FIXED POINTS AND DUALITY OF CLOSED CONVEX SETS IN BANACH SPACES

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Submitted to the Graduate Faculty of

the Dietrich School of Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2018

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

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July 30, 2018

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In the first chapter we construct a new example of an affine norm continuous mapping on a closed, convex, non-weakly compact set C that cannot be extended to a continuous linear map on the entire space X. Although often used in the field of the fixed point theory, most of the examples known in the literature are restrictions of continuous, linear mappings from X to C.

The second chapter focuses on the notion of the affine dual of a closed, convex, bounded set. Using a theorem of M. Krein and D. Milman from Studia Mathematica 1940, one can show that certain spaces like c_0 and $L^1[0,1]$ are not dual spaces. However, it turns out that we can see them as affine dual spaces.

In the third part of this thesis we provide a new proof that compactness in ℓ_1 for closed, bounded, convex sets is equivalent to the fixed point property for cascading nonexpansive mappings. We also prove an analogue of this result in $L^1[0,1]$.

The last part is dedicated to the study of the stability constant of the weak*-fixed point property for the dual of separable Lindenstrauss spaces. Initiated in 1980 and 1982 by P. Soardi and T.C. Lim for the space c_0, we now find a precise formula in the general case of an arbitrary predual of ℓ_1 that depends only on a geometrical property of the unit ball of ℓ_1 with respect to the predual considered. As a consequence, this formula establishes a quantitative result in terms of geometric properties.

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1.0 PREFACE

First and foremost, I would like to thank Dr. Christopher Lennard for teaching me beautiful mathematics over the years and, most important, for teaching me the incredible power of example, both in math and in real life. I also want to thank Mrs. Cathy Lennard for her great delicacy and uplifting optimism, for all the delicious meals and for always putting a smile on my face.

During my time at Pitt, I have met Łukasz Piasecki, to whom I want to thank for his beautiful seminar talks which introduced me to the theory of Lindenstrauss spaces as well as for teaching me the extraordinary values of focus and perseverence. I also want to thank Dr. Emanuele Casini and Dr. Enrico Miglierina for their sincere friendship and for inviting me to collaborate.

Next, I would like to thank my thesis committee members: Dr. Jason DeBlois, Dr. Patrick Dowling, Dr. Maria Japón, Dr. Paul Gartside, Dr. Piotr Hajłasz, Dr. Juan Manfredi and Dr. Barry Turett. I very much appreciate your valuable suggestions and the time that you are dedicating to helping me finish my thesis work. I thank Mrs. Angela Athanas for all the great conversations we had and for sharing with me her knowledge about teaching. Last but not least, I thank to all my friends for all the beautiful memories we shared together: Andra and Calin Ciobanu, Pamela Delgado, Irina Ilioaea, Soukaina Filali, Cezar Lupu, Luis Berlioz, Derek Orr, Jared Burns, Ivan Ramirez, Torrey Gallagher.

I thank deeply to my parents for their love and extraordinary care which I feel everyday. They have always put my well-being first place and encouraged me to follow my dreams. I thank my aunt Ioana and my uncle Nelu for taking such a great care of me, especially during my first days here, and for making my first trip to Pitt a memorable one.

Finally, I would like to thank my highschool math teacher, Prof. Anca Banica, for

opening my path and taste towards mathematics. For all the love, commitment, strong belief in me, and for the best lemonade I have ever had, this thesis is dedicated to her.

2.0 INTRODUCTION

We begin with a new example of an affine norm continuous mapping on a closed, convex, non-weakly compact set C of ℓ_∞ that cannot be extended to a continuous linear map on the entire space ℓ_∞. Although often used in the field of metric fixed point theory, most of the known examples presented in the literature are restrictions of continuous, linear mappings from a Banach space X to C. Starting from this, we construct a continuous and affine map $g : C \to C$, which does not admit a linear continuous + vector constant extension to the entire ℓ_∞.

Next, we continue by introducing the notion of the affine dual of a closed, bounded, convex subset C. The affine dual is connected in a natural way with the notion of the dual space by considering mappings which are now affine, continuous and bounded on the set C. It is known that spaces like $(c_0, \|\cdot\|_\infty)$ and $(L^1[0,1], \|\cdot\|_1)$ are not dual spaces. Indeed, in 1940, in Studia Mathematica, M. Krein and D. Milman proved that if X is a locally convex Hausdorff topological vector space, and K a compact convex subset of X, then K is the closed convex hull of its extreme points. Also, Banach-Alaoglu-Bourbaki had shown that in every dual Banach space the closed unit ball is compact in the weak* topology. By using these two theorems, one can easily prove by contradiction that the above two Banach spaces are not dual spaces, since their unit balls have no extreme points. However, it turns out that we can view them as affine dual spaces. This leads to the question: Can every Banach space X be seen as an affine dual of some closed, bounded, convex set C?

The third part of this thesis focuses mainly on the nonreflexive Banach space $(\ell_1, \|\cdot\|_1)$. In 1979, K. Goebel and T. Kuczumow constructed a family of subsets C of ℓ_1 irregular with respect to the fixed point property. Then, they used these sets to give an example of a nested sequence of closed, bounded, convex subsets of ℓ_1 which alternatively satisfy or fail the fixed point property for nonexpansive mappings. This construction showed that the fixed point property for nonexpansive mappings is very unstable in ℓ_1. In 2004 ([22]), W. Kaczor and S. Prus characterized exactly which ones of the above sets have the fixed point property for asymptotically nonexpansive mappings. Recent work of P. Dowling, C. Lennard and B. Turett ([15]) showed that in the case of the positive face S of the closed unit ball in ℓ_1, one can find a subset H of it with the property that all closed, bounded, convex sets G with $H \subseteq G \subseteq S$ fail the fixed point property for $\|\cdot\|_1$- nonexpansive maps. Using similar ideas, we will show that in the case when S is replaced by a certain class of the Goebel-Kuczumow sets, we can prove that all closed, bounded, convex sets G with $H \subseteq G \subseteq S$ fail the fixed point property for $\|\cdot\|_1$- uniformly Lipschitzian maps. This result suggested to us that the only closed, bounded, convex sets C in ℓ_1 which have the fixed point property for uniformly Lipschitzian mappings are the norm compact ones. Indeed, this result was proved in 2016 in JMAA by Tomas Dominguez-Benavides and Maria Japón ([5]). In section 5 we will give another proof of this result and we will extend it through a new class of mappings to the space $(L^1[0,1], \|\cdot\|_1)$. It remains an open question whether the result remains true for uniformly Lipschitzian mappings with Lipschitz constant $L \leq 2$.

The last chapter of this thesis contains recent work from two papers in collaboration with E. Casini, E. Miglierina and Ł. Piasecki and is devoted to the study of the stability constant of the weak*-fixed point property for the dual of separable Lindenstrauss spaces. Generally speaking, stability of the w*-fixed point property deals with the following question: let us suppose that a dual Banach space X^* has the $\sigma(X^*, X)$-fixed point property and let Y be a Banach space isomorphic to X with "small" Banach-Mazur distance from X. Does Y^* have the $\sigma(Y^*, Y)$-fixed point property? Initiated in 1980 and 1982 by P. Soardi and T.C. Lim for the space c_0, we will now find a precise formula in the general case of an arbitrary predual of ℓ_1 that depends only on a geometrical property of the unit ball of ℓ_1 with respect to the predual considered. Therefore, this formula establishes a quantitative result in terms of geometric properties.

2.1 PRELIMINARIES AND OVERVIEW

We begin with the definition of several classical Banach spaces that we will use throughout the next chapters. We define the Banach spaces $(\ell_1, \|\cdot\|_1)$, $(c_0, \|\cdot\|_\infty)$, and $(\ell_\infty, \|\cdot\|_\infty)$ by

$$\ell_1 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\} ;$$

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \lim_{n\longrightarrow\infty} x_n = 0 \right\} ;$$

where $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| \in \mathbb{R}$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$.

$$\ell_\infty := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\} .$$

The subspace $(c_{00}, \|\cdot\|_\infty)$ of $(c_0, \|\cdot\|_\infty)$ is defined as

$$c_{00} := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } x_n = 0 \text{ for all but finitely many } n \in \mathbb{N} \right\} .$$

It is known that ℓ_1 is the dual of c_0 and ℓ_∞ is the dual of ℓ_1.

Also, for all $n \in \mathbb{N}$, we define $e_n = (e_{n,k})_{k\in\mathbb{N}}$ by setting $e_{n,n} := 1$ and $e_{n,k} := 0$, for all $k \in \mathbb{N}$ with $k \neq n$.

Next, we introduce some different classes of mappings and the various connections between them. We will use these mappings to establish new fixed point results. We say that that a closed, bounded, convex (c.b.c) set C has the fixed point property (fpp) with respect to a certain class of mappings $\mathcal{E}$ if for all $T \in \mathcal{E}$, T has a fixed point in C.

Definition 2.1.1. Let C be a nonempty closed, bounded, convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : C \longrightarrow C$ be a mapping. Next, let $C_0 := C$ and for all $n \in \mathbb{N}$, define $C_n := \overline{\text{co}}(T(C_{n-1}))$. One can check that T maps every C_n into C_n.

(1) We say that T is *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\| \text{ , for all } x, y \in C \text{ .}$$

(2) We say that T is *asymptotically nonexpansive* if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ in $[1,\infty)$ such that $\lambda_n \to 1$, such that for all $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq \lambda_n \|x - y\| \ , \text{ for all } x, y \in C \ .$$

(3) We say that T is *eventually asymptotically nonexpansive* if there exists $\mu \in \mathbb{N}$ and a sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ in $[1,\infty)$ such that $\lambda_n \to 1$, and for all $n \geq \mu$,

$$\|T^n(x) - T^n(y)\| \leq \lambda_n \|x - y\| \ , \text{ for all } x, y \in C$$

(4) We say that T is *cascading nonexpansive* if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ in $[1,\infty)$ such that $\lambda_n \to 1$, and for all $n \in \mathbb{N}_0$,

$$\|T(x) - T(y)\| \leq \lambda_n \|x - y\| \text{ for all } x, y \in C_n \ .$$

(5) We say that T is *uniformly Lipschitzian* if there exists $M \in [1,\infty)$, such that for all $m \in \mathbb{N}$,

$$\|T^m(x) - T^m(y)\| \leq M \|x - y\| \ , \text{ for all } x, y \in C \ .$$

We call M a *uniform Lipschitz constant* for T.

(6) We define T to be *affine* if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \ ,$$

for all $x_1, x_2 \in C$ with $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

(7) We say that T has an *approximate fixed point sequence* if there exists $(x_n)_{n\in\mathbb{N}}$ in C such that $\|T(x_n) - x_n\| \underset{n}{\longrightarrow} 0$.

It is true that $[(1) \implies (2) \implies (3)]$, $[(1) \implies (2) \implies (5)]$, and $[(1) \implies (4) \implies (7)]$. Of course, the converses are not generally true. Cascading nonexpansive mappings are analogous to eventually asymptotically nonexpansive mappings, but the next two examples by Christopher Lennard and Veysel Nezir ([27]) show that neither of these two classes of mappings contain the other.

Example 2.1.2. ([27], Example 2.4)*(Not all Cascading Nonexpansive Mappings are Eventually Asymptotically Nonexpansive).* Let $(\gamma_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence in (0,1) that converges to 1. Also assume that $(\frac{\gamma_{n+1}}{\gamma_n})_{n\in\mathbb{N}}$ is a decreasing sequence in $(1,2]$ (for example $\gamma_n := \frac{8^n}{1+8^n}$). Next consider the Banach space ℓ_1 endowed with the equivalent norm

$$\|x\|_\sim := \sup_{\mu\in\mathbb{N}} \gamma_\mu \sum_{k=\mu}^{\infty} |x_k|$$

$\forall\ x \in \ell_1$. Let $K := \{x \in \ell_1 : \text{each } x_j \geq 0 \text{ and } \sum_{j=1}^{\infty} x_j = 1\}$.

Define the mapping $R : K \to K$ by $R(x) = (0, x_1, x_2, \ldots)$, $\forall\ x \in K$. The set $K_0 := K$ is closed, convex and each $K_n := \overline{\text{co}}(R(K_{n-1})) = \{(\underbrace{0,\ldots,0}_{\text{n}}, x_{n+1}, x_{n+2}, \ldots) \in \ell_1\text{: each } x_j \geq 0$ and $\sum_{j=n+1}^{\infty} x_j = 1\}$. Fix $n \in \mathbb{N}_0$. Fix $u, v \in K_n$.

Then $u = \{(\underbrace{0,\ldots,0}_{\text{n}}, x_{n+1}, x_{n+2}, \ldots)\}$ and $v = \{(\underbrace{0,\ldots,0}_{\text{n}}, y_{n+1}, y_{n+2}, \ldots)\}$, for some x, y as described above. Thus, by applying R we have that $Ru = \{(\underbrace{0,\ldots,0}_{\text{n+1}}, x_{n+1}, x_{n+2}, \ldots)\}$ and $Rv = \{(\underbrace{0,\ldots,0}_{\text{n+1}}, y_{n+1}, y_{n+2}, \ldots)\}$.

Then, we have

$$\|u - v\|_\sim = \sup_{\mu\geq n+1} \gamma_\mu \sum_{k=\mu}^{\infty} |x_k - y_k|$$

and

$$\|Ru - Rv\|_\sim = \sup_{\mu\geq n+2} \gamma_\mu \sum_{k=\mu-1}^{\infty} |x_k - y_k|$$

Therefore, we obtain

$$\|Ru - Rv\|_\sim \leq \|u - v\|_\sim \sup_{\mu\geq n+2} \frac{\gamma_\mu}{\gamma_{\mu-1}} = \frac{\gamma_{n+2}}{\gamma_{n+1}} \|u - v\|_\sim$$

Also, $e_{n+1}, e_{n+2} \in K_n$ and

$$\|Re_{n+1} - Re_{n+2}\|_{\sim} = \|e_{n+2} - e_{n+3}\| = 2\gamma_{n+2} = \frac{\gamma_{n+2}}{\gamma_{n+1}} \|e_{n+1} - e_{n+2}\|_{\sim}$$

So, R is a cascading nonexpansive mapping on K, with best constants $\Lambda_n = \frac{\gamma_{n+2}}{\gamma_{n+1}}$, for all $n \in \mathbb{N}_0$.

On the other hand, $\|e_1 - e_2\|_{\sim} = 2\gamma_1$, and for all $n \in \mathbb{N}$,

$$\|R^n e_1 - R^n e_2\|_{\sim} = \|e_{n+1} - e_{n+2}\| = 2\gamma_{n+1} = \frac{\gamma_{n+1}}{\gamma_1} \|e_1 - e_2\|_{\sim}$$

But $\frac{\gamma_{n+1}}{\gamma_1} \to \dfrac{1}{\gamma_1} > 1$. Therefore, R is not eventually asymptotically nonexpansive on K, which further implies that R is not asymptotically nonexpansive on K.

Example 2.1.3. ([27], Example 2.5)*(Not all Eventually Asymptotically Nonexpansive Mappings are Cascading Nonexpansive).* Consider the Banach space $(\mathbb{R}, |\cdot|)$ and the closed, bounded, convex set $K_0 := K := \left[0, \frac{1}{\sqrt{2}}\right]$. Let $a \wedge b := min\{a, b\}$, $\forall\ a, b \in \mathbb{R}$. Also, let $\mathbb{Q}$ denote the set of all rational numbers and $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$.

We define the mapping $U : K \to K$ by setting $Ux := \sqrt{2}x \wedge \left(\frac{1}{\sqrt{2}}\right)$, for all $x \in \mathbb{Q} \cap K$ and $Ux := 0$, for all $x \in \mathbb{I} \cap K$. One can check that $K_n := \overline{\text{co}}(U(K_{n-1})) = K$, for all $n \in \mathbb{N}$. Further, 0, $\frac{1}{2} \in K$ and

$$\left|U(0) - U\left(\frac{1}{2}\right)\right| = \left|0 - \frac{1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}} = \sqrt{2}\frac{1}{2} = \sqrt{2}\left|0 - \frac{1}{2}\right|$$

Thus, U fails to be a cascading nonexpansive mapping on K. On the other hand, $\forall\ n \geq 2$, for all $x \in K$, $U^n x = 0$. Therefore, U is eventually nonexpansive on K. Also, since U is not a continuous mapping on K, U is not asymptotically nonexpansive. Łukasz Piasecki ([29]) has constructed an asymptotically nonexpansive mapping T on a closed, bounded, convex set K in a Banach space $(X, \|\cdot\|)$ such that T is not cascading nonexpansive. It is also well known that [(1) $\implies$ (7)] and [(4) $\implies$ (7)]. (See, for example, [18] and [27]). We note that it is an open question as to whether [(2) $\implies$ (7)]. It is also well known that [(6) $\implies$ (7)]. We include a proof, for the sake of completeness.

Lemma 2.1.4. *Let $(X, \|\cdot\|)$ be a Banach space and $M \subseteq X$ be a nonempty, closed, bounded, and convex set. Let $T : M \longrightarrow M$ be an affine mapping. Then there exists an approximate fixed point sequence $(x_n)_{n\in\mathbb{N}}$ for T in M.*

Proof. Fix $x_0 \in M$. Define

$$x_n := \left(\frac{I + T + T^2 + \cdots + T^n}{n+1}\right)(x_0) \quad , \text{ for all } n \in \mathbb{N} \, .$$

Each x_n is in M, because M is convex. Let

$$d := \operatorname{diam}(M) := \sup_{u,v\in M} \|u - v\| \in [0, \infty) \, .$$

Since T is affine, we have that

$$\begin{aligned}
\|Tx_n - x_n\| &= \left\| T\left(\frac{I + T + T^2 + \cdots + T^n}{n+1}\right)(x_0) - \left(\frac{I + T + T^2 + \cdots + T^n}{n+1}\right)(x_0)\right\| \\
&= \frac{1}{n+1}\left\|(T + T^2 + T^3 + \cdots T^{n+1})(x_0) - (I + T + T^2 + \cdots + T^n)(x_0)\right\| \\
&= \frac{1}{n+1}\left\|T^{n+1}x_0 - x_0\right\| \\
&\leq \frac{d}{n+1} \longrightarrow 0
\end{aligned}$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence for T. □

In chapter 5 we use the notion of asymptotically isometric copy of ℓ_1 introduced in [13] by P. Dowling, C. Lennard and B. Turett.

Definition 2.1.5. We say that a Banach space $(X, \|\cdot\|)$ is *asymptotically isometric* to $(\ell_1, \|\cdot\|_1)$ if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that
$X = \overline{\text{linear span}}\{x_n : n \in \mathbb{N}\}$ and there exists a sequence of scalars $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0, 1)$ such that $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$ and $\forall \alpha \in c_{00}$,

$$\sum_{n=1}^{\infty}(1 - \varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\| \leq \sum_{n=1}^{\infty}(1 + \varepsilon_n)|a_n|$$

Note that

Remark 2.1.6. When each $\varepsilon_n = 0$, $(X, \|\cdot\|)$ is an isometric copy of $(\ell_1, \|\cdot\|_1)$.

3.0 AN INTERESTING SUBSET OF ℓ_∞

An example of a closed, convex, non-weakly compact set $C \subseteq \ell_\infty$ and an affine, norm continuous map $f : C \to \mathbb{R}$ such that $\nexists\ \alpha, L$ with $\alpha \in \mathbb{R}$, $L : \ell_\infty \to \mathbb{R}$ a continuous linear map for which $f(x) = L(x) + \alpha$, $\forall x \in C$.

Many times the first examples of continuous affine mappings on a closed, bounded, convex set that we provide are restrictions of a continuous linear+constant mappings from the entire space to the given set. So the question was to provide an example of closed, convex set and an affine map which does not have a continuous linear + constant extension to the entire space.

It was known already the existence of a closed, convex, norm compact subset C of ℓ_∞ and an affine continuous map $f : C \to \mathbb{R}$ with the above property.

The set and the map were defined in the following way:

$C := \{x \in \ell_\infty : |x_n - 1| \leqslant \frac{1}{n}, \forall n \in \mathbb{N}\}$ and $f : C \to \mathbb{R}$ by

$$f(x) := \sum_{n=1}^{\infty} \frac{x_n - 1}{n} \quad , \ \forall x \in C$$

Through a perturbation of this example, we constructed a new example of a closed, convex, non-weakly compact set C and an affine, norm continuous map $f : C \to \mathbb{R}$ such that $\nexists\ \alpha, L, \alpha \in \mathbb{R}, L : \ell_\infty \to \mathbb{R}$ continuous linear map for which $f(x) = L(x) + \alpha$, $\forall x \in C$.

3.1 A NEW EXAMPLE C

Our example is the following:

Let $C := \{x \in \ell_\infty : \left|\frac{x_1+\ldots+x_n}{n} - 1\right| \leqslant \frac{1}{n}, \forall n \in \mathbb{N}\}$ and define $f : C \to \mathbb{R}$ by

$$f(x) := \sum_{n=1}^{\infty} \frac{\frac{x_1+\ldots+x_n}{n} - 1}{n} \quad , \quad \forall x \in C$$

.

A few basic facts about the set C:

Fact 1. $\mathbf{1} = (\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}, \ldots) \in \mathbf{C}$, which implies that C is non-empty.

Fact 2. It is easy to prove that C is convex.

Fact 3. C is not norm-compact.

In order to prove this, first notice that $g_\nu := \mathbf{1} - e_\nu \in C$, $\forall \nu \in \mathbb{N}$. Indeed for $x = g_n$, $\forall i, 1 \leqslant i \leqslant n-1$, $\left|\frac{x_1+\ldots+x_i}{i} - 1\right| = 0 \leqslant \frac{1}{i}$ and $\forall i \geqslant n$, $\left|\frac{x_1+\ldots+x_i}{i} - 1\right| = \frac{1}{i} \leqslant \frac{1}{i}$.

So, $\forall \nu \neq \mu$ in $\mathbb{N}$, $\|g_\mu - g_\nu\|_\infty = \|e_\mu - e_\nu\|_\infty = 1$. Thus, C is not norm compact.

Fact 4. C is norm closed; in fact, C is closed in the coordinate-wise topology.

For this, let $(x^{(t)})_{t\in\mathbb{N}}$ be a sequence in C and $z \in \ell_\infty$ and suppose $\left\|x^{(t)} - z\right\|_\infty \underset{t}{\to} 0$ which implies that $x_j^{(t)} \underset{t}{\to} z_j$, $\forall j \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. Then

$\frac{1}{n} \geqslant \left|\frac{x_1^{(t)}+\ldots+x_n^{(t)}}{n} - 1\right| \underset{t}{\to} \left|\frac{z_1+\ldots+z_n}{n} - 1\right|$ which implies that $\left|\frac{z_1+\ldots+z_n}{n} - 1\right| \leqslant \frac{1}{n}$. Therefore, $z \in C$.

Fact 5. C is an $\|\cdot\|_\infty$ -norm bounded subset of ℓ_∞. Fix $x \in C$. Notice that $\forall n \in \mathbb{N}$, the following inequality holds:

$$\begin{aligned} |x_{n+1}| &= |x_1 + \ldots + x_n + x_{n+1} - (x_1 + \ldots + x_n)| \\ &\leq |x_1 + \ldots + x_n + x_{n+1} - (n+1)| + 1 + |n - (x_1 + \ldots + x_n)| \\ &\leq 3 \end{aligned}$$

So, $\|x\|_\infty \leqslant 3$, $\forall x \in C$.

Next, we will prove that:

Fact 6. C is not a weakly compact subset of ℓ_∞.

Proof. In Sequences and Series in Banach Spaces by Joseph Diestel, chapter 3, problem 2, says the following: weakly compact subsets of ℓ_∞ are norm-separable. First, we will prove that our set is not norm-separable and therefore, it is not weakly compact.

At the end of this chapter, we will also provide a proof of the above exercise. Now, denote by $\tilde{C}$ the translation of the set C by 1.

Basically,

$$\tilde{C} = \{z \in \ell_\infty : \left|\frac{z_1 + \ldots + z_n}{n}\right| \leqslant \frac{1}{n}, \forall n \in \mathbb{N}\} = \{z \in \ell_\infty : -1 \leqslant z_1 + \ldots + z_n \leqslant 1, \forall n \in \mathbb{N}\}$$

It is not hard to see that $\tilde{C}$ contains all sequences of the form $x := (x_1, x_2, \ldots, x_{2n-1}, x_{2n}, \ldots)$ where $(x_{2n-1}, x_{2n}) \in \{(1, -1), (-1, 1)\}$, $\forall n \in \mathbb{N}$. Indeed, $\forall n \in \mathbb{N}$ $x_1 + \cdots + x_n \in \{-1, 0, 1\}$. Moreover, there are $2^{\mathbb{N}}$ choices of such sequences. Any two distinct sequences differ on at least 2 positions, therefore they are all 2 units apart; i.e $\|x - y\|_\infty = 2$. But this implies that $\tilde{C}$ is not norm-separable and so, it cannot be weakly compact. Thus, C is not weakly compact.

□

As mentioned above, let us prove that weakly compact subsets of ℓ_∞ are norm-separable. Since ℓ_∞ can be embedded isometrically into $L^\infty[0,1]$, it is sufficient to prove that every weakly compact subset K of $L^\infty[0,1]$ is norm-separable. By the Krein-Smulian theorem and the fact that subspaces of separable metric spaces are separable, there is no loss of generality in assuming that K is also convex.

We will use the following theorem of A. Grothendieck ([20]):

Theorem 3.1.1. *Every continuous linear operator $W : (L^\infty[0,1], \|\cdot\|_\infty) \to (X, \|\cdot\|_X)$, where X is a separable Banach space, that is weakly compact (i.e. $\overline{W(B_{L^\infty[0,1]})}$ is weakly compact in X) is also completely continuous (i.e. $f_n \underset{n}{\to} 0$ weakly implies $W(f_n) \underset{n}{\to} 0$ in norm).*

Proof. Fix $1 \leq p < \infty$; let $X = L^p[0,1]$. Notice $W : L^\infty \to L^p$ given by $W(f) = f$ is a continuous, linear operator ($\|f\|_p \leq \|f\|_\infty$, $\forall f \in L^\infty[0,1]$) that is also weakly compact (for $1 < p < \infty$ this follows from the fact that $L^p[0,1]$ is reflexive and for $p = 1$ it follows from a theorem of Dunford and Pettis). So W is also completely continuous, which means that $\forall\ K \subseteq L^\infty[0,1]$ weakly compact, $K \subseteq L^p[0,1]$ is norm compact. Also $K \subseteq L^p[0,1]$ is automatically $\|\cdot\|_p$-separable, $\forall p \in [1,\infty)$.

Let $p = \nu \in \mathbb{N}$. Let $E_\nu = \{f_n^{(\nu)}, n \in \mathbb{N}\}$ be a countable dense sequence in $(K, \|\cdot\|_\nu)$. Let $E = \cup_{\nu\in\mathbb{N}} E_\nu$. Obviously, E is countable. We will show that a certain countable superset F of E is dense in $(K, \|\cdot\|_\infty)$.

Recall that $\forall\ u \in L^\infty[0,1]$, $\lim\limits_{p\to\infty} \|u\|_p = \|u\|_\infty$.

Indeed, first notice that $\|u\|_p \leq \|u\|_\infty$. Recall that

$$\|u\|_\infty = \text{ess-sup}_{0\leq t\leq 1}|u(t)| = \inf\{\tau > 0 : m(\{t \in [0,1] : |u(t)| > \tau) = 0\}$$

.

Without loss of generality, $\|u\|_\infty > 0$, and so $m(\{t \in [0,1] : |u(t)| > \|u\|_\infty - \varepsilon\}) > 0$

Fix $p \in [1,\infty)$;

$$\begin{aligned}
\|u\|_p &= \left(\int_{t\in[0,1]} |u(t)|^p \, dm(t)\right)^{\frac{1}{p}} \\
&\geq \left(\int_{t\in E_\varepsilon} |u(t)|^p \, dm(t)\right)^{\frac{1}{p}} \\
&\geq \left(\int_{t\in E_\varepsilon} (\|u\|_\infty - \varepsilon)^p dm(t)\right)^{\frac{1}{p}} \\
&= \left(\|u\|_\infty - \varepsilon\right)\left(m(E_\varepsilon)\right)^{\frac{1}{p}}
\end{aligned}$$

As p tends to ∞, we obtain that

$$\|u\|_\infty \geq \limsup_{p\to\infty} \|u\|_p \geq \liminf_{p\to\infty} \|u\|_p \geq \|u\|_\infty - \varepsilon$$

By letting $\varepsilon \to 0$, we get that $\lim\limits_{p\to\infty} \|u\|_p = \|u\|_\infty$.

Fix $f \in K$ arbitrary. From the previous observation, $\forall\ \nu \in \mathbb{N}$, $\exists\ g_\nu = f_{n_\nu}^{(\nu)} \in\ E_\nu \subseteq E$ such that

$$\|f - g_\nu\|_\nu < \frac{1}{2^\nu}$$

Therefore, $(g_\nu)_{\nu\in\mathbb{N}}$ is a sequence in K-weakly compact, which implies that there exists a subsequence $(g_{\nu_k})_{\nu_k\in\mathbb{N}}$ converging weakly to some $h \in K \subseteq L^\infty[0,1]$. By Grothendieck's theorem, $\lim\limits_{k\to\infty} \|g_{\nu_k} - h\|_p = 0$, $\forall p \in [1,\infty)$.

Let $k_0 = 1$; $\nu_{k_0} = \nu_1$.

Choose $\nu_{k_1} \in \mathbb{N}$, $k_1 > 1$ such that

$$\left\|g_{\nu_{k_1}} - h\right\|_{\nu_1} < \frac{1}{2}$$

Choose $\nu_{k_2} > \nu_{k_1}$ with $k_2 > k_1$ such that

$$\left\|g_{\nu_{k_2}} - h\right\|_{\nu_{k_1}} < \frac{1}{2^2}$$

In general, the following holds:

$$\left\|g_{\nu_{k_{l+1}}} - h\right\|_{\nu_{k_l}} < \frac{1}{2^{l+1}}$$

$\forall\ l \in \mathbb{N}$ with $k_{l+1} > k_l$.

Fix $\varepsilon > 0$, $\exists\ l_0 \in \mathbb{N}$ such that $\forall\ l \geq l_0$

$$\begin{aligned}
\|f - h\|_\infty &< \|f - h\|_{\nu_{k_l}} + \frac{\varepsilon}{2} \\
&< \left\|f - g_{\nu_{k_{l+1}}}\right\|_{\nu_{k_l}} + \left\|g_{\nu_{k_{l+1}}} - h\right\|_{\nu_{k_l}} + \frac{\varepsilon}{2} \\
&\leq \left\|f - g_{\nu_{k_{l+1}}}\right\|_{\nu_{k_{l+1}}} + \frac{1}{2^{l+1}} + \frac{\varepsilon}{2} \\
&< \frac{1}{2^{\nu_{l+1}}} + \frac{1}{2^{l+1}} + \frac{\varepsilon}{2} \\
&< \frac{1}{2^{l+1}} + \frac{1}{2^{l+1}} + \frac{\varepsilon}{2} = \frac{1}{2^l} + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\|f - h\|_\infty < \varepsilon$ implies that $h = f$ a.e. Denote by $h_k := g_{\nu_k}$. Thus, $\lim_{k\to\infty} h_k = f \in K$ weakly in $L^\infty[0,1]$. Every h_k is in K and K is assumed to be a convex set. By Mazur's theorem, there exists $(t_j)_{j\in\mathbb{N}}$ in $[0,\infty)$ and $(p_m)_{m\in\mathbb{N}}$ in N, strictly increasing, such that

$$A_m = \sum_{j=p_m+1}^{p_{m+1}} t_j h_j \underset{\text{m}}{\longrightarrow} h = f$$

in $\|\cdot\|_\infty$-norm and each $\sum_{j=p_m+1}^{p_{m+1}} t_j = 1$

The fact that K is convex implies that each $A_m \in K$. By perturbing each t_j a little, we may also assume that each $t_j \in \mathbb{Q}^+$. Note that $F = \{\sum_{k=1}^{\nu} s_k e_k : s_k \in \mathbb{Q}^+, e_k \in E, \sum_{k=1}^{\nu} s_k = 1\}$ is a countable subset of K and F is $\|\cdot\|_\infty$-norm dense in K. Let $\varepsilon > 0$ and f be as above. Then

$$\|f - A_m\|_\infty < \varepsilon$$

$\forall\, m \geq m_\varepsilon$. □

We have the following **open questions** regarding the set C:

Question 1: Does C have the fixed point property for affine nonexpansive maps?

Question 2: Does C have the fixed point property for nonexpansive maps?

3.2 AN AFFINE, NORM CONTINUOUS MAP

Next, we will continue with the proof of the second part of the statement.

Theorem 3.2.1 (R. Popescu (RP))**.** *There exists an affine, norm continuous map $f : C \to \mathbb{R}$ such that $\nexists\, \alpha, L$, with $\alpha \in \mathbb{R}$, $L : \ell_\infty \to \mathbb{R}$ a continuous, linear map for which $f(x) = L(x) + \alpha$, $\forall x \in C$.*

Proof. : Define $f : C \to \mathbb{R}$ by: $f(x) := \sum_{n=1}^{\infty} \frac{\frac{x_1+\ldots+x_n}{n} - 1}{n}$, $\forall x \in C$.

Clearly, f is well-defined since:

$$\sum_{n=1}^{\infty} \left| \frac{\frac{x_1+\ldots+x_n}{n} - 1}{n} \right| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \forall x \in C$$

.

Our next claim is that:

Claim 3.2.2. *: $f : C \to \mathbb{R}$ is norm-to-usual continuous on C.*

Proof. : The proof is straightforward.

□

Claim 3.2.3. *f is affine.*

Proof. : It is an easy fact to check. □

Claim 3.2.4. *f does not admit a linear continuous + constant extension to the entire space ℓ_∞, i.e $\nexists$ $\alpha, L, \alpha \in \mathbb{R}$, $L : \ell_\infty \to \mathbb{R}$ a continuous linear map for which $f(x) = L(x) + \alpha$, $\forall x \in C$.*

Proof. : We will prove it by contradiction. Suppose it does. Therefore, $\exists$ α, L, with $\alpha \in \mathbb{R}$, $L : \ell_\infty \to \mathbb{R}$ continuous linear map for which $f(x) = L(x) + \alpha$, $\forall x \in C$.

From $f(\mathbf{1}) = L(\mathbf{1}) + \alpha = 0 \Rightarrow -L(\mathbf{1}) = \alpha$.

Next, notice that $\forall n \in \mathbb{N}$,

$$x_n = \mathbf{1} - \frac{1}{n}e_1 - \ldots - \frac{1}{n}e_n = \left(\frac{n-1}{n}, \ldots, \frac{n-1}{n}, 1, 1, \ldots \right) \in C$$

.

Applying f, we obtain

$$f(x_n) = L\left(\mathbf{1} - \frac{1}{n}e_1 - \ldots\right) - L(\mathbf{1}) = -\frac{1}{n}(L(e_1) + L(e_2) + \ldots + L(e_n))$$

.

On the other hand,

$$f(x_n) = -\frac{1}{n} - \frac{1}{2n} - \ldots - \frac{1}{(n-1)n} - \frac{1}{n^2} - \frac{1}{(n+1)^2} - \ldots$$

Therefore,

$$-\frac{1}{n}(L(e_1) + L(e_2) + \ldots + L(e_n)) = -\frac{1}{n} - \frac{1}{2n} - \ldots - \frac{1}{(n-1)n} - \frac{1}{n^2} - \frac{1}{(n+1)^2} - \ldots$$

So,

$$L(e_1) + L(e_2) + \ldots + L(e_n) = 1 + \frac{1}{2} + \ldots + \frac{1}{(n-1)} + \frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \ldots$$

Let $s_n := \sum_{i=1}^{n} \frac{1}{i}$ and $v_n := \sum_{i=n+1}^{\infty} \frac{1}{i^2}$

Then, $L(e_1) + L(e_2) + \ldots + L(e_n) = L(e_1 + \ldots + e_n) = s_n + nv_n \geqslant s_n$

From our assumption, L is linear and continuous and so it is bounded. This implies that $L(e_1 + \ldots + e_n) \leq \|L\| \, \|e_1 + \ldots + e_n\|_\infty = \|L\|$, but $L(e_1 + \ldots + e_n) \geqslant \lim_{n\to\infty} s_n = \infty$. Thus, we have reached a contradiction.

□

At this point we can also prove the following:

Theorem 3.2.5. *(RP) There exists $g : C \to C$ a continuous and affine map, which does not admit a linear continuous + vector constant extension to the entire ℓ_∞, i.e $\nexists\ \alpha, L$, with $\alpha \in \ell_\infty, L : \ell_\infty \to \ell_\infty$ continuous linear map for which $g(x) = L(x) + \alpha$, $\forall x \in C$*

Proof. : We will use the map f in order to define the map g. We have seen that $|f(x)| \leqslant \sum_{n=1}^{\infty} |\frac{\frac{x_1+\ldots+x_n}{n}-1}{n}| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} := A < \infty, \forall x \in C$.

Thus, $|\frac{f(x)}{A}| \leqslant 1$, $\forall x \in C$, which implies that $0 \leqslant y := \frac{f(x)}{A} + 1 \leqslant 2$.

Next, notice that the set C can also be rewritten as $C := \{z \in \ell_\infty : n-1 \leqslant z_1+\ldots+z_n \leqslant n+1, \forall n \in \mathbb{N}\}$. Also, if $0 \leqslant y \leqslant 2$, then the vector $z := (y, 1, 1, \ldots, 1, \ldots) \in C$. Indeed, $\forall n \in \mathbb{N}$, $z_1 + \ldots + z_n = y + (n-1)$ and $n - 1 \leqslant y + (n-1) \leqslant (n+1)$.

Let $g : C \to C$ be defined by:

$$g(x) = \left(\frac{f(x)}{A} + 1, 1, 1, \ldots, 1, \ldots\right), \forall x \in C$$

The above observation assures us that g is well-defined. Moreover, it is easy to see that g is affine. The fact that g is continuous can be deduced from evaluating $\|g(x) - g(y)\|_\infty = \left|\frac{f(x)-f(y)}{A}\right|$, $\forall x, y \in C$. Now since f is continuous, g is also continuous.

Furthemore, suppose that there exist α and L, $\alpha \in \ell_\infty$ and $L : \ell_\infty \to \ell_\infty$ linear continuous map such that $g(x) = L(x) + \alpha$, $\forall x \in C$. This implies that $\frac{f(x)}{A} + 1 = L_1(x) + \alpha_1$, where by $L_1(x)$, respectively α_1, we mean the first coordinate of $L(x)$, respectively α. Thus, $f(x) = A \cdot L_1(x) + A(\alpha_1 - 1)$, $\forall x \in C$. But $A \cdot L_1 : \ell_\infty \to \mathbb{R}$ is a linear continuous map, $A(\alpha_1 - 1) \in \mathbb{R}$, and we have reached a contradiction at this point, since we have already proven that f does not admit a continuous linear+ constant extension to the entire space ℓ_∞.

□

Remark. The map g has a fixed point. In fact, it has infinitely many. Indeed, by setting $g(x) = x$ for $x = (x_1, x_2, \ldots, x_n, \ldots) \in C$, we see that $x_2 = x_3 = \ldots = x_n = \ldots = 1$ and $x_1 = \frac{f(x)}{A} + 1$. But now, from the definition of f, we obtain: $f(x) = f(x_1, 1, 1, \ldots) = (x_1 - 1)A$. Therefore, $x_1 = \frac{(x_1-1)A}{A} + 1$ a true equality, $\forall x_1 \in \mathbb{R}$ with $0 \leqslant x_1 \leqslant 2$. This last required inequality on x_1 guarantees that $x = (x_1, 1, 1, \ldots)$ is in C. In conclusion, all the vectors of the form $x = (x_1, 1, 1, \ldots)$, with $0 \leqslant x_1 \leqslant 2$, are fixed points for the map g.

3.3 TWO DIFFERENT SUBSETS OF C

We close this chapter with two examples and an open question. We will use again one of these examples in the next chapter, where we discuss about the notion of the affine dual of a closed, bounded, convex set.

In 2004, in Proc. of AMS, P. Dowling, C. Lennard and B. Turett (see [14]) proved that weak compactness is equivalent with the fixed point property in c_0.

Starting from our set C, we define $H \subseteq C$ by

$$H := \{x \in \ell_\infty : \left|\frac{x_1 + \ldots + x_n}{n} - 1\right| \leqslant \frac{1}{n}, \forall n \in \mathbb{N}; x_n \underset{n}{\to} 1\}$$

The vector $(2, 0, 2, 0, 2, 0, \ldots) \in C$, but is not an element of H, therefore H is a strict subset of C.

Also, we can write $H = \tilde{H} + \mathbf{1}$, where $\tilde{H} \subseteq c_0$. Basically, we can view $\tilde{H} = \{x \in c_0 : -1 \leq x_1 + x_2 + \cdots + x_n \leq 1, \forall n \in \mathbb{N}\}$. One can prove that $\tilde{H}$ is a closed, bounded, convex, non-weakly compact subset of c_0.

First, H is obviously bounded as a subset of the bounded set C. It is also easy to see that H is convex. We will prove now that H is norm-closed. For this, let $(x^p)_{p \in \mathbb{N}}$ be a sequence in H such that $||x^p - x||_\infty \underset{p}{\to} 0$. In particular, $|x_n^p - x_n| \underset{p}{\to} 0$, $\forall n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ arbitrary. Then $|x_1^p + \ldots + x_n^p - n| \leqslant 1$, $\forall p$ and so, passing p to infinity, we get $|x_1 + \ldots + x_n - n| \leqslant 1$. Since n was arbitrary chosen, the previous inequality is true for all n.

We are left to show that $x_n \underset{n}{\to} 1$, where $x = (x_1, x_2, \ldots, x_n, \ldots)$. Fix $\epsilon > 0$.

There exists p_ϵ such that $\forall p \geqslant p_\epsilon$, we have $\|x^p - x\|_\infty \leqslant \frac{\epsilon}{2}$.

This implies that $|x_n^{p_\epsilon} - x_n| \leqslant \frac{\epsilon}{2}, \forall n \in \mathbb{N}$. Since $x_n^{p_\epsilon} \underset{n}{\to} 1$, there exists N_ϵ such that $\forall n \geqslant N_\epsilon$, we have $|x_n^{p_\epsilon} - 1| \leqslant \frac{\epsilon}{2}$.

Applying the triangle inequality, we obtain the following: $\forall n \geqslant N_\epsilon$,

$$|x_n - 1| \leqslant |x_n^{p_\epsilon} - x_n| + |x_n^{p_\epsilon} - 1| \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

.

Therefore, $x_n \underset{n}{\to} 1$ and so $x \in H$.

To see that H is not weakly compact, consider the following sequence:

$$u^{(1)} = (2, 1, 1, 1, 1, \ldots, 1, \ldots) \in H$$

$$u^{(2)} = (2, 0, 1, 1, 1, \ldots, 1, \ldots) \in H$$

$$u^{(3)} = (2, 0, 2, 1, 1, \ldots, 1, \ldots) \in H$$

$$u^{(4)} = (2, 0, 2, 0, 1, \ldots, 1, \ldots) \in H$$

and so on...

This sequence converges coordinate-wise, and thus w$^* = \sigma(\ell_\infty, \ell_1)$ (since H is bounded), to $v = (2, 0, 2, 0, 2, 0, \ldots) \in C \setminus H$. Thus, H is not w*-closed and so it is not w*-compact, which further implies that H is not weakly compact in ℓ_∞.

Remark. The set H , being essentially a subset of c_0, is norm-separable and therefore we cannot apply the same separability argument as we did for the set C to prove non-weakly compactness.

Therefore, it fails fixed point property for nonexpansive maps. It turns out however, that it also fails the fixed point property for affine nonexpansive maps.

Indeed, if we consider the mapping $T : \tilde{H} \to \tilde{H}$ defined by

$T(x_1, x_2, \cdots) = (1, -1, x_1, x_2, \cdots)$, then T is a fixed point free affine isometry.

In contrast to this example, we will now present an example of a closed, bounded, convex, weakly compact, not norm-compact subset K of C.

First, to simplify things, it is important to notice again that an element $x \in C$ if and only if $|x_1 + \ldots + x_n - n| \leqslant 1$, $\forall\, n \in \mathbb{N}$. This last condition also implies the fact that $x \in \ell_\infty$.

We construct the following sequence:

$$\tau_1 := (0, 1, 1, 1, \ldots, 1, \ldots) = \mathbf{1} - e_1 \in C$$

$$\tau_2 := (1, 0, 1, 1, \ldots, 1, \ldots) = \mathbf{1} - e_2 \in C$$

$$\tau_3 := (1, 1, 0, 1, \ldots, 1, \ldots) = \mathbf{1} - e_3 \in C$$

and, in general, let

$$\tau_n := (1, 1, 1, 1, \ldots, 0, \ldots) = \mathbf{1} - e_n \in C$$

Now, $\mathbf{1} - \tau_n = e_n \in c_0$ and the sequence $e_n \underset{n}{\to} 0$ weakly in c_0 (it converges coordinatewise, which is the same with weak convergence for bounded sequences). Therefore, $\tau_n \underset{n}{\to} \mathbf{1}$ weakly in ℓ_∞ since $(\ell_\infty)^* = \ell_1 \oplus c_0^\perp$.

Using the Krein-Smulian Theorem, which states that in a Banach space, the closed convex hull of a weakly compact set is weakly compact, we obtain that

$$K := \overline{\text{co}}^{\|\cdot\|_\infty}\{\{\tau_n : n \in \mathbb{N}\} \cup \{\mathbf{1}\}\} = \overline{\text{co}}^{\|\cdot\|_\infty}\{\tau_n : n \in \mathbb{N}\} = \mathbf{1} - \overline{\text{co}}^{\|\cdot\|_\infty}\{e_n : n \in \mathbb{N}\}$$

is a weakly compact subset of C.

Here the second equality holds because a closed and convex set is also weakly closed.

K is not norm-compact. Indeed, $\|\tau_j - \tau_k\|_\infty = 2$, $\forall j \neq k$ and therefore, it is not sequentially compact which is equivalent with compactness in a metric space.

By a result due to B. Maurey (1981), it is known that the set K, which is essentially a translation of the weakly compact subset $E := \overline{\text{co}}^{\|\cdot\|_\infty}\{e_n : n \in \mathbb{N}\}$ of c_0, has the fixed point property for $\|\cdot\|_\infty$ non-expansive maps.

Also, by Mil'man and Mil'man's Theorem, whose proof we will present next for the sake of completeness, it is also known that K has the fixed point property for $\|\cdot\|_\infty$ norm-to-norm continuous, affine maps.

Theorem 3.3.1. *: Let $(X, \|\cdot\|)$ be a Banach space, C a non-empty, weakly compact, convex subset of X and $T : C \to C$ be a norm-to-norm continuous and affine mapping. Then T has a fixed point in C.*

Proof. Fix $x_0 \in C$.

$\forall\ n \geqslant 1$, let $x_n = \frac{x_0+T(x_0)+\ldots+T^n(x_0)}{n+1} \in C$.

By applying T, since T is affine, we obtain:

$$T(x_n) = \frac{T(x_0) + T^2(x_0) + \ldots + T^{n+1}(x_0)}{n+1}$$

Therefore, $\|T(x_n) - x_n\| = \left\| \frac{T^{n+1}(x_0)-x_0}{n+1} \right\| \leqslant \frac{diam(C)}{n+1} \underset{n}{\to} 0$, where diam(C):= $\sup\limits_{x,y\in C} \|x - y\|$ and it is bounded in this case, since C is weakly compact.

Thus, $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence.

By the Eberlein-Smulian Theorem, there exists of subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that $x_{n_k} \underset{k}{\to} w$, for some $w \in C$. Without loss of generality, we may assume that $x_n \underset{n}{\to} w$ weakly and $\|T(x_n) - x_n\| \underset{n}{\to} 0$. By Mazur's Theorem, there exists a strictly increasing sequence $(q_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{N}$ such that $y_\nu = \sum\limits_{j=q_\nu+1}^{q_{\nu+1}} \alpha_j x_j \in co\{x_n : n \in \mathbb{N}\}$ approximates w in norm; i.e $\|y_\nu - w\| \underset{\nu}{\to} 0$.

Moreover, $\|T(y_\nu) - y_\nu\| = \left\| \sum\limits_{j=q_\nu+1}^{q_{\nu+1}} \alpha_j (T(x_j) - x_j) \right\| \underset{\nu}{\to} 0$.

Indeed, let $\epsilon > 0$. Since $\|T(x_n) - x_n\| \underset{n}{\to} 0$, $\exists\ n_0$ such that $\forall\ n \geqslant n_0$, $\|T(x_n) - x_n\| \leqslant \epsilon$.

Choose $\nu_0 \in \mathbb{N}$ such that $q_{\nu_0} + 1 \geqslant n_0$. Therefore, $\forall\ \nu \geqslant \nu_0$

$$\begin{aligned} \|T(y_\nu) - y_\nu\| &= \left\| \sum_{j=q_\nu+1}^{q_{\nu+1}} \alpha_j (T(x_j) - x_j) \right\| \\ &\leqslant \sum_{j=q_\nu+1}^{q_{\nu+1}} \alpha_j \|T(x_j) - x_j\| \\ &\leqslant \sum_{j=q_\nu+1}^{q_{\nu+1}} \alpha_j \epsilon \\ &= \epsilon \end{aligned}$$

Thus, $\|T(y_\nu) - y_\nu\| \underset{\nu}{\rightarrow} 0$.

Finally, notice that $\|T(w) - w\| \leqslant \|T(y_\nu) - T(w)\| + \|T(y_\nu) - y_\nu\| + \|y_\nu - w\|$. Since T is continuous, the right hand side goes to 0 as $\nu \underset{\nu}{\rightarrow} \infty$.

In conclusion, $T(w) = w$ and so w is a fixed point for the map T.

□

It is still an **open question** whether there exists a closed, bounded, convex, non-weakly compact subset C of c_0 which has the fixed point property for affine nonexpansive maps, with respect to the usual norm.

4.0 AFFINE DUALS OF CERTAIN CLOSED, BOUNDED, CONVEX SETS

In 1940, in Studia Mathematica, M. Krein and D. Milman ([25]) proved the following theorem:

Theorem 4.0.1. *Let X be a locally convex Hausdorff topological vector space, and let K be a compact convex subset of X. Then, K is the closed convex hull of its extreme points.*

We recall that an extreme point of a convex set A, is a point $x \in A$ with the property that if $x = \theta y + (1-\theta)z$ with $y, z \in A$ and $\theta \in [0,1]$, then $y = x$ and $z = x$.

Also, Banach-Alaoglu-Bourbaki had shown that in every dual Banach space the closed unit ball is compact in the weak* topology.

By using these two theorems, one can easily prove by contradiction that the Banach spaces $(c_0, \|\cdot\|_\infty)$ and $(L^1[0,1], \|\cdot\|_1)$ are not dual spaces, since their unit ball has no extreme points.

However, it turns out that we can view them as affine dual spaces.

Indeed, we start with the following definition:

Definition 4.0.2. Let $C^* := \{\varphi : C \to \mathbb{R}$ such that φ is affine, bounded and norm-to-usual continuous on $C\}$. Naturally, we define $\|\varphi\|_{C^*} := \sup_{x \in C} |\varphi(x)|$.

$(C^*, \|\cdot\|_{C^*})$ is called the affine dual of C.

Notice that in the case when C is weakly compact, one can drop the boundedness condition from the above definition. Indeed, the following lemma is true:

Lemma 4.0.3. *Let C be a convex, weakly compact subset of a Banach space $(X, \|\cdot\|)$ and $\varphi : C \to \mathbb{R}$ an affine, norm-to-usual continuous on C. Then φ is in fact weak to-usual continuous on C, and therefore bounded on C.*

Proof. Let W be a closed set in $\mathbb{R}$. We will first check for intervals of the form $[a,b]$, $[a,\infty)$ and $(-\infty,b]$, with $a,b\in\mathbb{R}$ and $a<b$, that $\varphi^{-1}(W)$ is a weakly closed subset of C. Using the fact that φ is affine, continuous and W is convex, we obtain that $\varphi^{-1}(W)$ is a closed and convex subset of C, and therefore it is weakly closed. This means that $\varphi^{-1}(a,b)$, $\varphi^{-1}(a,\infty)$, $\varphi^{-1}(-\infty,b)$ are weakly open subsets of C, for all $a,b\in\mathbb{R}$ with $a<b$. Now every open subset σ of $\mathbb{R}$ can be written as the disjoint union of open intervals, that is $\sigma=\cup_{j\in A}(a_j,b_j)$. Thus, $\varphi^{-1}(\sigma)$ is weakly open. Choose $\sigma=\mathbb{R}\backslash W$. Since $\varphi^{-1}(\sigma)$ is weakly open, we now obtain that $\varphi^{-1}(W)$ is weakly closed. Therefore, φ is weak to-usual continuous and the proof is finished. □

4.1 $(c_0,\|\cdot\|_\infty)$ AS AN AFFINE DUAL

We consider the following closed, bounded, convex set:

$C:=\overline{\text{co}}^{\|\cdot\|_\infty}\{e_n : n\in\mathbb{N}\}\subseteq(c_0,\|\cdot\|_\infty)$.

It is not difficult to prove that $C=D$, where we define

$D:=\{x=\sum_{n=1}^{\infty}t_ne_n$: each $t_n\geqslant 0$ and $\sum_{n=1}^{\infty}t_n\leqslant 1\}$.

The following theorem is true:

Theorem 4.1.1 (C. Lennard, R. Popescu (LP))**.** $C^*=\Gamma:=\{\psi_y+r : y\in c_0, r\in\mathbb{R}\}$.

Here, $\forall y=(y_n)_{n\geq 1}\in c_0$,

$\psi_y(x):=\sum_{n=1}^{\infty}t_ny_n,\ \forall x=\sum_{n=1}^{\infty}t_ne_n\in C.$

Proof. • **Step 1**: $\Gamma\subseteq C^*$.

Fix $y\in c_0$ and $r\in\mathbb{R}$. By the previous observation about the set C, $\psi_y(x)$ is a well-defined member of $\mathbb{R}$, $\forall x\in C$ (because $C\subseteq\ell_1$).

Clearly, ψ_y is an affine map:

$$\psi_y((1-\lambda)x+\lambda z)=(1-\lambda)\psi_y(x)+\lambda\psi_y(z)$$

$\forall x,z\in C$, $\forall\lambda\in[0,1]$.

Also, $\|\psi_y\|_{C^*} = \|y\|_\infty$, ψ_y is norm-to-usual continuous on C and so we obtain that $\psi_y + r \in C^*$. Indeed, let$(x^{(i)})_{i\geq 1}$ be a sequence in C such that $\left\|x^{(i)} - x\right\|_\infty \to 0$ as $i \to \infty$. Since $y = (y_n)_{n\geq 1} \in c_0$, there exists N such that for all $n \geq N$, $|y_n| \leq \frac{\varepsilon}{4}$. Also, since $\left\|x^{(i)} - x\right\|_\infty \to 0$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $\left|t_n^{(i)} - t_n\right| \leq \frac{\varepsilon}{2N(\|y\|_\infty+1)}$. Therefore, for all $i \geq i_0$,

$$
\begin{aligned}
\left|\psi_y(x^{(i)} - \psi_y(x)\right| &= \left|\sum_{n=1}^{\infty}(t_n^{(i)} - t_n)y_n\right| \\
&\leq \sum_{n=1}^{N}\left|t_n^{(i)} - t_n\right| \|y\|_\infty + \sum_{n=N+1}^{\infty}\left|t_n^{(i)} - t_n\right| \frac{\varepsilon}{4} \\
&\leq N\frac{\varepsilon}{2N(\|y\|_\infty + 1)} + 2\frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

• **Step 2**: $C^* \subseteq \Gamma$

Fix $\varphi \in C^*$, $\varphi : C \to \mathbb{R}$, φ is affine and norm-to-usual continuous. Let $r := \varphi(0) \in \mathbb{R}$.

Each $e_k \in C$. Define $y_k := \varphi(e_k) - \varphi(0)$, $\forall k \in \mathbb{N}$. Each $y_k \in \mathbb{R}$.

To show: $\varphi(x) = \psi_y(x) + r$, $\forall x \in C$, where $y = (y_1, \ldots, y_n, \ldots)$.

We will also prove that $y \in c_0$.

First we will show that $\Omega := \sup_{k\in\mathbb{N}} |y_k| < \infty$.

Suppose not. Then $\sup_{k\in\mathbb{N}} |y_k| = \infty$.

Choose $\mu_1 \in \mathbb{N}$ such that $|y_{\mu_1}| \geq 1$. Choose $\mu_2 \in \mathbb{N}$, $\mu_2 \geq \mu_1$ such that $|y_{\mu_2}| > 4 + |y_{\mu_1}|$.

Then

$$\left|\frac{y_{\mu_2} + y_{\mu_1}}{2}\right| \geq \frac{|y_{\mu_2}| - |y_{\mu_1}|}{2} \geq \frac{4 + |y_{\mu_1}| - |y_{\mu_1}|}{2} = 2$$

Choose $\mu_3 \in \mathbb{N}$, $\mu_3 > \mu_2$ such that

$|y_{\mu_3}| \geq 3 \cdot 2^2 + |y_{\mu_2}| + |y_{\mu_1}|$

Then
$$\left|\frac{y_{\mu_3}+y_{\mu_2}+y_{\mu_1}}{3}\right| \geqslant \frac{|y_{\mu_3}|-|y_{\mu_2}|-|y_{\mu_1}|}{3} \geqslant \frac{3\cdot 2^2}{3} = 2^2$$
Continuing inductively, we build a subsequence $(y_{\mu_j})_{j\in\mathbb{N}}$, such that

$$\frac{\left|y_{\mu_1}+\ldots+y_{\mu_j}\right|}{j} \geqslant 2^{j-1} \quad \forall j\in\mathbb{N} \;\; (*)$$

Notice that

$$\frac{y_{\mu_1}+\ldots+y_{\mu_j}}{j} = \frac{\varphi(e_{\mu_1})+\ldots+\varphi(e_{\mu_j})}{j} - r = \varphi\left(\frac{e_{\mu_1}+\ldots+e_{\mu_j}}{j}\right) - r$$

$\forall j\in\mathbb{N}$.

Furthermore,
$$\left\|\frac{e_{\mu_1}+\ldots+e_{\mu_j}}{j}\right\|_\infty = \frac{1}{j} \underset{j}{\to} 0$$
We know that $\varphi : C \to \mathbb{R}$ is norm-to-usual continuous, so

$$\varphi\left(\frac{e_{\mu_1}+\ldots+e_{\mu_j}}{j}\right) \underset{j}{\to} \varphi(0) = r$$

Thus,
$$\frac{y_{\mu_1}+\ldots+y_{\mu_j}}{j} \underset{j}{\to} 0$$
which is a contradiction with $(*)$.

Next, we will prove that $y_k \underset{k}{\to} 0$.

Let $(y_{m_k})_{k\in\mathbb{N}}$ be any subsequence of $(y_n)_{n\in\mathbb{N}}$. By previous work, we know that $(y_{m_k})_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. So, by the Bolzano-Weierstrass Theorem, there exists a further subsequence $(y_{m_{k_j}})_{j\in\mathbb{N}}$ and $L\in\mathbb{R}$ such that $w_j := y_{m_{k_j}} \underset{j}{\to} L$.

Since $(w_j)_{j\in\mathbb{N}}$ is bounded, it follows that

$$\frac{w_1+\ldots+w_j}{j} \underset{j}{\to} L.$$

On the other hand,

$$\frac{w_1+\ldots+w_j}{j} = \frac{\varphi(e_{\mu_{k_1}})+\ldots+\varphi(e_{\mu_{k_j}})}{j} - r = \varphi\left(\frac{e_{\mu_{k_1}}+\ldots+e_{\mu_{k_j}}}{j}\right) - r$$

Now,

$$\left\|\frac{e_{\mu_{k_1}}+\ldots+e_{\mu_{k_j}}}{j}\right\|_\infty=\frac{1}{j}\underset{j}{\to}0$$

and φ is norm-to-usual continuous on C.

We have

$$\varphi\left(\frac{e_{\mu_{k_1}}+\ldots+e_{\mu_{k_j}}}{j}\right)\underset{j}{\to}\varphi(0)=r$$

.

Thus,

$$\frac{w_1+\ldots+w_j}{j}\underset{j}{\to}0$$

which now implies that $L=0$.

We have obtained that every subsequence $(y_{m_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ has a further subsequence that converges to 0.

Hence, $y_n\underset{n}{\to}0$.

The following step is to prove that $\Lambda:=\sup_{x\in C}|\varphi(x)|<\infty$.

Fix $x\in E$:=co$\{e_k:k\in\mathbb{N}\}$; $x=\sum_{j=1}^{\nu}s_je_j$, $\nu\in\mathbb{N}$, $s_j\geqslant 0$, $\sum_{j=1}^{\nu}s_j=1$.

Since φ is affine

$$\begin{aligned}\varphi(x) &= \varphi\left(\sum_{j=1}^{\nu}s_je_j\right)=\sum_{j=1}^{\nu}s_j\varphi(e_j)\\ &= \sum_{j=1}^{\nu}s_j(y_j+r)=\sum_{j=1}^{\nu}s_jy_j+r\\ &\leq \sum_{j=1}^{\nu}s_j\Omega+r=\Omega+r\end{aligned}$$

E is norm dense in C and $\varphi:C\to\mathbb{R}$ is norm-to-usual continuous, so

$$\Lambda=\sup_{x\in C}|\varphi(x)|=\sup_{x\in E}|\varphi(x)|\leqslant\Omega+|r|<\infty$$

Our final claim is that $\varphi=\psi_y+r$.

Fix $x=\sum_{n=1}^{\infty}t_ne_n\in C$, $t_n\geqslant 0$, $\sum_{n=1}^{\infty}t_n\leqslant 1$.

Fix $N \in \mathbb{N}$. Suppose that $\operatorname{supp}(x) \subset [0, N]$.

$$\begin{aligned}
|\varphi(x) - \psi_y(x) - r| &= \left|\varphi\Big(\sum_{n=1}^{\infty} t_n e_n\Big) - \sum_{n=1}^{\infty} t_n y_n - r\right| \\
&= \left|\varphi\Big(\sum_{n=1}^{N} t_n e_n + \sum_{n=N+1}^{\infty} t_n e_n\Big) - \sum_{n=1}^{\infty} t_n\big(\varphi(e_n) - \varphi(0)\big) - \varphi(0)\right| \\
&= \left|\varphi\Big(\sum_{n=1}^{N} t_n e_n + \sum_{n=N+1}^{\infty} t_n e_n + \Big(1 - \sum_{n=1}^{\infty} t_n\Big)0\Big)\right. \\
&\qquad \left. - \sum_{n=1}^{\infty} t_n(\varphi(e_n) - \varphi(0)) - \varphi(0)\right| \\
&= \left|\sum_{n=1}^{N} t_n \varphi(e_n) - \sum_{n=1}^{N} t_n \varphi(e_n)\right| = 0.
\end{aligned}$$

Now suppose x is infinitely supported.

$$\begin{aligned}
|\varphi(x) - \psi_y(x) - r| &= \left|\varphi\Big(\sum_{n=1}^{N} t_n e_n + \Big(\sum_{k=N+1}^{\infty} t_k\Big) \sum_{n=N+1}^{\infty} \frac{t_n}{\Big(\sum_{k=N+1}^{\infty} t_k\Big)} e_n\right. \\
&\qquad \left. + \Big(1 - \sum_{n=1}^{\infty} t_n\Big)0\Big) - \sum_{n=1}^{\infty} \varphi(e_n) + \Big(\sum_{n=1}^{\infty} t_n - 1\Big)\varphi(0)\ \right| \\
&\leq \left|\sum_{n=1}^{N} t_n \varphi(e_n) + \Big(\sum_{k=N+1}^{\infty} t_k\Big)\varphi\left(\sum_{n=N+1}^{\infty} \frac{t_n}{\Big(\sum_{k=N+1}^{\infty} t_k\Big)} e_n\right)\right. \\
&\qquad \left. - \sum_{n=1}^{\infty} t_n \varphi(e_n)\right| \\
&\leq \sum_{n=N+1}^{\infty} t_n\,|\varphi(e_n)| + \sum_{k=N+1}^{\infty} t_k \Lambda = 2\Lambda \sum_{n=N+1}^{\infty} t_n \underset{N}{\longrightarrow} 0.
\end{aligned}$$

In conclusion, $\varphi(x) = \psi_y(x) + r$, $\forall x \in C$.

• **Step 3**: $\|\psi_y\|_{C^*} = \|y\|_\infty$.

Fix $y \in c_0$; fix $x \in C$.

$$\psi_y(x) = \left|\sum_{n=1}^{\infty} t_n y_n\right| \leq \left(\sum_{n=1}^{\infty} t_n\right) \|y\|_\infty$$

which implies that $\|\psi_y\|_{C^*} \leq \|y\|_\infty$.

On the other hand, $\|\psi_y\|_{C^*} \geq |\psi_y(e_k)| = |y_k|, \forall k \in \mathbb{N}$ and therefore $\|\psi_y\|_{C^*} \geq \|y\|_\infty$.

The final step is now proved.

It is perhaps worth pointing out at this stage that if we consider $C^\bigstar := \{\varphi : C \to \mathbb{R}$ such that φ is affine and weak-to-usual continuous on $C\}$ then, by Mazur's theorem, which basically states that closed convex sets are weakly closed, we obtain that $C^\bigstar = C^*$. Therefore, the previous proof can be somewhat simplified now, since the set C which we defined in the beginning is a weakly compact subset of c_0 by the Krein-Smulian Theorem. So $\Lambda = \sup\limits_{x \in C} |\varphi(x)| < \infty$ and $e_n \underset{n}{\to} 0$ weakly in c_0 implies that $\varphi(e_n) \underset{n}{\to} \varphi(0)$, or equivalently, $y_n \underset{n}{\to} 0$. However, we preferred to present the previous proof because, although longer, it uses only elementary knowledge and it is, in our opinion, somewhat more instructive.

$\square$

4.2 $(\ell^1_{DIFF}, \|\cdot\|_{DIFF})$ AS AN AFFINE DUAL

Consider the set $\tilde{H} := \{x \in c_0 : -1 \leqslant x_1 + \ldots + x_n \leqslant 1, \forall n \in \mathbb{N}\} \subseteq (c_0, \|\cdot\|_\infty)$.
We already know that $\tilde{H}$ is a c.b.c, non-weakly compact subset of c_0.

Definition 4.2.1. Let $\ell^1_{diff} := \{y \in c_0 : \sum_{i=1}^{\infty} |y_i - y_{i+1}| < \infty\}$ and $\forall y \in \ell^1_{diff}$,

$$\|y\|_{diff} := \sum_{i=1}^{\infty} |y_i - y_{i+1}|$$

Claim 4.2.2. $(\ell^1_{diff}, \|\cdot\|_{diff})$ *is a Banach space.*

Proof. : First, we will prove that $\|\cdot\|_{diff}$ defined as above is a norm.

Suppose we have $\|y\|_{diff} = 0$, for some $y \in \ell^1_{diff}$. Then $y_1 = y_2 = y_3 = \ldots = y_n = \ldots$ and since y is also an element of c_0, i.e $y_n \underset{n}{\to} 0$, we obtain that $y = 0$.

It is easy to see that $\|\alpha y\|_{diff} = |\alpha \, \|y\|_{diff}$, $\forall \alpha \in \mathbb{R}$, $\forall y \in \ell^1_{diff}$. By applying the triangle inequality, we obtain that $\|x + y\|_{diff} \leqslant \|x\|_{diff} + \|y\|_{diff}$, $\forall x, y \in \ell^1_{diff}$.

Therefore, $\|\cdot\|_{diff}$ is a norm.

The following observation will be important:

$\forall n, p \in \mathbb{N}$, $\forall y \in \ell^1_{diff}$, $\|y\|_{diff} \geqslant \sum_{i=n}^{p} |y_i - y_{i+1}| \geqslant |y_n - y_{p+1}| \underset{p}{\to} |y_n|$.

Thus, $\|y\|_{diff} \geqslant |y_n|$, $\forall n \in \mathbb{N}$ and taking the supremum, we obtain $\|y\|_{diff} \geqslant \|y\|_\infty$, $\forall y \in \ell^1_{diff}$.

Now let $(y^m)_{m\in\mathbb{N}}$ be a Cauchy sequence in ℓ^1_{diff}; $\forall \epsilon > 0$, $\exists N_\epsilon$ such that $\forall m, n \geqslant N_\epsilon$, $\|y^m - y^n\|_{diff} \leqslant \epsilon$. From the above observation $\|y^m - y^n\|_\infty \leqslant \epsilon$, which implies that there exists $x \in c_0$ such that $\|y^m - x\|_\infty \underset{m}{\to} 0$. In particular, $|y^m_p - x_p| \underset{m}{\to} 0$, $\forall p \in \mathbb{N}$. Fix $p \in \mathbb{N}$, arbitrary. For all $m, n \geqslant N_\epsilon$, $\sum_{i=1}^{p} |y^m_i - y^m_{i+1} - y^n_i + y^n_{i+1}| \leqslant \epsilon$. Let $m \underset{m}{\to} \infty$. We obtain that $\sum_{i=1}^{p} |x_i - x_{i+1} - y^n_i + y^n_{i+1}| \leqslant \epsilon$, which implies $\sum_{i=1}^{\infty} |x_i - x_{i+1} - y^n_i + y^n_{i+1}| \leqslant \epsilon$. Since $\sum_{i=1}^{\infty} |y^n_i - y^n_{i+1}| < \infty$, by applying the triangle inequality, we obtain $\sum_{i=1}^{\infty} |x_i - x_{i+1}| < \infty$.

In conclusion, $x \in \ell^1_{diff}$ and $\|x - y^n\| \leq \varepsilon$, for all $n \geq N_\varepsilon$. $\square$

Definition 4.2.3. Let $\Gamma := \{\psi_y + r : y \in \ell^1_{diff}, r \in \mathbb{R}\}$, where

$$\psi_y(x) := \sum_{n=1}^{\infty} x_n y_n$$

$\forall x \in \tilde{H}$ and $\forall y \in \ell^1_{diff}$.

Theorem 4.2.4 ((LP))**.** $\tilde{H}^* = \Gamma$.

Proof. : We start with the following claim:

Claim 4.2.5. *:* $\Gamma \subseteq \tilde{H}^*$.

Step 1: First of all, $\forall r \in \mathbb{R}$, $\forall y \in \ell^1_{diff}$, $\psi_y + r$ is well-defined. Fix $z \in \tilde{H}$. Using an Abel-summation formula, if the limit below exists, then:

$$\begin{aligned}
\sum_{n=1}^{\infty} z_n y_n &= \lim_{N\to\infty} \big(z_1 y_1 + z_2 y_2 + \ldots + z_N y_N\big) \\
&= \lim_{N\to\infty} \Big[z_1\big(y_1 - y_2\big) + \big(z_1 + z_2\big)\big(y_2 - y_3\big) + \ldots + \big(z_1 + \ldots + z_{N-1}\big)\big(y_{N-1} - y_N\big) \\
&\qquad + \big(z_1 + \ldots + z_N\big) y_N\Big]
\end{aligned}$$

For $M < N$,

$$\begin{aligned}
\Big|\sum_{n=M+1}^{N} z_n y_n\Big| &= \Big|\sum_{n=1}^{N} z_n y_n - \sum_{n=1}^{M} z_n y_n\Big| \\
&= \Big|\sum_{n=1}^{N-1} \big(z_1 + \ldots + z_n\big)\big(y_n - y_{n+1}\big) + \big(z_1 + \ldots + z_N\big) y_N \\
&\qquad - \Big[\sum_{n=1}^{M-1} \big(z_1 + \ldots + z_n\big)\big(y_n - y_{n+1}\big) + \big(z_1 + \ldots + z_M\big) y_M\Big]\Big| \\
&= \Big|\sum_{n=M}^{N-1} \big(z_1 + \ldots + z_n\big)\big(y_n - y_{n+1}\big) + \big(z_1 + \ldots + z_N\big) y_N \\
&\qquad - \big(z_1 + \ldots + z_M\big) y_M\Big| \\
&\leqslant \sum_{n=M}^{N-1} \Big|y_n - y_{n+1}\Big| + \Big|y_N\Big| + \Big|y_M\Big|
\end{aligned}$$

the last inequality holds because $z \in \tilde{H}$ and the final quantity because $y \in c_0$ and it tends to 0 as $N > M \underset{M}{\longrightarrow} \infty$, because $\sum_{n=1}^{\infty} |y_n - y_{n+1}| < \infty$. Therefore, since the partial sums form a Cauchy sequence, $\lim_{N\to\infty} \big(z_1y_1 + z_2y_2 + \ldots + z_Ny_N\big)$ exists in $\mathbb{R}$.

Step 2: We will next prove that for every $y \in \ell^1_{diff}$, $\Big|\psi_y(z)\Big| = \Big|\sum_{n=1}^{\infty} z_ny_n\Big|$ is bounded from above, independent of $z \in \tilde{H}$.

Fix $N \in \mathbb{N}$, arbitrary. Then

$$\begin{aligned}
\Big|\sum_{n=1}^{N} z_ny_n\Big| &= \Big|\sum_{n=1}^{N-1} \big(z_1 + \ldots + z_n\big)\big(y_n - y_{n+1}\big) + \big(z_1 + \ldots + z_N\big)y_N\Big| \\
&\leqslant \sum_{n=1}^{N-1} |y_n - y_{n+1}| + |y_N| \underset{N}{\longrightarrow} \sum_{n=1}^{\infty} |y_n - y_{n+1}| = \|y\|_{diff}
\end{aligned}$$

Since N was arbitrary fixed, we get that $\Big|\psi_y(z)\Big| = \Big|\sum_{n=1}^{\infty} z_ny_n\Big| \leqslant \|y\|_{diff}$, $\forall z \in \tilde{H}$.

In conclusion, $\sup_{z\in\tilde{H}} \Big|\psi_y(z)\Big| < \infty$.

On the other hand, for a fixed $N \in \mathbb{N}$, arbitrary, we can choose $z \in \tilde{H}$ by inductively selecting z_i such that such that

$$z_1 + z_2 + \cdots + z_i = sgn(y_i - y_{i+1})$$

$\forall\ i = 1, \ldots N - 1$

Next, choose z_N such that $z_1 + \ldots + z_{N-1} + z_N = sgn(y_N)$. Further, let $z_n := 0$ for all $n \geq N + 1$. Next fix $\varepsilon > 0$, and choose $N_\varepsilon \in \mathbb{N}$ such that for all $N \geq N_\varepsilon$, $|y_N| \leq \varepsilon/2$ and $\sum_{n=1}^{N-1} |y_n - y_{n+1}| \geq \|y\|_{diff} - \varepsilon/2$. Fix $N \geq N_\varepsilon$. Then,

$$\begin{aligned}
\Big|\sum_{n=1}^{N} z_ny_n\Big| &= \Big|\sum_{n=1}^{N-1} \big(z_1 + \ldots + z_n\big)\big(y_n - y_{n+1}\big) + \big(z_1 + \ldots + z_N\big)y_N\Big| \\
&\geqslant \sum_{n=1}^{N-1} |y_n - y_{n+1}| - |y_N| \\
&\geqslant \|y\|_{diff} - \varepsilon
\end{aligned}$$

This means that $\left|\psi_y(z)\right| \geq \|y\|_{diff} - \varepsilon$, and so $\sup_{z\in\tilde{H}} \left|\psi_y(z)\right| \geq \|y\|_{diff}$.

Combining it with the previous result, we obtain that $\sup_{z\in\tilde{H}} \left|\psi_y(z)\right| = \|y\|_{diff}$.

Step 3: It is easy to check that $\psi_y + r$ is affine. Now we will prove that it is also norm-to-usual continuous. W.L.O.G. $y \neq 0$. Let $z \in \tilde{H}$ and $(w^{(j)})_{j\in\mathbb{N}}$ be a sequence in $\tilde{H}$ with $\left\|w^{(j)} - z\right\|_\infty \underset{j}{\to} 0$.

Fix $N \in \mathbb{N}$. Then

$$\begin{aligned}
\left|\psi_y(w^{(j)}) - \psi_y(z)\right| &= \left|\sum_{n=1}^{\infty} w_n^{(j)} y_n - \sum_{n=1}^{\infty} z_n y_n\right| \\
&= \left|\sum_{n=1}^{\infty} \left(w_1^{(j)} + \ldots + w_n^{(j)}\right)\left(y_n - y_{n+1}\right) - \sum_{n=1}^{\infty} \left(z_1 + \ldots + z_n\right)\left(y_n - y_{n+1}\right)\right| \\
&\leqslant \sum_{n=1}^{N} \left|\left(w_1^{(j)} + \ldots + w_n^{(j)}\right) - \left(z_1 + \ldots + z_n\right)\right|\left|y_n - y_{n+1}\right| \\
&\quad + \sum_{n=N+1}^{\infty} \left|\left(w_1^{(j)} + \ldots + w_n^{(j)}\right) - \left(z_1 + \ldots + z_n\right)\right|\left|y_n - y_{n+1}\right| \\
&\leqslant \left(\left\|w^{(j)} - z\right\|_\infty + 2\left\|w^{(j)} - z\right\|_\infty + \ldots + N\left\|w^{(j)} - z\right\|_\infty\right)\|y\|_{diff} \\
&\quad + \sum_{n=N+1}^{\infty} 2\left|y_n - y_{n+1}\right|
\end{aligned}$$

$$= \frac{N(N+1)}{2}\left\|w^{(j)} - z\right\|_\infty \|y\|_{diff} + \sum_{n=N+1}^{\infty} 2\left|y_n - y_{n+1}\right|$$

Fix $\epsilon > 0$. There exists N_ϵ such that $\forall N \geqslant N_\epsilon$, $\sum_{n=N_\epsilon+1}^{\infty} 2\left|y_n - y_{n+1}\right| < \frac{\epsilon}{2}$.

Also, there exists j_ϵ such that $\forall j \geqslant j_\epsilon$,

$$\left\|w^{(j)} - z\right\|_\infty < \frac{2\epsilon}{N_\epsilon(N_\epsilon + 1)\|y\|_{diff}}$$

Summarizing, $\forall j \geqslant j_\epsilon$, $\left|\psi_y(w^{(j)}) - \psi_y(z)\right| \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Thus $\psi_y + r$ is norm-to-usual continuous.

Claim 4.2.6. *:* $\tilde{H}^* \subseteq \Gamma$.

Fix $\varphi \in \tilde{H}$, $\varphi : \tilde{H} \to \mathbb{R}$, φ is affine and norm-to-usual continuous.

Since $0 \in \tilde{H}$, let $r := \varphi(0) \in \mathbb{R}$.

Each $e_k \in \tilde{H}$. Define $y_k := \varphi(e_k) - \varphi(0)$, $\forall k \in \mathbb{N}$. Obviously $y_k \in \mathbb{R}$.

To show: $\varphi(x) = \psi_y(x) + r$, $\forall x \in \tilde{H}$, where $y = (y_1, \ldots, y_n, \ldots)$. We will also prove that $y \in \ell^1_{diff}$.

Step 1: We will first prove that $D := \sup\limits_{k \in \mathbb{N}} |y_k| < \infty$.

Suppose not. Then $\sup\limits_{k \in \mathbb{N}} |y_k| = \infty$. Choose $\mu_1 \in \mathbb{N}$ such that $|y_{\mu_1}| \geq 1$ and then $\mu_2 \in \mathbb{N}$, $\mu_2 \geq \mu_1$ such that $|y_{\mu_2}| > 4 + |y_{\mu_1}|$.

Then

$$\left|\frac{y_{\mu_2} + y_{\mu_1}}{2}\right| \geq \frac{|y_{\mu_2}| - |y_{\mu_1}|}{2} \geq \frac{4 + |y_{\mu_1}| - |y_{\mu_1}|}{2} = 2$$

Choose $\mu_3 \in \mathbb{N}$, $\mu_3 > \mu_2$ such that

$|y_{\mu_3}| \geq 3 \cdot 2^2 + |y_{\mu_2}| + |y_{\mu_1}|$

Then

$$\left|\frac{y_{\mu_3} + y_{\mu_2} + y_{\mu_1}}{3}\right| \geqslant \frac{|y_{\mu_3}| - |y_{\mu_2}| - |y_{\mu_1}|}{3} \geqslant \frac{3 \cdot 2^2}{3} = 2^2$$

Continuing inductively, we build a subsequence $(y_{\mu_j})_{j \in \mathbb{N}}$, such that

$$\frac{\left|y_{\mu_1} + \ldots + y_{\mu_j}\right|}{j} \geqslant 2^{j-1} \ , \ \forall j \in \mathbb{N} \ (*)$$

.

On the other hand, notice that

$$\frac{y_{\mu_1} + \ldots + y_{\mu_j}}{j} = \frac{\varphi(e_{\mu_1}) + \ldots + \varphi(e_{\mu_j})}{j} - r = \varphi\left(\frac{e_{\mu_1} + \ldots + e_{\mu_j}}{j}\right) - r$$

$\forall j \in \mathbb{N}$.

Furthermore , $\left\| \frac{e_{\mu_1}+\ldots+e_{\mu_j}}{j} \right\|_\infty = \frac{1}{j} \underset{j}{\to} 0$;

$\varphi : \tilde{H} \to \mathbb{R}$ is norm-to-usual continuous, so

$$\varphi\left(\frac{e_{\mu_1} + \ldots + e_{\mu_j}}{j} \right) \underset{j}{\to} \varphi(0) = r$$

Thus,

$$\frac{y_{\mu_1} + \ldots + y_{\mu_j}}{j} \underset{j}{\to} 0$$

a contradiction with $(*)$.

Step 2: Next, we will prove that $y_k \underset{k}{\to} 0$.

Let $(y_{m_k})_{k\in\mathbb{N}}$ be any subsequence of $(y_n)_{n\in\mathbb{N}}$. By previous work, we know that $(y_{m_k})_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. So, by the Bolzano-Weierstrass Theorem, there exists a further subsequence $(y_{m_{k_j}})_{j\in\mathbb{N}}$ and $L \in \mathbb{R}$ such that

$$w_j := y_{m_{k_j}} \underset{j}{\to} L$$

Since $(w_j)_{j\in\mathbb{N}}$ is bounded, it follows that

$$\frac{w_1 + \ldots + w_j}{j} \underset{j}{\to} L$$

Notice also that

$$\frac{w_1 + \ldots + w_j}{j} = \frac{\varphi(e_{\mu_{k_1}}) + \ldots + \varphi(e_{\mu_{k_j}})}{j} - r = \varphi\left(\frac{e_{\mu_{k_1}} + \ldots + e_{\mu_{k_j}}}{j} \right) - r$$

and

$$\left\| \frac{e_{\mu_{k_1}} + \ldots + e_{\mu_{k_j}}}{j} \right\|_\infty = \frac{1}{j} \underset{j}{\to} 0$$

By definition, φ is norm-to-usual continuous on C. So,

$$\varphi\left(\frac{e_{\mu_{k_1}} + \ldots + e_{\mu_{k_j}}}{j} \right) \underset{j}{\to} \varphi(0) = r$$

This implies that

$$\frac{w_1 + \ldots + w_j}{j} \underset{j}{\to} 0$$

Hence, $L = 0$.

So every subsequence $(y_{m_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ has a further subsequence that converges to 0.

In conclusion, $y_n \underset{n}{\to} 0$.

Step 3: $\sum_{i=1}^{\infty} |y_i - y_{i+1}| < \infty$

$\forall k \in \mathbb{N}$, since φ is affine and $e_k, -e_k \in \tilde{H}$, we have:

$$\frac{1}{2}\varphi(e_k) + \frac{1}{2}\varphi(-e_k) = \varphi(0)$$

So

$$-\varphi(e_k) = \varphi(-e_k) - 2\varphi(0), \forall k \in \mathbb{N} \;\; (\star)$$

Furthermore,

$|y_1 - y_2| = |\varphi(e_1) - \varphi(e_2)|$ equals either

$$\varphi(e_1) - \varphi(e_2) = \varphi(e_1) + \varphi(-e_2) - 2\varphi(0)$$

or

$$\varphi(e_2) - \varphi(e_1) = \varphi(e_2) + \varphi(-e_1) - 2\varphi(0)$$

Similarly, $|y_2 - y_3| = |\varphi(e_2) - \varphi(e_3)|$ equals either

$$\varphi(e_2) - \varphi(e_3) = \varphi(e_2) + \varphi(-e_3) - 2\varphi(0)$$

or

$$\varphi(e_3) - \varphi(e_2) = \varphi(e_3) + \varphi(-e_2) - 2\varphi(0)$$

In general, $|y_n - y_{n+1}| = |\varphi(e_n) - \varphi(e_{n+1})|$ equals either

$$\varphi(e_n) - \varphi(e_{n+1}) = \varphi(e_n) + \varphi(-e_{n+1}) - 2\varphi(0)$$

or

$$\varphi(e_{n+1}) - \varphi(e_n) = \varphi(e_{n+1}) + \varphi(-e_n) - 2\varphi(0)$$

At each step, when computing $|y_n - y_{n+1}|$, e_n and e_{n+1} have different signs under φ.

Now let us compute $\sum_{i=1}^{n} |y_i - y_{i+1}|$, for some fixed $n \in \mathbb{N}$.

From above,

$$\sum_{i=1}^{n} |y_i - y_{i+1}| = \Big(\varphi(\pm e_1) + \varphi(\mp e_2)\Big) + \ldots + \Big(\varphi(\pm e_n) + \varphi(\mp e_{n+1})\Big) - 2n\varphi(0)$$

Dividing it by $2n$, we obtain:

$$\frac{\sum_{i=1}^{n} |y_i - y_{i+1}|}{2n} = \frac{\Big(\varphi(\pm e_1) + \varphi(\mp e_2)\Big) + \ldots + \Big(\varphi(\pm e_n) + \varphi(\mp e_{n+1})\Big)}{2n} - \varphi(0) \quad (1)$$

which further equals, since φ is affine,

$$\varphi\left(\frac{(\pm e_1 \mp e_2) + (\pm e_2 \mp e_3) + \ldots + (\pm e_n \mp e_{n+1})}{2n}\right) - \varphi(0)$$

Remark 4.2.7. At this point, it is important to notice that $(\pm e_1 \mp e_2) + (\pm e_2 \mp e_3) + \ldots + (\pm e_n \mp e_{n+1}) \in \tilde{H}$.

Indeed, the sum of each parenthesis is 0 and the sum of an odd number of elements is either 1 or -1.

Let $z := (\pm e_1 \mp e_2) + (\pm e_2 \mp e_3) + \ldots + (\pm e_n \mp e_{n+1})$.

Then

$$\varphi\left(\frac{z}{2n}\right) = \varphi\left(\frac{1}{2n}z + \frac{2n-1}{2n}0\right) = \frac{1}{2n}\varphi(z) + \frac{2n-1}{2n}\varphi(0) \quad (2)$$

.

From (1) and (2), we obtain :

$\sum_{i=1}^{n} |y_i - y_{i+1}| = 2n\left(\frac{1}{2n}\varphi(y) + \frac{2n-1}{2n}\varphi(0)\right) - 2n\varphi(0) = \varphi(y) - \varphi(0) < \infty$ - this last inequality holds because $\varphi \in \tilde{H}^*$ is bounded, i.e $\sup_{x\in\tilde{H}} |\varphi(x)| < \infty$.

Since n was arbitrary fixed $\Rightarrow \sum_{i=1}^{\infty} |y_i - y_{i+1}| \leqslant \sup_{x\in\tilde{H}} |\varphi(x)| - \varphi(0) \leq \|\varphi\|_{C^*} - \varphi(0)$.

It is time now for the last step of the proof, that is:

Step 4: $\varphi(x) = \psi_y(x) + r$, $\forall x \in \tilde{H}$

Fix $x \in \tilde{H}$. Since φ is continuous, $\varphi(x) = \lim_{n\to\infty} \varphi\left(\sum_{j=1}^{n} x_j e_j\right)$.

Therefore, it is sufficient to show that $\varphi\left(\sum_{j=1}^{n} x_j e_j\right) = \psi_y\left(\sum_{j=1}^{n} x_j e_j\right) + r$, for every $n \in \mathbb{N}$.

This is equivalent to proving the following:

$$\begin{aligned}
\varphi\left(\sum_{j=1}^{n} x_j e_j\right) &= x_1\big(\varphi(e_1) - \varphi(0)\big) + \ldots + x_n\big(\varphi(e_n) - \varphi(0)\big) + \varphi(0) \\
&= \big(x_1 + 1\big)\varphi(e_1) + \big(x_1 + x_2 + 1\big)\varphi(e_2) + \ldots + \big(x_1 + \ldots + x_n + 1\big)\varphi(e_n) \\
&\quad + \big(1 - x_1 - \ldots - x_n\big)\varphi(0) \\
&\quad - \varphi(e_1) - \big(x_1 + 1\big)\varphi(e_2) - \ldots - \big(x_1 + \ldots + x_{n-1} + 1\big)\varphi(e_n)
\end{aligned}$$

if and only if

$$\begin{aligned}
\sum_{i=1}^{n} \big(x_1 + \ldots + x_{i-1} + 1\big)\varphi(e_i) + 1\varphi\left(\sum_{j=1}^{n} x_j e_j\right) &= \big(x_1 + 1\big)\varphi(e_1) + \big(x_1 + x_2 + 1\big)\varphi(e_2) \\
&\quad + \ldots + \big(x_1 + \ldots + x_n + 1\big)\varphi(e_n) \\
&\quad + \big(1 - x_1 - \ldots - x_n\big)\varphi(0)
\end{aligned}$$

Notice that all the coefficients in front of φ are now positive and, moreover, the sum of the coefficients in the right hand side equals the sum of the coefficients in the left hand side.

Indeed,

$$\begin{aligned}
\sum_{i=1}^{n}\big(x_1+\ldots+x_{i-1}+1\big)+1 &= \big(x_1+1\big)+\big(x_1+x_2+1\big)+\ldots+\big(x_1+\ldots+x_{n-1}+1\big) \\
&\quad +2 \\
&= \big(x_1+1\big)+\big(x_1+x_2+1\big)+\ldots+\big(x_1+\ldots+x_{n-1}+1\big) \\
&\quad +\big(1-x_1-\ldots-x_n\big)+\big(1+x_1+\ldots+x_n\big) \\
&:= A
\end{aligned}$$

If we divide both sides by A and use the fact that φ is affine, we obtain that the equality is equivalent to

$$\varphi\left(\frac{\sum\limits_{i=1}^{n}(x_1+\ldots+x_i+1)e_i}{A}\right) = \varphi\left(\frac{\sum\limits_{i=1}^{n}(x_1+\ldots+x_i+1)e_i}{A}\right)$$

which is obviously true.

Thus $\varphi\left(\sum\limits_{i=1}^{n} x_i e_i\right) = \psi_y\left(\sum\limits_{i=1}^{n} x_i e_i\right) + r$, for all $n \in \mathbb{N}$.

By passing to the limit with n and taking into account that both φ and ψ_y are continuous, we obtain the following equality:

$$\varphi(x) = \psi_y(x) + r$$

for all $x \in \tilde{H}$ and the proof is now complete. □

Proposition 4.2.8. *$(\ell^1_{diff}, \|\cdot\|_{diff})$ and $(\ell_1, \|\cdot\|_1)$ are isometrically isomorphic.*

Proof. Consider the mapping $T : (\ell_1, \|\cdot\|_1) \to (\ell^1_{diff}, \|\cdot\|_{diff})$ given by

$$T(x_1, x_2, x_3, \dots) = \left(\sum_{i=1}^{\infty} x_i, \sum_{i=2}^{\infty} x_i, \sum_{i=3}^{\infty} x_i, \dots \right)$$

It is easy to check that $\|T(x)\|_{diff} = \|x\|_1$ and T is an onto isometry.

□

The next natural question would be: what is the dual of $(\ell^1_{diff}, \|\cdot\|_{diff})$?

Proposition 4.2.9. *The dual of* $(\ell^1_{diff}, \|\cdot\|_{diff})$ *is* $(\ell_{\infty,sum}, \|\cdot\|_{\infty,sum})$*, where*

$$\ell_{\infty,sum} = \{y \in \ell_\infty \quad and \quad \|y\|_{\infty,sum} = \sup_{n\in\mathbb{N}} |y_1 + y_2 + \dots + y_n| < \infty\}$$

Proof. First, for any $y \in \ell_{\infty,sum}$, define

$$f_y : (\ell^1_{diff}, \|\cdot\|_{diff}) \to \mathbb{R}$$

by

$$f_y(x) = (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \dots$$

It is easy to see that $\|f_y\| \le \sup_{n\in\mathbb{N}} |y_1 + y_2 + \dots + y_n|$.

Now choose $x = (sgn(y_1 + \dots + y_n), sgn(y_1 + \dots + y_n), \dots, sgn(y_1 + \dots + y_n), 0, 0, \dots)$, i.e. $x = sgn(y_1 + \dots + y_n)(e_1 + \dots + e_n)$. We have that $\|x\|_{diff} = 1$ and $f_y(x) = |y_1 + y_2 + \dots + y_n| \le \|f_y\|$, $\forall\, n \in \mathbb{N}$, which now implies that $\|f_y\| = \|y\|_{\infty,sum}$.

Next, consider $f \in (\ell^1_{diff}, \|\cdot\|_{diff})^*$. For all $n \in \mathbb{N}$ let $\tilde{e}_n = e_1 + \dots + e_n$.

$$f(x) = f\Big((x_1 - x_2)\tilde{e}_1 + (x_2 - x_3)\tilde{e}_2 + \dots\Big) = (x_1 - x_2)f(\tilde{e}_1) + (x_2 - x_3)f(\tilde{e}_2) + \dots$$

where

$$f(\tilde{e}_1) = f(e_1)$$

$$f(\tilde{e}_2) = f(e_1 + e_2) = f(e_1) + f(e_2)$$

$$\dots$$

$$f(\tilde{e}_n) = f(e_1 + e_2 + \ldots + e_n) = f(e_1) + f(e_2) + \ldots + f(e_n)$$

and so, if we define $y_n = f(e_n)$, we obtain that

$$f(x) = (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + \ldots + (x_n - x_{n+1})(y_1 + y_2 + \ldots + y_n) + \ldots$$

Fix $n \in \mathbb{N}$, arbitrary. Choosing $x = sgn(y_1 + \ldots + y_n)(1, 1, \ldots, 1, 0, 0, \ldots)$ with 1 on the first n positions, leads us to:

$$f(x) = |y_1 + \ldots + y_n| \leq \|f\|$$

Since n was arbitrary chosen, we get that $\sup\limits_{n\in\mathbb{N}} |y_1 + \ldots + y_n| \leq \|f\|$.

Therefore, $y \in \ell_{\infty,sum}$. $\square$

The spaces $(\ell^1_{diff}, \|\cdot\|_{diff})$ and $(\ell_1, \|\cdot\|_1)$ are isometrically isomorphic implies that their duals are also isometrically isomorphic, that is:

$T^* : (\ell_{\infty,sum}, \|\cdot\|_{\infty,sum}) \to (\ell_\infty, \|\cdot\|_\infty)$ is an onto isometry and the space $(\ell_{\infty,sum}, \|\cdot\|_{\infty,sum})$ is hyperconvex. In this space, the set $\tilde{C} = \{y \in \ell_\infty : -1 \leq y_1 + y_2 + \ldots + y_n \leq 1\}$ represents the unit ball and therefore, $\tilde{C}$ has the fixed point property for $\|\cdot\|_{\infty,sum}$-nonexpansive maps.

Another question would be what happens if we consider the following Banach space $c_{0,sum} = \{y \in c_0 : \sup\limits_{n\in\mathbb{N}} |y_1 + y_2 + \ldots + y_n| < \infty\}$ with the $\|\cdot\|_{\infty,sum}$ norm.

Is this space separable? Notice that our set $(\tilde{H}, \|\cdot\|_{\infty,sum})$ is the unit ball of this set. The answer is NO.

Indeed, the image of $c_{0,sum}$ through T^* is the space $Z = \{z \in \ell_\infty : (z_{n+1} - z_n) \to 0\}$.

To see this, one can define the mappings $T : c_{0,sum} \to Z$ by

$$T(y) = (y_1,\, y_1 + y_2,\, y_1 + y_2 + y_3, \ldots,\, y_1 + \ldots + y_n, \ldots)$$

and $S : Z \to c_{0,sum}$ by

$$S(z) = (z_1,\, z_2 - z_1,\, z_3 - z_2, \ldots,\, z_n - z_{n-1}, \ldots)$$

It is easy to check that T and S are mutually inverse linear mappings with the property that $\|T(y)\|_\infty = \|y\|_{\infty,sum}$, $\forall y \in c_{0,sum}$.

Now the space Z is not separable as it contains the following sequences:

$$z = \left(\pm\left(1, \frac{1}{2}\right), 0, \pm\left(\frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}\right), 0, \pm\left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}\right), 0, \dots \right)$$

One can see that by replacing the plus-minus symbol outside of each parenthesis by z_i where $(z_i)_{i\geq 1}$ is a sequence in ℓ_∞, we obtain an isometric copy of ℓ_∞. However, Z is not isometrically isomorphic to ℓ_∞ since the unit ball of $\ell_{\infty,sum}$ contains countably many extreme points, while the unit ball of ℓ_∞ has uncountably many extreme points. It turns out that Z is not only not isomorphic to ℓ_∞, but it is not isomorphic to any dual Banach space.

The following theorem is the main tool in the proof of the above statement:

Theorem 4.2.10. *Let X be a complemented Banach subspace of W and $W \approx Y^*$. Then $j(X)$ is complemented in X^{**}.*

Proof. By hypothesis, $\exists$ a continuous linear projection P with range X such that

$$P : (W, \|\cdot\|_W) \to (W, \|\cdot\|_W)$$

Also, $\exists$ a continuous linear map

$$\Gamma : (W, \|\cdot\|_W) \to (Y^*, \|\cdot\|_{Y^*})$$

that is one-to-one and onto; and therefore it has a continuous inverse

$$\Gamma^{-1} : (Y^*, \|\cdot\|_{Y^*}) \to (W, \|\cdot\|_W)$$

$\Gamma(X)$ is a complemented subspace of Y^*, complemented by $\Gamma P \Gamma^{-1} : Y^* \to Y^*$.

So, wlog, $(W, \|\cdot\|_W) = (Y^*, \|\cdot\|_{Y^*})$.

Consequently we have: $P : Y^* \to X \subseteq Y^*$, P is a continuous linear projection with $P(Y^*) = X$. Let $k : X \to Y^*$ be the inclusion mapping, i.e. $k(x) = x, \ \forall x \in X$.

Note that $k^* : Y^{**} \to X^*$ and $k^{**} : X^{**} \to Y^{***}$.

Let $m : Y \to Y^{**}$ be the natural inclusion map, i.e:

$$(my)\beta := \beta(y),\ \forall \beta \in Y^*,\ \forall y \in Y$$

Of course, $m^* : Y^{***} \to Y^*$. Let $j : X \to X^{**}$ be the natural inclusion map for X, i.e.:

$$(jx)\varphi := \varphi(x),\ \forall \varphi \in X^*,\ \forall x \in X$$

Consider

$$\eta := jPm^*k^{**} : X^{**} \to j(X) \subseteq X^{**}$$

It is easy to see that η is a continuous linear map and $\|\eta\|_{op} \leq \|P\|_{op}$. It remains to show that : $\eta(jx) = jx$, $\forall x \in X$. Fix an arbitrary $x \in X$. For all $\phi \in Y^{**}$:

$$\begin{aligned}
(k^{**}(jx))\phi &= (jx)(k^*\phi) \\
&= (k^*\phi)x \\
&= \phi(kx)
\end{aligned}$$

Thus, $\forall\, y \in Y$:

$$\begin{aligned}
[m^*(k^{**}(jx))]y &= [k^{**}(jx)](my) \\
&= (my)(kx) \\
&= (kx)y
\end{aligned}$$

In summary, $m^*k^{**}jx = kx = x$. Therefore, $Pm^*k^{**}jx = Px = x$ and so $\eta(jx) = jPm^*k^{**}(jx) = j(Pm^*k^{**}jx) = jx$. This is exactly what to wanted to show and the proof is now complete. □

Notice that if c_0 is isomorphic to a complemented subspace X of W and W is isomorphic to some Y^*, then $j(X)$ is a complemented subspace of X^{**}, which implies that $j(c_0)$ is a complemented subspace of c_0^{**}, so c_0 is a complemented subspace of ℓ_∞, which is a contradiction with Phillips' Theorem ([32], 7.5 on page 539).

Proposition 4.2.11. *Let* $X := \{p = (0, v_1, 0, v_2, 0, v_3, \ldots) : v = (v_1, v_2, v_3, \ldots) \in c_0\}$. $(X, \|\cdot\|_\infty)$ *is a Banach subspace of* $(Z, \|\cdot\|_\infty)$ *that is isometrically isomorphic to* $(c_0, \|\cdot\|_\infty)$.

Proof. Let $p = (0, v_1, 0, v_2, 0, v_3, \ldots)$, where $v = (v_1, v_2, v_3, \ldots) \in c_0$. Since

$$p_{2n} - p_{2n-1} = v_n - 0 \to 0$$

$$p_{2n+1} - p_{2n} = 0 - v_n \to 0$$

we have that $p_{n+1} - p_n \to 0$. Also, $p \in \ell_\infty$. Therefore, $X \subseteq Z$.

Clearly, $(X, \|\cdot\|_\infty) \cong (c_0, \|\cdot\|_\infty)$.

□

Proposition 4.2.12. $(X, \|\cdot\|_\infty)$ *is a complemented subspace of* $(Z, \|\cdot\|_\infty)$.

Proof. Define $U : Z \to X$ by

$$U(z) = (0,\, z_2 - z_1,\, 0,\, z_4 - z_3, \ldots, z_{2n} - z_{2n-1},\, 0, \ldots)$$

U is clearly linear and continuous from $(Z, \|\cdot\|_\infty)$ into $(X, \|\cdot\|_\infty)$.

Fix $z \in Z$. Let

$$\begin{aligned} p := Uz &= (p_1,\, p_2,\, p_3,\, p_4, \ldots) \\ &= (0,\, z_2 - z_1,\, 0,\, z_4 - z_3, \ldots, z_{2n} - z_{2n-1},\, 0, \ldots) \end{aligned}$$

Then,

$$\begin{aligned} U(p) &= (0,\, p_2 - p_1,\, 0,\, p_4 - p_3, \ldots, p_{2n} - p_{2n-1},\, 0, \ldots) \\ &= (0,\, z_2 - z_1 - 0,\, 0,\, z_4 - z_3 - 0,\, 0,\, \ldots,\, z_{2n} - z_{2n-1} - 0, \ldots) \\ &= p \end{aligned}$$

So, $U^2(z) = U(z)$, $\forall z \in Z$.

□

From Theorem 3.2.10, propositions 3.2.11 and 3.2.12, if Z were isomorphic to a dual space Y^*, then $(c_0, \|\cdot\|_\infty)$ would be a complemented subspace of $(\ell_\infty, \|\cdot\|_\infty)$; which is a contradiction.

We have just proven:

Theorem 4.2.13. $(Z, \|\cdot\|_\infty)$ *is not isomorphic to any dual Banach space* $(Y^*, \|\cdot\|_{Y^*})$.

In particular, $(Z, \|\cdot\|_\infty)$ is not isomorphic to $(\ell_\infty, \|\cdot\|_\infty)$.

4.3 $(L^1[0,1], \|\cdot\|_1)$ AS AN AFFINE DUAL

Now we will turn our attention to the space $(L^1[0,1], \|\cdot\|_1)$. Let Σ be the collection of Lebesgue-measurable subsets of $[0,1]$. Also, we will denote the Lebesgue measure on Σ by m.

We define the set:

$C := \{f \in L^1[0,1] : \mathbf{0} \leq f \leq \mathbf{1}\}.$

The following theorem holds:

Theorem 4.3.1. *(LP)* $C^* = \Gamma := \{\psi_u + r : u \in L^1[0,1], r \in \mathbb{R}\}$.

Here, $\forall u \in L^1[0,1], \psi_u(f) := \int_0^1 f u \, dm$, $\forall f \in C$.

Proof. • **Step 1**: $\Gamma \subseteq C^*$.

Fix $u \in L^1[0,1]$. ψ_u is clearly well-defined and affine. We want to prove that it is also norm-to-usual continuous on C. Fix $f \in C$ and $(f_j)_{j\in\Lambda}$ a net in C such that $\|f_j - f\|_1 \underset{j}{\to} 0$. Fix $\varepsilon > 0$. There exists $g \in L^\infty[0,1]$ such that $\|u - g\| < \frac{\varepsilon}{4}$. Fix $j \in \Lambda$.

$$\begin{aligned}
|\psi_u(f_j) - \psi_u(f)| &= \left| \int_0^1 f_j u \, dm - \int_0^1 f u \, dm \right| \\
&= \left| \int_0^1 (f_j - f) u \, dm \right| \\
&\leq \left| \int_0^1 (f_j - f) g \, dm \right| + \left| \int_0^1 (f_j - f)(u - g) \, dm \right| \\
&\leq \|f_j - f\|_1 \|g\|_\infty + \|f_j - f\|_\infty \|u - g\|_1 \\
&\leq \|f_j - f\|_1 \|g\|_\infty + 2\frac{\varepsilon}{4}
\end{aligned}$$

Since $\|f_j - f\|_1 \underset{j}{\to} 0$, there exists $j_0 \in \Lambda$ such that $\|f_j - f\|_1 \leq \frac{\varepsilon}{2(\|g\|_\infty + 1)}$, $\forall j \geq j_0$. Thus, $\forall j \geq j_0$,

$$|\psi_u(f_j) - \psi_u(f)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

.

Since C is a weakly compact, convex subset of $L^1[0,1]$ and ψ_u is norm-to-usual continuous on C, it follows from Lemma 3.0.3 that ψ_u is weak-to-usual continuous, and therefore norm bounded on C.

• **Step 2**: $C^* \subseteq \Gamma$

Fix $\sigma \in C^*$. So $\sigma : C \to \mathbb{R}$ is affine and weak-to-usual continuous on C. Without loss of generality, $\sigma(0) = 0$ (otherwise replace σ by $\sigma - \sigma(0)$).

Now $\forall E \in \Sigma$, $\chi_E \in C$. Define $\mu(E) := \sigma(\chi_E)$, $\forall E \in \Sigma$. Since σ is bounded, we obtain that $\sup_{E\in\Lambda} |\mu(E)| < \infty$.

Let $E, F \in \Lambda$ with $E \cap F = \emptyset$.

$$\mu(E \cup F) := \sigma(\chi_{E\cup F}) = \sigma(\chi_E + \chi_F)$$

Fix $f \in C$; $\frac{f}{2} \in C$.

To show: $\sigma(f) = 2\sigma(\frac{f}{2})$.

Indeed, $\mathbf{0} \in C$ and so

$$\sigma\left(\frac{f}{2}\right) = \sigma\left(\frac{f}{2} + \frac{\mathbf{0}}{2}\right) = \frac{1}{2}\sigma(f) + \frac{1}{2}\sigma(\mathbf{0}) = \frac{1}{2}\sigma(f) + 0$$

which now implies

$$\sigma(f) = 2\sigma\left(\frac{f}{2}\right)$$

Therefore,

$$\begin{aligned} \mu\big(E \cup F\big) : &= \sigma\left(\chi_{E\cup F}\right) = 2\sigma\left(\frac{1}{2}\chi_{E\cup F}\right) = 2\left(\frac{1}{2}\sigma(\chi_E) + \frac{1}{2}\sigma(\chi_F)\right) = \sigma\big(\chi_E\big) + \sigma\big(\chi_F\big) \\ &= \mu\big(E\big) + \mu\big(F\big) \end{aligned}$$

Inductively, for every finite pairwise disjoint sequence, one can show that

$$\mu\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} \mu\big(E_i\big)$$

Now let $(E_n)_{n\in\mathbb{N}}$ be a pairwise disjoint sequence in Σ. Denote by $E := \bigcup_{n\in\mathbb{N}} E_n$.

To show: $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$.

Fix $N \in \mathbb{N}$.

$$\mu(E) = \mu\left(\bigcup_{n=1}^{N} E_n \cup \bigcup_{n=N+1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{N} E_n\right) + \mu\left(\bigcup_{n=N+1}^{\infty} E_n\right) = \sum_{n=1}^{N} \mu(E_n) + \mu(Q_N)$$

where we denote by $Q_N := \bigcup_{n=N+1}^{\infty} E_n$. Of course, $\mu(Q_N) = \sigma(\chi_{Q_N})$.

Also note that $m(Q_N) \underset{N}{\rightarrow} 0$.

Fix $g \in L^\infty[0,1]$.

$$\left|\int_0^1 \chi_{Q_N} g \, dm\right| \le \|\chi_{Q_N}\|_1 \|g\|_\infty = m(Q_N) \|g\|_\infty \underset{N}{\rightarrow} 0$$

So, $\chi_{Q_N} \underset{N}{\rightarrow} 0$ weakly. Since σ is weak-to-usual continous on C we obtain that $\mu(Q_N) = \sigma(\chi_{Q_N}) \underset{N}{\rightarrow} \sigma(\mathbf{0}) = 0$.

In conclusion, $\mu(E) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$. This means that μ is a finite countably additive measure on Σ. Moreover, $E \in \Sigma$ and $m(E) = 0$ implies $\chi_E = \mathbf{0}$ m-a.e. and so $\mu(E) := \sigma(\chi_E) = \sigma(\mathbf{0}) = 0$. This means that μ is absolutely continuous with respect to the Lebesgue measure m on .

Now by the Radon-Nikodym Theorem, there exists $u \in L^1[0,1]$ such that $\mu(E) = \int_E u \, dm$, $\forall E \in \Sigma$. Therefore, $\sigma(\chi_E) = \int_E u \, dm$, $\forall E \in \Sigma$.

Next, let us consider $S := \{s = \sum_{j=1}^{n} \alpha_j \chi_{E_j} : (E_j)_{j=1}^n$ is a Σ measurable partition of [0,1] and each $\alpha_j \in [0,1]\}$. S is $\|\cdot\|_\infty$-dense, which implies it is $\|\cdot\|_1$-dense, and so $\sigma(L^1[0,1], L^\infty[0,1])$-dense in C. Fix $s = \sum_{j=1}^{n} \alpha_j \chi_{E_j} \in S$. Without loss of generality, $s \neq 0$ and so $\alpha := \sum_{j=1}^{n} \alpha_j > 0$.

Then $\sigma(s) = \alpha\sigma\left(\sum_{j=1}^{n} \frac{\alpha_j}{\alpha} \chi_{E_j}\right) = \alpha \sum_{j=1}^{n} \frac{\alpha_j}{\alpha} \sigma(\chi_{E_j}) = \sum_{j=1}^{n} \alpha_j \int_{E_j} u \, dm = \int_0^1 su \, dm = \psi_u(s)$.

Both σ and ψ_u are weak-to-usual continuous functions on C and S is weakly dense in C. Thus, $\sigma(f) = \psi_u(f), \forall f \in C$ and the proof is complete.

□

One may ask whether these results of affine duality hold in a more general case, that is: for every Banach space $(X, \|\cdot\|)$ does there exists a closed, bounded, convex set C such that its affine dual is X? The answer is YES and it is given by the following theorem:

Theorem 4.3.2. *(C.Lennard, R. Popescu [28]) Let X be a Banach space over* $\mathbf{R}$.

Then

(1) $\left(B_X,\, d_{\|\cdot\|}\right)^* = \{\varphi + r :\ \varphi \in X^*,\, r \in \mathbf{R}\}$

(2) $(B_{X^*},\, \sigma(X^*,\, X))^* = \{jx + r :\ x \in X,\, r \in \mathbf{R}\}$

Here $j : X \to X^{**}$ *is the natural inclusion map given by* $(jx)\varphi = \varphi(x)$, $\forall x \in X$, $\forall \varphi \in X^*$.

In part (1) of Theorem 3.3.2, for $C = B_X$, C^* is as defined earlier, i.e.:

$C^* := \{\varphi : C \to R$ such that φ is affine, bounded and norm-to-usual continuous on C$\}$.

In part (2), for $C = B_{X^*}$ and $\tau = \sigma(X^*, X)$, since (C, τ) is compact, C^* is naturally defined in this way:

$C^* := \{\varphi : C \to \mathbf{R}$ such that φ is affine and τ-to-usual continuous on C$\}$.

5.0 GOEBEL-KUCZUMOW SETS OF ℓ_1

5.1 INTRODUCTION

In 1979, K. Goebel and T. Kuczumow constructed a family of subsets C of ℓ_1 irregular with respect to the fixed point property. Then, they used these sets to give an example of a nested sequence of closed, bounded, convex subsets of ℓ_1 which alternatively satisfy or fail the fixed point property for nonexpansive mappings. This construction showed that the fixed point property for nonexpansive mappings is very unstable in ℓ_1.

Indeed, let us denote by $\Im$ the family of all sets

$$C = \{\sum_{j=1}^{\infty} \lambda_j(1+\alpha_j)e_j : \sum_{j=1}^{\infty} \lambda_j = 1, \lambda_j \geq 0, j = 1, 2, \ldots\}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a bounded sequence of nonnegative numbers.

The sets C are closed, bounded, convex, but not weak*-compact. We call such sets Goebel-Kuczumow sets. One can check that the weak*-closure of C is equal to

$$\overline{C}^{weak*} = \{\sum_{i=1}^{\infty} \mu_i(1+\alpha_i)e_i : \sum_{j=1}^{\infty} \mu_i \leq 1, \mu_i \geq 0, i = 1, 2, \ldots\}$$

Also, let

$$N_C = \{n \in \mathbb{N} : \alpha_n = \inf_{k\in\mathbb{N}} \alpha_k\}$$

In [19] the following result is proved:

Theorem 5.1.1. *A set $C \in \Im$ has the fixed point property for nonexpansive mappings if and only if N_C is nonempty and finite. In this case, for every nonexpansive mapping $f : C \to C$ there exists a nonexpansive retract from C onto $Fix(f)$.*

In 2004 ([22]) , W. Kaczor and S. Prus characterized exactly which ones of the above sets have the fixed point property for asymptotically nonexpansive mappings. Namely, part of the results they had obtained includes the following one:

Theorem 5.1.2. *Let $C \in \Im$ correspond to a bounded sequence of (α_n) of nonnegative numbers. The following conditions are equivalent:*

i) $\inf_{i \in \mathbb{N}} \alpha_i < \liminf_{n \to \infty} \alpha_n$

ii) C has the fixed point property for asymptotically nonexpansive mappings.

The main aim of this chapter is to show that certain sets among the Goebel-Kuczumow ones fail the fixed point property for uniformly lipschitzian mappings. Although the result for the sets we consider could also be inferred either from Goebel-Kuczumow theorem or the Kaczor-Prus theorem, we will offer below a new construction and, moreover, we will prove that if such a set C fails the fpp for uniformly Lipschitzian mappings, then the entire class of closed, bounded, convex subsets between H (which we will define below) and C, that is for all $H \subseteq G \subseteq C$, G fails the fpp for uniformly Lipschitzian mappings.

This first result indicated to us at the time that the only closed, bounded, convex subsets of ℓ_1 which have the fpp for uniformly Lipschitzian mappings are the compact ones. Indeed, as we will see in the next chapter, this turns out to be true, when we will prove the result in full generality and, in addition, we will show that compactness in ℓ_1 is equivalent not only with the fpp for uniformly Lipschitzian mappings, but also with the fpp for cascading nonexpansive mappings.

Finally, one should notice that this result does not contradict in any way the other two theorems with respect to the Goebel-Kuczumow sets mentioned above as we have showed in the preliminaries and overview section that the nonexpansive mappings, asymptotically nonexpansive, cascading nonexpansive and uniformly Lipschtizian mappings form different classes of mappings.

5.2 UNIFORMLY LIPSCHITZIAN FIXED POINT FREE

Let $C := \overline{\text{co}}\{(1+\varepsilon_n)e_n\}_{n\geq 1}$ with $\varepsilon_n \downarrow 0$ and each $\varepsilon_n < 1$. For all $n \in \mathbb{N}$ we define $x_n := (1+\varepsilon_n)e_n$, and $q = \sum_{j=1}^{\infty} \frac{1}{3^j} x_j$. Also, $\forall n \in \mathbb{N}$, let $y_n := q + u_n$, where $u_n = \left(\frac{1}{3^{2n}} + \frac{1}{2}\right) x_{2n-1} - \frac{1}{3^{2n}} x_{2n}$.

Define $H := \overline{\text{co}}\{y_n : n \in \mathbb{N}\} \subseteq C$. In [13], it is proved that H fails the fixed point property for nonexpansive mappings, when $\varepsilon_n \to 0$ fast enough (See [13], page 142). We define $l^{1,+}$ to be the set of all sequences in ℓ_1 with non-negative coordinates, and we define $c_0^{\downarrow}$ to be the set of all decreasing sequences in c_0 with non-negative coordinates.

Recent work of P. Dowling, C. Lennard and B. Turett ([15]) showed that in the case that C is the positive face S of the closed unit ball in ℓ_1, one can find a subset H' of it with the property that all closed, bounded, convex sets G with $H' \subseteq G \subseteq S$ fail the fixed point property for $\|\cdot\|_1$- nonexpansive maps. Using similar ideas to the ones in [15], we can prove the following theorem:

Theorem 5.2.1. *(1) There exists a $\|\cdot\|_1$-uniformly Lipschitzian mapping $R : C \to H$ such that*

$$R(u) = u, \quad \textit{for all} \quad u \in H$$

(2) For all closed, bounded, convex sets G with

$$H \subseteq G \subseteq C$$

there exists a $\|\cdot\|_1$-uniformly Lipschitzian map $U : G \to G$ that is fixed point free.

Proof. **(1)** Consider the mapping $\Lambda : C \to l^{1,+}$ defined by

$$\Lambda\left(\sum_{n=1}^{\infty} t_n x_n\right) = (v_n)_{n\geq 1}$$

where

$$
\begin{aligned}
v_{2n-1} &= t_{2n-1} \vee \left[\frac{1}{3^{2n-1}} + \frac{2}{3^{2n}}\left(t_{2n-1} + t_{2n} - \frac{4}{3^{2n}}\right)\right] - \left[\frac{1}{3^{2n-1}} + \frac{2}{3^{2n}}\left(t_{2n-1} + t_{2n} - \frac{4}{3^{2n}}\right)\right] \\
&\geq 0
\end{aligned}
$$

and

$$
v_{2n} = \left[t_{2n} + \frac{2}{3^{2n}}\left(t_{2n-1} + t_{2n} - \frac{4}{3^{2n}}\right)\right] \vee \frac{1}{3^{2n}} - \frac{1}{3^{2n}} \geq 0
$$

Then

$$
\begin{aligned}
\sum_{n=1}^{\infty} v_n &\geq \sum_{n=1}^{\infty} \left[t_{2n-1} - \frac{1}{3^{2n-1}} - \frac{2}{3^{2n}}\left(t_{2n-1} + t_{2n} - \frac{4}{3^{2n}}\right) + t_{2n} + \frac{2}{3^{2n}}\left(t_{2n-1} + t_{2n} - \frac{4}{3^{2n}}\right)\right. \\
&\qquad \left. - \frac{1}{3^{2n}}\right] \\
&= \sum_{n=1}^{\infty} \left[t_{2n-1} + t_{2n} - \frac{1}{3^{2n-1}} - \frac{1}{3^{2n}}\right] \\
&= \frac{1}{2}
\end{aligned}
$$

Therefore, $\text{Range}(\Lambda) \subseteq E$, where $E := \left\{ v = (v_n)_{n\geq 1} \in l^{1,+} : \sum_{n=1}^{\infty} v_n \geq \frac{1}{2} \right\}$.

(2) Now we define $\psi : \ell^{1,+} \to c_0^{\downarrow}$ by:

$$
\psi : x \to \left(\sum_{k=n}^{\infty} x_k\right)_{n\in 2\mathbb{N}-1}
$$

For $e \in \ell_\infty$, let $e := (1, 1, \ldots, 1, \ldots)$ and, for all sequences $v \in E$, let $w = (w_k)_{k\in\mathbb{N}}$, where:

$$
w := M(v) := \psi(v) \wedge \left(\frac{1}{2}e\right) = \left(\left[\sum_{j=2k-1}^{\infty} v_j\right] \wedge \frac{1}{2}\right)_{k\in\mathbb{N}}
$$

.

Note that $w \in F$ where $F := \{w = (w_k)_{k\geq 1} \in c_0^{\downarrow} : w_1 = \frac{1}{2}\}$

(3) Next, let $\Gamma : c_0^{\downarrow} \to l^1$ be given by:

$$
\Gamma(w) := \Bigg(\ldots, \underbrace{(w_n - w_{n+1})(1 + \frac{2}{3^{2n}})(1 + \varepsilon_{2n-1})}_{2n-1 \text{ position}}, \underbrace{(w_n - w_{n+1})(-\frac{2}{3^{2n}})(1 + \varepsilon_{2n})}_{2n \text{ position}}, \ldots\Bigg)
$$

Notice that

$$\begin{aligned} z : &= q + \Gamma(w) \\ &= q + \sum_{n=1}^{\infty} 2(w_n - w_{n+1}) \left[(\frac{1}{2} + \frac{1}{3^{2n}})(1 + \varepsilon_{2n-1})e_{2n-1} - \frac{1}{3^{2n}}(1 + \varepsilon_{2n})e_{2n} \right] \\ &= q + \sum_{n=1}^{\infty} 2(w_n - w_{n+1})u_n \in H \end{aligned}$$

since $\sum_{n=1}^{\infty} 2(w_n - w_{n+1}) = 2 \cdot \frac{1}{2} = 1$ and $w_n - w_{n+1} \geq 0$, $\forall n \in \mathbb{N}$

(4) Finally, we define $R : C \to H$ by:

$$R(u) := q + \Gamma \circ M \circ \Lambda(u)$$

$\forall u \in C$.

We claim that R is a uniformly Lipschitzian retraction from C onto H.

First, we show that for all $u \in H, R(u) = u$. Fix $u \in H$.

This implies that $u = q + \sum_{n=1}^{\infty} t_n u_n$, where $t_n \geq 0, \forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} t_n = 1$.

Therefore,

$$\begin{aligned} u &= \frac{1}{3}x_1 + \frac{1}{3^2}x_2 + \frac{1}{3^3}x_3 + \dots \\ &\quad + t_1 \left(\frac{1}{2} + \frac{1}{3^2} \right) x_1 - t_1 \frac{1}{3^2} x_2 + t_2 \left(\frac{1}{2} + \frac{1}{3^4} \right) x_3 - t_2 \frac{1}{3^4} x_4 + \dots \\ &= \sum_{n=1}^{\infty} t'_n x_n \end{aligned}$$

where

$$t'_{2n-1} = \frac{1}{3^{2n-1}} + t_n \left(\frac{1}{2} + \frac{1}{3^{2n}} \right) \geq 0$$

and

$$t'_{2n} = \frac{1}{3^{2n}} - t_n \frac{1}{3^{2n}} \geq 0$$

By applying Λ to u and noticing that $\frac{2}{3^{2n}}\left(t'_{2n-1}+t'_{2n}-\frac{4}{3^{2n}}\right)=\frac{t_n}{3^{2n}}$ and $t'_{2n}+\frac{t_n}{3^{2n}}=\frac{1}{3^{2n}}$, the following holds:

$$v_{2n-1}=t'_{2n-1}-\frac{1}{3^{2n-1}}-\frac{t_n}{3^{2n}}=\frac{t_n}{2}$$

and

$$v_{2n}=\frac{1}{3^{2n}}-\frac{1}{3^{2n}}=0$$

which further leads us to

$$\begin{aligned} M(\Lambda(v)) &= \left(\sum_{j=1}^{\infty} v_j \wedge \frac{1}{2}, \sum_{j=3}^{\infty} v_j \wedge \frac{1}{2}, \sum_{j=5}^{\infty} v_j \wedge \frac{1}{2}, \ldots\right) \\ &= \left(\frac{1}{2}, \frac{1}{2}\sum_{k=2}^{\infty} t_k, \frac{1}{2}\sum_{k=3}^{\infty} t_k, \ldots\right) \end{aligned}$$

So $w_n - w_{n+1} = \frac{t_n}{2}$. Therefore:

$$\begin{aligned} \Gamma(w) &= \Bigg(\ldots, \underbrace{\left(\frac{t_n}{2}\right)\left(1+\frac{2}{3^{2n}}\right)(1+\varepsilon_{2n-1})}_{2n-1 \text{ position}}, \underbrace{\left(\frac{t_n}{2}\right)\left(-\frac{2}{3^{2n}}\right)(1+\varepsilon_{2n})}_{2n \text{ position}}, \ldots\Bigg) \\ &= \Bigg(\ldots, \underbrace{2\left(\frac{t_n}{2}\right)\left(\frac{1}{2}+\frac{1}{3^{2n}}\right)(1+\varepsilon_{2n-1})}_{2n-1 \text{ position}}, \underbrace{2\left(\frac{t_n}{2}\right)\left(-\frac{1}{3^{2n}}\right)(1+\varepsilon_{2n})}_{2n \text{ position}}, \ldots\Bigg) \\ &= 2\frac{1}{2}t_1u_1 + 2\frac{1}{2}t_2u_2 + \ldots \end{aligned}$$

In conclusion, $R(u) = q + 2\frac{1}{2}t_1u_1 + 2\frac{1}{2}t_2u_2 + \ldots = q + \sum_{n=1}^{\infty} t_nu_n = u$.

Next, we will prove that R is Lipschitz, which actually will imply that R is uniformly Lipschitzian since R is a retraction.

Fix $u = \sum_{n=1}^{\infty} t_nx_n \in C$ and $v = \sum_{n=1}^{\infty} s_nx_n \in C$.

Then,

$$M(\Lambda(u)) = \left(\sum_{j=\alpha}^{\infty} p_j \wedge \frac{1}{2}\right)_{\alpha\in\mathbb{N}}$$

where

$$\begin{aligned} p_j \quad &= \quad t_{2j-1} \vee \left[\frac{1}{3^{2j-1}} + \frac{2}{3^{2j}}\left(t_{2j-1} + t_{2j} - \frac{4}{3^{2j}}\right)\right] - \left[\frac{1}{3^{2j-1}} + \frac{2}{3^{2j}}\left(t_{2j-1} + t_{2j} - \frac{4}{3^{2j}}\right)\right] \\ &+ \left[t_{2j} + \frac{2}{3^{2j}}\left(t_{2j-1} + t_{2j} - \frac{4}{3^{2j}}\right)\right] \vee \frac{1}{3^{2j}} - \frac{1}{3^{2j}} \end{aligned}$$

(basically $p_j = (\Lambda u)_{2j-1} + (\Lambda u)_{2j}$)

It follows that

$$\Gamma \circ M \circ \Lambda(u) = 2\sum_{\alpha=1}^{\infty}\left(\left[\sum_{j=\alpha}^{\infty} p_j \wedge \frac{1}{2}\right] - \left[\sum_{j=\alpha+1}^{\infty} p_j \wedge \frac{1}{2}\right]\right) u_\alpha$$

Similarly,

$$\Gamma \circ M \circ \Lambda(v) = 2\sum_{\alpha=1}^{\infty}\left(\left[\sum_{j=\alpha}^{\infty} q_j \wedge \frac{1}{2}\right] - \left[\sum_{j=\alpha+1}^{\infty} q_j \wedge \frac{1}{2}\right]\right) u_\alpha$$

where

$$\begin{aligned} q_j \quad &= \quad s_{2j-1} \vee \left[\frac{1}{3^{2j-1}} + \frac{2}{3^{2j}}\left(s_{2j-1} + s_{2j} - \frac{4}{3^{2j}}\right)\right] - \left[\frac{1}{3^{2j-1}} + \frac{2}{3^{2j}}\left(s_{2j-1} + s_{2j} - \frac{4}{3^{2j}}\right)\right] \\ &+ \left[s_{2j} + \frac{2}{3^{2j}}\left(s_{2j-1} + s_{2j} - \frac{4}{3^{2j}}\right)\right] \vee \frac{1}{3^{2j}} - \frac{1}{3^{2j}} \end{aligned}$$

We have the following facts:

(1) $\exists$ a unique $k = k_u \in \mathbb{N}$ such that:

$$\sum_{j=k}^{\infty} p_j \geq \frac{1}{2} \quad \text{and} \quad \sum_{j=k+1}^{\infty} p_j < \frac{1}{2}$$

(2) $\exists$ a unique $l = l_u \in \mathbb{N}$ such that:

$$\sum_{j=l}^{\infty} q_j \geq \frac{1}{2} \quad \text{and} \quad \sum_{j=l+1}^{\infty} q_j < \frac{1}{2}$$

We may assume that $k \leq l$.

Case 1: $[k = l]$ Since $(u_n)_{n\geq 1}$ are disjointly supported, we obtain:

$$\|R(u) - R(v)\|_1 = 2\left[\left|\sum_{j=k+1}^{\infty} q_j - \sum_{j=k+1}^{\infty} p_j\right| \|u_k\|_1 + |p_{k+1} - q_{k+1}| \, \|u_{k+1}\|_1 + \ldots\right]$$

In order to estimate $p_j - q_j$, we need the following lemma:

Lemma 5.2.2. *For any real numbers a,b,c, d we have that*

$$|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$$

Using this inequality, one can show that

$$\begin{aligned}
|p_j - q_j| \quad &\leq \quad |t_{2j-1} - s_{2j-1}| \vee \frac{2}{3^{2j}}\left|(t_{2j-1} - s_{2j-1}) + (t_{2j} - s_{2j})\right| + \\
&\quad + \left|t_{2j} - s_{2j}\right| + \frac{2}{3^{2j}}\left|(t_{2j-1} - s_{2j-1}) + (t_{2j} - s_{2j})\right| + \\
&\quad + \frac{2}{3^{2j}}\left|(t_{2j-1} - s_{2j-1}) + (t_{2j} - s_{2j})\right| \\
&\leq \quad \left(1 + \frac{6}{3^{2j}}\right)\left|(t_{2j-1} - s_{2j-1}) + (t_{2j} - s_{2j})\right| \\
&\leq \quad \left(1 + \frac{2}{3}\right)[|(t_{2j-1} - s_{2j-1})| + |(t_{2j} - s_{2j})|]
\end{aligned} \tag{5.2}$$

for all $j \geq 1$.

Also, it is easy to show that

$$\begin{aligned}
\|u_j\|_1 \quad &\leq \quad \left(\frac{1}{2} + \frac{1}{3^{2j}}\right)\|x_{2j-1}\|_1 + \frac{1}{3^{2j}}\|x_{2j}\|_1 \\
&\leq \quad \left(\frac{1}{2} + \frac{1}{3^{2j}}\right)2 + \frac{1}{3^{2j}}2 \\
&\leq \quad 1 + \frac{4}{9}
\end{aligned}$$

for all $j \geq 1$.

We obtain that

$$\begin{aligned}
\|R(u)-R(v)\|_1 &\leq 2\left(1+\frac{4}{9}\right)\left[\left|\sum_{j=k+1}^{\infty} q_j - \sum_{j=k+1}^{\infty} p_j\right| + \left|p_{k+1}-q_{k+1}\right| + \ldots\right] \\
&\leq 2\times 2\left(1+\frac{4}{9}\right)\sum_{j=k+1}^{\infty}\left|p_j - q_j\right| \\
&\leq 2\times 2\left(1+\frac{4}{9}\right)\left(1+\frac{2}{3}\right)\sum_{j=k+1}^{\infty}\left|(t_{2j-1}-s_{2j-1})\right| + \left|(t_{2j}-s_{2j})\right| \\
&\leq 2\times 2\left(1+\frac{4}{9}\right)\left(1+\frac{2}{3}\right)\sum_{j=1}^{\infty}\left|t_j - s_j\right| \\
&\leq 10\sum_{j=1}^{\infty}|t_j-s_j|\,\|x_j\|_1 \\
&= 10\,\|u-v\|_1
\end{aligned}$$

Case 2: $[k<l]$ Since $(u_n)_{n\geq 1}$ are disjointly supported, we obtain:

$$\begin{aligned}
\|R(u)-R(v)\|_1 &= 2\left[\left|\frac{1}{2}-\sum_{j=k+1}^{\infty} p_j\right| \|u_k\|_1 + p_{k+1}\ \|u_{k+1}\|_1 + \ldots + p_{l-1}\ \|u_{l-1}\|_1 + \right. \\
&\quad \left. + \left|p_l - \left(\frac{1}{2}-\sum_{j=l+1}^{\infty} q_j\right)\right| \|u_l\|_1 + |p_{l+1}-q_{l+1}|\ \|u_{l+1}\|_1 + \ldots\right] \\
&\leq 2\left(1+\frac{4}{9}\right)\left[\left(\frac{1}{2}-\sum_{j=k+1}^{\infty} p_j\right) + p_{k+1} + \ldots + p_{l-1}\right. \\
&\quad \left. + \left|p_l - \left(\frac{1}{2}-\sum_{j=l+1}^{\infty} q_j\right)\right| + |p_{l+1}-q_{l+1}| + |p_{l+2}-q_{l+2}| + \ldots\right]
\end{aligned}$$

Sub-Case 2.1

$$\frac{1}{2}-\sum_{j=l+1}^{\infty} q_j \leq p_l$$

Then

$$\begin{aligned}
\|R(u)-R(v)\|_1 &\leq 2\left(1+\frac{4}{9}\right)\Bigg[\left(\frac{1}{2}-\sum_{j=k+1}^{\infty} p_j\right)+p_{k+1}+\ldots+p_{l-1} \\
&\quad +\left(p_l-\left(\frac{1}{2}-\sum_{j=l+1}^{\infty} q_j\right)\right)+|p_{l+1}-q_{l+1}\Bigg|+|p_{l+2}-q_{l+2}|+\ldots\Bigg] \\
&= 2\left(1+\frac{4}{9}\right)\left[\left(\sum_{j=l+1}^{\infty} q_j-\sum_{j=l+1}^{\infty} p_j\right)+|p_{l+1}-q_{l+1}|+|p_{l+2}-q_{l+2}|+\ldots\right]
\end{aligned}$$

This is almost exactly the inequality from **Case 1** and so, we know that

$$\begin{aligned}
\|R(u)-R(v)\|_1 &\leq 2\left(1+\frac{4}{9}\right)\Bigg[\left(\sum_{j=l+1}^{\infty} q_j-\sum_{j=l+1}^{\infty} p_j\right)+|p_{l+1}-q_{l+1}| \\
&\quad +|p_{l+2}-q_{l+2}|+\ldots\Bigg] \\
&\leq 10\sum_{j=1}^{\infty}|t_j-s_j|\,\|x_j\|_1 \\
&= 10\,\|u-v\|_1
\end{aligned}$$

Sub-Case 2.2

$$\frac{1}{2}-\sum_{j=l+1}^{\infty} q_j > p_l$$

In this case,

$$\begin{aligned}
\|R(u)-R(v)\|_1 &\leq 2\left(1+\frac{4}{9}\right)\Bigg[\left(\frac{1}{2}-\sum_{j=k+1}^{\infty} p_j\right)+p_{k+1}+\ldots+p_{l-1} \\
&\quad +\left(\left(\frac{1}{2}-\sum_{j=l+1}^{\infty} q_j\right)-p_l\right)+|p_{l+1}-q_{l+1}|+|p_{l+2}-q_{l+2}|+\ldots\Bigg]
\end{aligned}$$

By the definition of l

$$\frac{1}{2}\leq\sum_{j=l}^{\infty} q_j$$

Therefore,

$$\begin{aligned}
\|R(u)-R(v)\|_1 &\leq 2\left(1+\frac{4}{9}\right)\left[\left(\frac{1}{2}-\sum_{j=k+1}^{\infty}p_j\right)+p_{k+1}+\ldots+p_{l-1}\right.\\
&\quad \left.+\left(\left(\frac{1}{2}-\sum_{j=l}^{\infty}q_j\right)+q_l-p_l\right)+|p_{l+1}-q_{l+1}|+|p_{l+2}-q_{l+2}|+\ldots\right]\\
&\leq 2\left(1+\frac{4}{9}\right)\left[\left(\sum_{j=l}^{\infty}q_j-\sum_{j=l}^{\infty}p_j\right)+\Big(0+q_l-p_l\Big)+\right.\\
&\quad \left.+|p_{l+1}-q_{l+1}|+|p_{l+2}-q_{l+2}|+\ldots\right]
\end{aligned}$$

Again, similar to **Case 1** this leads us to

$$\|R(u)-R(v)\|_1 \leq 10\,\|u-v\|_1$$

In conclusion, we have shown that $\forall\ u, v \in C$

$$\|R(u)-R(v)\|_1 \leq 10\,\|u-v\|_1$$

(2) Fix a closed, bounded, convex set G with $H \subseteq G \subseteq C$. Let $T : H \to H$ be a fixed point free $\|\cdot\|_1$-nonexpansive map, which exists when $\varepsilon_n \to 0$ fast enough. (See [13], page 142). We define $U : G \to G$ by

$$U(x) := T(R(x)), \text{ for all } x \in G$$

Clearly, U is a uniformly Lipschitzian mapping and fixed point free with Lipschitz constant 10.

□

6.0 A CHARACTERIZATION OF COMPACTNESS IN ℓ_1 AND $L^1[0,1]$

6.1 INTRODUCTION

In the field of metric fixed point theory we often try to characterize geometrical properties of a set C in a Banach space X in terms of fixed point results for different classes of mappings. As we will see below, there are several cases in which such characterizations represent necessary and sufficient conditions.

It is known for example that a convex closed subset C of a linear normed space is compact if and only if every continuous mapping defined from C into C has a fixed point. This result is due to Schauder, who proved that compactness is a sufficient condition, and to V. Klee ([24]), who showed this is also a necessary condition.

Later on, in 1985, P.K.Lin and Y. Sternfeld ([30]) improved this result when they showed that a convex closed subset C of a linear normed space is compact if and only if every Lipschitzian mapping T defined from C into C has a fixed point. Moreover, since the mapping $\lambda T + (1-\lambda)I$ has the same fixed points as T, by taking $\lambda \to 0$ one can show that the lipschitz constant can be made arbitrarily close to 1.

Nevertheless, for the case of non-expansive mappings (i.e. lipschitz constant $L = 1$), the situation is quite different. In Hilbert spaces (Browder (1965) [3] and W. Ray (1985) [33]) showed that a closed convex set satisfies the fpp for nonexpansive maps iff it is weakly compact.

Furthermore, if we look at the classical nonreflexive Banach spaces, it is known that in ℓ_1 the above theorem doesn't hold as there are convex non norm compact sets (for instance, all the weak*-compact sets) which have the fpp for nonexpansive maps. Since weak compactness is equivalent to norm compactness in ℓ_1 by Schur property, this implies that there are convex non weakly compact sets which have the fpp for nonexpansive maps.

On the other hand, in 2004, P. Dowling, C. Lennard, B. Turett ([14]) proved that in c_0 weak compactness is equivalent to the fpp for nonexpansive maps. The result does not hold for c, as we were able to show in [17] through an appropriate counterexample and using the notion of hyperconvexity.

The main aim of this chapter is to give a characterization of compactness in ℓ_1 for closed, bounded, convex sets in terms of the class of cascading nonexpansive mappings and uniformly lipschitzian mappings.

The notion of cascading nonexpansive mappings was introduced by Christopher Lennard and Veysel Nezir ([27]) where they used this notion to characterize reflexivity for Banach lattices or Banach spaces with unconditional basis. In particular, they have proved that any Banach space containing isomorphically ℓ_1, also contains a set K which fails the FPP for cascading non-expansive mappings.

Let $T : C \to C$ be a mapping. Next, let $C_0 := C$ and for all $n \in \mathbb{N}$, define $C_n := \overline{\text{co}}(T(C_{n-1}))$. One can check that T maps every C_n into C_n.

Definition 6.1.1. Let $(X, \|\cdot\|)$ be a Banach space, C be a closed bounded convex subset of X, $T : C \to C$ a mapping and $(C_n)_{n\in\mathbb{N}_0}$ be defined as above. We say that T is cascading nonexpansive if there exists a sequence $(\Lambda_n)_{n\in\mathbb{N}_0}$ in $[1, \infty)$ such that $\Lambda_n \to 1$, and for all $n \in \mathbb{N}_0$, for all $x, y \in C_n$, $\|T(x) - T(y)\| \leq \Lambda_n \|x - y\|$.

6.2 MAIN RESULT

In 2016, in the Journal of Mathematical Analysis and Applications, T. Dominguez-Benavides and Maria Japón ([5]) proved that compactness of a closed convex set in ℓ_1 is equivalent to the fixed point property for cascading nonexpansive mappings. In addition, the map constructed turned out to be also uniformly Lipschitz, where the Lipschitz constant L can be chosen arbitrarily close to 2 from above.

The main aim of this section is to provide a new proof of the above result for the case of closed, bounded, convex subsets of ℓ_1.

Theorem 6.2.1. *(M. Japón, C. Lennard, R. Popescu (JLP)[21]) Let K be a closed bounded convex subset of $(\ell_1, \|\cdot\|_1)$. Then K is norm compact if and only if K has the fixed point property for cascading nonexpansive maps if and only if K has the fixed point property for uniformly Lipschitzian maps.*

Proof. In this proof, for simplicity, we will often denote the norm $\|\cdot\|_1$ by $\|\cdot\|$. Let K be a non-weakly compact, closed, bounded, convex subset of $(\ell_1, \|\cdot\|_1)$. Then, by translating and rescaling K, we may assume W.L.O.G. that K contains a weak* null sequence $(x_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sum_{n=1}^{\infty}(1+\varepsilon_n)|t_n|$$

for all $(t_n)_{n\geq 1} \in \ell_1$, and for some decreasing null sequence $(\varepsilon_n)_{n\geq 1}$ in $(0,1)$. We call the closed linear span of such a sequence $(x_n)_{n\geq 1}$ an asymptotically isometric copy of ℓ_1 (see [13]).

Let $L > 2$ be arbitrary fixed and choose $\varepsilon \in (0,1)$ such that $2(\frac{1+\varepsilon}{1-\varepsilon})^2 < L$. Without loss of generality, we may assume that each $\varepsilon_n \leq \varepsilon$. Hence, from the above inequalities, $1-\varepsilon \leq 1-\varepsilon_n \leq \|x_n\| \leq 1+\varepsilon_n \leq 1+\varepsilon$, for all $n \in \mathbb{N}$.

First we solve the problem for the case when the x_i's are disjointly supported.

Let $C = \{\sum_{i=1}^{\infty} t_i x_i | t_i \geq 0, \sum_{i=1}^{\infty} t_i = 1\} \subseteq K$ and N_i=supp$\{x_i\}$, $\forall i \in \mathbb{N}$. For every $\Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_n, \dots) \in \ell_1$, consider the following non-negative convergent series :

$$0 \leq \frac{\sum_{i \in N_1} |\Lambda_i|}{\|x_1\|} + \frac{\sum_{i \in N_2} |\Lambda_i|}{\|x_2\|} + \cdots + \leq \frac{\|\Lambda\|_{\ell_1}}{1-\varepsilon}$$

There exists l-minimum such that

$$\sum_{j=l}^{\infty} \frac{\sum_{i \in N_j} |\Lambda_i|}{\|x_j\|} \leq 1$$

For all $j \in \mathbb{N}$, denote by

$$\alpha_j = \frac{\sum_{i \in N_j} |\Lambda_i|}{\|x_j\|}$$

Notice that if $\beta_j = \frac{\sum_{i \in N_j} |\mu_i|}{\|x_j\|}$ for some $\mu = (\mu_1, \mu_2, \cdots, \mu_n, \dots) \in \ell_1$, then

$$|\alpha_j - \beta_j| \leq \frac{\sum_{i \in N_j} |\Lambda_i - \mu_i|}{\|x_j\|} \leq \frac{1}{1-\varepsilon} \sum_{i \in N_j} |\Lambda_i - \mu_i|$$

We define $R : \ell_1 \to C$ by:

$R\Big(\Lambda_1, \Lambda_2, \cdots\Big) = \left(1 - \sum_{j=l}^{\infty} \alpha_j\right) x_l + \alpha_l x_{l+1} + \alpha_{l+1} x_{l+2} + \dots$

R is fixed point free. Indeed, suppose there exists $x \in \ell_1$ such that $R(x) = x$. Then $x \in C$, so x is of the form $x = \sum_{i=1}^{\infty} t_i x_i$, for some $t_i \geq 0$ and $\sum_{i=1}^{\infty} t_i = 1$. This further implies that $\alpha_i = t_i$ for all $i \in \mathbb{N}$. Therefore $l = 1$ and $R\left(\sum_{i=1}^{\infty} t_i x_i\right) = \left(1 - \sum_{i=1}^{\infty} t_i\right) x_1 + t_1 x_2 + t_2 x_3 + \dots,$ which leads us to $t_i = 0$, for all i, but this is a contradiction with $\sum_{i=1}^{\infty} t_i = 1$.

Next, we want to show that R is cascading nonexpansive.

Let $D_1 = \overline{\text{co}}\{x_n\}_{n \geq 2} = \{\sum_{i=1}^{\infty} t_i x_{i+1} | t_i \geq 0, \sum_{i=1}^{\infty} t_i = 1\}$. Then

$$R(C) = D_1 \subseteq C_1 := \overline{\text{co}}(R(K)) \subseteq D_0 := C$$

and, in general,

$$D_n \subseteq C_n \subseteq D_{n-1}$$

for all $n \geq 1$, where $D_n = \overline{\text{co}}\{x_j\}_{j \geq n+1}$ and $C_n = \overline{\text{co}}(R(C_{n-1}))$.

Consider $x, y \in C_n$, $n \geq 1$. Then $x, y \in D_{n-1}$ and so

$$x = t_1 x_n + t_2 x_{n+1} + \dots \tag{6.1}$$

and

$$y = s_1 x_n + s_2 x_{n+1} + \dots \tag{6.2}$$

From the way R is defined, we obtain that

$$R(x) = t_1 x_{n+1} + t_2 x_{n+2} + \dots \tag{6.3}$$

and

$$R(y) = s_1 x_{n+1} + s_2 x_{n+2} + \dots \tag{6.4}$$

Thus, $\|R(x) - R(y)\|_1 \leq \frac{1+\varepsilon_n}{1-\varepsilon_n} \|x - y\|_1$, for all $x, y \in C_n$, for every $n \geq 1$.

We next check that R is Lipschitz at the first level. Let $x, y \in \ell_1$, $x = (\Lambda_1, \Lambda_2, \dots)$, $y = (\mu_1, \mu_2, \dots)$; Denote by l the minimum for x and s the minimum for y.

It is sufficient to consider 2 cases:

Case 1: $l = s$

$$R\Big(\Lambda_1, \Lambda_2, \dots\Big) = \Big(1 - \sum_{j=l}^{\infty} \alpha_j\Big) x_l + \alpha_l x_{l+1} + \alpha_{l+1} x_{l+2} \dots \tag{6.5}$$

and

$$R\Big(\mu_1, \mu_2, \dots\Big) = \Big(1 - \sum_{j=l}^{\infty} \beta_j\Big) x_l + \beta_l x_{l+1} + \beta_{l+1} x_{l+2} \dots \tag{6.6}$$

Then

$$\begin{aligned} \|R(x) - R(y)\|_1 &\leq \frac{1}{1-\varepsilon}\Bigg[\Big(\sum_{j=l}^{\infty}\sum_{i\in N_j} |\Lambda_i - \mu_i|\Big)\|x_l\|_1 + \sum_{i\in N_l}|\Lambda_i - \mu_i|\,\|x_{l+1}\|_1 + \dots\Bigg] \\ &\leq 2\frac{1+\varepsilon}{1-\varepsilon}\|x - y\|_1 \end{aligned}$$

Case 2: $l < s$

$$
\begin{aligned}
\|R(x)-R(y)\|_1 &\leq (1+\varepsilon)\Big[|1-\sum_{j=l}^{\infty}\alpha_j| + \alpha_l + \cdots + \alpha_{s-2} + |\alpha_{s-1} - (1-\sum_{j=s}^{\infty}\beta_j)| \\
&\quad + |\alpha_s - \beta_s| + |\alpha_{s+1} - \beta_{s+1}| + \ldots\Big]
\end{aligned}
$$

Sub-Case 2.1

$$
1-\sum_{j=s}^{\infty}\beta_j \leq \alpha_{s-1} \tag{6.7}
$$

In this case we obtain that

$$
\begin{aligned}
\|R(x)-R(y)\|_1 &\leq (1+\varepsilon)\Big[1-\sum_{j=l}^{\infty}\alpha_j + \alpha_l + \cdots + \alpha_{s-2} + \alpha_{s-1} - (1-\sum_{j=s}^{\infty}\beta_j) \\
&\quad + |\alpha_s - \beta_s| + |\alpha_{s+1} - \beta_{s+1}| + \ldots\Big] \\
&= (1+\varepsilon)\left[\sum_{j=s}^{\infty}\beta_j - \sum_{j=s}^{\infty}\alpha_j + |\alpha_s - \beta_s| + |\alpha_{s+1} - \beta_{s+1}| + \ldots\right] \\
&\leq \frac{1+\varepsilon}{1-\varepsilon}\left[\sum_{j=s}^{\infty}\sum_{i\in\mathbb{N}_j}|\Lambda_i - \mu_i| + \sum_{i\in\mathbb{N}_s}|\Lambda_i - \mu_i| + \ldots\right] \\
&\leq 2\frac{1+\varepsilon}{1-\varepsilon}\|x-y\|_1 .
\end{aligned}
$$

Sub-Case 2.2

$$
1-\sum_{j=s}^{\infty}\beta_j \geq \alpha_{s-1} \tag{6.8}
$$

In this case,

$$
\begin{aligned}
\|R(x)-R(y)\|_1 &\leq (1+\varepsilon)\Big[1-\sum_{j=l}^{\infty}\alpha_j + \alpha_l + \cdots + \alpha_{s-2} + (1-\sum_{j=s}^{\infty}\beta_j) - \alpha_{s-1} \\
&\quad + |\alpha_s - \beta_s| + |\alpha_{s+1} - \beta_{s+1}| + \ldots\Big] \\
&= (1+\varepsilon)\Big[1-\sum_{j=s-1}^{\infty}\alpha_j + (1-\sum_{j=s}^{\infty}\beta_j) - \alpha_{s-1} \\
&\quad + |\alpha_s - \beta_s| + |\alpha_{s+1} - \beta_{s+1}| + \ldots\Big]
\end{aligned}
$$

By the definition of s (see above)

$$1 < \sum_{j=s-1}^{\infty} \beta_j$$

Therefore,

$$\begin{aligned}
\|R(x)-R(y)\|_1 &\leq (1+\varepsilon)\Big[1-\sum_{j=s-1}^{\infty}\alpha_j + (1-\sum_{j=s-1}^{\infty}\beta_j) + \beta_{s-1} - \alpha_{s-1} \\
&\quad +|\alpha_s - \beta_s| + \dots\Big] \\
&\leq (1+\varepsilon)\left[\sum_{j=s-1}^{\infty}\beta_j - \sum_{j=s-1}^{\infty}\alpha_j + 0 + \beta_{s-1} - \alpha_{s-1} + |\alpha_s - \beta_s| + \dots\right] \\
&\leq \frac{1+\varepsilon}{1-\varepsilon}\left[\sum_{j=s-1}^{\infty}\sum_{i\in\mathbb{N}_j}|\Lambda_i - \mu_i| + \sum_{i\in N_{s-1}}|\Lambda_i-\mu_i| + \sum_{i\in N_s}|\Lambda_i - \mu_i| \dots\right] \\
&\leq 2\frac{1+\varepsilon}{1-\varepsilon}\|x-y\|_1 .
\end{aligned}$$

A similar argument shows that for every $n \geq 1$, for all $x, y \in \ell_1$ (and therefore for all $x, y \in K = C_0$), $\|R^n(x) - R^n(y)\|_1 \leq \frac{1+\varepsilon}{1-\varepsilon}\|R(x)-R(y)\|_1 \leq 2(\frac{1+\varepsilon}{1-\varepsilon})^2 \leq L\|x-y\|_1$. Thus, in this [pairwise-disjoint-x_js]-case, K fails the fixed point property for cascading nonexpansive maps, and also for uniformly Lipschitzian maps for every $L > 2$.

For the general case, we will use the following lemma (see [5]):

Lemma 6.2.2. *Let $(x_n)_{n\geq 1}$ be a basic sequence in a Banach space X with basic constant M. Assume that $(y_n)_{n\geq 1}$ is another sequence in X such that*

$$2M\sum_{n=1}^{\infty}\frac{\|x_n - y_n\|}{\|x_n\|} =: \theta < 1$$

Let $(x_n^*)_{n\geq 1}$ denote the Hahn-Banach extensions of the biorthogonal functionals of $(x_n)_{n\geq 1}$ to the whole space X. Then,

$$A(x) = x + \sum_{n=1}^{\infty} x_n^*(x)(y_n - x_n)$$

is an invertible isomorphism from X to X with $A(x_n) = y_n$ for every $n \in \mathbb{N}$, $\|A\| \leq (1+\theta)$ and $\|A^{-1}\| \leq (1-\theta)^{-1}$.

We are now ready to prove the theorem in the general case.

Again, let K be a closed, bounded, convex subset of $(\ell_1, \|\cdot\|_1)$. Without loss of generality, we can assume K contains a sequence $(x_n)_{n\geq 1}$ which spans an asymptotically isometric copy of ℓ_1 and is weak*-null.

Then there exists a disjointly supported sequence $(x'_n)_{n\geq 1}$ such that the conditions of the previous lemma are satisfied for $x_n = x_n$ and $y_n = x'_n$. Since we have already solved the case for disjointly supported sequences, it is sufficient now to simply consider the following mapping:

$$\tilde{R} = A^{-1} \circ R \circ A : \ell_1 \to \overline{\text{co}}\{x_n\}_{n\geq 1}$$

where $R : \ell_1 \to \overline{\text{co}}\{x'_n\}_{n\geq 1}$ and $A : \ell_1 \to \ell_1$ is such that $A(x_n) = x'_n$.

It is not hard to check that the above map is fixed point free, cascading nonexpansive, and also uniformly Lipschitzian. Finally, we can consider the restriction of $\tilde{R}$ to K .

By using the previous arguments, the Lipschitz constant can be chosen arbitrarily close to 2 from above .

6.3 COMPACTNESS IN $(L^1[0,1], \|\cdot\|_1)$

We will extend the above construction to the case of $(L^1[0,1], \|\cdot\|_1)$. More precisely, we will prove that if K is non-weakly compact then there exists $R : K \to K$ a cascading nonexpansive mapping, uniformly Lipschitzian that is fixed point free. Notice that due to Alspach's example, who showed that there exists a weakly compact subset of $(L^1[0,1], \|\cdot\|_1)$ which fails the fixed point property for nonexpansive mapping, we cannot expect a similar result as to the one in $(\ell_1, \|\cdot\|_1)$ with implications in both directions.

However, at the end of this chapter we will give a necessary and sufficient condition for a convex set to be weakly compact in $(L^1[0,1], \|\cdot\|_1)$, but in terms of a new class of mappings, the so called eventually affine mappings.

Let K be a non-weakly compact, closed, bounded, convex subset of $(L^1[0,1], \|\cdot\|_1)$. Then, after a dilation, we may assume that there exists $f \in L^1[0,1]$ and a sequence $(h_m)_{m\geq 1} \in L^1[0,1]$ such that $C := f + \overline{\text{co}}\{h_n\} \subseteq K$ and the sequence $(h_n)_{n\geq 1}$ spans an a.i. copy of ℓ_1 ([11]).

Again, we will first solve the problem for the case when the sequence $(h_n)_{n\geq 1}$ is disjointly supported. Without loss of generality, by passing to a subsequence if necessary, we can assume that supp $h_1 = C_1$, supp $h_2 = C_2$, and in general supp $h_n = C_n$.

Now, for all $g \in L^1[0,1]$, there exists a j-minimum such that

$$\frac{\int_{C_j} |g-f|}{\|h_j\|} + \frac{\int_{C_{j+1}} |g-f|}{\|h_{j+1}\|} + \ldots + \leq 1$$

Define $R : L^1[0,1] \to C$ by

$$R(g) = f + \left(1 - \sum_{k=j}^{\infty} \frac{\int_{C_k} |g-f|}{\|h_k\|}\right) h_j + \left(\frac{\int_{C_j} |g-f|}{\|h_j\|}\right) h_{j+1} + \ldots$$

Using the same type of arguments one can show that R is cascading nonexpansive, uniformly Lipschitz and fixed point free, with Lipschitz constant as arbitrarly close to 2 from above. For the general case, let K be a non-weakly compact, closed bounded and convex subset of $(L^1[0,1], \|\cdot\|_1)$. We have a function $f \in L^1[0,1]$ and a sequence $(h_n)_{n\in\mathbb{N}}$ which spans an a.i copy of ℓ_1 such that $f + \overline{\text{co}}(h_n) \subseteq K$. Again, one can extract a disjointly supported sequence $(h'_n)_{n\in\mathbb{N}}$ such that the conditions of Lemma 1 are satisfied for $x_n = h_n$ and $y_n = h'_n$. So there exists $A : L^1[0,1] \to L^1[0,1]$ such that $A(h_n) = h'_n$.

Now let $A(f) = f'$ and the mapping $\tilde{R}$ be defined by:

$$\tilde{R} = A^{-1} \circ R \circ A : L^1[0,1] \to (f + \overline{\text{co}}(h_n))$$

$\tilde{R}$ is a fixed point free cascading nonexpansive map, uniformly Lipschitzian.

Here

$$R : L^1[0,1] \to (f' + \overline{\text{co}}(h'_n))$$

Finally, consider the restriction of $\tilde{R}$ to the set K.

Definition 6.3.1. Let $T : K \to K$ be a mapping and for all $n \in \mathbb{N}$, let $K_n = \overline{\text{co}}T(K_{n-1})$, where $K_0 := K$. We say that T is eventually affine if there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $T : K_n \to K_n$ is affine.

One can notice that the above map $\tilde{R}$ satisfies the definition and it is therefore eventually affine.

We have the following theorem:

Theorem 6.3.2. *(M. Japón, C. Lennard, R. Popescu (JLP)[21]) Let K be a closed, bounded, convex subset of $(L^1[0,1], \|\cdot\|_1)$. Then K is weakly compact iff K has the fixed point property for eventually affine mappings.*

Proof. If K is weakly compact, then every K_n is weakly compact. Let $T : K \to K$ be an eventually affine map. Therefore $T : K_{n_0} \to K_{n_0}$ is affine, for some $n_0 \in \mathbb{N}$. Using Mazur's Theorem, we obtain that T has a fixed point in K_{n_0}. For the other implication, assume K is a non-weakly compact, closed, bounded, convex subset of $(L^1[0,1], \|\cdot\|_1)$. The above construction provides an example of an eventually affine mapping $\tilde{R} : K \to K$ that is fixed point free.

□

It is an **open question** whether the result remains true for uniformly Lipschitzian mappings with Lipschitz constant $L \leq 2$.

7.0 STABILITY CONSTANT OF THE WEAK*-FPP FOR THE DUAL OF SEPARABLE LINDENSTRAUSS SPACES

7.1 INTRODUCTION

Let X be an infinite dimensional real Banach space. We say that X is a Lindenstrauss space if its dual is the space $L_1(\mu)$ for some measure μ.

A nonempty bounded closed and convex subset C of X has the fixed point property(or shortly fpp) if each nonexpansive mapping (i.e., the mapping $T : C \to C$ such that $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$) has a fixed point. A dual space X^* is said to have the $\sigma(X^*, X)$-fpp if every nonempty, convex, w*-compact subset C of X^* has fpp.

The study of $\sigma(X^*, X)$-fpp reveals to be of special interest whenever a dual space has different preduals. For instance, this situation occurs when we consider the space ℓ_1 and its preduals c_0 and c where it is well known (see [23]) that ℓ_1 has $\sigma(\ell_1, c_0)$ -fpp whereas it lacks the $\sigma(\ell_1, c)$-fpp.

The main purpose of the present chapter is to investigate the stability of $\sigma(\ell_1, X)$-fpp. Generally speaking, stability of the w*-fixed point property deals with the following question: let us suppose that a dual Banach space X^* has the $\sigma(X^*, X)$-fixed point property and let Y be a Banach space isomorphic to X with "small" Banach-Mazur distance from X. Does Y^* have the $\sigma(Y^*, Y)$-fixed point property?

We begin with the following definition:

Definition 7.1.1. A dual space X^* enjoys the stable $\sigma(X^*, X)$-fpp if there exists a real number $\gamma > 1$ such that Y^* has the $\sigma(Y^*, Y)$-fpp whenever $d(X, Y) < \gamma$, where $d(X, Y)$ is the Banach-Mazur distance between X and Y.

It is worth pointing out that every nonseparable dual of a separable Lindenstrauss space fails the w*-fpp -Corollary 3.4 in [7]. Indeed, in [7], Theorem 3.2, it was proved that if a separable Banach space contains a subspace isometric to c, then the dual fails the $\sigma(X^*, X)$-fixed point property. Since every separable Lindenstrauss space X with nonseparable dual contains a subspace isometric with $C(\Delta)$ -Theorem 2.3 in [26] (here Δ represents the Cantor set), we obtain that it also contains a subspace isometric to c. Therefore its dual fails weak*-fpp and the proof is now complete.

Thus, we restrict our attention only to preduals of ℓ_1. So let X be a predual of ℓ_1 such that ℓ_1 has the w*-fpp. We introduce two constants:

$$r^*(X) = \inf \left\{ r > 0 : (\mathrm{ext}(B_{\ell_1}))' \subset rB_{\ell_1} \right\}$$

$$\gamma^*(X) = \sup \left\{ \gamma \geq 1 : \text{ every } Y^* \text{ has } \sigma(Y^*, Y)\text{-fpp whenever } d(X, Y) \leq \gamma \right\}.$$

Here, by $d(X, Y)$ we understand the Banach-Mazur distance between the two spaces X and Y, that is

$$d(X, Y) = \inf\{\|T\| \left\|T^{-1}\right\| : T \in GL(X, Y)\}$$

.

Also, $(ext(B_{\ell_1}))'$ denotes the set of $\sigma(\ell_1, X)$-limit points of the extreme points of B_{ℓ_1}.

7.2 MAIN RESULT

It is well-known that if $r^*(X) = 0$, then $X = c_0$ and by results of Soardi [35] and Lim [31] we have $\gamma^*(c_0) = 2$. From Theorem 3.4 in [9] we know that if $r^*(X) = 1$, then $\gamma^*(X) = 1$.

Furthermore, if $r^*(X) \in (0, 1)$ then the inequality $\gamma^*(X) \geq \frac{2}{1+r^*(X)}$ follows from the proof of Theorem 3.4 in [9].

So the question was the following: What is a precise value for $\gamma^*(X)$?

It turns out that if $r^*(X) \in (0,1)$, then $\gamma^*(X) \leq \frac{2}{1+r^*(X)}$. In consequence, for every $r^*(X) \in [0,1]$, we have $\gamma^*(X) = \frac{2}{1+r^*(X)}$ and the following theorem is true:

Theorem 7.2.1 (E. Casini, E. Miglierina, Ł. Piasecki, R. Popescu [8]). *If X is a predual of ℓ_1 with $r^*(X) \in [0,1]$, then $\gamma^*(X) = \frac{2}{1+r^*(X)}$.*

The general proof involves a number of different techniques and results and so, for a more clear exposure, in the following lines we will focus our attention only on a particular example of such predual.

Let c be the space of convergent sequences, equipped with the sup norm $\|\cdot\|_\infty$, $\|x\|_\infty = \sup\limits_{i\geq 1} |x_i|$, and ℓ_1 the space of absolutely summable sequences equipped with the norm $|\cdot|_{\ell_1}$, $|x|_{\ell_1} = \sum\limits_{i=1}^{\infty} |x_i|$. For each sequence $x = (x_1, x_2, \dots)$, let $\tilde{x} = (0, x_2, x_3, \dots)$, x^+ and x^- be the positive and negative part of x, respectively. Also, denote by $x^+_\infty := \lim\limits_{n\to\infty} x^+_n$ and similarly, $x^-_\infty := \lim\limits_{n\to\infty} x^-_n$. Now $\|x^+\|_\infty = \max\limits_{n\in N\cup\{\infty\}} \left|x^+_n\right|$ and $\|x^-\|_\infty = \max\limits_{n\in N\cup\{\infty\}} \left|x^-_n\right|$.

Let $\alpha = (\alpha_1, \alpha_2, \dots) \in B_{\ell_1}$. We define the space

$$W_\alpha = \left\{ x = (x_1, x_2, \dots) \in c : \lim_{i\to\infty} x_i = \sum_{i=1}^{+\infty} \alpha_i x_i \right\}.$$

Theorem 7.2.2. *(CMPP [8])*

Let $e^* = (r, 0, 0, \dots) \in \ell_1$ with $0 < r < 1$. For all $x \in W_{e^*}$, define

$$\|x\| = \left(\left\|\tilde{x}^+\right\|_\infty \vee r \left\|\tilde{x}^-\right\|_\infty + \left\|\tilde{x}^-\right\|_\infty \vee r \left\|\tilde{x}^+\right\|_\infty \right) \vee (1+r)\,|x_1|\,.$$

Then

$$(W_{e^*}, \|\cdot\|)^* = (\ell_1, |\cdot|),$$

where

$$|f| = \max\left\{ \frac{r}{1+r}\left|\tilde{f}^+\right|_{\ell_1} + \frac{1}{1+r}\left|\tilde{f}^-\right|_{\ell_1}, \frac{1}{1+r}\left|\tilde{f}^+\right|_{\ell_1} + \frac{r}{1+r}\left|\tilde{f}^-\right|_{\ell_1} \right\} + \frac{1}{1+r}\,|f_1|\,,$$

and duality map $\phi : \ell_1 \to W^*_{e^*}$ is defined by

$$(\phi(f))(x) = \sum_{j=1}^{+\infty} x_j f_j,$$

where $f=(f_1,f_2,\dots)\in\ell_1$ *and* $x=(x_1,x_2,\dots)\in W_{e^*}$*.*

Proof. First, we begin by noticing that $\|\cdot\|$ is a norm (checked!) equivalent to the $\|\cdot\|_\infty$ norm,

$$(1+r)\,\|x\|_\infty=(1+r)\,\|\tilde{x}\|_\infty\vee(1+r)|x_1|\le\|x\|\le 2\,\|\tilde{x}\|_\infty\vee 2\,|x_1|=2\,\|x\|_\infty$$

so, from Theorem 4.3 in [6], we know that the dual of $(W_{e^*},\|\cdot\|)$ is representable by ℓ_1 with duality map $\phi:\ell_1\to W_{e^*}$ defined by

$$(\phi(f))(x)=\sum_{j=1}^{+\infty}x_jf_j,$$

where $f=(f_1,f_2,\dots)\in\ell_1$ and $x=(x_1,x_2,\dots)\in W_{e^*}$. Therefore it suffices to show that

$$|f|=\sup\left\{\sum_{i=1}^{\infty}x_if_i : x\in W_{e^*},\left(\left\|\tilde{x}^+\right\|_\infty\vee r\left\|\tilde{x}^-\right\|_\infty+\left\|\tilde{x}^-\right\|_\infty\vee r\left\|\tilde{x}^+\right\|_\infty\right)\vee(1+r)\,|x_1|\le 1\right\}$$

for each $f\in\ell_1$. As in [31], the supremum can be taken again over x satisfying $x_if_i\ge 0$. In case $x_if_i<0$, replace it by 0 when estimating from above. Also, notice that

$$\frac{1}{2}\,|f|_{\ell_1}=\frac{1}{2}\left|\tilde{f}\right|_{\ell_1}+\frac{1}{2}\,|f_1|\le|f|\le\frac{1}{1+r}\left(\left|\tilde{f}^+\right|_{\ell_1}+\left|\tilde{f}^-\right|_1\right)+\frac{1}{1+r}\,|f_1|=\frac{1}{1+r}\,|f|_{\ell_1}.$$

Now let $f\in\ell_1$ and $f(x)=x_1f_1+x_2f_2+x_3f_3+\dots$. Without loss of generality, one can assume that

$$|f|=\frac{r}{1+r}\left|\tilde{f}^+\right|_{\ell_1}+\frac{1}{1+r}\left|\tilde{f}^-\right|_{\ell_1}+\frac{1}{1+r}\,|f_1|\,,$$

and so

$$\left|\tilde{f}^-\right|_{\ell_1}\ge\left|\tilde{f}^+\right|_{\ell_1}\quad(\heartsuit).$$

There are three cases to consider. However, we will present only one of them since the proof for the other two is very similar.

Case 1. Assume that

$$\left\|\tilde{x}^+\right\|_\infty\vee r\left\|\tilde{x}^-\right\|_\infty=\left|\tilde{x}_i^+\right| \tag{7.1}$$

and

$$\left\|\tilde{x}^{-}\right\|_{\infty} \vee r\left\|\tilde{x}^{+}\right\|_{\infty}=\left|\tilde{x}_{j}^{-}\right| \tag{7.2}$$

for some $i, j \geq 2$. Then

$$\|x\|=\left(\left|\tilde{x}_{i}^{+}\right|+\left|\tilde{x}_{j}^{-}\right|\right) \vee(1+r)\left|x_{1}\right| .$$

SubCase 1.1. If $(1+r)\left|x_{1}\right| \leq\left|\tilde{x}_{i}^{+}\right|+\left|\tilde{x}_{j}^{-}\right|$, then

$$\begin{aligned}
(1+r) f(x) & = (1+r) x_{1} f_{1}+x_{2} f_{2}+x_{3} f_{3}+\cdots+r x_{2} f_{2}+r x_{3} f_{3}+\cdots \\
& \leq \left(\left|\tilde{x}_{i}^{+}\right|+\left|\tilde{x}_{j}^{-}\right|\right)\left|f_{1}\right|+\left|\tilde{x}_{i}^{+}\right|\left|\tilde{f}^{+}\right|_{\ell_{1}}+\left|\tilde{x}_{j}^{-}\right|\left|\tilde{f}^{-}\right|_{\ell_{1}}+r\left|\tilde{x}_{i}^{+}\right|\left|\tilde{f}^{+}\right|_{\ell_{1}}+r\left|\tilde{x}_{j}^{-}\right|\left|\tilde{f}^{-}\right|_{\ell_{1}} \\
& \leq \left(\left|\tilde{x}_{i}^{+}\right|+\left|\tilde{x}_{j}^{-}\right|\right)\left(\left|f_{1}\right|+r\left|\tilde{f}^{+}\right|_{\ell_{1}}+\left|\tilde{f}^{-}\right|_{\ell_{1}}\right)
\end{aligned}$$

and the last inequality holds since $r\left|\tilde{x}_{j}^{-}\right| \leq\left|\tilde{x}_{i}^{+}\right|$ by (7.1) and $\left|\tilde{f}^{+}\right|_{\ell_{1}} \leq\left|\tilde{f}^{-}\right|_{\ell_{1}}$ by (♡). Thus, $f(x) \leq\|x\||f|$.

SubCase 1.2. $\left|\tilde{x}_{i}^{+}\right|+\left|\tilde{x}_{j}^{-}\right| \leq(1+r)\left|x_{1}\right|$, then from (7.1) we obtain $r\left|\tilde{x}_{j}^{-}\right| \leq\left|\tilde{x}_{i}^{+}\right|$ and so $(1+r)\left|\tilde{x}_{j}^{-}\right| \leq(1+r)\left|x_{1}\right|$, or equivalently $\left|\tilde{x}_{j}^{-}\right| \leq\left|x_{1}\right|$. Now we have

$$f(x) \leq\left|x_{1}\right|\left|f_{1}\right|+\left|\tilde{x}_{i}^{+}\right|\left|\tilde{f}^{+}\right|_{\ell_{1}}+\left|\tilde{x}_{j}^{-}\right|\left|\tilde{f}^{-}\right|_{\ell_{1}} \leq\left|x_{1}\right|\left(\left|f_{1}\right|+r\left|\tilde{f}^{+}\right|_{\ell_{1}}+\left|\tilde{f}^{-}\right|_{\ell_{1}}\right) .$$

This time the last inequality holds since $\left|\tilde{x}_{i}^{+}\right|-r\left|x_{1}\right| \leq\left|x_{1}\right|-\left|\tilde{x}_{j}^{-}\right|$, $0 \leq\left|x_{1}\right|-\left|\tilde{x}_{j}^{-}\right|$ and $\left|\tilde{f}^{+}\right|_{\ell_{1}} \leq\left|\tilde{f}^{-}\right|_{\ell_{1}}$ by (♡). Therefore, we obtain again that $f(x) \leq\|x\||f|$.

Case 2. $\left\|\tilde{x}^{+}\right\|_{\infty} \vee r\left\|\tilde{x}^{-}\right\|_{\infty}=\left|\tilde{x}_{i}^{+}\right|$, $\left\|\tilde{x}^{-}\right\|_{\infty} \vee r\left\|\tilde{x}^{+}\right\|_{\infty}=r\left|\tilde{x}_{i}^{+}\right|$ for some $i \geq 2$ and so $\|x\|=(1+r) \tilde{x}_{i}^{+} \vee(1+r)\left|x_{1}\right|=(1+r)\|x\|_{\infty}$.

Case 3. $\left\|\tilde{x}^{+}\right\|_{\infty} \vee r\left\|\tilde{x}^{-}\right\|_{\infty}=r\left|\tilde{x}_{j}^{-}\right|$, $\left\|\tilde{x}^{-}\right\|_{\infty} \vee r\left\|\tilde{x}^{+}\right\|_{\infty}=\left|\tilde{x}_{j}^{-}\right|$ for some $i \geq 2$ and so $\|x\|=(1+r)\left|\tilde{x}_{j}^{-}\right| \vee(1+r)\left|x_{1}\right|=(1+r)\|x\|_{\infty}$. Case 3 can be solved using similar ideas as in Case 2.

In conclusion, we have shown that

$$\sup \left\{\sum_{i=1}^{\infty} x_{i} f_{i}: x \in W_{e^{*}},\left(\left\|\tilde{x}^{+}\right\|_{\infty} \vee r\left\|\tilde{x}^{-}\right\|_{\infty}+\left\|\tilde{x}^{-}\right\|_{\infty} \vee r\left\|\tilde{x}^{+}\right\|_{\infty}\right) \vee(1+r)\left|x_{1}\right| \leq 1\right\} \leq|f|$$

To prove the reversed inequality, one can choose appropriate values for $x \in W_{e^*}$. Consider $x_1 = (\mathrm{sgn} f_1)\frac{1}{1+r}$, $x_i = \frac{-1}{1+r}$ for $f_i \leq 0$, $x_i = \frac{r}{1+r}$ for $f_i \geq 0$ and $x_i = (\mathrm{sgn} f_1)\frac{r}{1+r}$ for i far away in the sequence.

□

Theorem 7.2.3. *(CMPP [8])*

$(W_{e^*}, \|\cdot\|)^* = (\ell_1, |\cdot|)$ fails the w^*-fpp.

Proof. Consider a set $C \subset \ell_1$ defined by

$$C = \left\{ (rt_1, t_2, \dots) : t_i \geq 0, \sum_{i=1}^{\infty} t_i = 1 \right\}.$$

The set C is convex and from the above theorem we conclude that it is weak* compact in $(W_{e^*}, \|\cdot\|)^* = (\ell_1, |\cdot|)$. Define $T : C \to C$ by

$$T(rt_1, t_2, \dots) = (0, t_1, t_2, \dots).$$

The map T is fixed point free and $|\cdot|$-nonexpansive. Indeed, let $t = (rt_1, t_2, \dots)$ and $s = (rs_1, s_2, \dots)$ be two elements of the set C. We consider two cases:

Case 1: $t_1 - s_1 \geq 0$.

This further implies $\left|\widetilde{t-s}^-\right|_{\ell_1} \geq \left|\widetilde{t-s}^+\right|_{\ell_1}$ and so

$$|t - s| = \frac{r}{1+r}\left|\widetilde{t-s}^+\right|_{\ell_1} + \frac{1}{1+r}\left|\widetilde{t-s}^-\right|_{\ell_1} + \frac{r}{1+r}|t_1 - s_1|.$$

Now

$$\begin{aligned} |T(t) - T(s)| &= \max\left\{ \frac{r}{1+r}(|t_1 - s_1| + \left|\widetilde{t-s}^+\right|_{\ell_1}) + \frac{1}{1+r}\left|\widetilde{t-s}^-\right|_{\ell_1}, \right. \\ &\qquad \left. \frac{1}{1+r}(|t_1 - s_1| + \left|\widetilde{t-s}^+\right|_{\ell_1}) + \frac{r}{1+r}\left|\widetilde{t-s}^-\right|_{\ell_1} \right\} \\ &= \frac{r}{1+r}\left|\widetilde{t-s}^+\right|_{\ell_1} + \frac{1}{1+r}\left|\widetilde{t-s}^-\right|_{\ell_1} + \frac{r}{1+r}|t_1 - s_1| \\ &= |t - s| \end{aligned}$$

and so T is $|\cdot|$-isometry.

Case 2: $t_1 - s_1 \leq 0$. The proof is similar with Case 1.

□

7.3 WEAK*-FPP FOR CONTRACTIVE MAPS IN THE SPACE ℓ_1

In 1997, in the Proc. AMS, P. Dowling and C. Lennard [12] proved that ℓ_1 with the Lim norm fails the w*-fpp with respect to the predual c_0 through an affine, contractive map. Using an adaptation of this example, M. Smyth proved that ℓ_1 fails the w*-fpp with respect to the predual c through an affine, contractive map [34].

Now consider again the same setting as in the previous section, where $e^* = (r, 0, 0, \dots) \in \ell_1$ and $0 < r < 1$.

Based on these two examples and the mapping constructions, we will show in what follows that the same result holds for the space $(W_{e^*}, \|\cdot\|)^* = (\ell_1, |\cdot|)$, that is:

Theorem 7.3.1. *$(W_{e^*}, \|\cdot\|)^* = (\ell_1, |\cdot|)$ fails the weak*-fixed point property with an affine contractive map.*

Proof. Consider $C := \{(rx_1, x_2, \dots) | x_i \geq 0, x_1 + x_2 + x_4 + \dots = x_3 \leq 1\}$.

The set C is $\sigma(\ell_1, W_e)$-compact.

Fix $(\varepsilon_n)_{n\in\mathbb{N}}$ such that each $\varepsilon_n \in (0, 1)$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Fix an arbitrary $x \in C$ and define $T : C \to C$ by

$$\begin{aligned} T(rx_1, x_2, x_3, \dots) \;=\; & \Big(r\Big(1 - x_3 + (1-r)x_1 + \varepsilon_2 x_2 + \varepsilon_4 x_4 + \varepsilon_5 x_5 + \dots\Big), rx_1, 1, \\ & (1-\varepsilon_2)x_2, (1-\varepsilon_4)x_4, (1-\varepsilon_5)x_5, \dots\Big) \end{aligned}$$

We claim that T is fixed point free. Indeed, suppose not. Then there is an $x \in C$ with $T(x) = x$. Thus

$$\Big(1 - x_3 + (1-r)x_1 + \varepsilon_2 x_2 + \varepsilon_4 x_4 + \varepsilon_5 x_5 + \dots\Big) = x_1$$

$$rx_1 = x_2$$

$$1 = x_3$$

$$(1-\varepsilon_2)x_2 = x_4$$

$$(1-\varepsilon_4)x_4 = x_5$$

$$(1-\varepsilon_5)x_5 = x_6$$

$$\dots$$

Consequently, for each $k \in \mathbb{N}$, $k \geq 4$, $x_k = x_2(1-\varepsilon_2)\prod_{j=4}^{k}(1-\varepsilon_j)$. There are two cases. If $x_2 = 0$, then $x_1 = 0$, $x_k = 0, \forall k \geq 4$ and $x_3 = 1$, contradiction with $x \in C$. If $x_2 \neq 0$, then $\prod_{j=4}^{\infty}(1-\varepsilon_j) = \lim_{k\to\infty} x_k = 0$. This is precisely equivalent with $\sum_{j=4}^{\infty} \varepsilon_j = \infty$; a contradiction.

Clearly that T is an affine map. Next we show that it is also $|\cdot|_1$- contractive.

Let $x, y \in C$ with $x = (rx_1, x_2, x_3, \dots)$ and $y = (ry_1, y_2, y_3, \dots)$. For all $i \in \mathbb{N}$, denote by $\alpha_i = x_i - y_i$. Also let $I = \{i \geq 2, i \neq 3 | \alpha_i \geq 0\}$ and $J = \{i \geq 2, i \neq 3 | \alpha_i < 0\}$.

$$\begin{aligned} T(x) - T(y) &= \Bigg(r\Big(-\alpha_3 + (1-r)\alpha_1 + \varepsilon_2\alpha_2, \varepsilon_4\alpha_4, \dots\Big), r\alpha_1, 0, \\ &\quad (1-\varepsilon_2)\alpha_2, (1-\varepsilon_4)\alpha_4, (1-\varepsilon_5)\alpha_5 \dots\Bigg) \end{aligned}$$

We will consider two cases.

Case 1: Assume that $\Big[-\alpha_3 + (1-r)\alpha_1 + \varepsilon_2\alpha_2, \varepsilon_4\alpha_4 + \dots\Big] \geq 0$.

This implies that $r\alpha_1 + (1-\varepsilon_2))\alpha_2 + (1-\varepsilon_4)\alpha_4 + (1-\varepsilon_5)\alpha_5 + \dots \leq 0$ (because the two add up to 0), and so

$$r\alpha_1 + \sum_{i\in I}(1-\varepsilon_i)\alpha_i \le \sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)$$

Subcase 1.1: $\alpha_1 > 0$

In this case $0 \le r\alpha_1 + \sum_{i\in I}(1-\varepsilon_i)\alpha_i \le \sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)$ (1)

Therefore we have:

$$\begin{aligned}
|T(x)-T(y)|_1 &= \frac{r}{1+r}\left[r\alpha_1 + \sum_{i\in I}(1-\varepsilon_i)\alpha_i\right] + \frac{1}{1+r}\left[\sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)\right] \\
&\quad + \frac{r}{1+r}\Big[-\alpha_3 + (1-r)\alpha_1 + \varepsilon_2\alpha_2 + \varepsilon_4\alpha_4 + \varepsilon_5\alpha_5 + \dots\Big] \\
&= \frac{r}{1+r}\left[\sum_{i\in I}\alpha_i\right] + \frac{1}{1+r}\left[\sum_{j\in J}(1-\varepsilon_j - r\varepsilon_j)(-\alpha_j)\right] + \frac{r}{1+r}(-\alpha_3) \\
&\quad + \frac{r}{1+r}\alpha_1 \\
&< \frac{r}{1+r}\left[\sum_{i\in I}\alpha_i\right] + \frac{1}{1+r}\left[\sum_{j\in J}(-\alpha_j)\right] + \frac{r}{1+r}(-\alpha_3) + \frac{r}{1+r}\alpha_1 \\
&\le \max\left\{\frac{r}{1+r}\widetilde{(x-y)}^{+} + \frac{1}{1+r}\widetilde{(x-y)}^{-}, \frac{1}{1+r}\widetilde{(x-y)}^{+} + \frac{r}{1+r}\widetilde{(x-y)}^{-}\right\} \\
&\quad + \frac{r}{1+r}|x_1-y_1| \\
&= |x-y|_1
\end{aligned}$$

Here equality happens iff $\alpha_j = 0$, for all $j \in J$, which by (1) implies that $\alpha_i = 0$, for all $i \in I$ and $x_1 = y_1$. Since $x, y \in C$, we further obtain that $x_3 = y_3$, and so $x = y$.

Subcase 1.2: $\alpha_1 < 0$

In this case $0 \le \sum_{i\in I}(1-\varepsilon_i)\alpha_i \le r(-\alpha_1)\sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)$ (2)

Therefore we have:

$$\begin{aligned}
|T(x)-T(y)|_1 &= \frac{r}{1+r}\left[\sum_{i\in I}(1-\varepsilon_i)\alpha_i\right] + \frac{1}{1+r}\left[r(-\alpha_1) + \sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)\right] \\
&\quad + \frac{r}{1+r}\Big[-\alpha_3 + (1-r)\alpha_1 + \varepsilon_2\alpha_2 + \varepsilon_4\alpha_4 + \varepsilon_5\alpha_5 + \dots\Big]
\end{aligned}$$

$$
\begin{aligned}
< \;& \frac{r}{1+r}\Big[\sum_{i\in I}(1-\varepsilon_i)\alpha_i\Big] + \frac{1}{1+r}\Big[r(-\alpha_1) + \sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)\Big] + \frac{r}{1+r}\Big[-\alpha_3 + \\
& \varepsilon_2\alpha_2 + \varepsilon_4\alpha_4 + \varepsilon_5\alpha_5 + \dots\Big] \\
= \;& \frac{r}{1+r}\Big[\sum_{i\in I}\alpha_i\Big] + \frac{1}{1+r}\Big[\sum_{j\in J}(1-\varepsilon_j - r\varepsilon_j)(-\alpha_j)\Big] + \frac{r}{1+r}(-\alpha_3) + \frac{r}{1+r}(-\alpha_1) \\
< \;& \frac{r}{1+r}\Big[\sum_{i\in I}\alpha_i\Big] + \frac{1}{1+r}\Big[\sum_{j\in J}(-\alpha_j)\Big] + \frac{r}{1+r}(-\alpha_3) + \frac{r}{1+r}(-\alpha_1). \\
\leq \;& \max\left\{\frac{r}{1+r}\widetilde{(x-y)}^{+} + \frac{1}{1+r}\widetilde{(x-y)}^{-}, \frac{1}{1+r}\widetilde{(x-y)}^{+} + \frac{r}{1+r}\widetilde{(x-y)}^{-}\right\} \\
& + \frac{r}{1+r}\left|x_1 - y_1\right| \\
= \;& |x-y|_1
\end{aligned}
$$

In this case equality happens iff $\alpha_1 = 0$ and $\alpha_j = 0$, for all $j \in J$, which by (2) implies that $\alpha_i = 0$, for all $i \in I$. Since $x, y \in C$, we further obtain that $x_3 = y_3$, and so $x = y$. □

Next we solve case 2, which is very similar to Case 1.

Case 2: Assume that $\Big[-\alpha_3 + (1-r)\alpha_1 + \varepsilon_2\alpha_2, \varepsilon_4\alpha_4 + \dots\Big] \leq 0.$

This now implies that $r\alpha_1 + (1-\varepsilon_2))\alpha_2 + (1-\varepsilon_4)\alpha_4 + (1-\varepsilon_5)\alpha_5 + \dots \geq 0$ and so

$$
r\alpha_1 + \sum_{i\in I}(1-\varepsilon_i)\alpha_i \geq \sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)
$$

Subcase 2.1: $\alpha_1 > 0$

In this case $r\alpha_1 + \sum_{i\in I}(1-\varepsilon_i)\alpha_i \geq \sum_{j\in J}(1-\varepsilon_j)(-\alpha_j) \geq 0$ (3)

Therefore we have:

$$
\begin{aligned}
|T(x)-T(y)|_1 &= \frac{1}{1+r}\left[r\alpha_1+\sum_{i\in I}(1-\varepsilon_i)\alpha_i\right]+\frac{r}{1+r}\left[\sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)\right] \\
&\quad +\frac{r}{1+r}\left[\alpha_3-(1-r)\alpha_1-\varepsilon_2\alpha_2-\varepsilon_4\alpha_4-\varepsilon_5\alpha_5-\dots\right] \\
&< \frac{1}{1+r}\left[r\alpha_1+\sum_{i\in I}(1-\varepsilon_i)\alpha_i\right]+\frac{r}{1+r}\left[\sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)\right] \\
&\quad +\frac{r}{1+r}\left[\alpha_3-\varepsilon_2\alpha_2-\varepsilon_4\alpha_4-\varepsilon_5\alpha_5-\dots\right] \\
&= \frac{1}{1+r}\left[\sum_{i\in I}(1-\varepsilon_i-r\varepsilon_i)\alpha_i\right]+\frac{1}{1+r}\left[\sum_{j\in J}(-\alpha_j)\right]+\frac{r}{1+r}(\alpha_3)+\frac{r}{1+r}\alpha_1 \\
&< \frac{1}{1+r}\left[\sum_{i\in I}\alpha_i\right]+\frac{r}{1+r}\left[\sum_{j\in J}(-\alpha_j)\right]+\frac{r}{1+r}(\alpha_3)+\frac{r}{1+r}\alpha_1 \\
&\le \max\left\{\frac{r}{1+r}\widetilde{(x-y)}^{+}+\frac{1}{1+r}\widetilde{(x-y)}^{-},\frac{1}{1+r}\widetilde{(x-y)}^{+}+\frac{r}{1+r}\widetilde{(x-y)}^{-}\right\} \\
&\quad +\frac{r}{1+r}|x_1-y_1| \\
&= |x-y|_1
\end{aligned}
$$

Here equality happens iff $\alpha_1 = 0$ and $\alpha_i = 0$, for all $i \in I$, which by (3) implies that $\alpha_j = 0$, for all $j \in J$. Since $x, y \in C$, we further obtain that $x_3 = y_3$, and so $x = y$.

Subcase 2.2: $\alpha_1 < 0$

In this case $\sum_{i\in I}(1-\varepsilon_i)\alpha_i \ge r(-\alpha_1)\sum_{j\in J}(1-\varepsilon_j)(-\alpha_j) \ge 0$ (4)

Therefore we have:

$$
\begin{aligned}
|T(x)-T(y)|_1 &= \frac{1}{1+r}\left[\sum_{i\in I}(1-\varepsilon_i)\alpha_i\right]+\frac{r}{1+r}\left[r(-\alpha_1)+\sum_{j\in J}(1-\varepsilon_j)(-\alpha_j)\right] \\
&\quad +\frac{r}{1+r}\left[\alpha_3+(1-r)(-\alpha_1)-\varepsilon_2\alpha_2-\varepsilon_4\alpha_4-\varepsilon_5\alpha_5-\dots\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1+r}\left[\sum_{i\in I}(1-\varepsilon_i - r\varepsilon_i)\alpha_i\right] + \frac{1}{1+r}\left[\sum_{j\in J}(-\alpha_j)\right] + \frac{r}{1+r}(\alpha_3) + \frac{r}{1+r}(-\alpha_1) \\
&< \frac{1}{1+r}\left[\sum_{i\in I}\alpha_i\right] + \frac{r}{1+r}\left[\sum_{j\in J}(-\alpha_j)\right] + \frac{r}{1+r}(\alpha_3) \\
&\quad + \frac{r}{1+r}(-\alpha_1) \\
&\leq \max\left\{\frac{r}{1+r}\widetilde{(x-y)}^{+} + \frac{1}{1+r}\widetilde{(x-y)}^{-}, \frac{1}{1+r}\widetilde{(x-y)}^{+} + \frac{r}{1+r}\widetilde{(x-y)}^{-}\right\} \\
&\quad + \frac{r}{1+r}\left|x_1 - y_1\right| \\
&= \left|x-y\right|_1
\end{aligned}
$$

Equality happens iff $\alpha_i = 0$, for all $i \in I$, which by (4) implies that $\alpha_j = 0$, for all $j \in J$ and $\alpha_1 = 0$. Since $x, y \in C$, we further obtain that $x_3 = y_3$, and so $x = y$.

As a final remark, notice that when $\varepsilon_i = 0$, for all $i \in \mathbb{N}$, the map T is another example of a non-expansive fixed point free map on the set C.

Of course, this is the solution only for a class of ℓ_1 preduals, but we considered it interesting since it represents a natural extension of the cases c_0 and c. From theorem 6.2.1 we know that ℓ_1 lacks the $\sigma(\ell_1, X)$- FPP for nonexpansive, with $T : C \to C$ nonexpansive and fixed point free. By following the idea of J. Burns, C. Lennard, and J. Sivek from Studia Mathematica ([4]), and considering the mapping $S : C \to C$ defined as $S(x) = \sum_{j=0}^{\infty} \frac{T^j(x)}{2^{j+1}}$, one can still show for the general case that ℓ_1 lacks the $\sigma(\ell_1, X)$- FPP for contractive maps.

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