following candidate finite element spaces:

- $\Sigma_{h,0} = \Sigma_h \cap H_0^1(\Omega)$.
- $V_{h,0} = \{v_h \in V_h;\ (v_h \cdot n)|_{\partial\Omega} = 0\}$.
- $W_{h,0} = W_h \cap L_0^2(\Omega) = \{q \in W_h;\ \int_\Omega q\, dx = 0\}$.

Note that $z \in \Sigma_h$ is in $\Sigma_{h,0}$ if and only if:

1. $z(a) = 0$, $\forall a \in \mathbb{V}_b$,
2. $\frac{\partial z}{\partial x}(a) = 0$, $\forall a \in \mathbb{V}_{b,y} \cup \mathbb{V}_c$,
3. $\frac{\partial z}{\partial y}(a) = 0$, $\forall a \in \mathbb{V}_{b,x} \cup \mathbb{V}_c$.

The number of constraints imposed on $\Sigma_{h,0}$ is: $|\mathbb{V}_b| + |\mathbb{V}_{b,y}| + |\mathbb{V}_c| + |\mathbb{V}_{b,x}| + |\mathbb{V}_c| = 2|\mathbb{V}_b| + |\mathbb{V}_c|$. As a result,

$$dim(\Sigma_{h,0}) = dim(\Sigma_h) - 2|\mathbb{V}_b| - |\mathbb{V}_c| = 4|\mathbb{V}| - 2|\mathbb{V}_b| - |\mathbb{V}_c|.$$

Next, we note that $v_h \in V_{h,0}$ if and only if:

1. $v_1(a) = 0$, $\forall a \in \mathbb{V}_{b,x} \cup \mathbb{V}_c$,
2. $\int_e v_1 ds = 0$, $\forall e \in \mathbb{E}^b_x$,
3. $v_2(a) = 0$, $\forall a \in \mathbb{V}_{b,y} \cup \mathbb{V}_c$,
4. $\int_e v_2 ds = 0$, $\forall e \in \mathbb{E}^b_y$.

The number of constraints imposed on $V_{h,0}$ is: $|\mathbb{V}_{b,x}|+|\mathbb{V}_{b,y}|+2|\mathbb{V}_c|+|\mathbb{E}^b|= |\mathbb{V}_b|+|\mathbb{V}_c|+|\mathbb{E}^b|$. Thus,

$$dim(V_{h,0}) = dim(V_h) - |\mathbb{V}_b|-|\mathbb{V}_c|-|\mathbb{E}^b|= 4|\mathbb{T}|+4|\mathbb{V}|+|\mathbb{E}|-|\mathbb{V}_b|-|\mathbb{V}_c|-|\mathbb{E}^b|.$$

where $\mathbb{E}$ denotes the number of edges.

**Lemma 9.** *There holds*

$$Ker(div(V_{h,0})) = curl(\Sigma_{h,0}). \tag{3.13}$$

*Proof.* Firstly, we show that

$$curl(\Sigma_{h,0}) \subseteq Ker(div(V_{h,0})).$$

Let $v \in curl(\Sigma_{h,0})$. Then there exists $z \in \Sigma_{h,0}$ such that $curl(z) = v$. Since the divergence of the curl operator is zero, we have $div(v) = 0$, i.e., $v \in Ker(div(V_{h,0}))$. Therefore, we have $curl(\Sigma_{h,0}) \subseteq Ker(div(V_{h,0}))$. Then we show

$$Ker(div(V_{h,0})) \subseteq curl(\Sigma_{h,0}).$$

Let $v \in Ker(div(V_{h,0}))$ so that $div(v) = 0$. Then, there exists $z \in H^1_0(\Omega)$ such that $v = curl(z)$. Since $v|_{\mathcal{Q}} \in Q_{2,3} \times Q_{2,3}$, we have $z|_{\mathcal{Q}} \in Q_{3,3}$. Furthermore, since $\frac{\partial v_1}{\partial x}$ and $\frac{\partial v_2}{\partial y}$ are continuous at the vertices, $\frac{\partial^2 z}{\partial x \partial y}$ is continuous at the vertices. Therefore, $z \in \Sigma_{h,0}$, and this implies $Ker(div(V_{h,0})) \subseteq curl(\Sigma_{h,0})$. Hence, $curl(\Sigma_{h,0}) = Ker(div(V_{h,0}))$. □

As a result of Lemma 9, we have,

$$\begin{aligned} dim(div(V_{h,0})) &= dim(V_{h,0}) - dim(\Sigma_{h,0}) \\ &= (4|\mathbb{T}|+4|\mathbb{V}|-|\mathbb{V}_b|-|\mathbb{V}_c|-|\mathbb{E}^b_h|+|\mathbb{E}|) - (4|\mathbb{V}|-2|\mathbb{V}_b|-|\mathbb{V}_c|) \\ &= 4|\mathbb{T}|+|\mathbb{V}_b|-|\mathbb{E}^b_h|+|\mathbb{E}|. \end{aligned}$$

It is easy to see that $|\mathbb{V}_b|= |\mathbb{E}^b|$. Thus, $dim(div(V_{h,0})) = 4|\mathbb{T}|+|\mathbb{E}|$.
Also, by Euler's formula

$$\begin{aligned} dim(W_{h,0}) &= dim(W_h) - 1 \\ &= |\mathbb{V}|+5|\mathbb{T}|-1 \\ &= |\mathbb{V}|+5|\mathbb{T}|-(|\mathbb{V}|+|\mathbb{T}|-|\mathbb{E}|) \\ &= 4|\mathbb{T}|+|\mathbb{E}|. \end{aligned}$$

**Theorem 5.** *$div : V_{h,0} \to W_{h,0}$ is a surjective map.*

*Proof.* Note that $div(V_{h,0}) = \{r : r \in Q_{2,2} \cap L_0^2,\ \exists v \in V_{h,0},\ div(v) = r\}$. Let $r \in div(V_{h,0})$. Then $\exists v \in V_{h,0}$ such that $div(v) = r$. Then, by the divergence theorem,

$$\int_\Omega r\,dx = \int_\Omega div(v)\,dx = \int_{\partial\Omega} v \cdot n\,ds = 0.$$

since $v \in V_{h,0}$. Moreover, $v = 0$ on $\partial\Omega$, i.e. $r = 0$ on $\partial\Omega$ and $r$ is continuous at the vertices since $v \in V_h$. Therefore, $r \in W_{h,0}$ and as a result, $div(V_{h,0}) \subseteq W_{h,0}$. From the dimension argument, it follows that $div(V_{h,0}) = W_{h,0}$. □

The dimension arguments above give:

$$dim(\Sigma_{h,0}) + dim(W_{h,0}) = dim(V_{h,0}).$$

By (3.13) and Theorem 5, it is easy to see that the complex $(\Sigma_{h,0}, V_{h,0}, W_{h,0})$ is exact.
Following the same method used in the proof of Theorem 4, we see that for $\forall q \in W_{h,0}$, there exists $v \in V_{h,0}$ such that $div(v) = q$ and $\|v\|_h \leq c\|q\|_{L^2(\Omega)}$, where $\|\cdot\|_h$ is the $H^1$ type norm defined in the next section.

**Lemma 10.** *There holds*

$$\|v\|^2_{H^1(\mathcal{Q})} + \sum_{e\in\partial\mathcal{Q}} \frac{1}{h_e}\|v\|^2_{L^2(e)} \leq c(\|\hat{v}\|^2_{H^1(\hat{\mathcal{Q}})} + \sum_{\hat{e}\in\partial\hat{\mathcal{Q}}} \frac{1}{h_e}\|\hat{v}\|^2_{L^2(\hat{e})}).$$

*for $\forall v \in V_{h,0}$. As a result, the newly defined finite element spaces satisfy the inf-sup stability condition for the problem with the homogeneous boundary conditions.*

$$\inf_{q\in W_{h,0}} \sup_{v\in V_{h,0}\setminus\{0\}} \frac{B(v,q)}{\|v\|_h\|q\|_{L^2(\Omega)}} \geq c > 0$$

*Proof.* By scaling, we have

$$\|v\|_{H^1(\mathcal{Q})} \leq c\|\hat{v}\|_{H^1(\hat{\mathcal{Q}})}.$$

By a change of variables, it's easy to show that for $e \in \partial\mathcal{Q}$,

$$\int_e |v|^2 ds = h_e \int_{\hat{e}} |\hat{v}| ds.$$

Therefore,

$$\sum_{e\in\partial\mathcal{Q}} \frac{1}{h_e}\|v\|^2_{L^2(e)} = \sum_{\hat{e}\in\partial\hat{\mathcal{Q}}} \frac{1}{h_e}\|v\|^2_{L^2(\hat{e})}.$$

□

Let $v$, $v^{(1)}$ and $v^{(2)}$ be defined as in the proof of Theorem 4 and $v(x) = \hat{v}(\hat{x})$, where $x = F(\hat{x})$.

$$\|v^{(1)} - I_h w\|^2_{H^1(\mathcal{Q})} + \sum_{e\in(\partial\mathcal{Q}\cap\partial\Omega)} \frac{1}{h_e}\|v^{(1)} - I_h w\|^2_{L^2(e)} \leq \|v^{(1)} - I_h w\|^2_{H^1(\mathcal{Q})} + \sum_{e\in\partial\mathcal{Q}} \frac{1}{h_e}\|v^{(1)} - I_h w\|^2_{L^2(e)} \leq |||\hat{v}^{(1)} - \widehat{I_h w}|||$$

$$\leq c(\|q\|^2_{L^2(\mathcal{Q})} + \|I_h w\|^2_{H^1(\mathcal{Q})} + h^{-2}\|w - I_h w\|_{L^2(\mathcal{Q})} + \|w\|_{H^1(\mathcal{Q})}).$$

Since $I_h w = 0$ on $\partial\Omega$, the triangle inequality yields

$$\|v^{(1)}\|^2_{H^1(\mathcal{Q})} + \sum_{e\in(\partial\mathcal{Q}\cap\partial\Omega)} \frac{1}{h_e}\|v^{(1)}\|^2_{L^2(e)} \leq c(\|q\|^2_{L^2(\mathcal{Q})} + \|I_h w\|^2_{H^1(\mathcal{Q})} + h^{-2}\|w - I_h w\|_{L^2(\mathcal{Q})} + \|w\|_{H^1(\mathcal{Q})}).$$

Define the following $H^1$-type norm,

$$\|v\|^2_h := \|\nabla v\|^2_{L^2(\Omega)} + \sum_{e\in\mathbb{E}^b} h_e \|\frac{\partial v}{\partial n_e}\|^2_{L^2(e)} + \sum_{e\in\mathbb{E}^b} \frac{1}{h_e}\|v\|^2_{L^2(e)}. \tag{3.14}$$

Note that

$$\|v^{(1)}\|^2_h \leq c(\|v^{(1)}\|^2_{H^1(\mathcal{Q})} + \sum_{e\in(\partial\mathcal{Q}\cap\partial\Omega)} \frac{1}{h_e}\|v^{(1)}\|^2_{L^2(e)})$$

Therefore,

$$\|v^{(1)}\|^2_h \leq c(\|q\|^2_{L^2(\mathcal{Q})} + \|I_h w\|^2_{H^1(\mathcal{Q})} + h^{-2}\|w - I_h w\|_{L^2(\mathcal{Q})} + \|w\|_{H^1(\mathcal{Q})}) \leq c\|q\|_{L^2(\Omega)}.$$

From Lemma 7, we know that $\|v^{(2)}\|_{H^1(\Omega)} \leq c(\|q\|_{L^2(\Omega)} + \|v^{(1)}\|_{H^1(\Omega)})$, by the equivalence of the norms, on $V_{h,0}$,

$$\|v\|_h \leq c\|q\|_{L^2(\Omega)}.$$

As a result,

$$\inf_{q\in W_{h,0}} \sup_{v\in V_{h,0}\setminus\{0\}} \frac{B(v,q)}{\|v\|_h \|q\|_{L^2(\Omega)}} \geq c > 0$$

## 3.2 NITSCHE'S METHOD

In this section, we apply Nitsche's method to the two-dimensional Stokes problem and show that there exists a unique solution. In this aspect, we define two bilinear forms $A$ and $B$ and restate the problem. Then, we verify that coercivity and continuity constraints are satisfied and therefore the existence of a unique solution is guaranteed.

Let $\Omega$, $u$, $f$, $p$ and $\gamma$ be defned as in section 2, i.e.,

$$\begin{aligned} -\gamma\Delta u + \nabla p = f \qquad & in \quad \Omega, \\ div(u) = 0 \qquad & in \quad \Omega, \\ u = 0 \qquad & on \quad \partial\Omega. \end{aligned} \tag{3.15}$$

Multiplying (3.12) by $v_h$, integrating over $\Omega$ and applying an integration by parts formula gives us:

$$\int_\Omega \nabla u : \nabla v_h dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v_h ds - \int_\Omega p\ div(v_h) dx + \int_\Omega p\ v_h \cdot n ds = \int_\Omega f \cdot v_h dx, \quad \forall v_h \in V_{h,0}, \tag{3.16}$$

where ":" denotes the Frobenius inner product. Let $n$ and $t$ denote the outward unit normal and the unit tangent vectors, respectively. Since $v_h \in V_{h,0}$, $(v_h \cdot n)|_{\partial\Omega} = 0$, and therefore $v_h = (v_h \cdot n)\cdot n + (v_h \cdot t)\cdot t = (v_h \cdot t)\cdot t$

and $\int_{\partial\Omega} p\ v_h \cdot n ds = 0$. Moreover,

$$(\frac{\partial u}{\partial n} \cdot v_h)|_{\partial\Omega} = \frac{\partial u}{\partial n} \cdot (v_h \cdot t)t = ((\frac{\partial u}{\partial n} \cdot n)n + (\frac{\partial u}{\partial n} t) \cdot t))(v_h \cdot t)t,$$
$$= (\frac{\partial u}{\partial n} \cdot t)t \cdot (v_h \cdot t)t = (\frac{\partial u}{\partial n} \cdot t)(v_h \cdot t).$$

Thus, (3.14) becomes:

$$\int_{\Omega} \nabla u : \nabla v_h dx - \int_{\partial\Omega} (\frac{\partial u}{\partial n} \cdot t)(v_h \cdot t) ds - \int_{\Omega} p\ div(v_h) dx = \int_{\Omega} f \cdot v_h dx. \tag{3.17}$$

Let $B$ denote the bilinear form and $F$ denote the linear operator defined by the following formulas:

$$B(v_h, p) = -\int_{\Omega} p\, div(v_h) dx,$$
$$F(v_h) = \int_{\Omega} f \cdot v_h dx,$$

Then (3.16) can be written as:

$$\int_{\Omega} \nabla u : \nabla v_h dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v_h ds - \int_{\Omega} p\ div(v_h) dx + B(v_h, p) = F(v_h), \tag{3.18}$$

$\forall v_h \in V_{h,0}$. Since $u|_{\partial\Omega=0}$, we can symmetrize and stabilize (3.17) as follows:

$$\int_{\Omega} \nabla u : \nabla v_h dx - \sum_{e \in \mathbb{E}^b} \int_e ((\frac{\partial u}{\partial n_e} \cdot t)(v_h \cdot t) + (\frac{\partial v_h}{\partial n_e} \cdot t)(u \cdot t) - \frac{\sigma}{h_e} u \cdot v_h) ds + B(v_h, q) = F(v_h), \tag{3.19}$$

where $h_e$ denotes the length of the edge $e$ and $\sigma$ is a penalization parameter.

Let $A$ denote the bilinear form defined as follows:

$$A(v, w) = \int_{\Omega} \nabla v : \nabla w\, dx - \sum_{e \in \mathbb{E}^b} \int_e ((\frac{\partial v}{\partial n_e} \cdot t)(w \cdot t) + (\frac{\partial w}{\partial n_e} \cdot t)(v \cdot t) - \frac{\sigma}{h_e} v \cdot w) ds.$$

Note that $A(u, v_h) + B(v_h, p) = F(v_h)$, since $u = 0$ on $\partial\Omega$ and $div(u) = 0$ in $\Omega$. Thus, the problem of finding the solution $u$ to (3.14) reduces to finding $u$ satisfying:

$$A(u, v_h) + B(v_h, p) = F(v_h),$$
$$B(u, q_h) = 0 \tag{3.20}$$

$\forall v_h \in V_{h,0}$ *and* $\forall q_h \in W_{h,0}$.

The finite element method applied to the problem given by (3.14) can be stated as:

Find $(u_h, p_h)$ that satisfy the following.

$$A(u_h, v_h) + B(v_h, p_h) = F(v_h),$$
$$B(u_h, q_h) = 0 \tag{3.21}$$

$\forall v_h \in V_{h,0}$ *and* $\forall q_h \in W_{h,0}$.

To verify the existence and uniqueness of the solution, $u$, to this system, we need to show that $B$ is continuous and coercive on $V_{h,0}$.

**Lemma 11.** *With respect to the $H^1$-type norm given by (3.14), $A$ is a continuous bilinear form on $(\prod_{\mathcal{Q}\in\mathcal{Q}_h} H^2(\mathcal{Q}))^2 \cap (H^1(\mathcal{Q}))^2$. Moreover, $A$ is coercive on $V_{h,0}$, provided $\sigma$ is sufficiently large.*

*Proof.* To prove the coercivity of $A$, we need to show $A(v,v) \geq c\,\|v\|_h^2$ holds for $\forall v \in V_{h,0}$. By definition, we have:

$$\begin{aligned} A(v,v) &= \int_\Omega |\nabla v|^2 dx - 2\sum_{e\in\mathbb{E}^b}\int_e (\frac{\partial v}{\partial n_e} v - \int_e \frac{\sigma}{h_e}|v|^2) ds, \\ &= \int_\Omega |\nabla v|^2 dx - 2\sum_{e\in\mathbb{E}^b}\int_e \frac{\partial v}{\partial n_e} v + \sum_{e\in\mathbb{E}^b}\int_e \frac{\sigma}{h_e}|v|^2 ds. \end{aligned}$$

By trace and inverse inequalities, $\sum\limits_{e\in\mathbb{E}^b} h_e\|\frac{\partial v}{\partial n_e}\|^2_{L^2(e)} \leq c\|\nabla v\|^2_{L^2(\Omega)}$. Therefore, we have,

$$\begin{aligned} 2|\sum_{e\in\mathbb{E}^b}\int_e \frac{\partial v}{\partial n_e} v ds| &\leq 2\sum_{e\in\mathbb{E}^b}|\int_e (h_e^{-1/2}v)(h_e^{1/2}\frac{\partial v}{\partial n_e}) ds| \\ &\leq 2(\sum_{e\in\mathbb{E}^b} h_e^{-1}\|v\|^2_{L^2(e)})^{1/2}(\sum_{e\in\mathbb{E}^b} h_e\|\frac{\partial v}{\partial n_e}\|^2_{L^2(e)})^{1/2} \\ &\leq 2c(\sum_{e\in\mathbb{E}^b} h_e^{-1}\|v\|^2_{L^2(e)})^{1/2}\|\nabla v\|_{L^2(\Omega)} \\ &\leq 2c(\frac{1}{2\epsilon}\sum_{e\in\mathbb{E}^b} h_e^{-1}\|v\|^2_{L^2(e)} + \frac{\epsilon}{2}\|\nabla v\|^2_{L^2(\Omega)}) \\ &= \frac{c}{\epsilon}\sum_{e\in\mathbb{E}^b}(h_e^{-1}\|v\|^2_{L^2(e)} + c\epsilon\|\nabla v\|^2_{L^2(\Omega)}). \end{aligned}$$

This yields:

$$\begin{aligned} A(v,v) &\geq \|\nabla v\|^2_{L^2(\Omega)} + \sigma\sum_{e\in\mathbb{E}^b}\frac{1}{h_e}\|v\|^2_{L^2(e)} - 2|\sum_{e\in\mathbb{E}^b}\frac{\partial v}{\partial n_e} v ds| \\ &\geq (1-c\epsilon)\|\nabla v\|^2_{L^2(\Omega)} + (\sigma - \frac{c}{\epsilon})\sum_{e\in\mathbb{E}^b}\frac{1}{h_e}\|v\|^2_{L^2(e)}. \end{aligned}$$

If we choose $\epsilon = \frac{1}{2c}$, then we have $\sigma - \frac{c}{\epsilon} = \sigma - 2c^2$. Thus, for $\sigma \geq 4c^2$,

$$A(v,v) \geq \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} + 2c^2\sum_{e\in\mathbb{E}^b}\frac{1}{h_e}\|v\|^2_{L^2(e)} \geq min\{\frac{1}{2}, 2c^2\}(\|\nabla v\|^2_{L^2(\Omega)} + \sum_{e\in\mathbb{E}^b}\frac{1}{h_e}\|v\|^2_{L^2(e)}) \geq c\, min\{\frac{1}{2}, 2c^2\})\|v\|_h^2.$$

Thus, we get $A(v,v) \geq c\|v\|_h^2$ and have proven the coercivity of $A$.

To prove the continuity of $A$, we need to show that $|A(v,w)| \leq c\|v\|_h\|w\|_h$. Applying the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|A(v,w)|^2 &= |\int_\Omega \nabla w : \nabla v dx - \sum_{e\in\mathbb{E}^b}\int_e (\frac{\partial w}{\partial n_e} v + \frac{\partial v}{\partial n_e} w - \frac{\sigma}{h_e} wv) ds|^2 \\
&\le c(|\int_\Omega \nabla w : \nabla v dx|^2 + \sum_{e\in\mathbb{E}^b} |\int_e (\frac{\partial w}{\partial n_e} v + \frac{\partial v}{\partial n_e} w + \frac{\sigma}{h_e} w\cdot v) ds|^2) \\
&\le c(\|\nabla v\|^2_{L^2(\Omega)}\|\nabla w\|^2_{L^2(\Omega)} + \sum_{e\in\mathbb{E}^b}|\int_e \frac{\partial w}{\partial n_e} v ds|^2 + \sum_{e\in\mathbb{E}^b}|\int_e \frac{\partial v}{\partial n_e} w ds|^2 + \sum_{e\in\mathbb{E}^b}|\int_e \frac{\sigma}{h_e} w\cdot v) ds|^2) \\
&\le c(\|\nabla v\|^2_{L^2(\Omega)}\|\nabla w\|^2_{L^2(\Omega)} + \sum_{e\in\mathbb{E}^b}(h_e\|\frac{\partial w}{\partial n_e}\|^2_{L^2(e)})(\frac{1}{h_e}\|v\|^2_{L^2(e)}) + \sum_{e\in\mathbb{E}^b}(h_e\|\frac{\partial v}{\partial n_e}\|^2_{L^2(e)})(\frac{1}{h_e}\|w\|^2_{L^2(e)}) \\
&+ \sum_{e\in\mathbb{E}^b}|\frac{\sigma}{h_e}|^2\|w\|^2_{L^2(e)}\|v\|^2_{L^2(e)}) \\
&\le c(\|\nabla v\|^2_{L^2(\Omega)} + \sum_{e\in\mathbb{E}^b} h_e\|\frac{\partial v}{\partial n_e}\|^2_{L^2(e)} + \sum_{e\in\mathbb{E}^b}\frac{1}{h_e}\|v\|^2_{L^2(e)})(\|\nabla w\|^2_{L^2(\Omega)} + \sum_{e\in\mathbb{E}^b} h_e\|\frac{\partial w}{\partial n_e}\|^2_{L^2(e)} + \sum_{e\in\mathbb{E}^b}\frac{1}{h_e}\|w\|^2_{L^2(e)}) \\
&= c\|v\|^2_h\|w\|^2_h.
\end{aligned}
$$

As a result $|A(v,w)| \le c\|v\|_h\|w\|_h$. □

Define the continuous and discrete kernels of $B$:

$$
\begin{aligned}
Z &= \{v \in (H^1_0(\Omega))^2 : (v\cdot n)|_{\partial\Omega} = 0, B(v,q) = 0, \quad \forall q \in L^2_0(\Omega)\}, \\
Z_h &= \{v_h \in V_{h,0} : B(v_h,q_h) = 0, \quad \forall q_h \in W_{h,0}\}.
\end{aligned}
$$

Note that $Z_h \subseteq Z$ and (3.20) implies,

$$
A(u_h, v_h) = F(v_h),
$$

$\forall v_h \in Z_h$. Since on $Z_h$, $A$ is a symmetric, elliptic and continuous bilinear form and $F$ is a continuous linear form, by Lax-Milgram theorem, there exists a unique solution $u \in Z_h$ satisfying (3.20). Furthermore, the inf-sup condition [4]:

$$
\inf_{q\in W_{h,0}} \sup_{v\in V_{h,0}\setminus\{0\}} \frac{B(v,q)}{\|v\|_h\|q\|_{L^2(\Omega)}} \ge \alpha > 0
$$

holds.

Suppose $u$ is the solution of the system given by (3.19).

By the Galerkin orthogonality,

$$
A(u-u_h, v_h) = A(u,v_h) - A(u_h,v_h) = F(v_h) - F(v_h) = 0, \tag{3.22}
$$

$\forall v_h \in Z_h$. Furthermore, by the coercivity and the continuity of $A$, we have:

$$
\begin{aligned}
\|u-u_h\|^2_h \le A(u-u_h, u-u_h) &= A(u-u_h, u-v_h) + A(u-u_h, v_h-u_h) \\
&= A(u-u_h, u-v_h) \le C\|u-u_h\|_h\|u-v_h\|_h, \quad \forall v_h \in Z_h
\end{aligned}
$$

since $(v_h - u_h) \in Z_h$.

As a result, we have,

$$\|u - u_h\|_h \leq C\|u - v_h\|_h, \quad \forall v_h \in Z_h.$$

Therefore, since the *inf-sup* condition holds, we see that [9]

$$\|u - u_h\|_h \leq C \inf_{v_h \in Z_h} \|u - v_h\|_h \leq C \inf_{v_h \in V_h} \|u - v_h\|_h. \tag{3.23}$$

From (3.22) – (3.23), we conclude that $(u - u_h)$ is orthogonal to $Z_h$ with respect to the bilinear form, $A$. Using the trace and inverse inequalities given below, we can derive the following lemma and approximate an upper bound on the error in the velocity and pressure approximations. Any polynomial $v$ satisfies [11]:

$$\begin{aligned}
\|v\|_{H^m(\Omega)} &\leq ch^{l-m}\|v\|_{H^l(\Omega)}, \\
\|v\|^2_{L^2(e)} &\leq c(\frac{1}{h}\|v\|^2_{L^2(\Omega)} + h\|v\|^2_{H^1(\Omega)}), \\
\|\frac{\partial v}{\partial n}\|^2_{L^2(\partial\Omega)} &\leq \frac{c}{h}\|\nabla v\|^2_{L^2(\Omega)}.
\end{aligned}$$

**Theorem 6.** *There holds*

$$\|u - u_h\|_h \leq ch^2\|u\|_{H^3(\Omega)}, \quad \|p - p_h\|_{L^2(\Omega)} \leq ch^2(h\,\|p\|_{H^3(\Omega)} + \|u\|_{H^3(\Omega)}).$$

*Proof.* Let $I_h : (H^1_0(\Omega))^2 \to (Q_{3,2} \times Q_{2,3}) \cap (H^1_0(\Omega))^2$ denote the nodal interpolant and $P_h : L^2 \to W_{h,0}$ the interpolant satisfying:

$$\int_\Omega P_h\, p\, dx = \int_\Omega p\, q\, dx, \quad \forall q \in W_{h,0}.$$

Then,

$$\begin{aligned}
\|u - I_h u\|^2_h &= \|\nabla(u - I_h u)\|^2_{L^2(\Omega)} + \sum_{e \in \mathbb{E}^b} h_e \|\frac{\partial(u - I_h u)}{\partial n_e}\|^2_{L^2(e)} + \sum_{e \in \mathbb{E}^b} h_e^{-1}\|u - I_h u\|^2_{L^2(e)} \\
&\leq ch^{2s}\|u\|^2_{H^{s+1}(\Omega)} + \sum_{e \in \mathbb{E}^b} c \cdot h_e(h^{-1}\|\frac{\partial(u - I_h u)}{\partial n_e}\|^2_{L^2(\Omega)} + h\|\frac{\partial(u - I_h u)}{\partial n_e}\|^2_{H^1(\Omega)}) \\
&+ \sum_{e \in \mathbb{E}^b} h_e^{-1}(h^{-1}\|(u - I_h u)\|^2_{L^2(\Omega)} + h\|(u - I_h u)\|^2_{H^1(\Omega)}) \\
&\leq ch^{2s}\|u\|^2_{H^{s+1}(\Omega)} + h^2\|(u - I_h u)\|^2_{H^2(\Omega)} + ch^{-2}\|u - I_h u\|^2_{L^2(\Omega)} + \|u - I_h u\|^2_{H^1(\Omega)} \\
&\leq ch^{2s}\|u\|^2_{H^{s+1}(\Omega)} + ch^2 h^{2(s-1)}\|u\|^2_{H^{s+1}(\Omega)} + ch^{-2}h^{2(s+1)}\|u\|^2_{H^{s+1}(\Omega)} \\
&\leq ch^{2s}\|u\|^2_{H^{s+1}(\Omega)},
\end{aligned}$$

where $1 \leq s \leq 2$. Taking $s = 2$ gives,

$$\|u - I_h u\|_h \leq ch^2\|u\|_{H^3(\Omega)}.$$

Using (3.23) and $I_h u \in V_{h,0}$, we see that the error in the velocity approximation of this scheme is as follows:

$$\|u - u_h\|_h \leq ch^2\|u\|_{H^3(\Omega)}. \tag{3.24}$$

On the other hand, by the triangle inequality,

$$\|p - p_h\|_{L^2(\Omega)} \le \|p - P_h p\|_{L^2(\Omega)} + \|P_h p - p_h\|_{L^2(\Omega)} \le ch^3 \|p\|_{H^3(\Omega)} + \|P_h p - p_h\|_{L^2(\Omega)}. \tag{3.25}$$

The *inf-sup* stability and the continuity of $B$ in the $H^1$-type norm yields:

$$c\|p_h - P_h p\|_{L^2(\Omega)} \le \sup_{v_h \in V_{h,0} \setminus \{0\}} \frac{B(v_h, p_h - P_h p)}{\|v_h\|_h}. \tag{3.26}$$

Since $P_h p \in W_{h,0}$, we have

$$A(u_h, v_h) + B(v_h, P_h p) = F(v_h).$$

$$A(u, v_h) + B(v_h, p_h) = F(v_h).$$

Subtracting the two equations yields,

$$A(u - u_h, v_h) + B(v_h, p_h - P_h p) = 0.$$

The continuity of $A$ in the weighted norm gives:

$$\begin{aligned} |B(v_h, p_h - P_h p)| &= |A(u - u_h, v_h)| \\ &\le \|u - u_h\|_h \|v_h\|_h. \end{aligned} \tag{3.27}$$

Using (3.26) in (3.27), we get:

$$\|p_h - P_h p\|_{L^2(\Omega)} \le \frac{c\|u - u_h\|_h \|v_h\|_h}{\|v_h\|_h} \le c\|u - u_h\|_h \le ch^2 \|u\|_{H^3(\Omega)}$$

As a result, by (3.25),

$$\|p - p_h\|_{L^2(\Omega)} \le ch^2 (h\, \|p\|_{H^3(\Omega)} + \|u\|_{H^3(\Omega)}).$$

□

# 4.0 CONCLUSION

In this thesis project, we have constructed a pair of divergence free, conforming and stable finite elements for the Stokes problem on two dimensional rectangular grids with no-slip boundary conditions. We used pointwise continuous $Q_{3,2} \times Q_{2,3}$ polynomials that are partially $C^1$ at the vertices and $Q_{2,2}$ polynomials that are continuous at the vertices as the functions forming the velocity and pressure spaces, respectively.
By defining appropriate norms and using an affine transformation and a scaled Piola transform, we showed that the *inf-sup* stability condition holds for the finite elements on each rectangular element and then in the entire domain with and without homogeneous boundary conditions.
By applying Nitsche's method to the problem and verifying that the related bilinear form is coercive and continuous, we proved that there exists a unique solution to the two-dimensional Stokes problem and proved the convergence of the finite element solution in a $H^1$-type norm.

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