THE RECONSTRUCTION AND REALIZATION OF TOPOLOGICAL GROUPS

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ABSTRACT

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A topological group is a group equipped with a topology so that the group operations are continuous. The symmetries (or automorphisms) of any given geometric object have a natural group structure, and often a canonical topology making them into a topological group.

Specifically, the automorphism groups of a countable structure is a topological group with the pointwise convergence topology, and the autohomeomorphism group of a compact space is a topological group with the compact-open topology. Here we investigate when these canonical topologies are minimal or the minimum amongst all Hausdorff group topologies.

Further, given an abstract topological group, we aim to realize it as the symmetry group of some geometric object with its canonical topology. We examine two such classes of objects: the automorphism groups of graphs with the pointwise convergence topology and the autohomeomorpism groups of continua (compact and connected spaces) with the compact-open topology.

Keywords: topological group, minimum group topology, minimal group topology, automorphism group, Cayley graph, locally finite graph, regular graph, rigid graph, autohomeomorphism group, profinite group, continuum, Cook continuum, rigid continuum.

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1.0 INTRODUCTION

The definitions of topology and group are among the first definitions that a modern day mathematician would learn. Historically, these two fundamental categories of mathematics had developed independently. In 1926, the idea of abstract topological groups were implicitly defined for the first time by Schreier [34]. Schreier implicitly used them to work on continuous groups of transformations. There is a well-written historical notes on topological groups by Higgins [20].

Formally, a *topological group* is a group equipped with a topology such that the group operations (multiplication and inversion) are both continuous under this topology. Group structure naturally arises from the symmetries of any geometric object. For example, the automorphism group of a graph, the permutation group of a set, the automorphism group of a countable model of a first order theory and the autohomeomorphism group of a topological space. Given a group, there are potentially many topologies that can be assigned to it making it into a topological group. We refer to this kind of topology as a *topological group topology*. Canonically, an automorphism group and a symmetry group are equipped with the *pointwise convergence topology*, whereas an autohomeomorphism group is equipped with the *compact-open topology*. See Chapter 2 for the definitions and basic theorems.

Since their introduction, topological groups have been studied extensively and applied to various mathematical branches. For example, dynamical systems, representation theory, functional analysis etc.

The interplay between the algebraic group structure and topological structure generates many surprising results, for example the Pontryagin Duality Theorem [29]. Informally one part of the theorem says: if G is a compact abelian group then its character group is an abelian group whose character group is a compact abelian group isomorphic to G. This theorem suggests that all the topological information is preserved by purely the algebraic structure of its character group and is retrievable. The full power of this theorem allows the study of compact abelian groups to be "reduced" to the purely algebraic study of abelian groups. Later, van Kampen [23] extended this theorem to locally compact abelian groups. A comprehensive discussion of these Duality Theorems can be found in Morris' book [28] or his collaboration with Hofmann [22].

Another example where the algebraic structure determines the topological structure is any group satisfying the Small Index Property (any subgroup of index less than $2^{\aleph_0}$ contains the stabilizer of a tuple). The formal definition says that a topological group G has such property if any subgroup is open in G if and only if it has at most countable index. A good introduction on the Small Index Property is written by Cameron [3] Section 4.8.

Our approaches to study the interplay between algebraic and topological structure are through reconstruction and realization of topological groups. We aim to reconstruct a topological group from a given symmetry group and we study the effect of the algebraic structure on the existence of the "smallest" topology maintaining Hausdorffness. The detailed description of our results is in Section 1.1. On the other hand, we try to realize topological groups as symmetry groups of some geometric objects so that the topological structure is preserved.

This illustrates how the topological structure affects the algebraic structure. We discuss these results in Section 1.2.

1.1 RECONSTRUCTION OF TOPOLOGICAL GROUPS

A topology τ on a group G is a *topological group topology* if the group operations on G are τ-continuous. The collection of all topological group topologies on a group G is partially ordered by set-inclusion, with the discrete topology as the maximum element, and the indiscrete topology the minimum. However, the indiscrete topology is not Hausdorff. The collection of all *Hausdorff* topological group topologies on G is then a sub-partial order of all topological group topologies. This sub-order always has a maximum element (the discrete topology) but may or may not have a minimum element (a Hausdorff group topology on G contained in all other Hausdorff group topologies), and also may or may not have minimal elements (a Hausdorff group topology such that no strictly coarser group topology is Hausdorff). Note that the minimum Hausdorff group topology (if it exists) is certainly minimal, but the converse is false in general.

Minimal group topologies have been extensively studied, see the survey by Dikranjan and Megrelishvili [9] for example, which cites over 200 sources. By contrast, minimum (Hausdorff) group topologies are less well understood. Gaughan [16] showed that the group $\mathrm{Sym}(X)$ of all permutations of an infinite set X has the minimum group topology, namely the topology of pointwise convergence. Later, Gartside and Glyn [15] showed that the group $H(X)$ of all homeomorphisms of a space X, where X is any metrizable one-manifold (with or without boundary), has the minimum group topology. In this case the minimum group topology is the usual compact-open topology on $H(X)$. Megrelishvili and Polev [27] have extended this last theorem to $H(X)$ for many compact *linearly ordered topological spaces (LOTS)* X.

We continue this line of investigation into the existence, or otherwise, of minimum Hausdorff group topologies. A number of questions raised in [9] are answered (notably, Questions 2.3, 4.28 and 4.41).

First it is established that many autohomeomorphism groups do not have a minimum Hausdorff group topology (Theorem 9). This leads to Corollary 10, which states that the autohomeomorphism groups of the Cantor set (zero-dimensional) and the Hilbert cube have no minimum group topology. In particular, for every compact metrizable space X containing an open n-cell for some $n \geq 2$, its autohomeomorphism group $H(X)$ has no minimum Hausdorff group topology (Corollary 11).

We then show how, in certain circumstances, it is possible to 'shrink' the compact-open topology τ_k on a autohomeomorphism group $H(X)$ around a closed subset C of X to obtain a new (Hausdorff) group topology $\tau_{k|C}$ on $H(X)$ (Theorem 13). This shrinking process is derived from an idea of Gamarnik [14]. It follows from Corollary 14, for example, that the compact-open topology on the autohomeomorphism group of the Mobius band is not minimal.

The real merit of the $\tau_{k|C}$ topology becomes apparent for compact metrizable spaces X containing a dense open one-manifold. Let I_X be the union of all "interval-like" components and S_X be the union of all "circle-like" components (see Section 2 for precise definitions). Then $O_X = I_X \cup S_X$ is dense in X and if $C = X \setminus O_X$, then we can prove that such $H(X)$ does have a minimum group topology, namely $\tau_{k|C}$ (Theorem 17). A wide range of spaces satisfy the hypotheses of this theorem. It is now interesting to discover when $\tau_{k|C}$ is equal to τ_k and when it is strictly coarser. Note that when $\tau_{k|C}$ is strictly coarser it follows that the compact-open topology τ_k is *not minimal*, while equality implies that τ_k is the minimum, and thus is the only *minimal*, topology. Consequently we derive large classes of compact metric spaces for which we know whether or not the compact-open topology is a minimal group topology on the autohomeomorphism group.

In Section 3.3 we give sufficient conditions for both cases. In Section 3.3.1, Propositions 19 offers sufficient conditions for $\tau_{k|C}$ to be strictly coarser than τ_k (thus demonstrating τ_k is not a minimal topology). This is illustrated in Example 20. In Section 3.3.2 we give sufficient conditions (Proposition 21 and Lemma 24) for the minimum topology to coincide with the compact-open topology (see Example 22 and Example 23). Naturally, this implies that the compact-open topology is minimal. The relevant spaces are similar to the class of (compact, metrizable) *graph-like* spaces recently introduced by Thomassen and Vella [35]. They defined a space X to be *graph-like* if there is a collection E of pairwise disjoint edges of X such that $X \setminus E$ is zero-dimensional, where an *edge* of X is an open subset homeomorphic to $(0, 1)$, whose closure is a simple arc.

However a complete classification seems elusive, and we present examples demonstrating the difficulties. From Proposition 19 and Proposition 21, we (essentially) know whether or not $\tau_{k|C} = \tau_k$ except in two cases: (i) when S_X is empty, C is zero-dimensional but not a convergent sequence (or finite) and (ii) S_X is empty, C is not zero-dimensional but every component in I_X has closure either an arc or a circle. Of particular interest in case (ii) is when C and/or X is connected, and preferably locally connected. In each case there do not seem to be simple conditions we can place on X and/or C that allow us to determine if $\tau_{k|C}$

is, or is not, equal to τ_k. We demonstrate this with Examples 25, 26, 27 and 28.

Finally we turn from autohomeomorphism groups to automorphism groups. For a countable model M of a first order theory, its automorphism group $\mathrm{Aut}(M)$ can be topologized as a subgroup of $\mathrm{Sym}(M)$ with the pointwise convergence topology. Question 2.3 of [9] asks which oligomorphic groups have a minimum group topology, mentioning $\mathrm{Sym}(\mathbb{N})$, $\mathrm{Aut}(\mathbb{Q}, <)$, and the autohomeomorphism group of the Cantor space, in particular. We present a reasonably broad answer to this question, encompassing the aforementioned groups.

We show that certain oligomorphic automorphism groups, including all strongly homogeneous automorphism groups, have a minimum (Hausdorff) group topology (Theorem 29). In most cases this minimum group topology is strictly coarser than the topology of pointwise convergence, and so this latter topology is not minimal. However, Example 30 shows that the automorphism group of the atomless countable Boolean algebra has no minimum group topology.

Question 4.41 from the survey [9] asks whether every Polish Roelcke precompact group is minimal. A group is *Roelcke precompact* if the lower uniformity is precompact. Uspenskij [36] has given a general method based on Roelcke precompactness for checking minimality for several large Polish groups, among others: $U(H)$, $H(C)$, and $Iso(U_1)$. It was natural then to ask if Roelcke precompactness combined with some nice property (Question 4.41 suggests Polish) always implies minimality. Our theorem gives a negative answer to this question (Corollary 31). This was proven independently using a different method by Ben Yaacov and Tsankov [2].

1.2 REALIZATION OF TOPOLOGICAL GROUPS

In the realm of topological group reconstruction, we always start with a concrete geometric object and study its symmetry group. The concrete examples provide useful visual insight into the topological and algebraic properties they possess. Thus it is desirable to be able to realize an abstract topological group as the symmetries of some geometric object while preserving both the algebraic and topological structure. Most of the related work so far has been done purely from an algebraic perspective.

One approach towards realization is through the automorphism group of an abstract graph Γ. Specifically, for a given finite group G, Frucht [12] constructed a finite graph Γ that realizes G algebraically, that is, the automorphism group $\mathrm{Aut}(\Gamma)$ is algebraically isomorphic to G. That is to say $\mathrm{Aut}(\Gamma) \cong G$. His proof involved a two-step strategy:

Step 1. construct a colored directed graph $\mathcal{C}$ such that $\mathrm{Aut}(\mathcal{C}) \cong G$;

Step 2. replace the colored edges by corresponding *rigid graphs* (graphs with only trivial automorphism) to obtain Γ.

In the finite case, Step 1. is achieved by utilizing Cayley graphs [6]. This two-step strategy is adopted to prove our results.

Later, Frucht [13] showed that one may also require Γ to be 3-regular, and Sabidussi [32] furthermore constructed r-regular graphs for every $r \geq 3$. A different generalization showing all countable groups can be realized was proven independently by De Groot [18] and Sabidussi [33]. Yet whereas these results consider algebraic isomorphisms only, we seek a *topological realization* of the given group G, that is $(\mathrm{Aut}(\Gamma), \tau_p)$ is *topologically isomorphic* to G. Since $\mathrm{Aut}(\Gamma)$ is the subgroup of the permutation group of its vertex set, $\mathrm{Aut}(\Gamma)$ inherits the pointwise convergence topology τ_p. Note that when Γ is countable $\mathrm{Aut}(\Gamma)$ becomes a Polish group (separable completely metrizable space) which is used extensively in the study of descriptive set theory.

Another alternative approach is to topologically realize a topological group as the autohomeomorphism group of a continuum (compact and connected space). In 2012, Hofmann and Morris [21] proved that if X is a Tychonoff compact space and the autohomeomorphism group $(H(X), \tau_k)$ is compact, then $H(X)$ is a profinite topological group. Naturally, they asked whether the converse is true: given a profinite group G, does there exist a continuum (or at least compact) space X whose autohomeomorphism group $H(X)$ is topologically isomorphic to G? The special case when G is a metrizable profinite group has been obtained by Gartside and Glyn [15]. We discuss our attempts in this direction in Section 1.2.3.

We give our results about realizing a topological group as the automorphism group of a locally finite graph in Section 1.2.1, the automorphism group of a non-locally finite graph in Section 1.2.2 and the autohomeomorphism group of a continuum in Section 1.2.3, respectively.

1.2.1 Locally Finite Graphs

If Γ is a locally finite graph and $v \in \Gamma$, then the stabilizer group of vertex v, $\mathrm{Aut}(\Gamma)_v$, is an open subgroup that is profinite (compact, Hausdorff and totally disconnected)[11]. Hence, for every connected, locally finite graph Γ, its automorphism group, $\mathrm{Aut}(\Gamma)$ is: a separable metrizable topological group, with an open profinite subgroup (of countable index). We show the converse is also true (Theorem 35): any separable metrizable topological group with an open, profinite subgroup can be topologically realized by a (countably) connected, locally finite graph. In Step 1 constructing the colored, directed graph $\mathcal{C}$, we use ideas related to those introduced by Krön and Möller [25].

Evans and Hewitt [11] construct two infinite, metric profinite groups G_1 and G_2 which are algebraically isomorphic but not topologically isomorphic. So applying Theorem 35 gives two connected locally finite graphs Γ_1 and Γ_2 such that $\mathrm{Aut}(\Gamma_1)$ is algebraically isomorphic to $\mathrm{Aut}(\Gamma_2)$ but $\mathrm{Aut}(\Gamma_1)$ and $\mathrm{Aut}(\Gamma_2)$ are not isomorphic as topological groups (Corollary 36). In particular they do not have the Small Index Property mentioned above.

We wish to impose the additional constraint of regularity on our realization as per Frucht [13] and Sabidussi [32]. To do so, we first show the following properties possessed by the automorphism group of a connected graph Γ of bounded degree at most $m+1$ (not necessarily regular) in Lemma 37: for any fixed v, in $\mathrm{Aut}(\Gamma)_v$ there is a decreasing sequence of open subgroups U_n, forming a neighborhood base at the identity, such that $|U_n/U_{n+1}| \leq m$ for all n. A remarkably strong converse holds (Theorem 38): not only can such groups be topologically realized as graphs with bounded degree, but in fact, with r-regular graphs Γ_G^r for all $r \geq \max(3, m+1)$. This result is obtained by controlling the degrees of vertices during both steps.

If we give any countable group the discrete topology then it satisfies the hypotheses of Theorem 38 with $m = 1$ (Corollary 39). In particular, we can realize $\mathbb{Q}$ as an r-regular graph for any $r \geq 3$ (see Section 4.4).

If Γ is a connected, locally finite graph then Γ is countable. More generally, the connected components of a locally finite graph partition it into countable, connected, locally finite subgraphs. We give a complete characterization of topological groups that can be realized as automorphism groups of such locally finite graphs (Theorem 40).

1.2.2 Non-Locally Finite Graphs

If Γ is a countable graph then $\mathrm{Aut}(\Gamma)$ is topologically isomorphic to a closed subgroup of the infinite symmetric group. Such a subgroup is topologically isomorphic to a Polish group that is non-Archimedean (it has local base at the identity consisting of open subgroups). Hence $\mathrm{Aut}(\Gamma)$ is non-Archimedean and Polish. We show the converse is true (Theorem 41): any Polish non-Archimedean topological group can be topologically realized as a graph.

The *weight* of a space X is the minimal size of a basis. So a space is separable metrizable if and only if it has countable weight. This result can be generalized to non-Archimedean and *(Raikov) complete* topological groups which arise as a closed subgroup in $\mathrm{Sym}(\kappa)$ for some cardinal κ. Here, a topological group G is *(Raikov) complete* if it is closed in every Hausdorff topological group into which it embeds. With this in mind the previous result is a special case of the characterization of topological groups arising as the automorphism group of a (connected) graph (Theorem 42).

The results of a recent paper by Dolinka, Gray, McPhee, Mitchell and Quick, [10], are phrased in terms of 'any group' which is isomorphic to the automorphism group of a countable graph. They observe that this includes, by results of Sabidussi and de Groot, all countable groups. The previous theorem makes precise what 'any group' is: they are the Polish non-Archimedean groups, or equivalently, the closed subgroups of the symmetric group on the natural numbers, or equivalently still, the automorphism group of some first order structure.

Reciprocally, the results of [10] enhance our own. For example, [10, Theorem 5.4] states that for every countable graph Γ there are continuum, $\mathfrak{c}$, many strongly algebraically closed, bipartite graphs whose automorphism group is isomorphic to $\mathrm{Aut}(\Gamma)$. It is straightforward to check that these bipartite graphs have automorphism groups which are topologically isomorphic to the automorphism group of Γ. Hence in Theorem 41 we can say that for every Polish non-Archimedean topological group, G, there are $\mathfrak{c}$-many strongly algebraically closed, bipartite graphs whose automorphism group is topologically isomorphic to G.

1.2.3 Continua

Gartside and Glyn [15] showed that when a profinite group G is also metrizable, then there is a continuum X_G whose autohomeomorphism group is isomorphic to G as a topological group. We can extend this result to arbitrary products of profinite groups that have a continuum realization (Theorem 45). The construction follows Frucht's strategy in a similar fashion. The difference is instead of replacing the colored, directed edges by rigid graphs, we replace them by *rigid continua* such as the Cook continuum [7].

Even in the case of $G = \mathbb{Z}^{\omega_1}$, this result is new (Section 5.2). But a general result seems to be elusive, this is due to the difficulty of compactifying non-locally compact spaces while keeping control of the autohomeomorphism groups. Discussion of potential approaches to resolve this converse are presented in Section 5.3.

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2.0 PRELIMINARIES

We establish the terminologies and basic facts about topological groups that are used in this thesis. The basic notions of topological groups can be found in any introductory book, for example, [8], [19], [20], [28] and [39]. Throughout this work, functions are usually assumed to be continuous and topologies are assumed to be Hausdorff, unless stated otherwise.

2.1 TOPOLOGICAL GROUPS

A *topological group* (G, τ) is a group G with a topology τ such that the group operations are τ-continuous, that is:

(a) the multiplication map $m : G \times G \to G$ defined by $m(x, y) = xy$ is continuous.

(b) and the inversion map $I : G \to G$ defined by $I(x) = x^{-1}$ is continuous.

We write G as shorthand of (G, τ) and $id = id_G$ for the identity element of group G. A standard example of a topological group is $\mathbb{R}$, with the usual topology and addition. Also, any group with the discrete topology is a topological group.

If $A \subseteq G, B \subseteq G$ and $x \in G$, the set $\{y \cdot z \mid y \in A, z \in B\}$ is denoted AB, and the sets A^{-1}, xA, Ax are similarly defined. The set $A^2 = AA$, in particular, is the set

$\{a \cdot a' \mid a, a' \in A\}$, not the set of all squares a^2 for $a \in A$.

The following standard results on topological groups can be found in [8] and [39]. The continuity conditions (a) and (b) can be expressed as follows:

(a′) for each neighborhood W of xy, there exists neighborhoods $U \ni x$ and $V \ni y$ such that $UV \subseteq W$.

(b′) for each neighborhood W of x^{-1}, there exists a neighborhood U of x such that $U^{-1} \subseteq W$.

Alternatively, the conditions (a) and (b) can be replaced by the single condition that the map $c : G \times G$ defined by $c(x, y) = x \cdot y^{-1}$ be continuous.

Let $a, b \in G$. Each of the maps $x \mapsto x^{-1}; x \mapsto ax; x \mapsto xb; x \mapsto axb$ is a autohomeomorphism of G onto G. If $\{U \mid U \in \mathcal{U}$ is a neighborhood base at $id\}$, then for any $x \in G$, $\{xU \mid U \in \mathcal{U}\}$ is a neighborhood base at x, and so is $\{Ux \mid U \in \mathcal{U}\}$. The open symmetric neighborhoods of the identity form a base. If U is open and a neighborhood of the identity, so is U^{-1} and thus so is $U \cap U^{-1}$.

Lemma 1. *[39, 13G.6] Let $\mathcal{U}$ be a neighborhood base of open sets at e in G. Then*

(a) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^2 \subseteq U$.

(b) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^{-1} \subseteq U$.

(c) for each $U \in \mathcal{U}$ and $x \in U$, there exists $V \in \mathcal{U}$ with $xV \subseteq U$.

(d) for each $U \in \mathcal{U}$ and $x \in G$, there exists $V \in \mathcal{U}$ with $xVx^{-1} \subseteq U$.

(e) for each $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ with $W \subseteq U \cap V$.

(f) $\{id\} = \bigcap\{U \mid U \in \mathcal{U}\}$.

The converse of the Lemma 1 is used in Chapter 3 in the proof of Theorem 13 so we state it explicitly here.

Lemma 2. *Let G be a group. Suppose $\mathcal{U}$ is a family of subsets of G all containing id satisfying the following conditions:*

(a) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^2 \subseteq U$;

(b) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^{-1} \subseteq U$;

(c) for each $U \in \mathcal{U}$ and $x \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $xV \subseteq U$;

(d) for each $U \in \mathcal{U}$ and $x \in G$, there exists $V \in \mathcal{U}$ with $xVx^{-1} \subseteq U$; and

(e) for each $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ with $W \subseteq U \cap V$.

Then $\{xU : x \in G,\ U \in \mathcal{U}\}$ is a basis for a topology τ on G making G into a topological group. The topology τ is T_1 (or equivalently, Hausdorff, or Tychonoff) if and only if $\bigcap \mathcal{U} = \{id\}$.

A topological space is said to be *totally disconnected* if the only connected subsets are singletons. A space X is *zero-dimensional* if every point of X has a neighborhood base consisting of clopen sets. Equivalently, X is zero-dimensional if for every $x \in X$ and A closed subset that doesn't contain x, there is a clopen set containing x and not meeting A. Note that every zero-dimensional T_1-space is totally disconnected. In fact, every locally compact Hausdorff space is zero-dimensional if and only if it is totally disconnected.

The *Cantor set* is a totally disconnected compact metric space. So the Cantor set is zero-dimensional.

Lemma 3. *Every compact metric space X is a continuous image of the Cantor set.*

The *Hilbert cube* is the topological product of $[0, 1/n]$ for all $n \geq 1$. It is homeomorphic to the countable infinite product of the unit interval $[0, 1]$. The Hilbert cube is a compact Hausdorff space.

A *Polish group* is a topological group that is also a Polish space. A *Polish space* is a separable completely metrizable space. A group is *non-Archimedean* if it has a local base at identity consisting of open groups. A group is *Raikov complete* if its two sided uniformity is complete, or equivalently it is closed in every Hausdorff topological group in which it embeds.

A group G is a *profinite group* if $G \cong \varprojlim G_\lambda$ where G_λ are finite groups, each endowed with the discrete topology and canonical group homomorphisms $\phi_{\lambda,\mu} : G_\lambda \to G_\mu$ if $G_\mu \leq G_\lambda$. Equivalently, a groups is profinite if and only if it is Hausdorff, compact, and totally disconnected.

2.2 MINIMUM TOPOLOGICAL GROUP TOPOLOGIES

A *partially ordered set* or *poset* is a set P with a binary relation $\leq$ that is reflexive, anti-symmetric, and transitive.

Given a group G, the collection of all topological group topologies $\mathcal{T}$ is partially ordered by set-inclusion $\subseteq$. The maximum element of this poset is the discrete topology $\tau_{\text{dis}} = \mathcal{P}(G)$ whereas the minimum element is the indiscrete topology $\tau_{\text{ind}} = \{\emptyset, G\}$.

Given a group G, a topological group topology τ is *Hausdorff* if (G, τ) forms a Hausdorff topological space. The collection of all Hausdorff topological group topologies on G, denoted $\mathcal{T}_H$ is a sub-partial order of $\mathcal{T}$. Since $\tau_{\text{dis}} \in \mathcal{T}_H$, this sub-poset always has a maximum element, namely τ_{dis}. For topological groups, the separation axioms have the following relation:

Lemma 4. *G is T_0 if and only if G is Hausdorff (T_2) if and only if G is regular if and only if G is Tychonoff.*

A Hausdorff topological group topology τ is a *minimal topology* if, there is no $\tau' \in \mathcal{T}_H$ with $\tau' \subsetneq \tau$. Minimality is a local property. A Hausdorff topological group topology is a *minimum topology* (or *absolute minimal*) τ exists if for all $\tau' \in \mathcal{T}_H$, $\tau \subseteq \tau'$.

A minimum topology is always a minimal topology, in fact, it is the unique minimal topology. However, a group may not have a minimal topology let alone a minimum topology.

2.3 AUTOHOMEOMORPHISM GROUPS

Let X be a compact metric space. The *autohomeomorphism group* (also known as the homeomorphism group in some literature), $H(X)$, is the set of all self-homeomorphisms of X together with composition as group operation.

(a) The basic neighborhoods in $H(X)$ of the identity in the *pointwise convergence topology* τ_p have the form $B_F^{\tau_p} = \{f \in H(X) : f(x) = x \text{ for all } x \in F\}$, for some finite subset F of X.

(b) The basic neighborhoods in $H(X)$ of the identity in the *compact-open topology* τ_k have the form $B_\varepsilon^{\tau_k} = \{f \in H(X) : d(f(x), x) < \varepsilon \text{ for all } x \in X\}$, for some $\varepsilon > 0$.

For example, for the autohomeomorphism group of the unit interval, $H(I)$, the basic open neighborhoods $B_\varepsilon^{\tau_k}, B_F^{\tau_p}$ can be illustrated as all autohomeomorphisms contained in the relevant blue region in Figure 1.

Graph-like spaces were recently introduced by Thomassen and Vella [35]. Given a topological space X, an *edge* of X is an open subset homeomorphic to $(0, 1)$, whose closure is a simple arc. Then X is *graph-like* if there is a collection E of pairwise disjoint edges of X

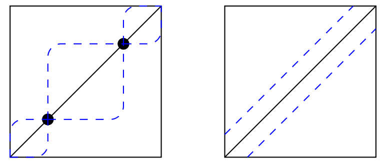

Figure 1: Example of τ_p and τ_k

such that $X \setminus E$ is zero-dimensional.

Let X be a space.

(a) Define O_X ('the one-manifold part of X') to be $O_X := \{x \in X :\ x$ has a neighborhood homeomorphic to $(0,1)\}$.

(b) Define the "circle-like" component S_X to be all points with a clopen neighborhood homeomorphic to the circle, S^1, and the "interval-like" component $I_X := O_X \setminus S_X$. Note that each point of I_X has a neighborhood homeomorphic to $(0,1)$ which is clopen in O_X.

(c) Define $C_X = X \setminus O_X$.

A *continuum* is a non-empty compact connected metric space. A continuum R is a *Cook continuum* if whenever $f : R \to R$ is continuous, then f is the identity or f is the constant map (see [24]). We say a continuum R is *rigid* if $H(R) = \{id\}$. Note that a Cook continuum is rigid. A *Peano continuum* is a continuum that is locally connected at each point.

2.4 OLIGOMORPHIC GROUPS

We adopt the following definitions related to oligomorphic groups from Cameron's book [5].

Let f_n denote the numbers of orbits of G on n-subsets. We say G is *highly homogeneous* iff $f_n = 1$ for all n. Let F_n denote the numbers of orbits of G on n-tuples of distinct elements. We say G is *highly transitive* if $F_n = 1$ for all n.

Two examples are:

(i) The symmetric group on an infinite set S, $\mathrm{Sym}(S)$, is highly transitive and hence highly homogeneous.

(ii) The group $\mathrm{Aut}(\mathbb{Q}, <)$ of order-preserving permutations of $\mathbb{Q}$ is highly homogeneous but not highly transitive.

Let M be a countable model of a first order theory. Then its automorphism group, $\mathrm{Aut}(M)$, considered as a topological subgroup of $\mathrm{Sym}(M)$ (the group of all permutations of M, with the topology of pointwise convergence) is a closed subgroup, and hence Polish. Conversely, every closed subgroup of $\mathrm{Sym}(\mathbb{N})$ can be identified as the automorphism group of a countable model of a first order theory. If G is a permutation group on a countable set Ω, then there is a structure M on Ω such that $G \leq \mathrm{Aut}(M)$ and for every n, G and $\mathrm{Aut}(M)$ hve the same orbits on Ω^n. This M is called the *canonical structure*. A theory is $\aleph_0$-*categorical* if it has a unique (up to isomorphism) countable model. If a theory is $\aleph_0$-categorical, that is, it has a unique (up to isomorphism) countable model M, then $\mathrm{Aut}(M)$ is said to be oligomorphic. Equivalently, a permutation group is *oligomorphic* if G is a group of permutations of an infinite set Ω with the property that, for all n, G has only a finite number of orbits on Ω^n.

If G is any permutation group on a countable set X, then there is a relational structure M on X (over a countable relational language) such that (i) $G \leq \mathrm{Aut}(M)$, and (ii) G and $\mathrm{Aut}(M)$ have the same orbits on X^n, for every n. In [3], such structure is referred to as the *canonical (relational) structure.*

Theorem 5 (Ryll-Nardzewski [31])**.** *A permutation group G is oligomorphic if and only if its canonical structure is $\aleph_0$-categorical.*

Examples of $\aleph_0$-categorical structures are $(\mathbb{Q}, <)$, countable atomless Boolean algebras, (countable) random graphs and the natural numbers with no relations [4]. By Theorem 5, the automorphism groups of these structures are oligomorphic - in particular, in the last case we obtain $\mathrm{Sym}(\mathbb{N})$.

Suppose M is a first-order countable structure and let $G := \mathrm{Aut}(M)$. A subgroup H of G is said to have *small index* in G if $[G : H] < 2^{\aleph_0}$. One can easily see that open subgroups of G have small index in G. We say $\mathrm{Aut}(M)$ has the *small index property* if every subgroup of $\mathrm{Aut}(M)$ of small index is open.

Definition 6. [30] A topological group G is called *Roelcke precompact* if for every neighborhood $U \ni id$, there exists a finite set $F \subseteq G$ such that $UFU = G$.

Equivalently, a group is Roelcke-precompact if the lower uniformity is precompact.

The notion of Roelcke precompactness was introduced by Roelcke and Dierolf [30] and later found a number of applications in the theory of topological groups.

From [17], A *Boolean algebra* $(A, \wedge, \vee, 0, 1, \neg)$ is a distributive lattice $(A, \wedge, \vee$ with least element 0 and greatest element 1 and with a unary operator $\neg$ (complementation) that satisfies $\neg x \wedge x = 0$ and $\neg x \vee x = 1$ for all $x \in A$. For example, the power set algebra of a set X is a Boolean algebra. An *atom* in a partial order $(P, \leq)$ with least element 0 is a nonzero element $a \in P$ such that there is no $x \in P$ with $0 < x < a$. Every power set algebra has its singletons as atoms.

Proposition 7. *[17] Any two countable, atomless Boolean algebras with more than one element are isomorphic.*

There is a unique countable, atomless Boolean algebra, for example, the clopen subset of Cantor set.

2.5 AUTOMORPHISM GROUPS OF GRAPHS

By a *graph* Γ is meant a set $V(\Gamma)$ (the *vertices* of Γ) together with a set $E(\Gamma)$ (the *edges* of Γ) of unordered pairs of *distinct* elements of $V(\Gamma)$. A *countable graph* is a graph with a countable vertex set.

Given two graphs Γ_1, Γ_2 a function $\alpha : V(\Gamma_1) \to V(\Gamma_2)$ is an *isomorphism* of Γ_1 onto Γ_2 if α is bijection and $(x, y) \in E(\Gamma_1)$ is mapped to $(\alpha(x), \alpha(y)) \in E(\Gamma_2)$. When there is no ambiguity, we omit V and E and simply say Γ. An *automorphism* of Γ is an isomorphism of Γ onto itself. We use $\mathrm{Aut}(\Gamma)$ to denote the group of all automorphisms of Γ. The identity element is the identity function id.

A *Cayley graph* $\mathcal{C}$ of a finite group G is a colored, directed graph with vertex set $V(\mathcal{C}) = G$ and edges of color $g \in G$ from vertex $u \in V(\mathcal{C})$ to $v \in V(\mathcal{C})$ if and only if $v = gu$. An automorphism α for a colored graph $\mathcal{C}$ is *color-preserving* if for every edge (x, y), the edges (x, y) and $(\alpha(x), \alpha(y)$ have the same color. For any finite group G, the group of color-preserving automorphisms of its Cayley graph $\mathrm{Aut}_c(\mathcal{C}(G))$, is isomorphic to G.

A graph R is *rigid* if and only if $\mathrm{Aut}(R) = \{id\}$. A graph is *r-regular* if and only if every vertex of the graph has degree r. A graph is *locally finite* if and only if every vertex of the graph has finite degree. In particular, any finite graph is locally finite.

Let G be a group acting on a set Y. The action is *transitive* if for any two points $x, y \in Y$ there is an element $g \in G$ such that $gx = y$. For a point $x \in Y$, the *stabilizer in* G of x is the subgroup $G_x = \{g \in G \mid gx = x\}$.

Suppose U is a subgroup of a group G.

Lemma 8. *[25] The group G acts on the set G/U of left cosets of U such that the image of a coset hU under an element $g \in G$ is $(gh)U$. This action is transitive. Conversely, if G acts transitively on some set Y and x is a point in Y, then Y can be identified with G/G_x.*

3.0 RECONSTRUCTION OF TOPOLOGICAL GROUPS

In Section 3.1, we establish sufficient conditions for when the autohomeomorphism group of a compact metrizable space, $H(X)$, does not have a minimum Hausdorff topology. This applies to (zero dimensional) Cantor space, the Hilbert cube or any X containing an open n-cell for $n \geq 2$.

In Section 3.2, we adapt Gamarnik's idea [14] to introduce a method of 'shrinking' the compact-open topology around a closed subset. Under reasonable conditions this will yield a topological group topology. Section 3.3 presents a sufficient condition – the closed subset of X being the complement of dense open one-manifold, for the shrinking technique to generate a minimum topology. We discuss in Section 3.3.2 sufficient conditions for τ_k to be strictly finer than $\tau_{k|C}$ (i.e. $\tau_k \supsetneq \tau_{k|C}$) and thus is not even minimal. Whereas, in Section 3.3.1, we present sufficient conditions for when these two topologies coincide ($\tau_{k|C} = \tau_k$) and hence τ_k becomes the minimum topology. We illustrate the difficulty of establishing necessary conditions for both directions through several examples in Section 3.3.3.

Adapting our results from $H(X)$ (Theorem 18) to $\mathrm{Aut}(M)$ in Section 3.4, we discuss some oligomorphic groups for which τ_p is the minimum topology and others for which it's not even minimal. This gives rise to an independent proof for the existence of a Polish Roelcke-precompact group that is not minimal [2].

3.1 AUTOHOMEOMORPHISM GROUPS WITH NO MINIMUM

Let (X, d) be a compact metric space. Recall that basic neighborhoods in $H(X)$ of the identity in τ_p have the form $B_F^{\tau_p} = \{f \in H(X) : f(x) = x \text{ for all } x \in F\}$, for some finite subset F of X, and basic neighborhoods in τ_k have the form $B_\varepsilon^{\tau_k} = \{f \in H(X) : d(f(x), x)) < \varepsilon$ for all $x \in X\}$, for some $\varepsilon > 0$. For any $S \subseteq X$, let $H(X|S) = \{h \in H(X) : h$ is the identity outside $S\}$. Note $H(X|S) \leq H(X)$, and it inherits any Hausdorff group topology from $H(X)$. Also, when T is all of X, $H(X|T)$ is simply $H(X)$.

Theorem 9. *Let (X, d) be a compact metric space. Let T be a non-empty open subset of X containing no isolated points. Suppose $H(X|T)$ has the following two properties:*

(T_k) ("transitivity w.r.t. τ_k") *for every $\varepsilon > 0$ and $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ distinct points in T such that $d(x_i, y_i) < \varepsilon$ for $i = 1, \ldots n$, there exists $h \in B_\varepsilon^{\tau_k} \cap H(X|T)$ such that $h(x_i) = y_i$ for $i = 1, \ldots, n$;*

(T_p) ("transitivity w.r.t. τ_p") *for every finite subset F of T, the set $B_F^{\tau_p} \cap H(X|T)$ is highly transitive on $T \setminus F$.*

Then if τ is a topological group topology on $H(X)$, $\tau \subseteq \tau_p \cap \tau_k$ and $id \in U \in \tau$; then $\{id\} \neq H(X|T) \subseteq U$. Hence, $H(X)$ does not have a minimum Hausdorff group topology.

Proof. Let τ be a topological group topology on $H(X)$. Take any U in τ containing id and any h in $H(X|T)$. We will show $h \in U$. Clearly by either condition (T_k) or (T_p) we have many non-identity $h \in H(X|T)$. Hence no τ-open set separates id from any such h, so τ is not Hausdorff, and $H(X)$ does not have a minimum Hausdorff group topology.

As τ is a group topology, we can find U' in τ such that U' is symmetric and $id \in U' \subseteq (U')^4 \subseteq U$.

As $U' \in \tau \subseteq \tau_p$, $U' \supseteq B_F^{\tau_p}$ for some finite subset F of X. Let us enumerate $F \cap T = \{x_1, \ldots, x_n\}$. As $U' \in \tau \subseteq \tau_k$, there is an $\varepsilon > 0$ such that $B_\varepsilon^{\tau_k} \subseteq U'$. We can suppose that $\varepsilon < \min\{d(x, x') : x \neq x', \, x, x' \in F \cup h^{-1}(F)\}$.

Pick h_1 in $H(X|T)$ such that: (1) $h_1 \in B_\varepsilon^{\tau_k}$ and (2) for $i = 1, \ldots, n$ the point $y_i = h_1(h^{-1}(x_i))$ is in T but not in F. (We may find such y_i since T has no isolated points, and such a h_1 exists by (T_k).)

Pick h_2 in $H(X|T)$ such that: (1) $h_2 \in B_F^{\tau_p}$ and (2) for $i = 1, \ldots, n$ the point $z_i = h_2(y_i)$ has $d(z_i, x_i) < \varepsilon$. As $\{y_1, \ldots, y_n\} \cap F = \emptyset$, existence of h_2 is guaranteed by (T_p).

Pick h_3 in $H(X|T)$ such that: (1) $h_3 \in B_\varepsilon^{\tau_k}$ and (2) $h(z_i) = x_i$ for $i = 1, \ldots n$. As $d(z_i, x_i) < \varepsilon$ for all i, the existence of h_3 is given by (T_k).

Let $h_4 = h(h_3 h_2 h_1)^{-1}$. Then evidently $h = h_4 h_3 h_2 h_1$. By construction h_1, h_2, h_3 are in U'. It remains to show that h_4 is in $B_F^{\tau_p}$ (which is contained in U'), for then $h \in (U')^4 \subseteq U$.

As h, h_3, h_2 and h_1 are in $H(X|T)$, so is h_4. Thus h_4 is in $B_F^{\tau_p}$ if and only if $h_4(x_i) = x_i$ for $i = 1, \ldots, n$, and this occurs if and only if $h_4^{-1}(x_i) = x_i$ for $i = 1, \ldots, n$. Fix an i. Then: $h_4^{-1}(x_i) = (h_3 h_2 h_1) h^{-1}(x_i) = (h_3 h_2)(h_1(h^{-1}(x_i))) = h_3(h_2(y_i)) = h_3(z_i) = x_i$. □

Corollary 10. *For the following spaces X the only group topology contained in both the topology of pointwise convergence and the compact-open topology is the indiscrete topology. In particular, $H(X)$ does not have a minimum Hausdorff group topology.*

(a) Every compact manifold (without boundary) of dimension at least 2*,*

(b) the Cantor set, and

(c) the Hilbert cube.

Proof. Apply the Theorem 9 with $T = X$ and it is clear from the definitions that the compact manifold, the Cantor set and the Hilbert cube all satisfy (T_k) and (T_p).

$\square$

Corollary 11. *If X is a compact metric space containing an open n-cell, for $n \geq 2$, then $H(X)$ does not have a minimum Hausdorff group topology.*

Proof. Let $T \subseteq X$ be the hypothesized open n-cell. In Corollary 10 (a), an open n-cell with $n \geq 2$ has properties (T_k) and (T_p). Let T to be the hypothesized open n-cell, for $n \geq 2$, then $H(X|T)$ has properties (T_k) and (T_p). By Theorem 9, $H(X)$ does not have a minimum Hausdorff group topology. $\square$

3.2 SHRINKING THE COMPACT-OPEN TOPOLOGY

Let (X, d) be compact metric. Let C be a closed subset of X. For $\varepsilon > 0$ define $C_\varepsilon = C_\varepsilon^d = \{x \in X : d(x, C) < \varepsilon\}$. Let $\tau_{k|C}$ be the collection of all unions of all translates of sets of the form: $B_\varepsilon = B_\varepsilon^{\tau_{k|C}} := \{h \in H(X) : d(h(x), x) < \varepsilon, d(h^{-1}(x), x) < \varepsilon \text{ for all } x \in X \setminus C_\varepsilon\}$.

Remark 12. If d' is another metric on X which is compatible with d, then by compactness, given $\varepsilon > 0$ there is a $\delta > 0$ such that $C_\delta^d \subseteq C_\varepsilon^{d'}$ and $C_\delta^{d'} \subseteq C_\varepsilon^d$. So $\tau_{k|C}$ is independent of the choice of compatible metric.

To verify that we have indeed defined a topological group topology, we use Lemma 2. Recall that we have the following conditions to check for $\mathcal{U}$ (a family of subsets of G all containing id):

(a) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^2 \subseteq U$;

(b) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V^{-1} \subseteq U$;

(c) for each $U \in \mathcal{U}$ and $x \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $xV \subseteq U$;

(d) for each $U \in \mathcal{U}$ and $x \in G$, there exists $V \in \mathcal{U}$ with $xVx^{-1} \subseteq U$; and

(e) for each $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ with $W \subseteq U \cap V$.

Also recall that such topology τ is T_1 (or equivalently, Hausdorff, or Tychonoff) if and only if $\bigcap \mathcal{U} = \{id\}$.

Theorem 13. *Let (X, d) be compact metric, C be a closed subset of X and define $\tau_{k|C}$ as above. Then the following hold:*

(1) $(H(X), \tau_{k|C})$ is a group topology provided *C is $H(X)$-invariant, i.e. $h(C) = C$ for all h in $H(X)$,*

(2) $\tau_{k|C}$ is T_0 if C is closed nowhere dense, and

(3) $\tau_{k|C} \subseteq \tau_k$ with equality if and only if for every $\varepsilon > 0$ there is an $0 < \delta \leq \varepsilon$ such that if h in $H(X)$ satisfies: $d(h(x), x) < \delta$ and $d(h^{-1}(x), x) < \delta$ for all $x \notin C_\delta$ then $d(h(x), x) < \varepsilon$ and $d(h^{-1}(x), x) < \varepsilon$ for all $x \in C_\delta$.

Proof.

For (1): We verify conditions (a) through (e) of the neighborhood characterization of a topological group topology (Lemma 2).

(a) We need to show: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $B_\delta^2 \subseteq B_\varepsilon$.

To this end fix $\varepsilon > 0$ and set $\delta = \varepsilon/2$. Take any $g_1, g_2 \in B_\delta$. Fix $x \notin C_\varepsilon$, then $x \notin C_\delta$. Since $d(g_1(x), x) < \delta$ we see $g_1(x) \notin C_\delta$, so $d(g_2 g_1(x), g_1(x)) < \delta$ and then $d(g_2 g_1(x), x) < 2\delta < \varepsilon$. Similarly $d(g_1^{-1} g_2^{-1}(x), x) < \varepsilon$. So for all $g_1, g_2 \in B_\delta, g_2 g_1 \in B_\varepsilon$, in other words $B_\delta^2 \subseteq B_\varepsilon$, as required.

(b) We need to show: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $B_\delta^{-1} \subseteq B_\varepsilon$. But by definition of B_ε, we have $B_\varepsilon = B_\varepsilon^{-1}$, so we can take $\delta = \varepsilon$.

(c) We need to show: for all $\varepsilon > 0$ and $f \in B_\varepsilon$, there is a $\delta > 0$ such that $fB_\delta \subseteq B_\varepsilon$.

Fix then $f \in B_\varepsilon$, and note $f^{-1} \in B_\varepsilon$. By compactness of X,

$$\varepsilon_1 := \max\{d(f(x), x), d(f^{-1}(x), x) : x \notin C_\varepsilon\} < \varepsilon.$$

Let $\varepsilon_2 := \varepsilon - \varepsilon_1$. Since f and f^{-1} are uniformly continuous on $X \setminus C_\varepsilon$, there is a $\delta' > 0$ such that for every $x, y \in X \setminus C_\varepsilon$ if $d(x, y) < \delta'$ then $d(f(x), f(y)) < \varepsilon_2$ and $d(f^{-1}(x), f^{-1}(y)) < \varepsilon_2$. Now let $\delta = \min\{\delta', \varepsilon_2\}$. Take any $x \notin C_\varepsilon$. Then $x \notin C_\delta$. Take any $g \in B_\delta$. Then $g^{-1} \in B_\delta$. So if $d(g(x), x) < \delta$ then $d(fg(x), f(x)) < \varepsilon_2$, and hence $d(fg(x), x) \leq d(fg(x), f(x)) + d(f(x), x) \leq \varepsilon_2 + \varepsilon_1 = \varepsilon$.

On the other hand, $d(f^{-1}(x), x) \leq \varepsilon_1$. Since $x \notin C_\varepsilon$, we know $f^{-1}(x) \notin C_\delta$, and so $d(g^{-1}f^{-1}(x), f^{-1}(x)) < \delta \leq \varepsilon_2$. Thus $d((fg)^{-1}(x), x) \leq d(g^{-1}f^{-1}(x), f^{-1}(x)) + d(f^{-1}(x), x) < \varepsilon_2 + \varepsilon_1 = \varepsilon$, and $fg \in B_\varepsilon$ for all $g \in B_\delta$, as required.

(d) We need to show: for every $\varepsilon > 0$, and $f \in H(X)$ there is a $\delta > 0$ such that $fB_\delta f^{-1} \subseteq B_\varepsilon$.

Fix $f \in H(X)$. We make two observations: (1) as C_ε is open in X, $f(C_\varepsilon)$ is open in X, and since C is $H(X)$-invariant (specifically $f(C) = C$) there is a δ_1 such that $C_{\delta_1} \subseteq f(C_\varepsilon)$; and (2) as f^{-1} is uniform continuous there is a δ_2 such that for all $x, y \in X$ if $d(x, y) < \delta_2$ then we have $d(f^{-1}(x), f^{-1}(y)) < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Observation (1) implies $x \notin C_\varepsilon, f(x) \notin C_\delta$. So for all $g \in B_\delta$, we have $d(gf^{-1}(x), f^{-1}(x)) < \delta$ and $d(g^{-1}f^{-1}(x), f^{-1}(x)) < \delta$. Observation (2) implies $d(fg^{-1}f^{-1}(x), ff^{-1}(x)) < \varepsilon$ and $d(fgf^{-1}(x), x) < \varepsilon$. Hence $fB_\delta f^{-1} \subseteq B_\varepsilon$, as required.

(e) We need to show: for all $\varepsilon_1, \varepsilon_2$ there is a $\delta > 0$ such that $B_\delta \subseteq B_{\varepsilon_1} \cap B_{\varepsilon_2}$. But after setting $\delta = \min\{\varepsilon_1, \varepsilon_2\}$ we are done.

For (2): Suppose C is closed nowhere dense. We show $\tau_{k|C}$ is T_0, by verifying that $\{id\} = \bigcap\{B_\varepsilon : \varepsilon > 0\}$.

Clearly $\{id\} \subseteq \bigcap\{B_\varepsilon : \varepsilon > 0\}$. If, for a contradiction, there is a g in the intersection such that $g \neq 1$, then there must be an $x \in X$ such that $d(g(x), x) = \varepsilon > 0$. If $x \in X \setminus C$, then $d(x, C) = \varepsilon' > 0$. Let $\delta = \min\{\varepsilon', \varepsilon\}$, then $d(g(x), x) > \delta/2$, so $g \notin B_{\delta/2}$. Contradiction! It follows that $g(x) \neq x$, for some $x \in C$ and $g(y) = y$ for all $y \notin C$. Since g is uniformly continuous, there is a δ' such that if $d(x, y) < \delta'$ then $d(g(x), g(y)) < \varepsilon/3$. Let $\delta = \min\{\delta', \varepsilon/3\}$. As C is closed and nowhere dense, C does not contain any open ball in X, in particular $\mathcal{B}_\delta(x) := \{t \in X : d(t, x) < \delta\} \not\subseteq C$. So there is a $y \notin C$ such that $d(x, y) < \delta$. Then $\varepsilon = d(f(x), x) \leq d(x, y) + d(y, f(y)) + d(f(y), f(x)) < \varepsilon/3 + 0 + \varepsilon/3 < \varepsilon$. Contradiction again, and Claim (2) is established.

For (3): Clearly $\tau_{k|C} \subseteq \tau_k$. We verify that equality holds if and only if for all $\varepsilon > 0$ there is a $0 < \delta \leq \varepsilon$ such that if h in $H(X)$ satisfies: $d(h(x), x) < \delta$ and $d(h^{-1}(x), x) < \delta$ for all $x \notin C_\delta$ then $d(h(x), x) < \varepsilon$ and $d(h^{-1}(x), x) < \varepsilon$ for all $x \in C_\delta$ $(*)$.

($\Rightarrow$) Suppose equality holds, and in particular $\tau_k \subseteq \tau_{k|C}$. Then given any $\varepsilon > 0$ we know $B_\varepsilon^{\tau_k}$ is open in $\tau_{k|C}$. So there is a $\delta > 0$ such that $B_\delta^{\tau_{k|C}} \subseteq B_\varepsilon^{\tau_k}$. Of course we may assume $\delta \leq \varepsilon$. We show δ satisfies $(*)$.

So take any h in $H(X)$ such that $d(h(x), x) < \delta$ and $d(h^{-1}(x), x) < \delta$ for all $x \notin C_\delta$. Then h is in $B_\delta^{\tau_{k|C}}$. So h is in $B_\varepsilon^{\tau_k}$, which means $d(h(x), x) < \varepsilon$ and $d(h^{-1}(x), x) < \varepsilon$ for all $x \in X$, and hence certainly all x in C_δ.

(⇐) Now suppose we have $(*)$. Take any basic $B_\varepsilon^{\tau_k}$. We need to show it is in $\tau_{k|C}$ – in other words, we need to find a $0 < \delta$ so that $B_\delta^{\tau_{K|C}} \subseteq B_\varepsilon^{\tau_k}$, in other words: if h is in $H(X)$ and $d(h(x), x) < \delta$ and $d(h^{-1}(x), x) < \delta$ for all $x \notin C_\delta$ then $d(h(x), x) < \varepsilon$ and $d(h^{-1}(x), x) < \varepsilon$ for all $x \in X$.

However, let δ be as given by $(*)$. If $h \in H(X)$ and $d(h(x), x) < \delta$ and $d(h^{-1}(x), x) < \delta$ for all $x \notin C_\delta$ we have $d(h(x), x) < \varepsilon$ and $d(h^{-1}(x), x) < \varepsilon$ for all $x \in C_\delta$. Since $\delta \leq \varepsilon$ we also have $d(h(x), x) < \varepsilon$ and $d(h^{-1}(x), x) < \varepsilon$ for all $x \notin C_\delta$. And so the relevant inequalities hold for all x in X – as needed. □

Corollary 14. *The compact-open topology on the autohomeomorphism group of a compact manifold with non-trivial boundary is not minimal.*

Proof. Let M be a compact manifold with non-empty boundary, C. Applying Theorem 13 we get that $\tau_{k|C}$ is the minimum Hausdorff topological group topology on $H(M)$. Clearly condition 3) fails, so τ_k is not minimal.

□

Remark 15. In particular, the compact-open topology on the autohomeomorphism group of the Mobius band is not minimal, answering Question 4.28 of [9].

3.3 AUTOHOMEOMORPHISM GROUPS WITH MINIMUM

Let X be a space. Recall that O_X ('the one-manifold part of X') is $\{x \in X : x$ has a neighborhood homeomorphic to $(0,1)\}$. Recall that S_X is all points with a clopen neighborhood homeomorphic to the circle, S^1, and $I_X = O_X \setminus S_X$. Recall that $C_X = X \setminus O_X$. For any homeomorphism h of a space X, define $\text{Move}(h) = \{x \in X : h(x) \neq x\}$.

Remark 16. Suppose X is compact and metrizable. Then observe:

(a) I_X is a countable (possibly empty) disjoint sum of copies of $(0, 1)$, and S_X is a countable (possibly empty) disjoint sum of circles, and

(b) O_X is open, and the disjoint sum of I_X and S_X.

Our key result on the existence of a minimum Hausdorff group topology on certain autohomeomorphism groups is the following.

Theorem 17. *If X is a compact metrizable space such that O_X is dense in X, then $\tau_{k|C_X}$ is the minimum Hausdorff topological group topology on $H(X)$.*

However, to deal with automorphism groups in the Section 3.4 we prove a strengthening of Theorem 17 to include certain subgroups of the autohomeomorphism group. Note that $H([0, 1])$ is non-Abelian so the next result is indeed stronger.

Theorem 18. *Let X be a compact metrizable space such that O_X is dense in X. Let G be a subgroup of $H(X)$ such that for every open subset U of X homeomorphic, via h, to $(0, 1)$ there are non-commuting p and q in G such that Move(p) and Move(q) are subsets of $h^{-1}[1/3, 2/3]$.*

Then the topology $\tau_{k|C_X}$ on $H(X)$ restricted to G is the minimum Hausdorff group topology on G.

Proof. Let d be a compatible metric on X. Let $C = C_X$ and recall C is the set of points in X with no neighborhood homeomorphic to $(0, 1)$. Clearly C is $H(X)$-invariant. As O_X is open and dense by hypothesis, C is closed and nowhere dense. Hence by Theorem 13, $\tau_{k|C}$ is a Hausdorff topological group topology on $H(X)$, contained in the compact-open topology τ_k. We show its restriction to G is the minimum Hausdorff topological group topology on G.

We denote the closed unit interval, $[0,1]$, by I and the unit circle, $\{(\cos(2\pi\theta), \sin(2\pi\theta)) : \theta \in [0,1)\}$, by S^1. Given $a < b$ we write $[a,b]$ for the subset of S^1 given by $\{(\cos(2\pi\theta), \sin(2\pi\theta)) : \theta \in [a,b]\}$. It will always be clear from context if '$[a,b]$' is an interval in $\mathbb{R}$ or a subset of S^1.

Fix $\varepsilon > 0$, let $C_\varepsilon = \{x \in X : d(x, C) < \varepsilon\}$. We need to show $B = B_\varepsilon^{\tau_k|C} \cap G = \{h \in G : d(h(x), x) < \varepsilon \text{ and } d(h^{-1}(x), x) < \varepsilon \text{ for all } x \notin C_\varepsilon\}$ is open in every Hausdorff group topology on G.

By compactness of X, $X \setminus C_\varepsilon \subseteq \bigoplus_{n=1}^{N} I_n \oplus \bigoplus_{m=1}^{M} S_{N+m} \subseteq X \setminus C$, where each I_n is homeomorphic to a closed interval and each S_{N+m} is homeomorphic to a circle. Each I_n is contained in a J_n homeomorphic to $\mathbb{R}$, and we fix a homeomorphism f_n of J_n with $\mathbb{R}$ so that f_n maps I_n to $I \subseteq \mathbb{R}$. For each S_{N+m} fix a homeomorphism f_{N+m} of S_{N+m} to S^1.

Then by the hypothesis on G for any $a < b$ and $k \leq N + M$ we can find non-commuting $p_k^{(a,b)}$ and $q_k^{(a,b)}$ in G such that $f_k(\mathrm{Move}(p_k^{(a,b)}))$ and $f_k(\mathrm{Move}(q_k^{(a,b)}))$ are subsets of $[a,b]$. Fix $k \leq N + M$ and define $T(a,b,k) = \{g \in G : g p_k^{(a,b)} g^{-1} \text{ and } q_k^{(a,b)} \text{ do not commute and } g^{-1} p_k^{(a,b)} g \text{ and } q_k^{(a,b)} \text{ do not commute}\}$.

Claim. In any Hausdorff group topology on G the set $T(a,b,k)$ is open and contains id.

For notational convenience, write p for $p_k^{(a,b)}$ and q for $q_k^{(a,b)}$. That id is in $T(a,b,k)$ is immediate since $id \circ p \circ id^{-1} = p = id^{-1} \circ p \circ id$, and $p = p_k^{(a,b)}$ does not commute with $q = q_k^{(a,b)}$.

Now consider the maps $\varphi, \psi : G \to G$ defined by $\varphi(g) = gpg^{-1}q(gpg^{-1})^{-1}q^{-1}$ and $\psi(g) = g^{-1}pgq(g^{-1}pg)^{-1}q^{-1}$. These maps are continuous for every topological group topology on G. Note that $\varphi(g) = id$ if and only if gpg^{-1} and q commute. Similarly, $\psi(g) = id$ if and only if $g^{-1}pg$ and q commute. Since $\{id\}$ is closed in any Hausdorff topology on G, the set $V = G \setminus \{id\}$ is open, and then the set $\varphi^{-1}(V) \cap \psi^{-1}(V) = T(a, b, k)$ is open in any Hausdorff topological group topology, as required.

Claim. If $g \in T(a, b, k)$, then there exists x, y in $f_k^{-1}[a, b]$ such that $g(x)$ and $g^{-1}(y)$ are in $f_k^{-1}[a, b]$.

Let $U_k = f_k^{-1}[a, b]$. We prove by contradiction that there is an x in U_k such that $g(x)$ is in U_k. By symmetry in the definition of $T(a, b, k)$, as g is in $T(a, b, k)$, so is g^{-1}, and the existence of the required y follows.

For notational simplicity, write p for $p_k^{(a,b)}$ and q for $q_k^{(a,b)}$. For concreteness let us suppose $k = N+m$, so U_k is a subset of S_k. (The case when $k \leq N$ is identical except each occurrence below of 'S_k' should be replaced with 'J_k'.) For the desired contradiction suppose that for all x in U_k we have $f_k(g(x)) \notin [a, b]$.

Note that if $z \in g(S_k \setminus U_k)$, then $g^{-1}(z) \in S_k \setminus U_k$. This implies $pg^{-1}(z) = g^{-1}(z)$, because $f_k(\mathrm{Move}(g)) \subseteq [a, b]$, and $gpg^{-1}(z) = gg^{-1}(z) = z$. Therefore $z \notin \mathrm{Move}(gpg^{-1})$, so $\mathrm{Move}(gpg^{-1}) \subseteq g(U_k)$.

Since $g(U_k) \subseteq X \setminus U_k$, we have $\mathrm{Move}(gpg^{-1}) \subseteq X \setminus U_k$. Then $\mathrm{Move}(gpg^{-1})$ and $\mathrm{Move}(q)$ are disjoint.

Fix $x \in S_k$. If $x \in U_k$, then $q(gpg^{-1})(x) = q(x) \subseteq U_k$, so $(gpg^{-1})q(x) = q(x)$. Thus q and gpg^{-1} commute at x. Similarly, if $x \in S_k \setminus U_k$, then q and gpg^{-1} commute at x.

Hence gpg^{-1} and q commute on S_k. By choice of $p = p_k^{(a,b)}$ and $q = q_k^{(a,b)}$, gpg^{-1} and q certainly commute outside S_k (where they are the identity). Since g is in $T(a, b, k)$, we have our desired contradiction.

We now show that some finite intersection of $T(a,b,k)$'s is contained in the given basic B. Since all $T(a,b,k)$'s are open neighborhoods of the identity in any Hausdorff group topology on G, this completes the proof that $\tau_{k|C_X}$ restricted to G is the minimum Hausdorff topological group topology on G.

To this end, pick $t \in \mathbb{N}$ such that for all $k \leq N+M$ and $i = -2, 0, \ldots, t$ the d-diameter of $f_k^{-1}[i/t, (i+1)/t]$ is $< \varepsilon/3$. Define

$$T = \bigcap \{T(i/t, (i+1)/t, k) : k \leq N+M,\ i = -2, 0, \ldots, t\}.$$

Claim. If $h \notin B$ then $h \notin T$.

Take any h in G which is not in B. So there is an $x \in X \setminus C_\varepsilon$ such that $d(h(x), x) \geq \varepsilon$ or $d(h^{-1}(x), x) \geq \varepsilon$. The sets $T(a,b,k)$ in the definition of T are symmetric – if h is in $T(a,b,k)$ then so is h^{-1} – so we may assume, without loss of generality, that $d(h(x), x)) \geq \varepsilon$. Then $x \in I_n$ for some n, in which case set $k = n$ and $A = I_k$, or $x \in S_{N+m}$ for some m, in which case set $k = N+m$ and $A = S_k$.

Consider an interval $K = [i/t, (i+1)/t]$ in the circle S^1. Let $K_- = [i_-/t, (i_-+1)/t]$ and $K_+ = [i_+/t, (i_++1)/t]$, where $i_- = (i-1) \pmod t$ and $i_+ = (i+1) \pmod t$. So K_- is the interval 'preceding' K and K_+ is the interval 'succeeding' K in the natural cyclic order. Similarly, consider an interval $K = [i/t, (i+1)/t]$ in the closed unit interval I. Let $K_- = [i_-/t, (i_-+1)/t]$ and $K_+ = [i_+/t, (i_++1)/t]$, where $i_- = i-1$ and $i_+ = i+1$.

There is a unique interval $K = [i/t, (i+1)/t]$ such that $f_k(x)$ is in K but not in K_-. Note that $-1 \leq i \leq t-1$. We will show h is not in (at least) one of $T(i_-/t, (i_-+1)/t, k)$, $T(i/t, (i+1)/t, k)$, or $T(i_+/t, (i_++1)/t, k)$, and hence is not in T, as desired.

Otherwise, from the preceding Claim, we can pick x_{-1} in $f_k^{-1}K_-$ such that $h(x_{-1})$ is also in $f_k^{-1}K_-$, x_0 in $f_k^{-1}K$ such that $h(x_0)$ is in $f_k^{-1}K$, and $x_1 \in f_k^{-1}K_+$ such that $h(x_1) \in f_k^{-1}K_+$. Observe that it follows that h maps J_k (respectively, S_k) homeomorphically to J_k (respectively, S_k) if $k \leq N$ (respectively, $k > N$).

Indeed as $f_k(x_{-1}) < f_k(x_0) < f_k(x_1)$ (either in the cyclic order on S^1, or standard order on $\mathbb{R}$) h maps $f_k^{-1}[x_{-1}, x_1]$ into $f_k^{-1}(K_- \cup K \cup K_+)$. In particular $h(x)$ is in $f_k^{-1}(K_- \cup K \cup K_+)$. But as the d-diameter of each of $f_k^{-1}K_-$, $f_k^{-1}K$ and $f_k^{-1}K_+$ is no more than $\varepsilon/3$, this means $d(h(x), x) \leq 2(\varepsilon/3)$, contradicting $d(h(x), x) \geq \varepsilon$. □

3.3.1 Conditions on X ensuring $\tau_{k|C} \neq \tau_k$

Here we give sufficient conditions for the minimum topology to be strictly smaller than the compact-open topology. Note that this implies that the compact-open topology is not minimal.

Proposition 19. *Let X be compact metrizable, and suppose O_X is dense in X. Let $C = C_X$. In the following cases $\tau_{k|C} \neq \tau_k$, and the compact-open topology on $H(X)$ is not minimal:*

(a) C contains at least two points which are the limit of clopen circles ($|\overline{S_X} \cap C| \geq 2$), or

(b) there is a component I of I_X whose closure meets C in at least three points ($|\overline{I} \cap C| \geq 3$).

Proof. Fix a compatible metric d for X. In both cases we verify that the condition for equality of $\tau_{k|C}$ and τ_k in Theorem 13 (3) fails.

For (a): Let x_1 and x_2 be distinct points in C which are the limit of circles in S_X. Let $\varepsilon = d(x_1, x_2)/2$. Take any $0 < \delta \leq \varepsilon$. Then we can find circles S_1 and S_2 in S_X so that $S_i \subseteq B_d(x_i, \delta/4)$ for $i = 1, 2$. Define h to be a homeomorphism of X which is the identity outside $S_1 \cup S_2$ and which switches S_1 and S_2. Then h is the identity outside C_δ but moves points of C_δ (namely all those in $S_1 \cup S_2$) at least $d(x_1, x_2) - (\delta/4 + \delta/4) \geq d(x_1, x_2)/2 = \varepsilon$.

For (b): Fix the open interval I in I_X such that $|\overline{I} \cap C| \geq 3$. The set $\overline{I} \setminus I$, which is the remainder of a compactification of an open interval, has either one or two components. By hypothesis we can pick two distinct points of C, say x_1 and x_2, which are in the same component. Then x_1 and x_2 are in the closure of a ray, J, (homeomorphic to $[1/2, 1)$) contained in I. Let $\varepsilon = d(x_1, x_2)/2$. Take any $0 < \delta \leq \varepsilon$. Without loss of generality we can assume the ray J is contained in C_δ. We can pick y_1, y_1' and y_2, y_2' in the ray J so that: y_1 precedes y_1' in J, y_1' precedes y_2', y_2' precedes y_2 in J, $d(x_i, y_i) < \delta/4$ and $d(x_i, y_i') < \delta/4$ for $i = 1, 2$. Pick a homeomorphism h of X which is the identity outside the closed subinterval of J between y_1 and y_2, but which moves y_1' to y_2'. Then as in case (a), h is the identity outside C_δ but moves a point (namely, y_1') of C_δ at least $d(x_1, x_2)/2 = \varepsilon$. □

Example 20. (a) The autohomeomorphism group of the disjoint sum of two convergent sequences of circles, with the two limit points, ($X = \bigcup_n S_n(0,0) \cup \bigcup_n S_n(2,0) \cup \{(0,0), (2,0)\}$, where $S_n(\mathbf{x})$ is the circle of radius $1/n$ centered at $\mathbf{x}$), has a minimum group topology, but the compact-open topology is not minimal.

(b) The autohomeomorphism group of the topologist's sine curve has a minimum group topology, but the compact-open topology is not minimal.

(c) For every compact metric space K there is a compact metric $X = X_K$ such that O_X is dense and the countably infinite disjoint sum of circles, and C_X is homeomorphic to K. This space X_K has a minimum group topology, but the compact-open topology is not minimal.

(d) For every non-zero dimensional compact metric space K there is a compact metric $X =$

X_K such that O_X is dense and the countably infinite disjoint sum of open intervals, and C_X is homeomorphic to K. This space X_K has a minimum group topology, but the compact-open topology is not minimal.

Proof. Examples (a) and (b) are easy exemplars of cases (a) and (b) of Proposition 19. We sketch (c) and (d).

For (c) recall that every compact metric space is the remainder of a compactification of $\mathbb{N}$. It is clear we can replace each n in $\mathbb{N}$ with a circle. This gives X_K. That $\tau_{k|C} \neq \tau_k$ is immediate from (a) of the preceding proposition.

For (d) first note that every compact metric space, K, is the remainder of a compactification, Z_K, of a countably infinite disjoint sum of open intervals (see Lemma 24 below). Our hypothesis is that K is not zero-dimensional. So it is not totally disconnected, and contains a non-trivial component K'. Now we can add an open interval, I, to Z_K, to get a new compact metric space X_K, so that $\overline{I} \setminus I = K'$. Then X_K has the required topological properties, and has $\tau_{k|C} \neq \tau_k$ by case (b) of the preceding proposition.

□

3.3.2 Conditions on X ensuring $\tau_{k|C} = \tau_k$

Here we give sufficient conditions for the minimum topology to coincide with the compact-open topology. Note that this implies that the compact-open topology is minimal.

Proposition 21. *Let X be compact metrizable, and suppose O_X is dense in X. Let $C = C_X$. In the following cases $\tau_{k|C} = \tau_k$, and the compact-open topology on $H(X)$ is minimal:*

(a) C is zero-dimensional, $X \setminus S_X$ has only finitely many components, and S_X has at most

one limit point in C, or

(b) C is a convergent sequence and S_X has at most one limit point in C.

Proof. Fix a compatible metric d for X.

For (a): We will prove this in three steps. In all cases $C = C_X$ is zero-dimensional.

Let us start by assuming that X is connected and S_X is empty. We claim that X is path connected. This is not difficult to prove directly. Alternatively we can argue as follows. Since O_X is dense but S_X is empty, we have that I_X is dense. Let I_X^1 be all components (open intervals) in I_X whose closure is a circle and I_X^2 be all open intervals in I_X whose closure is an arc. As C is zero-dimensional, every open interval in I_X has closure either a circle or an arc. As X is connected it is clear that $X' = X \setminus I_X^1$ is also connected. Then X' is a graph-like metric continuum, so [35, 2.1] is locally connected. Then we see that X' is a Peano continuum, and so path connected. Reattaching all of the open intervals in I_X^1 (and their unique limit point in C) preserves path connectedness, so X is path connected.

Now we show that $\tau_{k|C} = \tau_k$ provided $S_X = \emptyset$ and X has finitely many components. Note that from above each component of X is path connected. We verify that the condition in Theorem 13 (3) for equality of $\tau_{k|C}$ and τ_k is satisfied.

Fix $\varepsilon > 0$. Since C is zero-dimensional and X has finitely many components, we can partition C into finitely many pieces, say $C_1, \ldots, C_n$, each of d-diameter strictly less that $\varepsilon/3$, ensuring that each component of X contains at least two pieces of the partition. Pick positive δ smaller than $\varepsilon/3$ and $\frac{1}{3}\min\{d(C_i, C_j) : i \neq j\}$. Note that C_δ is the disjoint union of $(C_i)_\delta = \{x \in X : d(C_i, x) < \delta\}$ for $i = 1, \ldots, n$.

Take any h in $H(X)$ such that $d(h(x), x) < \delta$ and $d(h^{-1}(x), x) < \delta$ for all x not in C_δ. Suppose, for a contradiction, that for some x in C_δ either $d(h(x), x) \geq \varepsilon$ or $d(h^{-1}(x), x) \geq \varepsilon$.

We can assume, without loss of generality, that in fact $d(h(x),x) \geq \varepsilon$.

Suppose $h(x)$ is not in C_δ. Then let $y = h(x)$. Now $y \notin C_\delta$ but $d(h^{-1}(y),y) = d(x,h(x)) \geq \varepsilon > \delta$, contradicting the hypothesis on h. So $h(x)$ is in C_δ. Fix i such that x is in $(C_i)_\delta$. Fix j such that $h(x)$ is in $(C_j)_\delta$. Since the diameter of C_i is $< \varepsilon/3$ and $\delta < \varepsilon/3$, the diameter of $(C_i)_\delta$ is $< \varepsilon$. Since $d(h(x),x) \geq \varepsilon$ we see $i \neq j$.

As each component of X is path connected and contains at least two pieces of the partition of C, there is an arc A from x to some point not in $(C_i)_\delta$ (indeed to a point in some C_k where $k \neq i$). As we travel along the arc A from x there is a (unique) first point, x', where A exits $(C_i)\delta$. Then x' is not in C_δ, so $d(h(x'),x') < \delta$, but $d(x',C_i) \leq \delta$, and hence $h(x')$ cannot be in $(C_j)_\delta$.

Now we see that $h(A)$ is an arc starting at $y = h(x)$ which exits $(C_j)_\delta$, say for the first time at y'. Then y' is not in C_δ, so $d(h^{-1}(y'),y') < \delta$, but $d(y',(C_j)_\delta) \leq \delta$, and hence $x'' = h^{-1}(y')$ cannot be in $(C_i)_\delta$. But x'' is in $(C_i)_\delta$, since x'' is on A before it exits $(C_i)_\delta$ for the first time at x'. Contradiction!

To complete the proof of (a), suppose S_X has at most one limit point in C and $X \setminus S_X$ has finitely many components. Note that every h in $H(X)$ maps each circle in S_X to another circle in S_X, and if S_X has a unique limit point then it is a fixed point of h. Take any $\varepsilon > 0$. If S_X has no limit in C (so it is a finite union of circles) then pick $\delta_0 > 0$ so that C_{δ_0} is disjoint from S_X. Otherwise, let x_0 be the unique limit in C of S_X. In this case pick $\delta_0 > 0$ so that every circle in S_X meeting $B_\delta(x_0)$ has diameter $< \varepsilon/3$. Now apply the previous step to $X' = X \setminus S_X$ and get a relevant $\delta_1 > 0$. Let $\delta = \min(\delta_0, \delta_1)$. It is straightforward to check that if h in $H(X)$ is such that $d(h(x),x) < \delta$ and $d(h^{-1}(x),x) < \delta$ for all $x \in X \setminus C$, then $d(h(x),x)$ and $d(h^{-1}(x),x)$ are both $< \varepsilon$ for all x in C_δ.

For (b): We assume $S_X = \emptyset$. Otherwise we can deal with S_X as we did at the end of case (a). By hypothesis C is a convergent sequence, say with unique limit point c.

Fix $\varepsilon > 0$. Then (replacing ε with something smaller, if necessary) we can find N and write $C = \{c_n : n \in \mathbb{N}\}$ so that c_n is not in $\overline{B_{\varepsilon/3}(c)}$ for $n \leq N$ but $c_n \in B_{\varepsilon/3}(c)$ for $n > N$. Pick $\delta > 0$ satisfying the following conditions: (i) $\delta < \varepsilon/3$, (ii) $B_{2\delta}(c_i) \cap B_{2\delta}(c_j) = \emptyset$ for distinct $i, j \leq N$, (iii) $B_{2\delta}(c_i) \cap \overline{B_{\varepsilon/3}(c)} = \emptyset$ for all $i \leq N$, (iv) for $i \leq N$ if I is a component of I_X (so, an open interval) such that $I \cap B_{2\delta}(c_i) \neq \emptyset$ then $c_i \in \overline{I}$, and (v) for $i \leq N$ if I is a component of I_X (so, an open interval) with two endpoints, one of which is c_i then $B_\delta(c_i) \cap I$ is a subinterval of I. (Note for (iv) and (v) there are only finitely many open intervals I to deal with for each $i \leq N$.)

Fix $h \in H(X)$ such that $d(h(x), x)$ and $d(h^{-1}(x), x)$ are both $< \delta$ for all x not in C_δ. Suppose, for a contradiction, that for some x in C_δ either $d(h(x), x) \geq \varepsilon$ or $d(h^{-1}(x), x) \geq \varepsilon$. We can assume, without loss of generality, that in fact $d(h(x), x) \geq \varepsilon$.

As C is h-invariant, h must take c to $h(c)$ and any c_i to some c_j. By choice of δ and restriction on h we see that in fact h is the identity on $\{c\} \cup \{c_1, \ldots, c_N\}$. Then for $i > N$ we must have that $h(c_i) = c_j$ where $j > N$. It follows (as all $d(c_i, c_j) < \varepsilon$ for all $i, j > N$) that x cannot be in C. Further, $y = h(x)$ is in C_δ, for if y is not in C_δ then $d(h^{-1}(y), y) = d(x, h(x)) \geq \varepsilon > \delta$, contradicting the hypothesis on h.

Thus x is in an open interval, I, a component of I_X. Let $\{c_i, c_j\} = \overline{I} \backslash I$ be the endpoint(s) of this interval. If both $i, j > N$ then $h(x)$ is not in any $B_\delta(c_k)$ for any $k \leq N$, but is in C_δ, and hence x and $h(x)$ are in $\varepsilon/3$-ball around c – contradicting $d(h(x), x) \geq \varepsilon$.

If $i = j \leq N$ then $\overline{I}$ is a circle containing c_i. Then $h(\overline{I})$ is also a circle containing I, and so meets $B_\delta(c_i)$. By conditions (iii) and (iv) $h(\overline{I})$ meets no part of C_δ. Hence $h(x)$ must be in $B_\delta(c_i)$, and $d(h(x), x) < \varepsilon$, contradiction.

Otherwise, at least one of i and j is $\leq N$. Say i. By condition (v), the arc A, which is the subinterval of the arc $\overline{I}$ obtained by starting at c_i and traveling along the arc to x, is contained in $B_\delta(c_i)$. Then $h(A)$ is an arc traveling from c_i to $h(x)$ which is in C_δ but not in $B_\delta(c_i)$. So let y' be the point on the boundary of C_δ (not in C_δ) when $h(A)$ leaves the complement of C_δ near $h(x)$. Then, as $h^{-1}(y')$ is in A which is contained in $B_\delta(c_i)$ but y' is near $h(x)$, we see $d(y', h^{-1}(y')) > \delta$, contradicting the hypothesis on h. □

The following examples are all easily seen to satisfy condition (a) above and indeed have finite C.

Example 22. Examples of finite C (hence C is zero-dimensional) that satisfy condition Proposition 21 (a).

(a) The autohomeomorphism group of a convergent sequence of circles (with unique limit) has the minimum (and hence minimal) group topology, which is the compact-open topology.

(b) For every finite graph, the autohomeomorphism group with the compact-open topology is the minimum group topology (and so minimal).

(c) The autohomeomorphism group of the Hawaiian earring has the minimum (and hence minimal) group topology, which is the compact-open topology.

(d) More generally, if C is finite and S_X has a unique limit point in C, then $X \setminus S_X$ has only finitely many components and X can be obtained in the following way. Take any finite graph, which need not be connected, and may have multiple edges between vertices and from a vertex back to itself. Possibly 'decorate' one vertex by adding a sequence

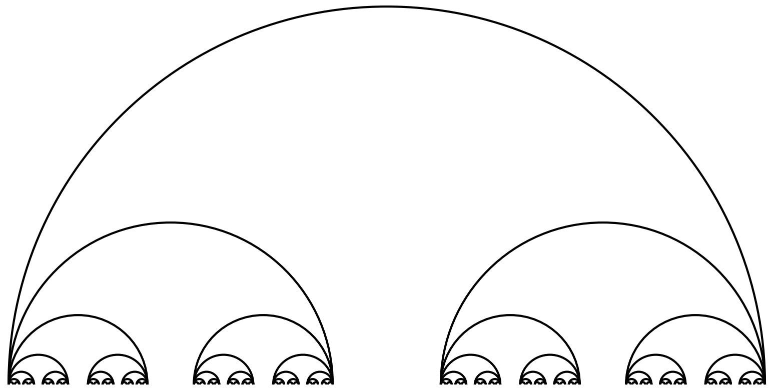

Figure 2: Cantor Bouquet of Semicircles

of circles converging to that vertex. Possibly decorate any other vertex by adding a Hawaiian earring at that vertex. Then the autohomeomorphism group of such a space has the minimum group topology.

Taking a convergent sequence, along with the limit point, of finite graphs or Hawaiian earrings (or spaces as in (d) above, but with no sequence of circles) will give spaces satisfying condition (b) of the preceding proposition.

Example 23. Many examples also arise when C is uncountable.

(a) The autohomeomorphism group of the Cantor bouquet of semi-circles has the minimum (and hence minimal) group topology, which is the compact-open topology.

(b) More generally, for any compact, connected metrizable graph-like space X, we see that $H(X)$ with the compact-open topology is the minimum group topology.

From (a) we deduce a general result.

Lemma 24. *For every compact metric space K there is a compact, metric connected $X =$*

X_K *such that* O_X *is dense,* $C_X = K$*, and* $\tau_{k|C} = \tau_k$*.*

Proof. Let Z be the space (Figure 2) from Example 23 (a). Every compact metric space K is the continuous image of the Cantor set. Applying this quotient to the copy of the Cantor set in Z gives $X = X_K$. It is clear that O_X is dense and $C_X = K$. Modifying the argument for Z easily shows that $\tau_{k|C} = \tau_k$ for X_K. □

3.3.3 The Remaining Cases

Let X be compact metrizable, and suppose O_X is dense in X. Let $C = C_X$. From Proposition 19 and Proposition 21, we (essentially) know whether or not $\tau_{k|C} = \tau_k$ except in two cases: (i) when S_X is empty, C is zero-dimensional but not a convergent sequence (or finite) and (ii) S_X is empty, C is not zero-dimensional but every component I in I_X has closure either an arc or a circle. Of particular interest in case (ii) is when C and/or X is connected, and preferably locally connected.

In each case there do not seem to be simple conditions we can place on X and/or C that allow us to determine if $\tau_{k|C}$ is, or is not, equal to τ_k. We demonstrate this with some examples, starting with case (i).

Example 25. Examples with C two convergent sequences

(a) $\tau_{k|C} \neq \tau_k$: two convergent sequences of Hawaiian earrings (Figure 3); two convergent sequences of arcs; two convergent sequences of double circles.

(b) $\tau_{k|C} = \tau_k$: a convergent sequence of $2n$-circles plus a convergent sequence of $2n+1$-circles; a convergent sequence of $2n$-ods plus a convergent sequence of $2n+1$-ods (Figure 4).

Similar examples exist when C is the Cantor set.

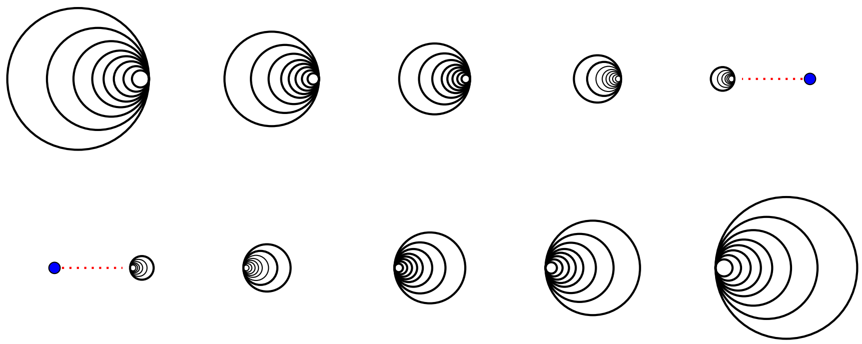

Figure 3: Convergent Sequences of Hawaiian Earrings

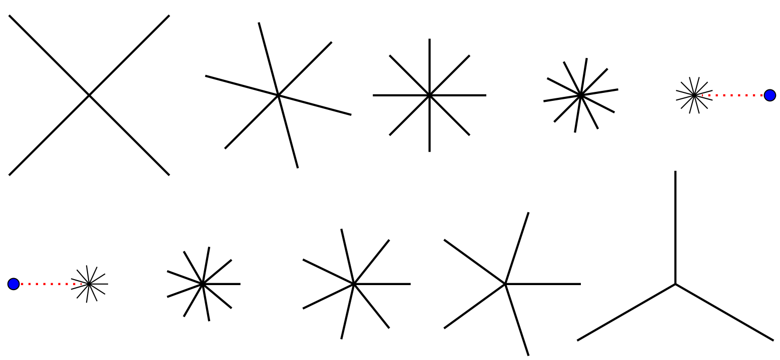

Figure 4: Convergent Sequences of Snowflakes

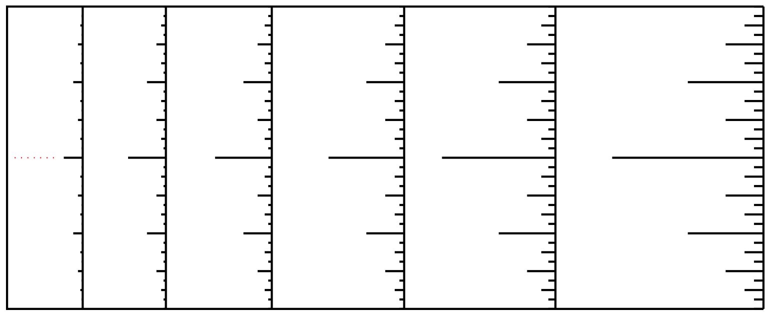

Figure 5: Rational Comb

So the remaining interesting case is when X and C are connected, S_X is empty and every I in I_X has closure an arc or a circle.

Example 26. Examples where $\tau_{k|C} = \tau_k$.

(a) Bouquet of circles over closed interval. In this case both X and C are connected and locally connected.

(b) More generally, apply Lemma 24 to any continuum K to get many examples with both X and C connected.

We conclude with two examples when $\tau_{k|C} \neq \tau_k$. Both are such that X is connected, and the second is additionally locally connected. We include both as the first example is a stepping stone to the second.

Example 27. Path connected but not locally connected example where $\tau_{k|C} \neq \tau_k$.

Proof. Each vertical line with attached horizontal lines forms a 'rational comb'. By shifting and scaling we can find a homeomorphism of the rational comb taking any given horizontal line to any other.

All the vertical lines are in C. The height of the boundary rectangle is 1 unit. Set $\varepsilon = 1/2$. Then for any $\delta > 0$, C_δ will contain one of the vertical lines distinct from the left edge, call it V, and all of the horizontal lines attached to it (in other words, a complete rational comb). Now we can find a homeomorphism of the continuum which is the identity outside the δ-neighborhood of the vertical line V, but inside that δ-neighborhood moves the rational comb around so some point is moved a distance at least $1/2$ (in fact we can move things any distance < 1). This establishes $\tau_{k|C} \neq \tau_k$. □

Note that we can modify Example 27 so that the horizontal 'teeth' of the rational combs in fact point out of the page. And we can replace each tooth with a circle (an arbitrarily small deformation of the original line segment).

Example 28. Connected and locally connected example with $\tau_{k|C} \neq \tau_k$.

Proof. We start by constructing the key building block, B, of the example. It is obtained by adjoining countably many circles (triangles in the Figure 6) to the unit square in the x-y plane ($[0,1]^2 \times \{0\}$). The circles have radii converging to zero, and are tangent to the points in the unit square whose x and y co-ordinates are both dyadic rationals. See the illustration. Note that each line, $\{x\} \times [0,1] \times \{0\}$, where x is a dyadic rational is homeomorphic to the variant of the rational comb described in the paragraph above.

The key property of our building block, B, is that we can find a homeomorphism h_0 of B moving some point on the line $\{1/2\} \times I \times \{0\}$ a distance at least $1/2$ along that line, but which is the identity on the boundary of the unit square. Note also that B is a subset of the triangular cylinder (dotted in the Figure 7), and C_B is the unit square.

To obtain the example X start with the unit square in the x, y-plane. Take countably

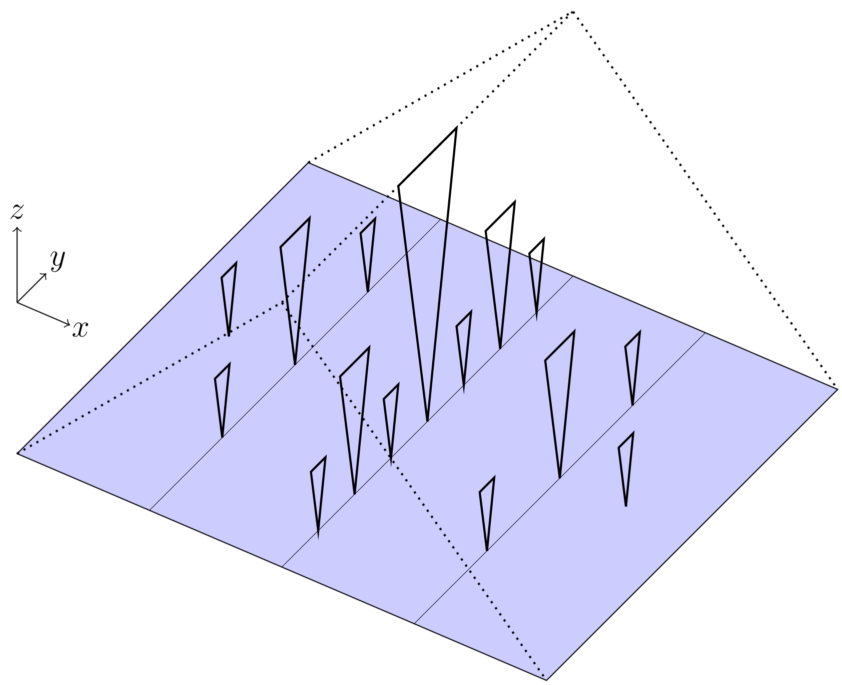

Figure 6: Cantor Triangles on Unit Square

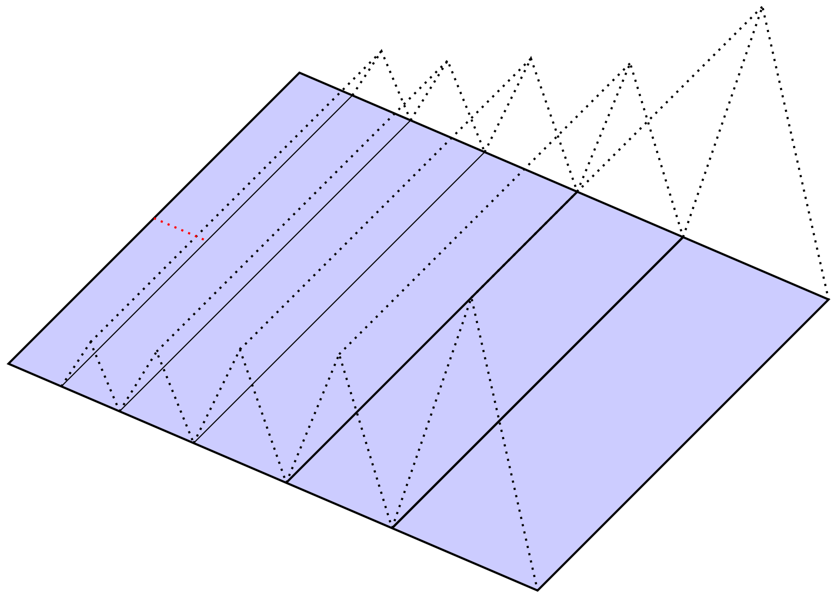

Figure 7: Steps of Cantor Triangles on Unit Square

many copies, B_n, of B. Scale (in the x and z directions only) and translate the B_n's to form a sequence, with heights shrinking to zero, converging to the left edge, $I \times \{0, 0)\}$ of the unit square.

This space X has the requisite properties. It is connected and locally connected. The set C_X is the unit square in the x, y-plane. While the proof that $\tau_{k|C} \neq \tau_k$ is similar to that for the Example 27.

Indeed let $\varepsilon = 1/4$. Then for any $\delta > 0$, we can find a copy of B completely contained in the δ-neighborhood of C. Extend the homeomorphism h_0 of B over all of X by making it the identity outside B. Then this extended homeomorphism, h_0, moves nothing outside C_δ, but moves – in the y-direction only – at least one point a distance $\geq \varepsilon$. $\square$

3.4 AUTOMORPHISM GROUPS

Recall that a closed subgroup of Sym($\mathbb{N}$) is oligomorphic if and only if for each $n \geq 1$ its natural action on $\mathbb{N}^n$ has finitely many orbits. Question 2.3 of [9] asks which oligomorphic groups have a minimum group topology, mentioning Sym($\mathbb{N}$), the automorphism group of the countable dense linear order, and the autohomeomorphism group of the Cantor space, in particular. We present a reasonably broad answer to this question, encompassing the mentioned groups.

Let $Q = \mathbb{Q} \cap (0,1)$. Then $\mathrm{Aut}(\mathbb{Q},<)$ (the automorphism group of the countable dense linear order) and $\mathrm{Aut}(Q,<)$ are isomorphic. Let S^1_Q – the 'rational circle' – be any countable dense subset of the circle, S^1.

Theorem 29. *Every highly homogeneous automorphism group G has a minimum group topology.*

That minimum group topology is strictly coarser than τ_p, except when $G =$ Sym($\mathbb{N}$), and so τ_p is not minimal.

Proof. Recall that for Sym($\mathbb{N}$) the topology of pointwise convergence is the minimum Hausdorff group topology [16], and note Sym($\mathbb{N}$) is the only automorphism group which is transitive.

Cameron showed [5] that the highly homogeneous non-transitive automorphism groups are precisely $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Aut}(S^1_Q,<)$.

The group $\mathrm{Aut}(Q,<)$ naturally embeds, say as G, in $H([0,1])$ (perhaps this is most clear if we think of elements of $[0,1]$ as Dedekind cuts of Q). Further G clearly satisfies the 'non-commuting pairs' condition of Theorem 18. Hence (G and its isomorph) $\mathrm{Aut}(Q,<)$

has the minimum Hausdorff group topology, τ_m. It is not difficult to see that in this case τ_m has the following sets as a neighborhood of the identity: $B_\varepsilon = \{\alpha \in \mathrm{Aut}(Q,<) : \forall x \in Q,\ d(\alpha(x),x) < \varepsilon\}$, where d is the usual metric on $\mathbb{R}$. No set of this form, B_ε, is a subset of $\{\alpha \in \mathrm{Aut}(Q,<) : \alpha(1/2) = 1/2\}$ which is open in the pointwise convergence topology, τ_p, on $\mathrm{Aut}(Q,<)$. Hence τ_m is strictly contained in τ_p, and so τ_p is not minimal.

Similar considerations apply to the following oligomorphic groups: all order preserving or order reversing bijections of Q, all bijections of the rational circle which preserve the cyclic order and all bijections of the rational circle which preserve or reverse the cyclic order $\mathrm{Aut}(S^1_Q,<)$. They are all oligomorphic. They all embed either in $H(I)$ or $H(S^1)$, satisfy the 'non-commuting pairs' condition of Theorem 18, and so they have a minimum Hausdorff group topology which is easily seen to be strictly finer than the pointwise convergence topology. □

However not all oligomorphic groups have a minimum Hausdorff group topology. Let B denote the (unique) atomless countable Boolean algebra. For example (see [38]), it can be described as the set of all periodic $\{0,1\}$-valued sequences $(x_n)_{n\in\mathbb{N}}$.

Example 30. The oligomorphic group $\mathrm{Aut}(B)$ has no minimum Hausdorff group topology.

Proof. The group $\mathrm{Aut}(B)$ is well known to be isomorphic to the autohomeomorphism group of the Cantor set, $H(C)$, and we have proved, Corollary 10, that this group does not have a minimum Hausdorff group topology. □

However, every oligomorphic group is Polish and Roelcke-precompact, so $\mathrm{Aut}(\mathbb{Q},<)$ gives a negative answer:

Corollary 31. *There is a Polish Roelcke-precompact group for which the minimum group topology is strictly coarser than* τ_p.

4.0 REALIZATION OF TOPOLOGICAL GROUPS AS GRAPHS

We adapt the two-step strategy that Frucht [12] used for realizing any arbitrary finite groups G algebraically as the automorphism group of a finite graph Γ_G to preserve topological isomorphism as follows:

Step 1. constructing a colored directed graph $\mathcal{C}_G$ such that $\mathrm{Aut}(\mathcal{C}_G) \cong G$ topologically;

Step 2. replacing the colored edges by corresponding rigid graphs to obtain Γ_G.

We topologically realize groups as the automorphism groups of locally finite (Theorem 35), regular (Theorem 38) and non-locally finite graphs (Theorem 41). The construction of graphs adopts the same two-steps strategy from Frucht. Instead of giving independent constructions for our theorems, we reorganize the contents in the two-step strategy format by proving technical theorems for the two steps first. In Section 4.1, similar to Step 1 constructing a colored and directed graph $\mathcal{C}$ such that $\mathrm{Aut}(\mathcal{C}) \cong G$ topologically, we prove Theorem 32) and Proposition 33. Next in Section 4.2, we describe the construction of a standard graph Γ from $\mathcal{C}$ and prove that $\mathrm{Aut}(\mathcal{C}) \cong \mathrm{Aut}(\Gamma)$ topologically (Theorem 34). Finally in Section 4.3, the main theorems and some corollaries are summarized and proved by using the technical results mentioned in the previous sections. To demonstrate the power of Theorem 38, we give an explicit construction of a regular graph Γ such that $\mathrm{Aut}(\Gamma) \cong \mathbb{Q}$ with the discrete topology in Section 4.4.

4.1 STEP 1: CONSTRUCTING USEFUL COLORED, DIRECTED GRAPHS

Let G be a separable metrizable group, with a decreasing sequence of open, profinite subgroups, $(U_n)_n$, forming a local base at the identity id. Let $T = \{t_0, t_1, \ldots, t_n, \ldots\}$ be a dense subset of G. For each n let T_n be a finite subset of T such that $T = \bigcup_n T_n$, and every t in T is in infinitely many T_n. Select, for each n, an injection c_n of T_n into the even natural numbers so that $c_n(T_n) \cap c_m(T_m) = \emptyset$ when $n \neq m$.

Define a colored, directed graph $\mathcal{C} = \mathcal{C}_G$ as follows. For each n let $\mathcal{C}_n = G/U_n$, the set of all left cosets of a U_n in G, and $\mathcal{C}_{\leq n} = \bigcup_{m \leq n} \mathcal{C}_m$. Then $\mathcal{C}$ is $\bigcup_n \mathcal{C}_n$. The colors of edges of $\mathcal{C}$ will be natural numbers. The directed, colored edges of $\mathcal{C}$ are of two types, defined as follows. For each left coset hU_n and t in T_n there is a 'horizontal' edge from hU_n to htU_n with color $c_n(t)$. For each pair of cosets hU_n and $h'U_{n+1}$ such that $hU_n \supseteq h'U_{n+1}$ there is a 'vertical' edge from $h'U_{n+1}$ to hU_n of color $2n + 1$.

We define a colored, directed graph to be connected if any two vertices are connected by a path of edges, ignoring color and orientation; and to be locally finite if for every vertex the number of edges leaving from or entering to it is finite.

Theorem 32. *Let $\mathcal{C}$ be the colored, directed graph associated with G a separable metrizable topological group, sequence, $(U_n)_n$ of open, profinite subgroups, and dense $T = \bigcup_n T_n$.*

Then $Aut(\mathcal{C}_G)$ is topologically isomorphic to G, and $\mathcal{C}$ has the following properties: it is connected, locally finite and can be written $\mathcal{C} = \bigcup_n \mathcal{C}_n$, where every v in $\mathcal{C}_n$ has: $|T_n|$-many 'horizontal' edges entering v, $|T_n|$-many 'horizontal' edges exiting v, $|U_n/U_{n+1}|$-many 'vertical' edges entering v, and if $n \geq 1$, exactly one 'vertical' edge exiting v, and these are *all edges entering or exiting v.*

Proof. We first check the properties of the graph $\mathcal{C}$, then verify some properties of its automorphisms, and complete the proof by showing $\mathrm{Aut}(\mathcal{C})$ and G are topologically isomorphic. We use id_G and $id_\mathcal{C}$ for the identity elements of G and $\mathrm{Aut}(\mathcal{C})$, respectively.

Properties of $\mathcal{C}$. Clearly, from the definition, $\mathcal{C}$ is countable and the claimed list of edges entering or exiting a given vertex is as stated. Hence $\mathcal{C}$ is locally finite.

Note that each $\mathcal{C}_n$ is a subgraph, with only horizontal edges, which may not be connected (T_n is not, necessarily, a generating set for G). Nevertheless, $\mathcal{C}$ is connected. To see this we show that for any hU_n in $\mathcal{C}$ there is an edge path from U_0 to hU_n. Pick t in $T \cap hU_n$, so $tU_n = hU_n$, and pick $N \geq n$ so that $t \in T_N$. Then there is a path of vertical edges from U_0 up to U_N, and an edge from U_N to tU_N (as t is in T_N), and then a path of vertical edges down from tU_N to $tU_n = hU_n$.

Properties of Automorphisms of $\mathcal{C}$. First note that, for each n, the nth level, $\mathcal{C}_n$, is invariant under $\mathrm{Aut}(\mathcal{C})$, and $\mathcal{C}_{\leq n}$ is also invariant. Conversely, if α is a map from $\mathcal{C}$ to $\mathcal{C}$ which, for every n, is an automorphism when restricted to $\mathcal{C}_{\leq n}$, then α is an automorphism of $\mathcal{C}$.

Claim (Key Properties)**.** Take any α from $\mathrm{Aut}(\mathcal{C})$ and hU_n in $\mathcal{C}$. Then there is an $N = N(hU_n) \geq n$ such that (1) for every g in $\alpha(U_N)$ we have $\alpha(hU_n) = ghU_n$, (2) $\alpha(hU_n) = \alpha(U_N)hU_n$, and (3) if $\alpha(U_N) = U_N$ then $\alpha(hU_n) = hU_n$.

To see this pick t in $T \cap hU_n$, so $tU_n = hU_n$, and pick $N \geq n$ so that $t \in T_N$. Take any g in $\alpha(U_N)$, so $\alpha(U_N) = gU_N$. As t is in T_N there is an edge from U_N to tU_N colored $c_N(t)$. As α is an automorphism there is an edge from $\alpha(U_N) = gU_N$ to $\alpha(tU_N)$ colored $c_N(t)$.

Hence $\alpha(hU_N) = \alpha(tU_N) = gt(U_N)$. So $\alpha(hU_N)$ is contained in both $\alpha(hU_n)$ (as $N \geq n$, and α takes vertical edges to vertical edges) and $ghU_n = ghU_NU_n = gtU_NU_n \supseteq gtU_N$. Thus $\alpha(hU_n) = ghU_n$, and (1) holds.

Now $\alpha(U_N)hU_n = \bigcup_{g\in\alpha(U_N)} ghU_n = \bigcup_{g\in\alpha(U_N)} \alpha(hU_n) = \alpha(hU_n)$, which is (2). While, if $\alpha(U_N) = U_N$, then take $g = id_G$ in $\alpha(U_N)$ and apply (1) to get (3).

Aut($\mathcal{C}$) is topologically isomorphic to G via Ψ and Φ. Define $\Psi : G \to \mathrm{Aut}(\mathcal{C})$ by $\Psi(g)(hU_n) = ghU_n$. Define $\Phi : \mathrm{Aut}(\mathcal{C}) \to G$ by $\Phi(\alpha) = \mathrm{pt}(\bigcap_n \alpha(U_n))$ where pt is the function that extracts the element from a singleton set.

Since for a 'horizontal' edge from hU_n to htU_n with color $c_n(t)$, we have $\Psi(g)(hU_n) = ghU_n$ and $\Psi(g)(htU_n) = ghtU_n$ with color $c_n(t)$, and so $\Psi(g)$ preserves the 'horizontal' colored edges. As for a 'vertical' edge from $h'U_{n+1}$ to hU_n of color $2n+1$, we have $\Psi(g)(h'U_{n+1}) = gh'U_{n+1}$ and $\Psi(g)(hU_n) = ghU_n$ with color $2n+1$, and so $\Psi(g)$ preserves the 'vertical' colored edges. A direct computation shows $\Psi(id_G) = id_{\mathcal{C}}$ and $\Psi(g)\Psi(g') = \Psi(gg')$. So $id_{\mathcal{C}} = \Psi(gg^{-1}) = \Psi(g)\Psi(g^{-1})$, and $\Psi(g)$ has an inverse. Thus, $\Psi(g)$ is an automorphism of $\mathcal{C}$, and Ψ is a homomorphism.

The vertical edges ensure that for all n we have $\alpha(U_n) \supseteq \alpha(U_{n+1})$. Since $\bigcap_n U_n = \{id_G\}$, by compactness of U and $\alpha(U)$, we see $\bigcap_n \alpha(U_n)$ contains precisely one element, and Φ is well-defined. Further $\Phi \circ \Psi$ is the identity: $(\Phi \circ \Psi)(g) = \Phi(\Psi(g)) = \mathrm{pt}(\bigcap_n \Psi(g)(U_n)) = \mathrm{pt}(\bigcap_n gU_n) = g$ because $gU_n \supseteq gU_{n+1}$ for all n and $\bigcap_n U_n = \{id_G\}$ implies $\bigcap_n gU_n = \{g\}$.

Next we show $\Psi \circ \Phi$ is the identity. Take any α in $\mathrm{Aut}(\mathcal{C})$. Take any hU_n in $\mathcal{C}$. We need to show $\Psi(\Phi(\alpha))(hU_n) = \alpha(hU_n)$. By the Key Property (2), there is an N such that $\alpha(hU_n) = \alpha(U_N)hU_n$. But, by definition, $\Phi(\alpha)$ is in $\alpha(U_N)$. Hence $\alpha(hU_n) = \Phi(\alpha)hU_n = \Psi(\Phi(\alpha))(hU_n)$, by definition of Ψ.

Therefore, Φ and Ψ are mutually inverse (algebraic isomorphisms).

It remains to show Ψ and Φ are continuous. Let $W = \mathrm{Aut}(\mathcal{C})_{U_0}$. Then W is an open profinite subgroup of $\mathrm{Aut}(\mathcal{C})$. For any α in W, we have $\alpha(U_0) = U_0$. In fact, for all n we have $\alpha(U_n) \subseteq U_0$, and so $\Phi(\alpha)$ is in U_0. So $\Phi(W) \subseteq U$. But conversely, for any g in U, $\Psi(g)(U_0) = gU_0 = U_0$, and $\Psi(U_0) \subseteq W$. Thus Ψ maps U_0 (algebraically) isomorphically to W.

As U_0 is open in G, W is open in $\mathrm{Aut}(\mathcal{C})$, Ψ and Φ are homomorphisms, and G and $\mathrm{Aut}(\mathcal{C})$ are topological groups, Ψ and Φ are continuous if and only if their restrictions to U and W, respectively, are continuous. Further, by compactness of U_0 and Hausdorffness of W, it suffices to check that $\Psi|_{U_0} : U_0 \to W$ is continuous at e, the identity of G.

A basic neighborhood of the identity in W has the form $B_F = \{\alpha \in W : \alpha(hU_n) = hU_n \text{ for all } hU_n \text{ in } F\}$, where F is any finite subset of $\mathcal{C}$. Pick N sufficiently large that $N \geq N(hU_n)$, where $N(hU_n)$ is given by Key Property (3) applied to each hU_n in F. Now U_N is a neighborhood of the identity contained in U_0. For any g in U_N and hU_n in F, since $N \geq N(hU_n)$ we have $g \in U_N \subseteq U_{N(hU_n)}$ and so $\Psi(g)(U_{N(hU_n)}) = gU_{N(hU_n)} = U_{N(hU_n)}$, but by Key Property (3) on page 54 we then have $\Psi(g)(hU_n) = hU_n$. Thus $\Psi(U_N) \subseteq B_F$, as required. ∎

Proposition 33. *If G is a closed subgroup of $Sym(\kappa)$ then there is a connected, colored, directed graph $\mathcal{C}$ such that G is topologically isomorphic to $Aut(\mathcal{C})$ and $\mathcal{C}$ of size, and number of colors, no more than κ.*

Proof. For notational convenience, let $\kappa^0 = \{\emptyset\}$. Then for each $k \geq 0$, $\mathrm{Sym}(\kappa)$ acts on κ^k in the natural way, namely $\alpha \cdot \langle s_1, s_2, \ldots, s_k\rangle = \langle \alpha(s_1), \ldots, \alpha(s_k)\rangle$. For a fixed k, the orbits of the restriction of this action to elements of G partition κ^k say into $\mathcal{P}_k$. Let $\mathcal{P} = \bigcup_{k\geq 0} \mathcal{P}_k$. Then it is well known in the countable case, and straightforward to check in general, that $G = \{\alpha \in \mathrm{Sym}(\kappa) : \forall P \in \mathcal{P}\ \alpha \cdot P = P\}$.

Let the vertices of our colored, directed graph $\mathcal{C}$ be $\bigcup_{k\geq 0} \kappa^k$. The colors will be elements of κ. To help assign colors to edges fix, for each $k \geq 1$, injections $c_k : \mathcal{P}_k \times \{1, \ldots, k\} \to \kappa \backslash \{0\}$, with pairwise disjoint images. For $k \geq 1$, each $P \in \mathcal{P}_k \subseteq \mathcal{P}$, $\sigma = \langle s_1, \ldots, s_k\rangle$ from P and $1 \leq i \leq k$, add an edge to $\mathcal{C}$ of color $c_k(P, i)$ from σ in κ^k to $\langle s_i\rangle$ in κ^1. For each $\langle s\rangle$ in κ^1 add an edge to $\mathcal{C}$ from $\emptyset$ in κ^0 to $\langle s\rangle$ of color 0. Clearly $\mathcal{C}$ is connected (every σ in κ^k, where $k \geq 1$, is in some P in $\mathcal{P}_k$ and so connected to some element of κ^1, and every element of κ^1 is connected to $\emptyset$ in κ^0) and has size, and number of colors, no more than κ.

Observe that if α is an automorphism of $\mathcal{C}$, then it maps κ^1 bijectively with κ^1, and – since it must take an edge to an edge of the same color – by construction α takes an element P of the partition $\mathcal{P}_k$ back to P. Hence, after identifying κ^1 and κ, we see that $\alpha|_\kappa$ is in $\{\alpha \in \mathrm{Sym}(\kappa) : \forall P \in \mathcal{P}\ \alpha \cdot P = P\}$, which is G. (Note that the edges connecting $\emptyset$ in κ^0 to elements of κ^1 are all the same color, namely 0, and so place no additional restriction on automorphisms of $\mathcal{C}$ – their sole purpose is to ensure $\mathcal{C}$ is connected.) Conversely, if g is in $G = \{\alpha \in \mathrm{Sym}(\kappa) : \forall P \in \mathcal{P}\ \alpha \cdot P = P\}$ then the action of g on the κ^k lifts g to a bijection of $\mathcal{C} = \bigcup_k \kappa^k$ which – by construction – respects all the colored, directed edges of $\mathcal{C}$, and so is an automorphism of $\mathcal{C}$.

Hence if we define $\Psi : G \to \mathrm{Aut}(\mathcal{C})$ by $\Psi(g)(\sigma) = g \cdot \sigma$ and $\Phi : \mathrm{Aut}(\mathcal{C}) \to G$ by $\Phi(\alpha)(s) = \alpha(\langle s\rangle)$, these are well-defined maps, which are easily checked to be mutually inverse homomorphisms. It remains to show that Ψ and Φ are continuous. Since Φ and Ψ are homomorphisms it suffices to check continuity at the identity elements, id_κ of $G \subseteq \mathrm{Sym}(\kappa)$

and $id_{\mathcal{C}}$ of $\text{Aut}(\mathcal{C})$, and with respect to subbasic open neighborhoods.

A subbasic open neighborhood of id_κ has the form $B(s) = \{g \in G : g(s) = s\}$. Then $B(\langle s \rangle) = \{\alpha \in \text{Aut}(\mathcal{C}) : \alpha(\langle s \rangle) = \langle s \rangle\}$ is a (subbasic) neighborhood of $id_{\mathcal{C}}$, and $\Phi(B(\langle s \rangle)) = B(S)$. Thus Φ is continuous.

Now take a subbasic open neighborhood of $id_{\mathcal{C}}$, which has the form $B(\langle s_1, \ldots, s_k \rangle) = \{\alpha \in \text{Aut}(\mathcal{C}) : \alpha(\langle s_1, \ldots, s_k \rangle) = \langle s_1, \ldots, s_k \rangle\}$. Then $B(s_1, \ldots, s_k) = \{g \in G; g(s_i) = s_i \text{ for } i = 1, \ldots, k\}$ is a (basic) open neighborhood of id_κ, and if g is in $B(s_1, \ldots, s_k)$ then $\Psi(g)(\langle s_1, \ldots, s_k \rangle) = \langle g(s_1), \ldots, g(s_k) \rangle = \langle s_1, \ldots, s_k \rangle$, and so is in $B(\langle s_1, \ldots, s_k \rangle)$. Hence Ψ is continuous. □

4.2 STEP 2: FROM COLORED, DIRECTED GRAPHS TO STANDARD GRAPHS

Let $\mathcal{C}$ be a colored and directed graph. Let C be the set of colors of $\mathcal{C}$ (possibly uncountable). Let $\mathcal{E}$ be a family of connected, rigid graphs, such that the size of $\mathcal{E}$ is at least the size of C. Fix an injection $c \mapsto E_c$ of the colors into $\mathcal{E}$. For each E in $\mathcal{E}$ let there be distinct vertices a_E and b_E from E. We term a_E the 'starting vertex' and b_E the 'ending vertex', and *informally* we think of E as being directed from its start vertex to its end vertex.

From the colored, directed graph $\mathcal{C}$ we make three standard graphs Γ_0, Γ and Γ^*. Here Γ_0 has the essential ideas of the construction, but has a potential flaw which is fixed in Γ, and Γ^* is a generalization of Γ which enables us to construct the r-regular graphs promised in Theorem 35.

First, let all vertices of $\mathcal{C}$ be vertices to Γ_0. Now add edges and vertices as follows. For each edge in $\mathcal{C}$ of color c, from vertex v to vertex w, add to Γ_0 a distinct copy of E_c, identifying v with a_{E_c} and renaming it v, and identifying w with b_{E_c} and renaming it w. The intent is that each edge of $\mathcal{C}$ is 'realized' by a corresponding rigid graph, so that automorphisms of Γ_0 map elements of $\mathcal{C}$ back to elements of $\mathcal{C}$ and 'directed, colored' rigid graphs to copies of the same rigid graph. A potential pitfall is that an element of $\mathcal{C}$ in Γ_0 may be mapped to a vertex in a rigid 'edge', rather than another member of $\mathcal{C}$. This problem is avoided by expanding each vertex in $\mathcal{C}$ into a subgraph distinguishable from elements of $\mathcal{E}$, as follows.

To the conditions on members of $\mathcal{E}$ add the requirement that if E is in $\mathcal{E}$ then no vertex of E has degree one (in other words, is terminal). For each vertex v of $\mathcal{C}$ fix a graph Γ_v which has two vertices, p_v and q_v, and an edge connecting them. To define the graph Γ, first take the disjoint sum of all Γ_v for v in $\mathcal{C}$. Let $\mathcal{C}' = \{p_v : v \in \mathcal{C}\}$. Now add edges and vertices as follows. For each edge in $\mathcal{C}$ of color c, from vertex v to vertex w, add to Γ a distinct copy of E_c, identifying p_v with a_{E_c} and renaming it p_v, and identifying p_w with b_{E_c} and renaming it p_w. Observe that for each v in $\mathcal{C}$ the vertex q_v of Γ has degree one. By the restriction on $\mathcal{E}$ any automorphism of Γ must take a q_v to a q_w, so p_v to a p_w. Hence automorphisms of Γ carry $\mathcal{C}'$ to $\mathcal{C}'$, and rigid graphs to rigid graphs, as desired to yield isomorphism of $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Aut}(\Gamma)$. We call Γ_v the "vertex graph" and E_c the "edge graph".

More generally, fix a finite, connected graph γ, and p a vertex of γ. For each v in $\mathcal{C}$ let Γ_v be a copy of γ and denote by p_v the copy of p. To define the graph Γ^*, first take the disjoint sum of all Γ_v for v in $\mathcal{C}$. Let $\mathcal{C}' = \{p_v : v \in \mathcal{C}\}$. Add edges and vertices as follows. For each edge in $\mathcal{C}$ of color c, from vertex v to vertex w, add to Γ^* a distinct copy of E_c, and attach it to Γ_v. At this point, precisely how the attachement is made is not specified. However, it is required that γ, p, the members of $\mathcal{E}$ and the method of attaching, is such that, for each v in $\mathcal{C}$, and automorphism of Γ^*, the subgraph Γ_v is isomorphically mapped to a subgraph Γ_w.

Identify $\mathcal{C}$ with $\mathcal{C}'$. Define $\rho : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\mathcal{C})$ by $\rho(\beta) = \beta \restriction \mathcal{C}'$. Define $\epsilon : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(\Gamma)$, the 'extension map', as follows. Let α be in $\mathrm{Aut}(\mathcal{C})$. Define $\epsilon(\alpha)$ to carry Γ_v isomorphically to $\Gamma_{\alpha v}$. Further, if, in $\mathcal{C}$, there is an edge of color c from v to w, then $\epsilon(\alpha)$ takes the copy of E_c attached from Γ_v to Γ_w to the copy of E_c attached from $\Gamma_{\alpha(v)}$ to $\Gamma_{\alpha(w)}$.

Theorem 34. *Let $\mathcal{C}$ be a colored, directed graph. Let Γ and Γ^* be standard graphs as defined above. Then Aut($\mathcal{C}$) and Aut(Γ^*) are topologically isomorphic via ρ and ϵ. In particular, Aut($\mathcal{C}$) and Aut(Γ) are topologically isomorphic.*

Further,

(a) if $\mathcal{C}$ is connected then Γ^* is connected,

(b) $|\Gamma^*| \leq |\mathcal{C}|. \sup\{|E| : E \in \mathcal{E}\}$, and

(c) if $\mathcal{C}$ is locally finite, and every E in $\mathcal{E}$ is locally finite then Γ^* is locally finite.

Proof. Claims (a) through (c) are immediate from the definitions.

By construction, ρ and ϵ are well-defined homomorphisms, which are mutual (algebraic) inverses. The map ρ is a restriction map, so it is clearly continuous.

To complete the proof we show the extension map, ϵ, is continuous (unlike for the restriction map this is not automatic). It suffices to check this at the identity, *id* of $\mathrm{Aut}(\mathcal{C})$. A basic neighborhood of the identity *id* of $\mathrm{Aut}(\Gamma)$ (respectively, $\mathrm{Aut}(\mathcal{C})$) has the form $B_F = \{\beta : \beta(v) = v \text{ for all } v \in F\}$, where F is a finite subset of Γ (respectively, $\mathcal{C}$). Each v in F which is not in $\mathcal{C}$ is a vertex in a connector, with start and end points v^- and v^+, say. Let $F' = (F \cap \mathcal{C}) \cup \{v^-, v^+ : v \in F \setminus \mathcal{C}\}$, which is a finite subset of $\mathcal{C}$. Then the Key Properties on page 54 for automorphisms of Γ shows that ϵ maps $B_{F'}$ into B_F, establishing continuity at the identity.

□

4.3 THEOREMS AND COROLLARIES

Theorem 35. *Let G be a topological group. Then G is a separable metrizable topological group with an open, profinite subgroup if and only if there exists a (countable) connected, locally finite graph Γ_G such that $Aut(\Gamma_G)$ is topologically isomorphic to G.*

Proof. Take any separable metrizable group G with an open profinite subgroup U. Then let $U_0 = U$, and find (using profiniteness) a decreasing sequence of open (hence profinite) subgroups, $(U_n)_n$, of U forming a local base at the identity of G. Fix any dense subset $T = \{t_n\}_n$ of G, and set $T_n = \{t_0, \ldots, t_n\}$.

Then G, $(U_n)_n$ and $T = \bigcup_n T_n$ satisfy the conditions of Theorem 32, and so there is a colored, directed $\mathcal{C}$ such that G is topologically isomorphic to $\mathrm{Aut}(\mathcal{C})$. Now apply Theorem 34 to get that standard graph $\Gamma = \Gamma_G$ such that $\mathrm{Aut}(\Gamma_G)$ is topologically isomorphic to $\mathrm{Aut}(\mathcal{C})$, and hence G. □

According to Evans and Hewitt [11], there are two metrizable profinite groups G_1 and G_2 which are algebraically isomorphic but not topologically isomorphic. Observe that both groups are homeomorphic to a Cantor set. So applying Theorem 35 gives the following result:

Corollary 36. *There are two connected locally finite graphs Γ_1 and Γ_2 such that $Aut(\Gamma_1)$ is* *algebraically isomorphic to* $Aut(\Gamma_2)$ *but* $Aut(\Gamma_1)$ *and* $Aut(\Gamma_2)$ *are not topologically isomorphic. In particular they do not have the Small Index Property.*

In a similar spirit as Frucht [13] and Sabidussi [32], who realized groups as r-regular graphs for $r \geq 3$, we can show the following result by controlling the degrees by strengthening both steps. The following lemma provides the necessary conditions for groups that can be topologically realized as connected graphs of bounded degree.

Lemma 37. *Suppose* Γ *is a connected graph of bounded degree, say* $\deg(v) \leq m + 1$ *for all vertices* v *of* Γ*. Then, for any fixed* v*, in* $Aut(\Gamma)_v$ *there is a decreasing sequence of open subgroups* U_n*, forming a neighborhood base at the identity, such that* $|U_n/U_{n+1}| \leq m$ *for all* n*.*

Proof. To see this, pick some v_0 in Γ, a neighbor v_1, enumerate $\Gamma \setminus \{v_0, v_1\}$ as $\{v_n\}_{n\geq 2}$ such that if the distance from v_0 to v_i is strictly less than the distance from v_0 to v_j then $i < j$, and define $U_n = \{\alpha \in \mathrm{Aut}(\Gamma) : \alpha(v_i) = v_i, \text{ for } i = 0, \ldots, n\}$. Then, for a fixed n, the vertex v_{n+1} has a neighbor v_i where $i < n + 1$, which in turn has a neighbor v_j where $j < i$, and so each α in U_n has no more than m possibilities for the value at v_{n+1} (one of the neighbors of $v_i = \alpha(v_i)$, other than $v_j = \alpha(v_j)$). Since the elements of U_{n+1} are those members of U_n fixing v_{n+1}, we see U_{n+1} has index no more than m in U_n. $\square$

Remarkably, a single condition of this type is sufficient for realizability by *regular* graphs:

Theorem 38. *Let* G *be a separable metrizable topological group with an open, profinite subgroup* U_1 *and a decreasing sequence of open subgroups* $(U_n)_n$*, forming a neighborhood base at the identity, such that* $|U_n/U_{n+1}| \leq m$ *for all* n*. Then for every* $r \geq \max(3, m + 1)$ *there is a connected,* r*-regular graph* Γ_G^r *such that* $Aut(\Gamma_G^r)$ *is topologically isomorphic to* G*.*

Proof. Let G be a separable metrizable topological group with a decreasing sequence of open, profinite subgroups $(U_n)_n$, forming a neighborhood base at the identity, such that $|U_n/U_{n+1}| \leq m$ for all n. Let $\mathcal{C} = \mathcal{C}_G$ be the colored, directed graph given by Theorem 32, with $T = \{t_n\}_n$, and $T_n = \{t_n\}$, an enumeration of a countable dense subset of G, so that each element is repeated infinitely often. This choice of T and T_n's ensures that every vertex in the nth level, $\mathcal{C}_n$, of $\mathcal{C}$, has exactly one horizontal edge entering it, and exactly one leaving. The bound on the indices means that each vertex in $\mathcal{C}_n$ has no more than m edges entering it from $\mathcal{C}_{n+1}$. Every vertex in a level $\mathcal{C}_n$ – other than $\mathcal{C}_0$ – also has exactly one edge leaving it for a vertex in the level below. The edges mentioned above are all the edges entering or leaving a vertex of $\mathcal{C}$.

Take some $r \geq \max(3, m+1)$. We first construct from $\mathcal{C}$ a connected graph Γ^* such that Theorem 34 ensures $\mathrm{Aut}(\mathcal{C}) \cong \mathrm{Aut}(\Gamma^*)$ topologically, and every vertex has degree no more than r. Then we modify Γ^* to get Γ_G^r which is connected, r-regular and $\mathrm{Aut}(\Gamma_G^r) \cong \mathrm{Aut}(\Gamma^*)$ topologically, and hence $\mathrm{Aut}(\Gamma_G^r)$ is topologically isomorphic to G.

Let γ' be a rigid, r-regular graph. Fix two pairs of neighboring vertices, a_1, a_2 and b_1, b_2. Remove the edge between a_1 and a_2, add an edge from a_1 to a new vertex a_1'; and remove the edge between b_1 and b_2, to get a new graph γ. All vertices of γ have degree r, other than a_1' which has degree one, and a_2, b_1, b_2 which all have degree $r-1$.

For any graph μ define $\tilde{\mu}$ to be the graph obtained from μ by expanding every vertex v of μ, say of degree d, into a complete graph on d vertices.

Set $\gamma_0' = \gamma'$, and recursively $\gamma_{n+1}' = \tilde{\gamma_n'}$. This gives an infinite family of rigid, r-regular graphs, $\mathcal{E}' = \{\gamma_n' : n \in \mathbb{N}\}$. For each γ_n' in $\mathcal{E}'$ fix a pair of neighboring vertices, c_1, c_2. Remove the edge between c_1 and c_2 to get a new graph γ_n^*. All vertices of γ_n^* have degree r, other than c_1 and c_2, which have degree $r-1$. Add two new vertices, c_1' and c_2', and edges between c_1, c_1' and between c_2, c_2', to get a new graph γ_n. All vertices of γ_n have degree r, other than c_1' and c_2', which have degree 1. Let $\mathcal{E} = \{\gamma_n : n \in \mathbb{N}\}$.

Define the kth 'streamer', Σ_k, as follows. First take the disjoint sum of all the γ_n^* for $n \geq k$. Then for each $n \geq k$, add an edge from the copy of c_2 in γ_n^* to the copy of c_1 in γ_{n+1}^*. Further add a new vertex c_1', and connect it via an edge to the copy of c_1 in γ_k^*. Then Σ_k is connected and all vertices of Σ_k have degree r, except c_1' which has degree 1.

Fix v in $\mathcal{C}_n$. Let Γ_v is a copy of γ. To complete the definition of Γ^* we need to specify how to attach 'edge' graphs (elements of $\mathcal{E}$) to the 'vertex' graphs, Γ_v.

There is a (unique) horizontal edge entering v in $\mathcal{C}$, say of color c. There is in Γ^* a corresponding copy of E_c. Attach it to Γ_v by identifying the copy of c_2' in E_c with the copy of b_1 in Γ_v. Note this vertex has degree r. There is also a unique horizontal edge exiting v in $\mathcal{C}$, also of color c. There is in Γ^* a corresponding copy of E_c. Attach it to Γ_v by identifying the copy of c_1' in E_c with the copy of a_2 in Γ_v. Note this vertex has degree r.

There are ℓ-many vertical edges entering v (from $\mathcal{C}_{n+1}$) in $\mathcal{C}$, where $\ell \leq m$, all of the same color, say c. There are in Γ^* corresponding copies of E_c. Attach each one to Γ_v by identifying the copy of c_2' in that copy of E_c with the copy of a_1' in Γ_v. Note this vertex has degree $\ell + 1$, which is no more than r.

If $n \geq 1$ there is a (unique) vertical edge exiting v in $\mathcal{C}$ (ending in a vertex in $\mathcal{C}_{n-1}$, say of color c. There is in Γ^* a corresponding copy of E_c. Attach it to Γ_v by identifying the copy of c_1' in E_c with the copy of b_2 in Γ_v. Note this vertex has degree r. (If $n = 0$ then leave b_2 alone, and note it still has degree $r - 1$.)

Thus Γ^* has the requisite properties.

It remains to modify Γ^* to get Γ^r_G. As noted above all vertices of Γ^* have degree r except:

(a) the copies of vertex b_2 in Γ_v for v in the bottom level, $\mathcal{C}_0$, and (potentially)

(b) the 'connector' vertices a'_1 for every Γ_v.

For each v in $\mathcal{C}_0$, add to Γ^* a copy of the streamer Σ_1, and identify its copy of c'_1 with the copy of b_2 in Γ_v. For each v in $\mathcal{C}$, consider the copy of a'_1 in Γ_v. If its degree is strictly less than r, successively attach copies of streamers $\Sigma_1, \Sigma_2, \ldots$, until a'_1 has degree r, by identifying the copy of c'_1 in the streamer Σ_k with a'_1. Together these modifications give Γ^r_G, and ensure it has degree r. It is also clear that Γ^r_G has automorphism group topologically isomorphic to that of Γ^*.

□

Corollary 39. *For every countable group G and $r \geq 3$ there is a connected, r-regular graph Γ^r_G such that $Aut(\Gamma^r_G)$ is isomorphic to G.*

Proof. Apply Theorem 38 with $m = 1$. □

We deduce the following characterization of topological groups arising as the automorphism group of an arbitrary locally finite graph.

Theorem 40. *Let G be a topological group. Then G is topologically isomorphic to a product $\prod_{\alpha \in \mathfrak{c}} H_\alpha$, where $H_\alpha = Sym(\kappa_\alpha) \times G_\alpha^{\kappa_\alpha}$, G_α is a separable metrizable group with an open profinite subgroup, and each κ_α is a cardinal (possibly 0) if and only if G is topologically isomorphic to the automorphism group of a locally finite graph.*

Proof. There are, up to isomorphism, $\mathfrak{c} = |\mathbb{R}|$ many rigid connected, locally finite graphs, and $\mathfrak{c}$ many connected locally finite graphs in total. Enumerate these latter as $\{\Gamma_\alpha : \alpha \in \mathfrak{c}\}$.

Hence if Γ is any locally finite graph, then by inspecting its connected components we see $\Gamma = \bigoplus_{\alpha \in \mathfrak{c}} \left(\bigoplus_{\beta \in \kappa_\alpha} \Gamma_\alpha \right)$ where each κ_α is a cardinal, possibly zero. It follows that $\mathrm{Aut}(\Gamma) = \prod_{\alpha \in \mathfrak{c}} (\mathrm{Sym}(\kappa_\alpha) \times \mathrm{Aut}(\Gamma^\alpha)^{\kappa_\alpha})$. □

A topological group is a non-Archimedean Polish group if and only if it is topologically isomorphic to a closed subgroup of the infinite symmetric group. If Γ is a countable graph then $\mathrm{Aut}(\Gamma)$ is topologically isomorphic to a closed subgroup of the infinite symmetric group, and hence is non-Archimedean and Polish. The next result establishes the converse using the same idea of construction established by Frucht.

Theorem 41. *Let G be a Polish non-Archimedean topological group. Then there is a countable, connected graph Γ_G such that $Aut(\Gamma_G)$ is topologically isomorphic to G.*

Recall that the *weight* of a space X is the minimal size of a basis. So a space is separable metrizable if and only if it has countable weight. Also recall that a topological group, G, is (Raikov) *complete* if its two sided uniformity is complete, or equivalently it is closed in every Hausdorff topological group in which it embeds. It is convenient to recall: every closed subgroup of a complete group is complete, every symmetric group $\mathrm{Sym}(S)$ is complete (and hence every automorphism group is complete), and a separable metrizable topological group is Raikov complete if and only if it is Polish. With this in mind Theorem 41 is a special case of the following (Theorem 42), which characterizes exactly which topological groups arise as the automorphism group of a (connected) graph.

Theorem 42. *Let G be a topological group. Then the following are equivalent:*

(1) G is non-Archimedean and complete,

(2) G embeds as a closed subgroup in $Sym(\kappa)$ for some cardinal κ (no more than the weight

of G), and

(3) G is topologically isomorphic to $Aut(\Gamma_G)$ for some connected graph Γ_G (of size no more than the weight of G).

Proof.

(2) $\Rightarrow$ (1) As, for every cardinal κ, the topological group $\mathrm{Sym}(\kappa)$ is non-Archimedean and complete, so is every closed subgroup, and so (2) implies (1).

(1) $\Rightarrow$ (2) Let G be non-Archimedean and complete. Let κ be the weight of G. Fix $\mathcal{B}_1$ a local base at the identity, id in G, consisting of open subgroups, such that $|\mathcal{B}_1| \leq \kappa$. Let $\mathcal{B}$ be the collection of all cosets of members of $\mathcal{B}_1$. Then $|\mathcal{B}| \leq \kappa$. Note that $\mathrm{Sym}(\mathcal{B})$ embeds as a closed subgroup of $\mathrm{Sym}(\kappa)$. Define $e : G \to \mathrm{Sym}(\mathcal{B})$ by $e(g)(hU) = ghU$. We show that e is an isomorphic embedding of G into $\mathrm{Sym}(\mathcal{B})$. By completeness, the image must be closed, and we have (2).

First we check that for each g in G we have that $e(g)$ is a permutation of $\mathcal{B}$ (and so e is well-defined). To see $e(g)$ is injective, note $ghU = e(g)(hU) = e(g)(h'U) = gh'U$ if and only if $hU = h'U$. For surjectivity observe, $e(g)(g^{-1}hU) = hU$ for any hU in $\mathcal{B}$. It is immediate that e is a homomorphism.

Next we verify e is injective. To see this take any distinct g and g' from G. Then there is a U in $\mathcal{B}_1$ such that $gU \neq g'U$. Now we have, $e(g)(U) = gU \neq g'U = e(g')(U)$, and $e(g) \neq e(g')$.

For continuity of e it suffices to check at id in G and subbasic neighborhoods of $id_{\mathcal{B}}$ in $\mathrm{Sym}(\mathcal{B})$. A subbasic neighborhood of $id_{\mathcal{B}}$ has the form $B(hU) = \{\alpha \in \mathrm{Sym}(\mathcal{B}) : \alpha(hU) = hU\}$. By continuity of $x \mapsto h^{-1}xh$, find an open subgroup V such that $h^{-1}Vh \subseteq U$, so $Vh \subseteq hU$. Then $e(V)(hU) = VhU \subseteq hUU = hU$, and so $e(V) \subseteq B(hU)$.

Finally we need to check that $e^{-1} : e(G) \to G$ is continuous. It suffices to show this at $id_{\mathcal{B}}$ and for basic neighborhoods of 1 in G. To this end take any U in $\mathcal{B}_1$. Then $B(U) = \{\alpha \in \mathrm{Sym}(\mathcal{B}) : \alpha(U) = U\}$ is a basic neighborhood of $id_{\mathcal{B}}$ in $\mathrm{Sym}(\mathcal{B})$. So $B(U) \cap e(G) = \{e(g) : g \in G \text{ and } e(g)(U) = U\} = \{e(g) : g \in U\} = e(U)$, is a neighborhood of $id_{\mathcal{B}}$ in $e(G)$. Hence, e^{-1} maps $B(U) \cap e(G) = e(U)$ into U, as desired.

Clearly, since $\mathrm{Aut}(\Gamma)$ is a closed subgroup of $\mathrm{Sym}(|\Gamma|)$, (3) implies (2).

(2) $\Rightarrow$ (3) So suppose G is a closed subgroup of $\mathrm{Sym}(\kappa)$. By Proposition 33 there is a colored, directed graph $\mathcal{C}$ of size and colors no more that κ such that $\mathrm{Aut}(\mathcal{C})$ is topologically isomorphic to G.

Since there is a κ-sized family of rigid graphs of size κ, none with a terminal edge, we can apply Theorem 34 to get a standard graph, Γ of size κ such that $\mathrm{Aut}(\Gamma)$ is topologically isomorphic to $\mathrm{Aut}(\mathcal{C})$, and hence G. $\square$

4.4 EXAMPLE OF $\mathbb{Q}$

The rationals $\mathbb{Q}$ with the discrete topology forms a separable metrizable topological group with a trivial open profinite subgroup. Following Theorem 38, we give a detailed construction of a regular graph Γ^r satisfying $\mathrm{Aut}(\Gamma^r) \cong \mathbb{Q}$.

Construction of $\mathcal{C}$ with $\mathrm{Aut}(\mathcal{C}) \cong \mathbb{Q}$. Let $\mathcal{C} = \mathcal{C}_G$ be the colored, directed graph given by Theorem 32. We enumerate $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ so that each element is repeated infinitely often, then we let $T = \mathbb{Q}$ and $T_n = \{q_n\}$. This choice of T and T_n's ensures that every vertex in the nth level, $\mathcal{C}_n$, of $\mathcal{C}$, has exactly one horizontal edge entering it, and exactly one leaving. Since $\mathbb{Q}$ is discrete, we can take $\{0\} = U_0 = U_1 = \cdots = U_n = \ldots$. Thus for all

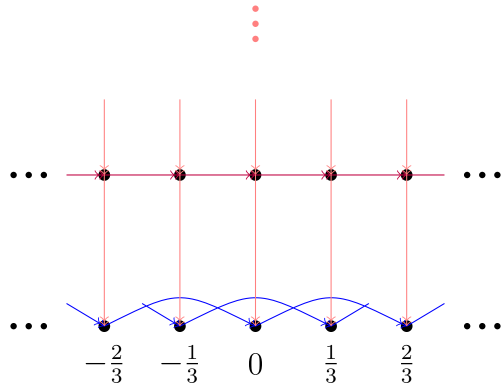

Figure 8: Realization of $\mathbb{Q}$ as a colored, directed graph

n, $|U_{n+1}/U_n| = 1$ which means that each vertex in $\mathcal{C}_n$ has exactly one edge entering it from $\mathcal{C}_{n+1}$. Every vertex in a level $\mathcal{C}_n$ – other than $\mathcal{C}_0$ – also has exactly one edge leaving it for a vertex in the level $\mathcal{C}_{n-1}$ below. The edges mentioned above are all the edges entering or leaving a vertex of $\mathcal{C}$.

More specifically (see Figure 8) , each $\mathcal{C}_n$ has $\mathbb{Q}$ as its vertex set. For vertex x in $\mathcal{C}_n$ with $n \geq 1$ and $T_n = \{q\}$, it has one horizontal edge from $x - q$, one horizontal edge going to $x + q$, one vertical edge from $x \in \mathcal{C}_{n+1}$ and one vertical edge going to $x \in \mathcal{C}_{n-1}$. For the case $x \in \mathcal{C}_0$, it has all but the last edge mentioned above.

From $\mathcal{C}$ to regular graph Γ^r. This is just applying the second half of the proof of Theorem 38 to obtain the desired Γ^r. More specifically, we use r-rigid graphs to replace colored directed edges from $\mathcal{C}$ and attach one r-streamer to every $x \in \mathcal{C}_0$.

By Theorem 38, this Γ^r topologically realizes $\mathbb{Q}$ (i.e., $\mathrm{Aut}(\Gamma^r) \cong \mathbb{Q}$).

4.5 FUTURE RESEARCH

Krön and Möller [25] defined two connected graphs X and Y to be *quasi-isometric* if there are functions $\phi : VX \to VY$ and $\psi : VY \to VX$ and constants a, b, c and d such that for all x, x_1 and x_2 in VX and y, y_1 and y_2 in VY, the following conditions hold:

1. $d_Y(\phi(x_1), \phi(x_2)) \leq a \cdot d_X(x_1, x_2)$ (boundedness of ϕ),
2. $d_X(\psi(y_1), \psi(y_2)) \leq b \cdot d_Y(y_1, y_2)$ (boundedness of ψ),
3. $d_X(\psi\phi(x), x) \leq c$ (quasi-injectivity of ϕ),
4. $d_Y(\phi\psi(y), y) \leq d$ (quasi-surjectivity of ϕ).

In their paper, they constructed an (algebraic) realization of any given compactly generated totally disconnected locally compact group. They then proved that any two different such realizations for the same group are quasi-isometric. Quasi-isometry is a weakened form of isometry which preserves many geometric properties. The same authors wrote a paper on this [26].

In the context of Theorem 35, it is possible to generate *different* graphs realizing the same group G. More precisely, the construction depended on the specific presentation of the open profinite $U \leq G$ as an inverse limit $\varprojlim U_n$, which is far from unique (for example, take any subsequence of the $\{U_n\}$). So similar questions arise with our realizations, most fundamentally:

Question 1. Given a group G that can be topologically realized as graphs Γ_1 and Γ_2, must Γ_1 and Γ_2 be quasi-isometric?

5.0 REALIZATION OF TOPOLOGICAL GROUPS AS CONTINUA

Recall that Hofmann and Morris [21] proved that a group may be realized as the autohomeorphism group of a compact space only if it is a profinite topological group. The converse remains open.

Question 2. For any profinite topological group G, does there exist a continuum/compact space X_G such that $H(X_G)$ is topologically isomorphic to G?

A partial result was obtained by Gartside and Glyn [15].

Theorem 43. *For every* metric *profinite group G, there exists a continuum X_G such that $H(X_G)$ is topologically isomorphic to G.*

Some partial results will be established in this chapter. In Section 5.1 a technical tool is created to expand the variety of rigid spaces. This is then used for the first result: realizing G, a (not necessarily countable) product of finite groups, as the autohomeomorphism group of a continuum. The proof of this result is in Section 5.2 and is broken down into two parts: we first construct this continuum X_G and then offer the technical proof that $H(X_G)$ is topologically isomorphic to G. In Section 5.3, we discuss some ideas that may lead towards a full solution.

5.1 THE LONG COOK CONTINUA

Let R be the Cook continuum [7]. Then the continuous maps on R to itself are $\{id\} \cup \{c_y : y \in R\}$, where c_y is the constant map for some $y \in R$. In particular, the only autohomeomorphism of R is the identity, and R is rigid. Let κ be an uncountable regular cardinal. We construct a *κ-long Cook continuum* R_κ as follows:

(i) For each α in κ fix distinct (α, a) and (α, b) from $\{\alpha\} \times R$.

(ii) Take the disjoint sum of all $\{\alpha\} \times R$ for α in κ, then for each α, identify (α, b) with $(\alpha+1, a)$ and for limit λ give (λ, a) the natural limit topology from below. Denote by R'_κ the resulting space.

(iii) Let R^*_κ be the one-point compactification of R_κ with $*$ the point at infinity.

(iv) Finally identify, $(0, a)$ and $*$ in R^*_κ, renaming it $*$. This gives R_κ.

Then R_κ is continuum.

Proposition 44. *The space R_κ defined above is rigid.*

Proof. Take an arbitrary homeomorphism $\varphi : R_\kappa \to R_\kappa$. We show it is the identity.

Note first that the point $*$ has character (minimal size of a local base) κ, and as κ is regular and uncountable, is the unique point with character κ. So it is fixed by φ.

Let $\varphi_0 = \varphi \mid_{\{\alpha\}\times R}$. Let $S_0 = \varphi_0(\{0\} \times R)$. Note this is metric continuum. Hence $S_0 \setminus \{*\}$ must be contained in some 'initial segment', of $R^*_\kappa \setminus \{*\}$. We show that $S_0 = \{0\} \times R$, and so φ_0 is just the identity of $\{0\} \times R$.

Well, S_0 cannot be properly contained in $\{0\} \times R$ (or φ_0 would be constant). While if S_0 contains $\{0\} \times R$ then it must equal S_0 (otherwise consider φ_0^{-1} on $\{0\} \times R$).

So the remaining case is when S_0 is neither contained in nor contains $\{0\} \times R$. Observe that every point $(\alpha + 1, a)$ (which is also (α, b)) is a cut point of R^*_κ. In particular, by considering $(1, a)$, we see that if S_0 is not contained in $\{0\} \times R$ then it must contain $(1, a)$ (or we get a disconnection of S_0). But now we see that the hypothesized point in $\{0\} \times R$ not contained in S_0 is disconnected from the image of φ by $(1, a)$.

In general, for each $\alpha \in \kappa$, let $\varphi_\alpha = \varphi \mid_{\{\alpha\} \times R}$. Inductively (using continuity at limits) we see that every φ_α is the identity. Hence φ is the identity, as required. □

5.2 CONSTRUCTION OF INFINITE PRODUCT

Theorem 45. *Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of profinite groups such that for each $\lambda \in \Lambda$*

(a) there exists a continuum Y_λ such that $H(Y_\lambda) \cong G_\lambda$;

(b) there exists $y_\lambda \in Y_\lambda$ such that $\forall \varphi \in H(Y_\lambda), \varphi(y_\lambda) = y_\lambda$.

Then there exists a continuum X_G and a point $x \in X_G$ such that $H(X_G) \cong \prod_{\lambda \in \Lambda} G_\lambda$ and x is a fixed point of $H(X_G)$.

Construction: Given $\{G_\lambda : \lambda \in \Lambda\}$, for all $\lambda \in \kappa$ we have $Y_\lambda = Y_{G_\lambda}$ and $y_\lambda = y_{G_\lambda}$ such that $H(Y_{G_\lambda}) \cong G_\lambda$ topologically and y_λ are fixed points correspondingly.

For each λ, consider a distinct rigid continuum R_λ. (Possible by using the long Cook continua of Section 5.1.) Choose distinct pairs of points $p'_\lambda, y'_\lambda \in R_\lambda$. Then $y'_\lambda \neq y'_\mu$ for all $\lambda \neq \mu$. For all $\lambda \in \Lambda$, let E_λ be R_λ with "starting point" y'_λ and "end point" p'_λ. For each $\lambda \in \Lambda$,

(i) identify $y_\lambda \in Y_\lambda$ with $y'_\lambda \in E_\lambda$ and call the resulting continuum X_λ and the resulting point x_λ. So $X_\lambda = Y_\lambda \oplus E_\lambda/(y_\lambda \sim y'_\lambda)$.

(ii) define $X'_\lambda = X_\lambda \setminus \{p'_\lambda\}$ for each $\lambda \in \Lambda$.

Define X_G as the one-point compactification of $\bigoplus_{\lambda\in\Lambda} X'_\lambda$. Let $p = X_G \setminus \bigoplus_{\lambda\in\Lambda} X'_\lambda$ be the point at infinity. Note that $X'_\lambda \cup \{p\} \cong X_\lambda$, so we consider X_λ as a subspace of X_G. Therefore (p', λ) are identified as the point p. It is clear that $\bigoplus_{\lambda\in\Lambda} X_\lambda$ is connected. Since X_λ is a subspace of X_G for each $\lambda \in \Lambda$, X_G is the unique one-point compactification of $\{X'_\lambda \mid \lambda \in \Lambda\}$. By construction, X_G is a Hausdorff continuum.

Proof. Let $\varphi \in H(X_G)$. Since X_G is the one-point compactification, $\varphi(p) = p$. Therefore p is the fixed point for X. For all $\lambda \in \Lambda$, φ fixes $(p', \lambda) \in E_\lambda$ hence $\varphi\mid_{E_\lambda} = id_{E_\lambda}$. Then $\forall \lambda \in \Lambda$, $\varphi(x_\lambda) = x_\lambda = y_\lambda \in Y_\lambda$ implies $\varphi\mid_{Y_\lambda} \in H(Y_\lambda)$.

Define $\Phi : H(X_G) \to \prod_{\lambda\in\Lambda} H(Y_\lambda)$ as $\varphi \mapsto (\varphi\mid_{Y_\lambda})_{\lambda\in\Lambda}$ and $\Psi : \prod_{\lambda\in\Lambda} H(Y_\lambda) \to H(X)$ as $(\varphi_\lambda)_{\lambda\in\Lambda} \mapsto \varphi$ by $\varphi(x) = \varphi_\lambda(x), \forall x \in Y_\lambda$ and $\varphi(x) = x, \forall x \in E_\lambda$.

Claim. φ is a homeomorphism on X_G.

Proof. **(1) φ is continuous.** For all U open in X_G, it can be written as $W \sqcup \left(\bigsqcup_{\lambda\in\Lambda} U_\lambda\right)$, where $U_\lambda = U \cap Y_\lambda$ for each $\lambda \in \Lambda$. Note that W and U_λ are all open.

Since $\varphi_\lambda \in H(Y_\lambda)$, there exists V_λ open in Y_λ and thus open in X_G, such that $\varphi(V_\lambda) = \varphi_\lambda(V_\lambda) \subseteq U_\lambda$. For all $x \in W$, then $x \notin X_\lambda$ for all $\lambda \in \Lambda$. Therefore, $\varphi(x) = x$, so $\varphi(W) = W$ makes W a valid open preimage set. Then define $V = W \cup \left(\bigcup_{\lambda\in\Lambda} V_\lambda\right)$. By definition, V is open in X_G and $\varphi(V) = \varphi(W \cup \left(\bigcup_{\lambda\in\Lambda} V_\lambda\right))) \subseteq (W \cup \left(\bigcup_{\lambda\in\Lambda} V_\lambda\right)) = U$.

If $x \in Y_\lambda \cap E_\lambda$ for some $\lambda \in \Lambda$, then x is the fixed point of Y_λ. Hence $\varphi_\lambda(x) = x = id(x)$.
(2) φ is surjective. For all $y \in X$, if $y \in Y_\lambda$ for some $\lambda \in \Lambda$, then $\varphi_\lambda \in H(Y_\lambda)$ implies that there is a unique $x \in Y_\lambda \subseteq X_G$ such that $\varphi(x) = \varphi_\lambda(x) = y$.

If $y \notin Y_\lambda$ for all $\lambda \in \Lambda$, then $\varphi(y) = id(y) = y$ provides y as the only preimage of y.

(3) φ is injective. Suppose $\varphi(u) = \varphi(v)$ for some $u, v \in X_G$. If $u \in Y_\lambda$ for some $\lambda \in \Lambda$, then since $\varphi(u) = \varphi_\lambda(u) \in Y_\lambda$ then $\varphi(v) \in Y_\lambda$. Thus $\varphi(v) = \varphi_\lambda(v)$ and since $\varphi_\lambda \in H(Y_\lambda)$, then φ_λ is injective and hence $u = v$.

If $u \notin Y_\lambda$ for all $\lambda \in \Lambda$, then since $\varphi(u) = id(u) = u \in E_\lambda$ for some $\lambda \in \Lambda$. Thus $\varphi(v) \in E_\lambda$ implies $\varphi(v) = id(v) = v$. Since the identity map is injective, $u = v$.

Therefore φ is a homeomorphism.

□

It suffices to show that Ψ (1) is a bijection (2) is continuous at identity; (3) takes compact set to compact set.

Φ, Ψ are homomorphisms. Let $\varphi, \varphi' \in H(X_G)$, Y_λ's are pairwise disjoint implies $\varphi\mid_{Y_\lambda}$, $\varphi'\mid_{Y_\lambda} \in H(Y_\lambda)$, $\Phi(\varphi) \circ \Phi(\varphi') = (\varphi\mid_{Y_\lambda})_{\lambda\in\Lambda} \circ (\varphi'\mid_{Y_\lambda})_{\lambda\in\Lambda} = ((\varphi \circ \varphi')\mid_{Y_\lambda})_{\lambda\in\Lambda} = \Phi(\varphi \circ \varphi')$.

Let $(\varphi_\lambda)_{\lambda\in\Lambda}, (\varphi'_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} H(Y_\lambda)$, $\Psi((\varphi_\lambda)_{\lambda\in\Lambda} \circ (\varphi'_\lambda)_{\lambda\in\Lambda}) = \Psi((\varphi_\lambda \circ \varphi'_\lambda)_{\lambda\in\Lambda}) = \varphi \circ \varphi' = \Psi((\varphi_\lambda)_{\lambda\in\Lambda}) \circ \Psi((\varphi'_\lambda)_{\lambda\in\Lambda})$.

Φ, Ψ are inverses. Let $\varphi \in H(X)$. Then $\Psi \circ \Phi(\varphi) = \Psi((\varphi\mid_{Y_\lambda})_{\lambda\in\Lambda}) = \varphi$.

Let $(\varphi_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} H(Y_\lambda)$. Then $\Phi \circ \Psi((\varphi_\lambda)_{\lambda\in\Lambda}) = \Phi(\varphi) = (\varphi\mid_{X_\lambda})_{\lambda\in\Lambda} = (\varphi_\lambda)_{\lambda\in\Lambda}$.

Ψ is continuous at identity. We need to show that for every open $U \subseteq H(X)$ with $id \in U$, there exists $V \subseteq \prod_{\lambda\in\Lambda} H(Y_\lambda)$ such that $(id_\lambda)_{\lambda\in\Lambda} \in V$ and $\Psi(V) = U$.

$H(X_G)$ has the compact-open topology with subbasic open sets $[K, W] = \{\varphi \in H(X_G) : \varphi(K) \subseteq W\}$ for any compact $K \subseteq X_G$ and any open $W \subseteq X_G$. It suffices to show that for all subbasic open sets $U \subseteq H(X_G)$ with $id \in U$, there exists open set $V \subseteq \prod_{\lambda\in\Lambda} H(Y_\lambda)$ such that $(id_\lambda)_{\lambda\in\Lambda} \in V$ and $\Psi(V) \subseteq U$. This is because for $x \in X_G$ not in $\bigcup_{\lambda\in\Lambda} Y_\lambda$, for all $\varphi \in H(X_G)$ then $\varphi(x) = x$ fixes the "edges".

Note that $id \in [K, W]$ if and only if $K \subseteq W$. Take any $U = [K, W]$ such that $K \subseteq W \subseteq X_G$ where K is compact and W is open in X_G. Define $K_\lambda = K \cap Y_\lambda$ and $W_\lambda = W \cap Y_\lambda$.

Claim. There exists a basic neighborhood V of $(id_\lambda)_{\lambda\in\Lambda}$ in $\prod_{\lambda\in\Lambda} H(Y_\lambda)$ such that $\Psi(V) \subseteq [K, W]$. It suffices to show that if $(\varphi_\lambda)_{\lambda\in\Lambda} \in V$ then for all $\lambda \in \Lambda, \varphi_\lambda(K_\lambda) \subseteq W_\lambda$.

Case 1. **K meets finitely many Y_λ.** Define $V_\lambda = [K_\lambda, W_\lambda] = \{\varphi_\lambda \in H(Y_\lambda) : \varphi_\lambda(K_\lambda) \subseteq W_\lambda\}$. There exists $F = \{\lambda_1, \ldots, \lambda_n\}$ such that $K_\lambda \neq \emptyset$ if $\lambda \in F$ and $K_\lambda = \emptyset$ for $\lambda \notin F$.

Define $V = \bigcap_{\lambda\in F} \pi_\lambda^{-1}(V_\lambda)$ where $\pi_\lambda : \prod_{\lambda\in\Lambda} H(Y_\lambda) \to H(Y_\lambda)$ is the canonical projection map and $V_\lambda = [K_\lambda, W_\lambda]$ for $\lambda \in F$. Take any $(\varphi_\lambda)_{\lambda\in\Lambda} \in V$. Then $\Psi((\varphi_\lambda)_{\lambda\in\Lambda}) = \varphi$ such that $\varphi(x) = x$ for all $x \in \bigoplus E_\lambda$ and $\varphi(x) = x$ for all $x \in Y_\lambda$ for $\lambda \notin F$; while $\varphi(x) = \varphi|_{Y_\lambda}(x) = \varphi_\lambda(x)$ for all $x \in Y_\lambda$ and $\lambda \in F$. Therefore for all $x \in K$:

(i) if $x \in K_\lambda$ for some $\lambda \in F$, then $\varphi(x) = \varphi|_{Y_\lambda}(x) = \varphi_\lambda(x) \in V_\lambda$ implies $\varphi_\lambda(x) \in W_\lambda \subseteq W$ for all $\lambda \in F$; and

(ii) if $x \notin K_\lambda$ for all $\lambda \in \Lambda$, then $\varphi(x) = x$ by definition of Ψ.

Hence $\varphi(K) \subseteq W$ and $\Psi(V) \subseteq [K, W]$ as desired.

Case 2. **K meets infinitely many Y_λ.** Define $V_\lambda = [K_\lambda, W_\lambda] = \{\varphi_\lambda \in H(Y_\lambda) : \varphi_\lambda(K_\lambda) \subseteq W_\lambda\}$. Since K is compact and closed in X, we have $p \in K$ and thus $p \in W$. Therefore W contains all but finitely many X_λ. So for some $F = \{\lambda_1, \ldots, \lambda_n\}$ such that $X_\lambda \nsubseteq W$ for $\lambda \in F$ and $X_\lambda \subseteq W$ for $\lambda \notin F$.

Define $V = \bigcap_{\lambda\in F} \pi_\lambda^{-1}(V_\lambda)$ where $\pi_\lambda : \prod_{\lambda\in\Lambda} H(Y_\lambda) \to H(Y_\lambda)$ is the canonical projection map and $V_\lambda = [K_\lambda, W_\lambda]$ for $\lambda \in F$. Take any $(\varphi_\lambda)_{\lambda\in\Lambda} \in V$. Then $\Psi((\varphi_\lambda)_{\lambda\in\Lambda}) = \varphi$ such that $\varphi(x) = x$ for all $x \in \bigoplus E_\lambda$ and $\varphi(x) = \varphi|_{Y_\lambda}(x) = \varphi_\lambda(x)$ for all $x \in Y_\lambda$ and $\forall\lambda \in \Lambda$. Therefore for all $x \in K$:

(i) if $x \in K_\lambda$ for some $\lambda \in F$, then $\varphi(x) = \varphi|_{Y_\lambda}(x) = \varphi_\lambda(x) \in V_\lambda$ implies $\varphi_\lambda(x) \in W_\lambda \subseteq W$ for all $\lambda \in F$;

(ii) if $\lambda \in F$, then $\varphi(K_\lambda) = \varphi_\lambda(K_\lambda) \subseteq W_\lambda \subseteq X_\lambda \subseteq W$ because $\varphi_\lambda \in V_\lambda = [K_\lambda, W_\lambda] \subseteq V$; and

(iii) if $x \notin K_\lambda$ for all $\lambda \in \Lambda$, then $\varphi(x) = x$ by definition of Ψ.

Ψ **is a homeomorphism** Since $H(Y_\lambda) \cong G_\lambda$ is compact for all $\lambda \in \Lambda$, $\prod_{\lambda\in\Lambda} H(Y_\lambda)$ is compact. Since Ψ is a continuous bijection with inverse Φ, we have Ψ is a homeomorphism. Since p is the infinity point in X, for all $\varphi \in H(X_G), \varphi(p) = p$ making p a fixed point of the continuum. □

Corollary 46. *Let $\{G_\lambda : \lambda \in \Lambda\}$ be a family of metrizable profinite groups. Here $|\Lambda| = \mathfrak{c}$. Then there exists a continuum X_G, such that $H(X_G) \cong \prod_{\lambda\in\Lambda} G_\lambda$.*

Proof. Let G be the product of groups $\prod_{\lambda\in\Lambda} G_\lambda$ where G_λ are metrizable, profinite groups and $\lambda \in \Lambda$. Then G is profinite. Applying Theorem 43, there exists a continuum X_G that topologically realizes G. □

In particular, this gives us a construction of a continuum X such that topologically $H(X) \cong \mathbb{Z}_2^{\omega_1}$.

5.3 FUTURE WORK

Varopoulos [37] proved the following theorems:

Theorem 47 (Varopoulos)**.** *If G is a semisimple compact totally disconnected group then $G \cong \prod_{\alpha\in A} S_\alpha$ where S_α are all finite simple groups.*

Theorem 48 (Varopoulos)**.** *If G is a compact totally disconnected group, then there exists a series of compact subgroups*

$$G = G_0 \rhd G_1 \rhd G_2 \cdots \rhd G_n \rhd \cdots ,$$

such that $\bigcap_{n=1}^{\infty} G_n = e$ and G_n/G_{n+1} is semisimple (that is $G_n/G_{n+1} \cong \prod_{\alpha \in A_n} S_\alpha$).

From Theorem 45, we can already realize arbitrary products of finite simple groups. It seems plausible that we can realize $G = A \rtimes B$ where A is an arbitrary product of finite simple groups and B is a countable product of finite simple groups. The idea for this construction should be a combination of our separate realization construction techniques for metric profinite groups and arbitrary products of profinite groups.

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