# TRANSFERABLE CREDIT DEFAULT SWAPS WITH COUNTERPARTY RISKS

by

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A *credit default swap* (CDS) is a financial contract between two parties who exchange cash flows based on an occurrence of an underlying credit default (or more general, event). Under a CDS, the seller will pay a compensation to the buyer at the time of the credit event, if it happens before the expiry; in return, the buyer agrees to pay a continuous premium to the seller until the occurrence of the credit event or the expiry, whichever is earlier. *Counterparty risk* refers to the risk caused by the default of one party of an active contract. By *transferable*, it means that one party of a CDS, at the time of his default, can sell the contract to a third party. We consider three kinds of transferability: (i) transferable only by the seller at most one time; (ii) transferable by both parties at most one time; and (iii) transferable by both parties any number of times. The problem here is to price transferable CDSs with counterparty risks. We study an intensity model where the credit event and default times are described by arrival times of Poisson processes with variable intensities depending on a state variable, for definiteness, chosen as the interest rate. We model the interest rate by a classical Cox-Ingersoll-Ross model. The pricing problem is then modeled by an initial value problem of a degenerate partial differential equation (PDE) on an unbounded domain. We prove that the PDE problem is well-poseded. We also derive certain useful estimates on the bounds of the solutions. Besides the intensity model, another primary model is the structure model. We establish the connection between these two models; in particular, we show that the solution of the structure model is the limit of a sequence of solutions of the intensity models. Certain numerical simulations are also provided.

**Keywords:** Credit default swaps, Transferable, Counterparty risk, Intensity model, Structure model, Well-posedness, CIR model.

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## PREFACE

This dissertation is submitted to fulfill of the requirements for the degree of Doctor of Philosophy at the University of Pittsburgh. The research is supervised by Professor Xinfu Chen.

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# 1.0 INTRODUCTION

A swap is a financial contract between two parties who exchange cash flows. A credit default swap, CDS for short, is a swap based on the occurrence of an underlying credit event, typically, a default of an underlying entity. Under a CDS contract, the seller will pay a lump sum of compensation to the buyer at the time of the credit event, if it happens before the expiry; in return, from the date the contact was signed and until the occurrence of the credit event or the expiry, whichever earlier, the buyer agrees to pay a continuous premium to the seller. The CDS is like an insurance: the buyer purchases it through paying the premium to get rid of the possible risk of huge loss caused by the credit event, and the seller takes the risk of losing the compensation paid to the buyer at the occurrence of the credit event to earn the premium. In this chapter, we shall first describe in detail several particular types of CDSs, then we derive mathematical models and finally state our main results.

## 1.1 CREDIT EVENT, COUNTERPARTY RISK, AND TRANSFERABILITY

There are two types of models for credit events: the intensity model and the structure model. The first one describes credit events by first arrival times of Poisson processes, whereas the second one describes a credit event by the crossing of a boundary on a state space of state variables. Dates back to 1974, Merton [18] proposed a structure model to study the risk for financial derivatives. Then in 1976, Black and Cox [3] used the first passage time to model the price of risky bonds. In 1991, Cooper and Mello [7] studied the equilibrium model for the default risk of swaps. The invention of credit default swaps (CDS) dates back to

1994 by Blythe Masters from JP Morgan [23]. From then, CDS is a very popular topic. In 1994, Duffie and Singleton [9] (updated version in 1999) studied the pricing for claims with symmetric counterparty risk. Then in 1996, Duffie and Huang [8] extended it to the asymmetric case through theory and numerical examples. In 1997, Zhou [26] studied the credit default through modeling the evolution of the firm's value by jump-diffusion processes. Later, Lando [16] (1998), Duffie and Singleton [9] (1999) and Brigo and Alfonsi [4] (2005) studied the credit risk with intensity models.

Besides the credit risk of the underlying credit event, there is another kind of risk involving in a CDS contract, which is called the counterparty risk. The counterparty risk is the risk of defaults of the buyer and/or the seller of the CDS contract. Here again, regarding default as credit event, we can describe the defaults by either an intensity model or a structure model.

We call a contract *nontransferable* if the contract automatically ends at the default time of one party. When one party, say the seller, defaults, the CDS contract may be in favor of the seller and therefore he would like to sell the contract to a third party. Hence, we use the word *transferable* to designate a CDS which allows a party to sell his contract to a third party at time of his default. We consider three types of transferability:

1. The CDS can be sold by the original seller to a third party at time of his default, and the new contract becomes non-transferable;
2. The CDS can be sold by the original seller or buyer to a third party at time of his default, and the new contract becomes non-transferable;
3. The CDS can be sold by the original seller or buyer to a third party at time of his default, and the new contract is still transferable any number of times.

Pricing a non-transferable CDS is typically modeled by a partial differential equation of one unknown function and has been investigated by Hu, Jiang, Liang, and Wei [14] and by Chen, He, Liu, and Zhao [6, 13, 25].

In this dissertation, we study for the first time transferable CDSs. We model a transferable CDS by a system of partial differential equations of multiple unknown functions. The resulting mathematical problem is thoroughly investigated. As a by product, we improve the results in [13, 25] in a significantly way that the results can carry from a scalar unknown function to vector unknown functions.

## 1.2 EXPECTED VALUE OF CDS

In this section, we consider the expected values of cash flows of various kinds of CDSs.

### 1.2.1 CDS without counterparty risks

A CDS is a financial contract between two parties who exchange cash flows based on an occurrence of a particular credit event. A prototype example of a credit event is a default of a due payment. The contract covers a time period denoted by $[0, T]$ (current time is $t = 0$). We call $T$ the expiry. We use $\tau$ to denote the time of the occurrence of the credit event. We call two parties the seller and the buyer respectively. First we assume that neither party will default before $T$; i.e., we assume that there is no counterparty risk. The contract ends at $\tau \wedge T := \min\{T, \tau\}$. We assume that the buyer pays to the seller a continuous rate, $Kq(\$/year)$, cash flow in time period $[0, \tau \wedge T]$. As a compensate, if $\tau \leqslant T$, then at time $\tau$, the seller pays the buyer $K$ (\$). We put the key ingredients of the basic CDS as below.

*BASIC CDS CONTRACT*

Let $T$ be the expiry and $\tau$ be the time of occurrence of the well-defined credit event.

1. *The buyer's right and the seller's obligation:* If the credit event happens before the expiry, i.e. if $\tau < T$, at the credit event occurrence time $\tau$, the seller pays the buyer a lump sum of compensation of $K$ dollars and the contract terminates.
2. *The seller's right and the buyer's obligation:* Till the credit event or the expiry, whichever is earlier, i.e. $T \wedge \tau$, the buyer pays the seller continuous premium at the rate $Kq$ dollars/year.

We use a stochastic process $\{r_t\}_{t \geq 0}$ to denote the short interest rate and assume that the discount factor from $t = t_1$ to $t = t_2$ is described by

$$\exp\left(-\int_{t_1}^{t_2} r_t dt\right).$$

From buyer's point of view, the present value of the contract is

$$p_{10} := K\ \mathbb{E}\left[e^{-\int_0^\tau r_t dt}\mathbf{1}_{\{\tau\le T\}} - \int_0^{\tau\wedge T} q e^{-\int_0^t r_\theta d\theta} dt \middle| \tau > 0\right]. \tag{1.2.1}$$

### 1.2.2 Non-transferable CDS with counterparty risks

It is possible that one of the two parties in the contract defaults before expiry. We use counterparty risk to denote the possible loss when the contract is in the money but the counterparty defaults. We call a contract *non-transferable* if the contract automatically ends when one of the two parties defaults. This problem has been studied; see [14, 6, 13, 25]. Here we follow their derivation.

Now we assume that there is counterparty risk and the CDS contract is non-transferable. As in Subsection 1.2.1, we still denote the expiry by $T$ and the time of the occurrence of the credit event by $\tau$. We use $\tau_1$ and $\tau_2$ to denote the default time of the seller and the buyer, respectively, of the CDS. The contract will terminate at the credit event occurrence time $\tau$, the seller's default time $\tau_1$, the buyer's default time $\tau_2$, or the expiry $T$, whichever happens the earliest. That is, the contract ends at $\tau \wedge \tau_1 \wedge \tau_2 \wedge T$. We assume that the buyer pays to the seller a continuous rate, $Kq$(\$/year), cash flow in the time period $[0, \tau \wedge \tau_1 \wedge \tau_2 \wedge T]$. As a compensate, if $\tau \le \tau_1 \wedge \tau_2 \wedge T$, then at time $\tau$, the seller pays the buyer $K$ (dollars). To describe the key ingredients of the non-transferable CDS with counterparty risk, we need to add the following amendment as the third item to the basic CDS contract.

**AMENDMENT**

**CDS WITH COUNTERPARTY RISKS**

Let $T$ be the expiry, $\tau$ be the time of occurrence of the well-defined credit event, $\tau_1$ the default time of the seller, and $\tau_2$ the default time of the buyer. The following amendment is to add to the basic CDS contract.

3. *Counterparty Risks:* If $\tau_1 < \tau_2 \wedge \tau \wedge T$, the contract terminates at time $\tau_1$ and the buyer and seller have no further rights or obligations; if $\tau_2 \leq \tau_1 \wedge \tau \wedge T$, the contract terminates at time $\tau_2$ and the buyer and seller have no further rights or obligations.

From buyer's point of view, the expected present value of the contract is

$$p_{20} := K\,\mathbb{E}\left[e^{-\int_0^\tau r_t dt}\mathbf{1}_{\{\tau<\tau_1\wedge\tau_2\wedge T\}} - \int_0^{\tau\wedge\tau_1\wedge\tau_2\wedge T} q e^{-\int_0^t r_\theta d\theta} dt \,\middle|\, \tau\wedge\tau_1\wedge\tau_2 > 0\right]. \qquad (1.2.2)$$

### 1.2.3 CDS transferable by seller at most one time

When one party, say the seller, defaults, the CDS contract may be in favor of the seller and therefore he would like to sell the contract to a third party. Hence, we use the word *transferable* to designate a CDS which allows a party to sell his contract to a third party at time of his default. The simplest type of transferable CDS is the one that can be sold by the original seller or buyer to a third party at time of his default, and the new contract becomes non-transferable.

Now we consider a CDS that is only transferable by the seller at most one time. The CDS transferable by the buyer at most one time can be discussed in a similar way. As in Subsection 1.2.2, we still denote the expiry by $T$, the time of the occurrence of the credit event by $\tau$, the default time of the seller by $\tau_1$ and the default time of the buyer by $\tau_2$. If the seller's default time is not the earliest, i.e. $\tau_1 \geq \tau_2 \wedge \tau \wedge T$, then the contract ends at the default time of the buyer $\tau_2$, the credit event occurrence time $\tau$ or the expiry $T$, whichever is earlier; but if the seller's default time is the earliest, i.e. $\tau_1 < \tau_2 \wedge \tau \wedge T$, then the contract ends at $\tau_1$, except that a new seller would like to buy the original seller's contract. We assume that the buyer pays to the seller a continuous rate, $Kq$(\$/year), cash flow from the initial time $t = 0$ to the end time of the contract. As a compensate, if the credit event happens before the end time of the contract, then at the credit event occurrence time $\tau$, the seller pays the buyer $K$ (dollars). To describe the key ingredients of the CDS Transferable by the seller at most one time, we need to add the following amendment as the third item to the basic CDS contract.

> **AMENDMENT**
> **CDS TRANSFERABLE BY SELLER**
> **AT MOST ONE TIME**
>
> Let $T$ be the expiry, $\tau$ be the time of occurrence of the well-defined credit event, $\tau_1$ the default time of the seller, and $\tau_2$ the default time of the buyer. The following amendment is to add to the basic CDS contract.
>
> 3. *Counterparty risks and transferring:* If $\tau_1 < \tau_2 \wedge \tau \wedge T$, at $\tau_1$ the contract terminate, except that a new seller would like to take over all the rights and obligations of the old seller; if $\tau_2 \leq \tau_1 \wedge \tau \wedge T$, the contract terminates at time $\tau_2$ and the buyer and seller have no further rights or obligations.

For simplicity, we assume that the new party never default.

To evaluate the CDS Transferable by the seller at most one time, we will do it in the following two steps:

**(1)** Let us firstly figure out the value of the CDS at time $t = \tau_1$ from the point of view of the new seller if such a new seller came into the contract. The transferring to a new seller of the CDS, though auction for example, only could happen when the original seller default before the buyer's default time, the credit event occurrence time and the expiry, whichever is the earliest. That is, the possible transferring of the CDS to the new seller only could happen when $\tau_1 < \tau_2 \wedge \tau \wedge T$.

Now we consider the case when $\tau_1 < \tau_2 \wedge \tau \wedge T$. At the time when $t = \tau_1$, if a new seller take over the old seller's position in the CDS contract, he will gain the rights to receive continuous premium $q$ from the buyer from current time $t = \tau_1$ till the time of $\tau \wedge \tau_2 \wedge T$, whose value at time $t = \tau_1$ is:

$$\int_t^{\tau \wedge \tau_2 \wedge T} Kqe^{-\int_t^s r_\theta d\theta} ds;$$

and at the same time, he will also take the obligation from the old seller that if the credit event happened before the expiry or the buyer's default time, i.e. if $\tau < \tau_2 \wedge T$, he should pay a lump sum of $K$ as the compensation to the buyer at time $\tau$, whose value at time $t = \tau_1$ is:

$$Ke^{-\int_t^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_2 \wedge T\}}.$$

Therefore, at time $t = \tau_1$, under the condition that $\tau_1 < \tau_2 \wedge \tau \wedge T$, the (expected) value of the CDS from the new seller's point of view is $Kw_{3\tau_1}$ where

$$w_{3t} := \mathbb{E}\left[ \int_t^{\tau \wedge \tau_2 \wedge T} e^{-\int_t^s r_\theta d\theta} q ds - e^{-\int_t^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_2 \wedge T\}} \middle| \tau_2 \wedge \tau > t \right]. \tag{1.2.3}$$

**(2)** At time $t = \tau_1$ when the old seller default, only under the condition that the contract is in favor of the party of seller, i.e. only when $Kw_{3\tau_1} > 0$, will there possibly be a new seller would like to take the seller's position.

Now let us see the value of the CDS contract at the initial time $t = 0$ from the buyer's point of view. Firstly, if the credit event happen before the expiry $T$ and also before the default time $\tau_1$ and $\tau_2$, that is, when $\tau < \tau_1 \wedge \tau_2 \wedge T$, the buyer will receive a lump sum of $K$ from the initial seller at time $\tau$. The present value of such an income at $t = 0$ is:

$$Ke^{-\int_0^\tau r_\theta d\theta}.$$

In return, the buyer should pay the initial seller the continuous premium $qK$, from $t = 0$ to the credit event occurrence time, the buyer's default time $\tau_1$, the seller's default time $\tau_2$ or the expiry $T$, whichever is the earliest, that is, from $t = 0$ to $t = \tau_1 \wedge \tau_2 \wedge \tau \wedge T$. The present value of such a payment at time $t = 0$ is:

$$Kq \int_0^{\tau \wedge \tau_1 \wedge \tau_2 \wedge T} e^{-\int_0^t r_\theta d\theta}.$$

Suppose $\tau_1 < \tau_2 \wedge \tau \wedge T$. If $Kw_{3\tau_1} > 0$, the CDS contract is favorable to the seller who can sell the contract to a new party to obtain a payment of present value

$$Ke^{-\int_0^{\tau_1} r_\theta d\theta} w_{3\tau_1}^+. \qquad (x^+ := \max\{0, x\}).$$

With the above discussion, the expected present value, from the buyer's point of view, of the CDS Transferable by the seller at most one time is

$$\begin{aligned} p_{30} :=& K\ \mathbb{E}\Bigg[ e^{-\int_0^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_1 \wedge \tau_2 \wedge T\}} - q \int_0^{\tau \wedge \tau_1 \wedge \tau_2 \wedge T} e^{-\int_0^t r_\theta d\theta}\, dt \\ & - \mathbf{1}_{\{\tau_1 < \tau \wedge \tau_2 \wedge T\}} e^{-\int_0^{\tau_1} r_\theta d\theta} w_{3\tau_1}^+ \Bigg| \tau \wedge \tau_1 \wedge \tau_2 > 0 \Bigg]. \end{aligned} \tag{1.2.4}$$

### 1.2.4 CDS transferable by seller and buyer at most one time

Now we consider a second type of transferable CDS, which is transferable by both the original seller and the original buyer and the new contract becomes non-transferable by the new seller and the new buyer. As in Subsection 1.2.2, we still denote the expiry by $T$, the time of the occurrence of the credit event by $\tau$, the default time of the seller by $\tau_1$ and the default time of the buyer by $\tau_2$. If neither seller's default time nor the buyer's default time is not the earliest, i.e. $\tau_1 \wedge \tau_2 \geq \tau \wedge T$, then the contract ends at the credit event occurrence time $\tau$ or the expiry $T$, whichever is earlier; if the seller's default time is the earliest, i.e. $\tau_1 < \tau_2 \wedge \tau \wedge T$, then the contract ends at $\tau_1$, except that a new seller would like to buy the original seller's contract; if the buyer's default time is the earliest, i.e. $\tau_2 < \tau_1 \wedge \tau \wedge T$, then the contract ends at $\tau_2$, except that a new seller would like to buy the original seller's contract. We assume that the buyer pays to the seller a continuous rate, $Kq$(\$/year), cash flow from the initial time $t = 0$ to the end time of the contract. As a compensate, if the credit event happens before the end time of the contract, then at the credit event occurrence time $\tau$, the seller pays the buyer $K$ (dollars). To describe the key ingredients of the CDS Transferable by seller and buyer at most one time, we need to add the following amendment as the third item to the basic CDS contract.

**AMENDMENT**

**CDS TRANSFERABLE BY SELLER AND BUYER AT MOST ONE TIME**

Let $T$ be the expiry, $\tau$ be the time of occurrence of the well-defined credit event, $\tau_1$ the default time of the seller, and $\tau_2$ the default time of the buyer. The following amendment is to add to the basic CDS contract.

3. *Counterparty risks and transferring:* If $\tau_1 < \tau_2 \wedge \tau \wedge T$, at $\tau_1$ the contract terminates, except that a new seller would like to take over all the rights and obligations of the old seller; if $\tau_2 \leq \tau_1 \wedge \tau \wedge T$, at $\tau_2$ the contract terminates, except that a new buyer would like to take over all the rights and obligations of the old buyer.

Here again, we assume that the new parties do not default.

To evaluate this CDS, the analysis is similar to that in Subsection 1.2.3. The following Figure 1 lists all possible scenarios.

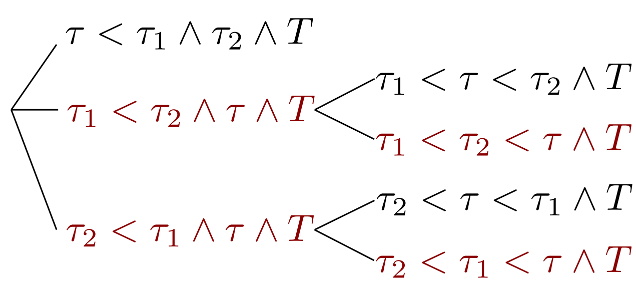


Figure 1: Possible scenarios

From Figure 1, we see that to evaluate the CDS, in the case when the seller default time arrives first ($\tau_1 < \tau_2 \wedge \tau \wedge T$), we need to evaluate first the value of CDS from the new buyer's point of view at $\tau_2$ with $\tau_1 < \tau_2 < \tau \wedge T$ (the buyer default after the seller but before the credit event and the expiry); in the case when the buyer default time arrives first ($\tau_2 < \tau_1 \wedge \tau \wedge T$), we need to evaluate first the value of the CDS from the new seller's point of view at $\tau_1$ with $\tau_2 < \tau_1 < \tau \wedge T$ (the seller default after the buyer but before the credit event and the expiry). So we will model the value of the CDS transferable by two parties backward from the terminal time in the following five steps.

**(1)** Suppose $\tau_1 < \tau_2 < \tau \wedge T$. At time $t = \tau_2$, the new seller has already taken over the position of the old seller at $\tau_1$, and the expected value of the CDS is $p_{1\tau_2}$ where

$$p_{1t} := \mathbb{E}\left[ Ke^{-\int_t^\tau r_\theta d\theta} \mathbf{1}_{\{\tau \leqslant T\}} - Kq \int_t^{\tau \wedge T} e^{-\int_t^s r_\theta d\theta} ds \middle| t < \tau \wedge T \right]. \tag{1.2.5}$$

Then at time $t = \tau_2$ with $\tau_1 < \tau_2 < \tau \wedge T$, there is a new buyer taking over the original defaulted buyer's position if and only if $p_{1\tau_2} > 0$.

**(2)** Assume that $\tau_1 < \tau_2 \wedge \tau \wedge T$. With the analysis in step (1), at $t = \tau_1$, the expected value of the CDS from the possible new seller's point of view is $w_{4\tau_1}$ where

$$\begin{aligned} w_{4t} := \mathbb{E}\Bigg[ & Kq \int_t^{\tau \wedge \tau_2 \wedge T} e^{-\int_t^s r_\theta d\theta} ds - Ke^{-\int_t^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_2 \wedge T\}} \\ & - \mathbf{1}_{\{\tau_2 < \tau \wedge T\}} e^{-\int_t^{\tau_2} r_\theta d\theta} p_{1\tau_2}^+ \Bigg| t < \tau \wedge \tau_2 \Bigg]. \end{aligned} \tag{1.2.6}$$

There is a new seller taking over the original defaulted buyer's position if and only if $w_{4\tau_1} > 0$.

**(3)** Suppose $\tau_2 < \tau_1 < \tau \wedge T$. At time $t = \tau_1$, the new buyer has already taken over the position of the old buyer at $\tau_2$, and the expected value of the CDS from the new seller's point of view at time $t = \tau_1$ is $-p_{1\tau_1}$. Then at time $t = \tau_1$, there is a new seller taking over the original defaulted seller's position if and only if $p_{1\tau_1} < 0$. We shall use the notation $x^- := (-x)^+ = \max\{0, -x\}$.

**(4)** Assume that $\tau_2 < \tau_1 \wedge \tau \wedge T$. With the analysis in step (3), at $t = \tau_2$, the expected value of the CDS from the possible new buyer's point of view is $v_{4\tau_2}$ where

$$\begin{aligned} v_{4t} :=\mathbb{E}\Bigg[ & Ke^{-\int_t^\tau r_\theta d\theta} \mathbf{1}_{\{\tau < \tau_1 \wedge T\}} - Kq \int_t^{\tau \wedge \tau_1 \wedge T} e^{-\int_t^s r_\theta d\theta} ds \\ & - \mathbf{1}_{\{\tau_1 < \tau \wedge T\}} e^{-\int_t^{\tau_1} r_\theta d\theta} p_{1\tau_1}^- \Bigg| \tau \wedge \tau_1 > t \Bigg]. \end{aligned} \tag{1.2.7}$$

Then at time $t = \tau_2$, there is a new buyer taking over the original defaulted buyer's position if and only if $v_{4\tau_2} > 0$.

**(5)** Now, with the analysis in step (2) and (4), at $t = 0$, from the buyer's point of view, the expected present value of the CDS is

$$p_{40} := \mathbb{E}\bigg[Ke^{-\int_0^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_1\wedge\tau_2\wedge T\}} - Kq\int_0^{\tau\wedge\tau_1\wedge\tau_2\wedge T} e^{-\int_0^s r_\theta d\theta} ds \qquad (1.2.8)$$

$$+ \mathbf{1}_{\{\tau_2<\tau\wedge\tau_1\wedge T\}} v^+_{4\tau_2} - \mathbf{1}_{\{\tau_1<\tau\wedge\tau_2\wedge T\}} w^+_{4\tau_1} \bigg| \tau\wedge\tau_1\wedge\tau_2 > 0\bigg].$$

### 1.2.5 CDS transferable by seller and buyer any number of times

Now we consider a third type of transferable CDS, which is transferable by both the sellers and the buyers any number of times. This type of CDS can be sold by the original seller or buyer to a third party at the time of his default, and the new contract is still transferable any number of times. As in Subsection 1.2.2, we still denote the expiry by $T$, the time of the occurrence of the credit event by $\tau$, the default time of the seller by $\tau_1$ and the default time of the buyer by $\tau_2$. The contract will end at the time when a buyer or seller default but no third party would like to buy his contract, or the expiry, whichever is earlier. We assume that the buyer pays to the seller a continuous rate, $Kq$($/year), cash flow from the initial time $t = 0$ to the end time of the contract. As a compensate, if the credit event happens before the end time of the contract, then at the credit event occurrence time $\tau$, the seller pays the buyer $K$ (dollars). For simplicity, we assume that buyers have the same credit rating and so are all sellers. To describe the key ingredients of the CDS Transferable by seller and buyer any number of times, we need to add the following amendment as the third item to the basic CDS contract.

*AMENDMENT*

*CDS TRANSFERABLE ANY NUMBER OF TIMES*

Let $T$ be the expiry, $\tau$ be the time of occurrence of the well-defined credit event, $\tau_1$ the default time of the seller, and $\tau_2$ the default time of the buyer. The following amendment is to add to the basic CDS contract.

3. *Counterparty risks and transferring:* The seller's and the buyer's rights and obligations can be repeatedly transferred when sellers and buyers default. Only a fraction $R_1 \in [0,1]$ of the expected value is received when sellers default and a fraction of $R_2 \in [0,1]$ of the expected value is received when buyers default.

Let $v_t$ denote the expected value of the CDS from the buyer's point view at time $t$, let $w_t$ denote the expected value of the CDS from the seller's point view at time $t$. Then with similar discussion as in Subsection 1.2.4,

$$v_t = \mathbb{E}\Big[Ke^{-\int_t^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_1\wedge\tau_2\wedge T\}} - Kq\int_t^{\tau\wedge\tau_1\wedge\tau_2\wedge T} e^{-\int_t^s r_\theta d\theta}ds + \mathbf{1}_{\{\tau_2<\tau\wedge\tau_1\wedge T\}}R_2 v_{\tau_2}^+ - \mathbf{1}_{\{\tau_1<\tau\wedge\tau_2\wedge T\}}w_{\tau_1}^+ \Big| \tau\wedge\tau_1\wedge\tau_2 > t\Big], \tag{1.2.9}$$

$$w_t = \mathbb{E}\Big[-Ke^{-\int_t^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_1\wedge\tau_2\wedge T\}} + Kq\int_t^{\tau\wedge\tau_1\wedge\tau_2\wedge T} e^{-\int_t^s r_\theta d\theta}ds - \mathbf{1}_{\{\tau_2<\tau\wedge\tau_1\wedge T\}} v_{\tau_2}^+ + \mathbf{1}_{\{\tau_1<\tau\wedge\tau_2\wedge T\}}R_1 w_{\tau_1}^+ \Big| \tau\wedge\tau_1\wedge\tau_2 > t\Big]. \tag{1.2.10}$$

## 1.3 MATHEMATICAL FORMULATIONS

In this section, we further specify credit event and default times to derive mathematical models.

### 1.3.1 Intensity model for credit events and default times

We first use the intensity model for credit events and default times, as in [14]. The discussion for the structure model will be in Subsection 1.5.5.

Let $\{\mathscr{F}_t\}_{t\geq 0}$ be the filtration generated by the interest rate process $\{r_t\}_{t\geq 0}$ following the modified CIR model (1.3.4). Let the credit event occurrence time, the seller's default time and the buyer's default time be modeled by the $\mathscr{G}_t$-stopping times $\tau$, $\tau_1$ and $\tau_2$, respectively, where the filtration $\{\mathscr{G}_t\}_{t\geq 0}$ is the information flow of the market and

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\{\tau < t\}) \vee \sigma(\{\tau_1 < t\}) \vee \sigma(\{\tau_2 < t\}).$$

We assume that the filtration $\{\mathscr{G}_t\}_{t\geq 0}$ is augmented so that it satisfies the usual conditions. Throughout this dissertation, $\mathbb{P}$ is the probability and $\mathbb{E}$ is the corresponding expectation under $\mathbb{P}$.

As in [14] and [1], we model the credit event and counterparty default times $\tau$, $\tau_1$ and $\tau_2$ by the arrival times of Poisson processes with variable intensities $\{\lambda_t\}$, $\{\lambda_{1t}\}$, $\{\lambda_{2t}\}$, respectively. Hence,

$$\begin{aligned} \mathbb{P}\{\tau > t | \mathscr{F}_t, \tau > s\} &= e^{-\int_s^t \lambda_\theta d\theta},\ 0 \leq s < t, \\ \mathbb{P}\{\tau_1 > t | \mathscr{F}_t, \tau_1 > s\} &= e^{-\int_s^t \lambda_{1\theta} d\theta},\ 0 \leq s < t, \\ \mathbb{P}\{\tau_2 > t | \mathscr{F}_t, \tau_2 > s\} &= e^{-\int_s^t \lambda_{2\theta} d\theta},\ 0 \leq s < t. \end{aligned} \tag{1.3.1}$$

Moreover, we assume that $\tau$, $\tau_1$ and $\tau_2$ are conditional independent in the sense that

$$\mathbb{P}\{\tau \wedge \tau_1 \wedge \tau_2 > t | \mathscr{F}_t, \tau \wedge \tau_1 \wedge \tau_2 > s\} = e^{-\int_s^t \lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta} d\theta},\ 0 \leq s < t. \tag{1.3.2}$$

By (1.3.2), we can have

$$\mathbb{P}\{\tau \in [s, s+ds), \tau_1 \wedge \tau_2 > t | \mathscr{F}_t, \tau \wedge \tau_1 \wedge \tau_2 > s\} = \lambda_t e^{-\int_s^t \lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta} d\theta} dt,\ 0 \leq s < t.$$

### 1.3.2 The state variable

For simplicity, we assume that there is only one state variable, taking as $\{r_t\}$. For this we assume that there are given functions $\Lambda(\cdot), \Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ such that the intensities are given by

$$\lambda_t = \Lambda(r_t),\ \ \lambda_{1t} = \Lambda_1(r_t),\ \ \lambda_{2t} = \Lambda_2(r_t),$$

for some non-negative functions $\Lambda(\cdot)$, $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$. A typical example of the structure functions $\Lambda, \Lambda_1, \Lambda_2$ is

$$\Lambda(r) = a + br, \quad \Lambda_1(r) = c + dr, \quad \Lambda_2(r) = e + fr$$

where $a, b, c, d, e, f$ are constants that can be obtained by empirical calibration.

To complete the model, it remains to describe the process $\{r_t\}$.

### 1.3.3 The CIR model

A popular model to describe the interest rate is the Cox-Ingersoll-Ross (CIR) model (reference: [22])

$$dr_t = (\kappa - \beta r_t)dt + \sigma\sqrt{r_t}dW_t,$$

where $\{W_t\}$ is the standard Brownian motion, $\kappa, \beta, \sigma$ are positive constants. To get rid of the possibility that $r_t < 0$, usually the following condition is imposed to the CIR model:

$$2\kappa \geq \sigma^2. \tag{1.3.3}$$

But as shown by the calibration in [13], in reality, there exist cases when (1.3.3) is not satisfied. So in [13] and [25] they drop this restriction by studying the following modified CIR model

$$dr_t = (\kappa - \beta r_t)dt + \sigma\sqrt{\max\{r_t, 0\}}dW_t, \tag{1.3.4}$$

which do not need the restriction (1.3.3). We will use this version of CIR model in this dissertation. By [22], [15] and [16], we know that without taking the positive part of the interest rate in the $dW_t$ term of the CIR model, the process cannot continue when $r_t = 0$. The modified CIR mode (1.3.4) fix this problem by making a reflection when the interest rate touches zero.

Moreover, as shown in [13] and [25], when using this modified CIR model with the constraint (1.3.3) dropped, adding the condition

$$\frac{\partial u}{\partial r} \in L^\infty$$

to the PDE problem for CDS evaluation problem is a good practice and can guarantee the well-posedness of the solution. We will follow this setup in this dissertation.

Using the above models for the credit events, default times and the interest rate, we derive the resulting mathematical problems in the following subsections.

### 1.3.4 CDS without counterparty risks

Using the models for the credit events, default times and the interest rate in Subsections 1.3.1, 1.3.2, and 1.3.3, we can express $p_1$ in (1.2.1) as

$$p_{10} = K\,\mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta+\lambda_\theta)d\theta}\,(\lambda_s - q)\,ds\middle|\mathscr{F}_0\right]. \tag{1.3.5}$$

The derivation is given in the Appendix A.1. Thus, $p_{10} = Ku_1(r_0, T)$ where $T$ is time to expiry and

$$u_1(r,T) = \mathbb{E}\left[\int_0^T e^{-\int_0^s [r_\theta+\Lambda(r_\theta)]d\theta}\,(\Lambda(r_s) - q)\,\middle|r_0 = r\right].$$

By the Feynman–Kac formula, $u_1(\cdot,\cdot)$ is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) u_1 = \Lambda - q & \text{in } (0,\infty)\times(0,\infty), \\ u_1(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial u_1}{\partial r} \in L^\infty((0,\infty)^2). \end{cases} \tag{1.3.6}$$

Here $\mathcal{L}$ is the operator defined by

$$\mathcal{L} = -\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r, \tag{1.3.7}$$

where $\kappa$, $\beta$, $\sigma$ are positive constants as in the CIR model.

**Remark 1.** *Problem (1.3.6) is not a standard initial value problem of a parabolic partial differential equation. Here there are two fundamental difficulties:*

1. *The domain $(0, \infty)$ is unbounded.*
2. *The coefficients of $\mathcal{L} + \Lambda$ are not bounded as $r \to \infty$.*
3. *The elliptic operator $\mathcal{L}$ is degenerate at $r = 0$. Thus one would like to know if it is suitable to impose a boundary condition at $r = 0$.*

*The key contribution of the dissertation is that we show that the boundary condition at $r = \infty$ and $r = 0$ can be replaced by the spatial Lipschitz continuity of the solution.*

### 1.3.5 Non-transferable CDS with counterparty risks

Using the models for the credit events, default times and the interest rate in Subsection 1.3.1, 1.3.2, 1.3.3, one can derive (using an approach similar to the derivation for (1.3.5)) that $p_{20}$ in (1.2.2) can be written as

$$p_{20} = K\mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta + \lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta})d\theta} (\lambda_s - q) \middle| \mathscr{F}_0\right].$$

Thus, $p_{20} = Ku_2(r_0, T)$ where

$$u_2(r, T) := \mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta + \Lambda(r_\theta) + \Lambda_1(r_\theta) + \Lambda_2(r_\theta)d\theta} (\Lambda(r_s) - q) \middle| r_0 = r\right].$$

By the Feynman–Kac formula, $u_2 = u_2(r, T)$ where $u_2(r, T)$ is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u_2 = \Lambda - q & \text{in } (0, \infty) \times (0, \infty), \\ u_2(\cdot, 0) = 0 & \text{in } (0, \infty), \\ \frac{\partial u_2}{\partial r} \in L^\infty((0, \infty)^2). \end{cases} \tag{1.3.8}$$

Here $\mathcal{L}$ is the operator defined by (1.3.7), with $\kappa$, $\beta$, $\sigma$ being positive constants as in the CIR model. This problem has been studied in [14, 6, 13, 25].

### 1.3.6 CDS transferable by seller at most one time

Using the models for the credit events, default times and the interest rate in Subsection 1.3.1, 1.3.2, 1.3.3, we can derive (presented in Appendix A.1) that $w_{3t}$ in (1.2.3) is given by

$$w_{3t} = \mathbb{E}\left[\int_t^T (\lambda_s - q) e^{-\int_t^s (r_\theta + \lambda_\theta + \lambda_{2\theta}) d\theta} ds \middle| \mathscr{F}_t \right]. \tag{1.3.9}$$

By the Feynman–Kac formula (reference, for example, [22]), $w_{3t} = w_3(r_t, T - t)$, where $w_3(\cdot, \cdot)$ is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_2\right) w_3 = \Lambda - q & \text{in } (0,\infty) \times (0,\infty), \\ w_3(\cdot, 0) = 0 & \text{in } (0,\infty). \end{cases} \tag{1.3.10}$$

Here $\mathcal{L}$ is the operator defined by (1.3.7), with $\kappa$, $\beta$, $\sigma$ being positive constants as in the CIR model.

Next we can derive (presented in Appendix A.1) that

$$p_{30} = K\, \mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta + \lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta}) d\theta} \left(\lambda_s - q - \lambda_{1s} w_3^+(r_s, T-s)\right) ds \middle| \mathscr{F}_0 \right]. \tag{1.3.11}$$

Hence, $p_{30} = K u_3(r_0, T)$ where

$$u_3(r, T) = \mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta + \lambda_\theta + \lambda_{1\theta} + \lambda_{2\theta}) d\theta} \left(\lambda_s - q - \lambda_{1s} w_3^+(r_s, T-s)\right) ds \middle| r_0 = r \right].$$

By the Feynman–Kac formula (reference, for example, [22]), $u_3(\cdot, \cdot)$ is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u_3 = \Lambda - q - \Lambda_1 w_3^+ & \text{in } (0,\infty) \times (0,\infty), \\ u_3(\cdot, 0) = 0 & \text{in } (0,\infty). \end{cases} \tag{1.3.12}$$

As discussed in Section 1.3.3, to accommodate the scenario $2\kappa < \sigma^2$ for the CIR model, we need the following extra conditions

$$\frac{\partial}{\partial r} u_3 \in L^\infty((0,\infty) \times (0,\infty)), \; \frac{\partial}{\partial r} w_3 \in L^\infty((0,\infty) \times (0,\infty)). \tag{1.3.13}$$

By (1.3.10), (1.3.12) and (1.3.13), for $u = u_3$ and $v = w_3$, $[u, v]$ is the solution of the following partial differential equations system:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_2\right) v = \Lambda - q & \text{in } (0,\infty)\times(0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u = \Lambda - q - \Lambda_1 v^+ & \text{in } (0,\infty)\times(0,\infty), \\ u(\cdot,0) = 0,\ v(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} u \in L^\infty((0,\infty)\times(0,\infty)), \\ \frac{\partial}{\partial r} v \in L^\infty((0,\infty)\times(0,\infty)), \end{cases} \tag{1.3.14}$$

where $\kappa$, $\beta$, $\sigma$, $K$ and $q$ are all positive constants, $\Lambda$, $\Lambda_1$ and $\Lambda_2$ are given non-negative functions of $r$, $\mathcal{L}$ is the operator defined as in (1.3.7).

### 1.3.7 CDS transferable by seller and buyer at most one time

**(1)** By (1.2.5), with similar derivations as in Appendix A.1, we have

$$p_{1t} = K\ \mathbb{E}\left[\int_t^T e^{-\int_t^s [r_\theta + \lambda_\theta] d\theta}\big(\lambda_s - q\big) ds \middle| \mathscr{F}_t\right].$$

By Feynman–Kac formula, $p_{1t} = K u_1(r_t, T-t)$, where $u(\cdot,\cdot)$ is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) u_1 = \Lambda - q & \text{in } (0,\infty)^2, \\ u_1(\cdot,0) = 0 & \text{in } (0,\infty). \end{cases} \tag{1.3.15}$$

**(2)** By (1.2.6), with similar derivations as in Appendix A.1, we have

$$w_{4t} = K\ \mathbb{E}\left[\int_t^T e^{-\int_t^s [r_\theta + \lambda_\theta + \lambda_{2\theta}] d\theta}\Big(q - \lambda_s - \lambda_{2s} u_1^+(r_s, T-s)\Big) ds \middle| \mathscr{F}_t\right].$$

This implies that $w_{4t} = K w_4(r_t, T-t)$ where

$$w_4(r,T) = \mathbb{E}\left[\int_t^T e^{-\int_t^s [r_\theta + \lambda_\theta + \lambda_{2\theta}] d\theta}\Big(q - \lambda_s - \lambda_{2s} u_1^+(r_s, T-s)\Big) ds \middle| r_0 = r\right].$$

By Feynman–Kac formula, $w_4(\cdot,\cdot)$ is the solution of

$$\begin{cases} \big(\mathcal{L} + \Lambda + \Lambda_2\big) w_4 = q - \Lambda - \Lambda_2 u_1^+ & \text{in } (0,\infty)^2, \\ w_4(\cdot,0) = 0 & \text{in } (0,\infty). \end{cases} \tag{1.3.16}$$

**(4)** By (1.2.7), with similar derivations as in Appendix A.1, we have

$$v_{4t} = K\mathbb{E}\left[\int_t^T e^{-\int_t^s [r_\theta+\lambda_\theta+\lambda_{1\theta}]d\theta}\Big(\lambda_s - q - \lambda_{1s}u_1^-(r_s, T-s)\Big)ds\Big|\mathscr{F}_t\right].$$

By the Feynman–Kac formula, $v_{4t} = Kv_4(r_t, T-t)$, where $v_4(\cdot,\cdot)$ is the solution of

$$\begin{cases} \big(\mathcal{L}+\Lambda+\Lambda_1\big)v_4 = \Lambda - q - \Lambda_1 u_1^- & \text{in } (0,\infty)^2, \\ v_4(\cdot, 0) = 0 & \text{in } (0,\infty). \end{cases} \tag{1.3.17}$$

**(5)** By (1.2.8), with similar derivations as in Appendix A.1, we have

$$\begin{aligned} p_{40} = K\,\mathbb{E}\Bigg[\int_0^T & e^{-\int_0^s [r_\theta+\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta}]d\theta}\Big(\lambda_s - q \\ & + \lambda_{2s}v_4^+(r_s, T-s) - \lambda_{1s}w_4^+(r_s, T-s)\Big)ds\Big|\mathscr{F}_0\Bigg]. \end{aligned}$$

By Feynman–Kac formula, $p_{40} = Ku_4(r_0, T)$, where $u_4(\cdot,\cdot)$ is the solution of

$$\begin{cases} (\mathcal{L}+\Lambda+\Lambda_1+\Lambda_2)\,u_4 = \Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+ & \text{in } (0,\infty)^2, \\ u_4(\cdot, 0) = 0 & \text{in } (0,\infty). \end{cases} \tag{1.3.18}$$

As discussed in Section 1.3.3, to accommodate the scenario $2\kappa < \sigma^2$ for the CIR model, we need the following extra conditions

$$\frac{\partial}{\partial r}u_1,\ \frac{\partial}{\partial r}w_4,\ \frac{\partial}{\partial r}v_4,\ \frac{\partial}{\partial r}u_4 \in L^\infty((0,\infty)\times(0,\infty)). \tag{1.3.19}$$

In summary, in view of (1.3.15), (1.3.16), (1.3.17), (1.3.18) and (1.3.19), $[u_1, w_4, v_4, u_4]$ is the solution of the following partial differential equations system:

$$\begin{cases} \big(\frac{\partial}{\partial T}+\mathcal{L}+\Lambda+\Lambda_1+\Lambda_2\big)\,u_4 = \Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+ & \text{in } (0,\infty)^2, \\ \big(\frac{\partial}{\partial T}+\mathcal{L}+\Lambda+\Lambda_1\big)\,v_4 = \Lambda - q - \Lambda_1\,u_1^- & \text{in } (0,\infty)^2, \\ \big(\frac{\partial}{\partial T}+\mathcal{L}+\Lambda+\Lambda_2\big)\,w_4 = q - \Lambda - \Lambda_2 u_1^+ & \text{in } (0,\infty)^2, \\ \big(\frac{\partial}{\partial T}+\mathcal{L}+\Lambda\big)\,u_1 = \Lambda - q & \text{in } (0,\infty)^2, \\ u_1(\cdot,0) = w_4(\cdot,0) = v_4(\cdot,0) = u_4(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial u_1}{\partial r}, \frac{\partial w_4}{\partial r}, \frac{\partial v_4}{\partial r}, \frac{\partial u_4}{\partial r} \in L^\infty((0,\infty)\times(0,\infty)), \end{cases} \tag{1.3.20}$$

where $\kappa$, $\beta$, $\sigma$, $K$ and $q$ are all positive constants, $\Lambda$, $\Lambda_1$ and $\Lambda_2$ are given non-negative functions of $r$, $\mathcal{L}$ is the operator defined as in (1.3.7).

### 1.3.8 CDS transferable by seller and buyer any number of times

With similar calculation as in Section 1.3.6, by (1.2.9) and (1.2.10),

$$v_t = \mathbb{E}\bigg[\int_t^T e^{-\int_0^s [r_\theta+\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta}]d\theta}\Big(K\lambda_s - Kq + \lambda_{2s}R_2 v_s^+ - \lambda_{1s}w_s^+\Big)ds\bigg| r_t = r, \tau\wedge\tau_1\wedge\tau_2 > t\bigg]$$
$$w_t = \mathbb{E}\bigg[\int_t^T e^{-\int_0^s [r_\theta+\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta}]d\theta}\Big(Kq - K\lambda_s - \lambda_{2s}v_s^+ + \lambda_{1s}R_1 w_s^+\Big)ds\bigg| r_t = r, \tau\wedge\tau_1\wedge\tau_2 > t\bigg]$$

Then, by Feynman–Kac formula, $v_t = Kv(r_t, T-t)$ and $w_t = Kw(r_t, T-t)$ where $[v, w]$ solves the system

$$\begin{cases} \big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\big)v = \Lambda - q + R_2\Lambda_2 v^+ - \Lambda_1 w^+ & \text{in } (0,\infty)^2, \\ \big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\big)w = q - \Lambda - \Lambda_2 v^+ + R_1\Lambda_1 w^+ & \text{in } (0,\infty)^2, \\ v(\cdot,0) = w(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial v}{\partial r},\ \frac{\partial w}{\partial r} \in L^\infty((0,\infty)^2). \end{cases} \tag{1.3.21}$$

**Remark 2.** *In the special case $R_1 = R_2 = 1$, we obtain $v = -w$. Using $v = v^+ - v^-$ we then obtain*

$$\begin{cases} \big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\big)v + \Lambda_1 v^+ - \Lambda_2 v^- = \Lambda - q & \text{in } (0,\infty)^2, \\ v(\cdot,0) = 0 & \text{in } (0,\infty). \end{cases}$$
$$\begin{cases} \big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\big)w + \Lambda_2 w^+ - \Lambda_1 w^- = q - \Lambda & \text{in } (0,\infty)^2, \\ w(\cdot,0) = 0 & \text{in } (0,\infty). \end{cases}$$

*In this case $v$ and $w$ can be solved separately. But in general they depend on each other.*

## 1.4 THE PRICE OF CDS

Let's denote by $u(r,T,q)$ the expected value of the CDS. We search for $q^*$ such that

$$u(r,T,q^*) = 0. \tag{1.4.1}$$

We call $q^*$ the price of the CDS. Typically we multiply $q^*$ by $10^4$ to obtain base points (%%). Denote the solution of $q$ in (1.4.1) by $q^* = Q(r,T)$. Then at any fixed time $t \in [0,T]$ subject to observes state variable value $r_t$, the price of the CDS is $Q(r_t, T-t)$.

It is interesting to notice that the price of CDS are different from different point of view for the Fourth type of CDS.

## 1.5 MAIN RESULTS

Here we state main results on mathematical analysis of the problems derived in the previous section.

### 1.5.1 A prototype mathematical problem

To study Problem (1.3.14), we begin with a more general PDE problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u = F & \text{in } (0,\infty)\times(0,\infty), \\ u(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial u}{\partial r} \in L^\infty((0,\infty)\times(0,\infty)), \end{cases} \tag{1.5.1}$$

where $u$ is the unknown function and $c$ and $F$ are given known functions. Then Problem (1.3.6) is Problem (1.5.1) with $c = \Lambda$ and $F = \Lambda - q$; Problem (1.3.8) is Problem (1.5.1) with $c = \Lambda + \Lambda_1 + \Lambda_2$ and $F = \Lambda - q$; Problem (1.3.14) is the joint of Problem (1.5.1) with $c = \Lambda + \Lambda_2$, $F = \Lambda - q$, and that with $c = \Lambda + \Lambda_1 + \Lambda_2$, $F = \Lambda - q - \Lambda_1 v^+$; Problem (1.3.20) is the joint of Problem (1.5.1) with $c = \Lambda$, $F = \Lambda - q$, that with $c = \Lambda + \Lambda_2$, $F = q - \Lambda - \Lambda_2 u_1^+$, that with $c = \Lambda + \Lambda_1$, $F = \Lambda - q - \Lambda_1 u_1^-$, and that with $c = \Lambda + \Lambda_1 + \Lambda_2$,

$\Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+$; Problem (1.3.21) is the system formed by Problem (1.5.1) with $c = \Lambda + \Lambda_1 + \Lambda_2$, $F = \Lambda - q + R_2\Lambda_2 v^+ - \Lambda_1 w^+$ and that with $c = \Lambda + \Lambda_1 + \Lambda_2$, $F = q - \Lambda - \Lambda_2 v^+ + R_1\Lambda_1 w^+$.

We will use the following notation throughout this dissertation: For any function $f(r)$ on $(0,\infty)$,

$$|||f||| := \sup_{r>0} \left| \frac{f(r)}{1+r} \right|.$$

For any function $F(r,T)$ on $(0,\infty)\times(0,\infty)$,

$$|||F||| := \sup_{r>0,T>0} \left| \frac{F(r,T)}{1+r} \right|.$$

**Theorem 1** (The Protetype Mathematical Problem)**.** *Let $\mathcal{L}$ be defined as in (1.3.7) where $\kappa$, $\beta$, and $\sigma$ are positive constants. Let $c(r)$ be a function of $r \in (0,\infty)$ and $F(r,T)$ be a function of $(r,T) \in (0,\infty)\times(0,\infty)$. Assume that*

$$|||F||| < \infty, \quad \left|\left|\left| \frac{\partial F}{\partial T} \right|\right|\right| < \infty, \quad c \geq 0 \text{ and } c \in L^\infty([0,\ell]) \ \forall\, \ell > 0. \tag{1.5.2}$$

*Then Problem (1.5.1) admits a unique solution $u$. Moreover, $u$ satisfies*

$$\|u\|_{L^\infty((0,\infty)\times(0,\infty))} \leq C_{\kappa,\beta,\sigma} |||F|||, \tag{1.5.3}$$

$$\left\| \frac{\partial u}{\partial r} \right\|_{L^\infty((0,\infty)\times(0,\infty))} \leq C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}} \left( |||F||| + \left|\left|\left| \frac{\partial F}{\partial T} \right|\right|\right| \right), \tag{1.5.4}$$

$$\left|\left|\left| \frac{\partial u}{\partial T} \right|\right|\right| \leq C_{\kappa,\beta,\sigma} \left( |||F||| + \left|\left|\left| \frac{\partial F}{\partial T} \right|\right|\right| \right), \tag{1.5.5}$$

$$\left\| \frac{\partial u}{\partial T} \right\|_{L^\infty((0,\infty)\times(0,\infty))} \leq C_{\kappa,\beta,\sigma} \left( \|F(\cdot,0)\|_{L^\infty((0,\infty))} + \left|\left|\left| \frac{\partial F}{\partial T} \right|\right|\right| \right), \tag{1.5.6}$$

*where $C_{\kappa,\beta,\sigma}$ is a positive constant depending only on $\kappa$, $\beta$, $\sigma$, and $C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}}$ is a positive constant depending only on $\kappa$, $\beta$, $\sigma$, $\ell$, and $\|c\|_{L^\infty([0,\ell])}$.*

This dissertation studies the well-posedness the solutions for Problem (1.3.14), Problem (1.3.20) and Problem (1.3.21) under certain conditions. We also study the relation between the intensity model and the structure model for the CDS transferable by the seller.

**Remark 3.** *This theorem 1 not only is important theoretically for the proof of the following three theorems, but also has practical meaning. Applying this theorem to Problem 1.3.14, 1.3.20 and 1.3.21, the boundedness of the intensity functions will give us the boundedness of the solutions and their derivatives w.r.t. the interest rate and the expiry. So as long as the frequency of credit event and counterparty defaults have some boundedness property, which is the usual case, the value of the CDS is bounded and changes w.r.t. the interest rate and expiry smoothly. In other words, if the interest rate and the expiry do not have a big jump, the value of the CDS won't change dramatically. But if the interest rate changes dramatically in a short period, the corresponding CDS value will also change with big jumps.*

### 1.5.2 CDS transferable by seller at most one time

**Theorem 2** (CDS transferable by seller at most one time)**.** *Let $\kappa$, $\beta$, $\sigma$ and $q$ be positive constants and $\mathcal{L}$ be as in (1.3.7). Let $\Lambda(\cdot)$, $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ be given non-negative functions on $(0,\infty)$. Assume that*

$$|||\Lambda||| < \infty, \quad |||\Lambda_1||| < \infty, \quad \Lambda, \Lambda_1, \Lambda_2 \in L^\infty([0,\ell]) \; \forall \, \ell > 0, \tag{1.5.7}$$

*and that*

$$\|\Lambda\|_{L^\infty((0,\infty))} < \infty \text{ or } \|\Lambda_1\|_{L^\infty((0,\infty))} < \infty. \tag{1.5.8}$$

*Then there is a unique solution to Problem (1.3.14).*

### 1.5.3 CDS transferable by seller and buyer at most one time

**Theorem 3** (CDS transferable by seller and buyer at most one time)**.** *Let $\kappa$, $\beta$, $\sigma$ and $q$ be positive constants and $\mathcal{L}$ be the operator defined as (1.3.7). Let $\Lambda$, $\Lambda_1$ and $\Lambda_2$ be non-negative functions of $r$ s.t.*

$$\|\Lambda\|_{L^\infty((0,\infty))} < \infty, \|\Lambda_1\|_{L^\infty((0,\infty))} < \infty, \|\Lambda_2\|_{L^\infty((0,\infty))} < \infty. \tag{1.5.9}$$

*Then there is a unique solution to the following Problem (i.e. (1.3.20))*

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u_4 = \Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+ & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1\right) v_4 = \Lambda - q - \Lambda_1 u_1^- & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_2\right) w_4 = q - \Lambda - \Lambda_2 u_1^+ & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) u_1 = \Lambda - q & \text{in } (0,\infty)^2, \\ u_1(\cdot,0) = w_4(\cdot,0) = v_4(\cdot,0) = u_4(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial u_1}{\partial r}, \frac{\partial w_4}{\partial r}, \frac{\partial v_4}{\partial r}, \frac{\partial u_4}{\partial r} \in L^\infty((0,\infty)\times(0,\infty)). \end{cases}$$

### 1.5.4 CDS transferable by seller and buyer any number of times

**Theorem 4** (CDS transferable by seller and buyer any number of times)**.** *Let $\kappa$, $\beta$, $\sigma$ and $q$ be positive constants, and $\mathcal{L}$ be the operator defined as (1.3.7). Let $\Lambda$, $\Lambda_1$ and $\Lambda_2$ be given non-negative functions satisfying*

$$|||\Lambda||| < \infty, \quad \Lambda \in L^\infty([0,\ell]) \; \forall \, \ell > 0, \tag{1.5.10}$$

$$\|\Lambda_1\|_{L^\infty((0,\infty))} < \infty, \qquad \|\Lambda_2\|_{L^\infty((0,\infty))} < \infty. \tag{1.5.11}$$

*Then there is a unique solution to the following Problem (i.e. (1.3.21))*

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) v = \Lambda - q + R_2\Lambda_2 v^+ - \Lambda_1 w^+ & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) w = q - \Lambda - \Lambda_2 v^+ + R_1\Lambda_1 w^+ & \text{in } (0,\infty)^2, \\ v(\cdot,0) = w(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial v}{\partial r}, \; \frac{\partial w}{\partial r} \in L^\infty((0,\infty)^2). \end{cases}$$

With the above theorems, we are guaranteed the existence and uniqueness of solutions for problems (1.3.14), (1.3.20) and (1.3.21). Therefore the numerical computation of the CDS values based on PDE problems (1.3.14), (1.3.20) and (1.3.21) will have theoretical support.

### 1.5.5 Relation between structure model and intensity model

Until now, we are using the intensity model to describe the seller's default time $\tau_1$ and the buyer's default time $\tau_2$, that is, we model them by (1.3.1) and (1.3.2) using the intensity functions $\Lambda_1$ and $\Lambda_2$ of the interest rate. There is another kind of model to describe the default times, which is called the structure model. With the structure model, instead of the intensity, a level of the interest rate is set so that the seller or the buyer will default if the interest rate increase to that level.

We will discuss the case for $\tau_2$ in this subsection. The discussion for $\tau_1$ can be done similarly. With the structure model,

$$\tau_2^* = \min\left\{t \geq 0 | r_t > R\right\}, \tag{1.5.12}$$

and the corresponding PDE problem for the CDS transferable by seller at most one time is

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) v^* = \Lambda - q & \text{in } (0,R)\times(0,\infty), \\ v^* = 0 & \text{in } [R,\infty)\times(0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1\right) u^* = \Lambda - q - \Lambda_1 v^{*+} & \text{in } (0,R)\times(0,\infty), \\ u^* = 0 & \text{in } [R,\infty)\times(0,\infty), \\ v^*(\cdot,0) = 0,\ u^*(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} v^* \in L^\infty((0,\infty)^2),\ \frac{\partial}{\partial r} u^* \in L^\infty((0,\infty)^2), & \end{cases} \tag{1.5.13}$$

One can use the stopping time under intensity model to approximate that under the structure model. In fact, if

$$\Lambda_2(r) = \gamma H(r - R), \tag{1.5.14}$$

where $H(x)$ is the Heaviside function which equals 1 if $x > 0$ and equals 0 otherwise. Then as $\gamma \to \infty$, the default intensity approaches to $\infty$ when $r_t$ reaches the level $R$, and the time corresponds to $\tau_2^*$ defined in (1.5.12). That is

$$\lim_{\gamma\to\infty} \tau_2^\gamma = \tau_2^*,$$

where $\tau_2^\gamma$ is the default time modeled by intensity model with intensity (1.5.14). Similarly, we want to see that, as $\gamma \to \infty$, whether the solution of the Problem (1.5.13) (the structure model) could be approximated by the solutions of the following Problem (1.5.15) (the intensity model)

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \gamma H(r-R)\right) v^\gamma = \Lambda - q & \text{in } (0,\infty)\times(0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \gamma H(r-R)\right) u^\gamma = \Lambda - q - \Lambda_1 (v^\gamma)^- & \text{in } (0,\infty)\times(0,\infty), \\ v^\gamma(\cdot,0) = 0,\ u^\gamma(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} v^\gamma \in L^\infty((0,\infty)\times(0,\infty)), \\ \frac{\partial}{\partial r} u^\gamma \in L^\infty((0,\infty)\times(0,\infty)), \end{cases} \tag{1.5.15}$$

which is the Problem (1.3.14) with $\Lambda_2$ defined as (1.5.14).

**Theorem 5.** *Let $\kappa$, $\beta$, $\sigma$, $q$ and $R$ be positive constants. Let $\mathcal{L}$ be as in (1.3.7). Let $\Lambda$, $\Lambda_1$ and $\Lambda_2$ satisfy*

$$|||\Lambda_1||| < \infty,\ |||\Lambda_2||| < \infty, \quad \Lambda_1, \Lambda_2 \in L^\infty([0,\ell])\ \forall\, \ell > 0, \tag{1.5.16}$$

$$\|\Lambda\|_{L^\infty((0,\infty))} < \infty. \tag{1.5.17}$$

*Let $[v^*, u^*]$ be the solution of the structure model (1.5.13). For each $\gamma > 0$, let $[v^\gamma, u^\gamma]$ be the solution of the intensity model (1.5.15). Then*

$$\lim_{\gamma\to\infty} v^\gamma = v^* \textit{ locally uniformly in } (0,\infty)\times[0,\infty), \tag{1.5.18}$$

$$\lim_{\gamma\to\infty} u^\gamma = u^* \textit{ locally uniformly in } (0,\infty)\times[0,\infty). \tag{1.5.19}$$

This theorem builds the link between the structure model and the intensity model, for the CDS transferable by the seller. So even though we need the frequency of defaults to be somewhat bounded to have the well-posedness of Problem (1.3.14), we can still deal with the case when the frequency of defaults tends to infinity. In fact, with the structure model, the default will happen immediately when the interest rate attains some level; this corresponds to the default frequency equaling to infinity when the interest rate attains that level. This Theorem 5 provides a theoretical proof for this insight.

The structure of this dissertation is as follows. This introduction is Chapter 1. Chapter 2 proves Theorem 1. Chapter 3 proves other theorems. Chapter 4 provides numerical computation. Chapter 5 is the conclusion.

## 2.0 A FUNDAMENTAL PROBLEM

In this section, we investigate the following Problem (i.e. (1.5.1))

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u = F & \text{in } (0,\infty)\times(0,\infty), \\ u(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial u}{\partial r} \in L^\infty((0,\infty)\times(0,\infty)), \end{cases}$$

where $\mathcal{L}$ is defined as in (1.3.7), $\kappa$, $\beta$, and $\sigma$ are positive constants, $c(r)$ is a function of $r \in (0,\infty)$ and $F(r,T)$ is a function of $(r,T) \in (0,\infty)\times(0,\infty)$. Namely, we prove Theorem 1. In Theorem 1, we assume that

$$|||F||| < \infty, \quad \left|\left|\left|\frac{\partial F}{\partial T}\right|\right|\right| < \infty, \quad c \geq 0 \text{ and } c \in L^\infty([0,\ell]) \; \forall\, \ell > 0. \tag{2.0.1}$$

Note that we only assume that $c$ is locally bounded. So Theorem 1 works for some unbounded functions like the linear functions. The conclusions of Theorem 1 are that Problem (1.5.1) admits a unique solution $u$ and that $u$ satisfies (1.5.3), (1.5.4), (1.5.5) and (1.5.6):

$$\begin{aligned} \|u\|_{L^\infty((0,\infty)\times(0,\infty))} &\leq C_{\kappa,\beta,\sigma} |||F|||, \\ \left\|\frac{\partial u}{\partial r}\right\|_{L^\infty((0,\infty)\times(0,\infty))} &\leq C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}} \left(|||F||| + \left|\left|\left|\frac{\partial F}{\partial T}\right|\right|\right|\right), \\ \left|\left|\left|\frac{\partial u}{\partial T}\right|\right|\right| &\leq C_{\kappa,\beta,\sigma} \left(|||F||| + \left|\left|\left|\frac{\partial F}{\partial T}\right|\right|\right|\right), \\ \left\|\frac{\partial u}{\partial T}\right\|_{L^\infty((0,\infty)\times(0,\infty))} &\leq C_{\kappa,\beta,\sigma} \left(\|F(\cdot,0)\|_{L^\infty((0,\infty))} + \left|\left|\left|\frac{\partial F}{\partial T}\right|\right|\right|\right), \end{aligned}$$

where $C_{\kappa,\beta,\sigma}$ is a positive constant depending only on $\kappa$, $\beta$, $\sigma$, and $C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}}$ is a positive constant depending only on $\kappa$, $\beta$, $\sigma$, $\ell$, and $\|c\|_{L^\infty([0,\ell])}$. The proof of Theorem 3.1 in [6] uses ideas and approaches similar to those presented here.

## 2.1 APPROXIMATION BY REGULAR PROBLEMS

To prove the existence of the solution for Problem (1.5.1), we need to study a corresponding regularized problem. In fact, since $\mathcal{L} = -\frac{\sigma^2 r}{2}\frac{\partial}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r$, Problem (1.5.1) may degenerate when $r = 0$ and may have possible problem when $r \to \infty$. To avoid these possible problems, we firstly restrict it to a finite region and study the following regularized problem:

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u^\epsilon = F & \text{in } (\epsilon, \frac{1}{\epsilon}) \times (0, \infty), \\ u^\epsilon(\cdot, 0) = 0 & \text{in } (\epsilon, \frac{1}{\epsilon}), \\ \frac{\partial u^\epsilon}{\partial r} = 0 & \text{on } \left\{\epsilon, \frac{1}{\epsilon}\right\} \times (0, \infty). \end{cases} \tag{2.1.1}$$

Note that here $F$ is a function on $(0, \infty) \times (0, \infty)$. For further use as in (2.2.12), (2.4.31) and (2.4.38), we need the following restriction on $\epsilon$:

$$0 < \epsilon < \min\left\{1, \frac{3a}{2b}, \frac{b}{a}\right\}, \tag{2.1.2}$$

where $a$ and $b$ are positive constants depending only on $\kappa$, $\beta$, $\sigma$ defined later in (2.4.28) and (2.4.30). From the standard parabolic equation theory in PDE (for example, [11]), we know that (2.1.1) has a unique regular solution, which we denote it by $u^\epsilon$.

Note that for any $\delta > 0$ and any $0 < \epsilon \le \frac{\delta}{2}$, for any $p > 1$, by the $L^p$ interior estimate for parabolic equation (2.1.1) [24] 212, there exists a constant $C_{\delta,p}$ which only depends on $\delta$ and $p$ s.t.

$$\|u^\epsilon\|_{W_p^{2,1}\left([\delta, \frac{1}{\delta}] \times [0, \frac{1}{\delta^2}]\right)} \le C_{\delta,p}\left(\|F\|_{L^p\left([\frac{\delta}{2}, \frac{2}{\delta}] \times [0, \frac{4}{\delta^2}]\right)} + \|u^\epsilon\|_{L^\infty\left([\frac{\delta}{2}, \frac{2}{\delta}] \times [0, \frac{4}{\delta^2}]\right)}\right).$$

If we can show that $u^\epsilon$ is bounded and the bound does not depend on $\epsilon$ and $\|F\|_{L^p\left([\frac{\delta}{2}, \frac{2}{\delta}] \times [0, \frac{4}{\delta^2}]\right)}$ is finite, then since $W_p^{2,1}\left([\delta, \frac{1}{\delta}] \times [0, \frac{1}{\delta^2}]\right)$ is compact embedded in $C^{1,\frac{1}{2}}\left([\delta, \frac{1}{\delta}] \times [0, \frac{1}{\delta^2}]\right)$ ([24] page 20), we can find a function $u$ in $C^{1,\frac{1}{2}}\left([\delta, \frac{1}{\delta}] \times [0, \frac{1}{\delta^2}]\right)$ satisfying the first two equations of (1.5.1). Then if we furthermore had shown that $\frac{\partial u^\epsilon}{\partial r}$ has a uniform bound which does not depend on $\epsilon$, we will have the existence of the solution for PDE (1.5.1). So we need to firstly do a priori estimates for $u^\epsilon$, $\frac{\partial u^\epsilon}{\partial r}$ and $\frac{\partial u^\epsilon}{\partial T}$. These a priori estimates are also important for the proof of the boundedness results (1.5.3), (1.5.4), (1.5.5) and (1.5.6).

## 2.2 $L^\infty$ BOUND OF $u^\epsilon$

**Lemma 1.** *Under the assumptions of Theorem 1,*

$$\|u^\epsilon\|_{L^\infty\left((\epsilon,\frac{1}{\epsilon})\times(0,\infty)\right)} \le C_{\kappa,\beta,\sigma}|||F|||, \tag{2.2.1}$$

*where $C_{\kappa,\beta,\sigma}$ is a constant depending only on $\kappa$, $\beta$ and $\sigma$.*

*Proof.* The idea to to construct some super-solution $\phi$ with $-\phi$ being the sub-solution of Problem (2.1.1) so that $\phi = C_{\kappa,\beta,\sigma}|||F|||\varphi$ for some bounded function $\varphi$. Let us firstly try the easy case that $\phi$ and $\varphi$ only depends on $r$. Since we want $\phi$ be the super-solution, we want

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\phi = -\frac{\sigma^2 r}{2}\phi'' - (\kappa - \beta r)\phi' + (r+c)\phi \ge F.$$

So we want

$$C_{\kappa,\beta,\sigma}\left(-\frac{\sigma^2 r}{2}\varphi'' - (\kappa - \beta r)\varphi' + (r+c)\varphi\right) \ge \frac{F}{|||F|||}.$$

Note that $|||F||| \ge \frac{F}{1+r}$, which implies $\frac{F}{|||F|||} \le 1 + r$, so we just need to show

$$-\frac{\sigma^2 r}{2}\varphi'' - (\kappa - \beta r)\varphi' + (r+c)\varphi \ge C_{\kappa,\beta,\sigma}(1+r).$$

If we have $c > c_0$ for some constant $c_0$ depending only on $\kappa, \beta, \sigma$, we can just take $\varphi = 1$ and we are done; but unfortunately, we only know $c > 0$ but do not have a positive lower bound for $c$. So we want to try to "borrow from $r$" and separate $r + c$ to be $(\frac{2}{3}r - c_0) + (\frac{1}{3}r + c_0)$ for some $c_0 > 0$. Then if we can find a bounded function $\varphi$ s.t.

$$-\frac{\sigma^2 r}{2}\varphi'' - (\kappa - \beta r)\varphi' + (\frac{2}{3}r - c_0)\varphi \ge 0,$$

then we are nearly done. Now let's denote

$$\mathcal{A}\varphi := -\frac{\sigma^2 r}{2}\varphi'' - (\kappa - \beta r)\varphi' + (\frac{2}{3}r - c_0)\varphi \tag{2.2.2}$$

and try to construct some bounded function $\varphi$ so that $\mathcal{A}\varphi \ge 0$.

Firstly, if $\varphi$ is some positive constant, then we need $(\frac{2}{3}r - c_0)\varphi \ge 0$. This cannot in general be true, since we don't want our bound depend on $\epsilon$. But we can still choose $\varphi$ so that $\varphi(r)$ is a constant when $r > b$ for some constant $b$. For example, we can just choose $\varphi = 1$ when $r > b$. And for the part when $r < b$, considering the format of $\mathcal{A}$ in (2.2.2), since the coefficient $-(\kappa - \beta r)$ of the $\varphi'$ term is negative when $r$ small enough, we want $\varphi'$ be negative when $r < b$, which motivate me to try to include $(r-b)^2$ in the construction of $\varphi$. But we want $\varphi$ be continuous, so we can try to define $\varphi$ in the following way:

$$\varphi(r) = 1 + k(r-b)^2 \mathbf{1}_{\{r<b\}},\ r > 0,$$

where $b > 0$ and $k > 0$ are constants to be determined. Note that

$$|\varphi(r)| \leq 1 + kb^2$$

is a bounded function, as we desired.

Now we want to see whether such a $\varphi$ satisfying $\mathcal{A}\varphi > 0$ or not and what are the conditions needed for $b$, $k$ and $c_0$. For now, let us fix an arbitrary $b > 0$.

When $r \geq b$,

$$\mathcal{A}\varphi = \frac{2}{3}r - c_0 \geq \frac{2}{3}b - c_0,$$

which is non-negative if

$$c_0 \leq \frac{2}{3}b.$$

When $r < b$,

$$\begin{aligned}
\mathcal{A}\varphi &= -\frac{\sigma^2 r}{2}2k - (\kappa - \beta r)2k(r-b) + (\frac{2}{3}r - c_0)(1 + k(r-b)^2) \\
&= -\sigma^2 rk - 2k\kappa r + 2k\kappa b + 2k\beta r^2 - 2k\beta rb + (\frac{2}{3}r - c_0)(1 + k(r-b)^2) \\
&= 2k\beta r^2 + 2k\kappa b - k(\sigma^2 + 2\kappa + 2b\beta)r + (\frac{2}{3}r - c_0)(1 + k(r-b)^2).
\end{aligned}$$

Here, $2k\beta r^2$ is very small when $r$ small, so don't depend on it to make $\mathcal{A}\varphi$ positive. So we drop it and have

$$\mathcal{A}\varphi \geq \big(2k\kappa b - k(\sigma^2 + 2\kappa + 2b\beta)r\big) + \Big((\frac{2}{3}r - c_0)(1 + k(r-b)^2\Big). \tag{2.2.3}$$

Now, for these two terms, $(2k\kappa b - k(\sigma^2 + 2\kappa + 2b\beta)r)$ is positive when $r$ is small enough, $\big((\frac{2}{3}r - c_0)(1 + k(r-b)^2\big)$ is positive when $r$ is big enough. Based on this point of view, we can find the conditions for $r_0$ and $k$ w.r.t. $b$ so that $\mathcal{A}\varphi > 0$.

Firstly, let us consider the case when $r$ is large, i.e. when $r \geq 3c_0$. Note that our discussion is under the condition that $r < b$, so we need

$$c_0 \leq \frac{b}{4} < \frac{b}{3}. \tag{2.2.4}$$

With this condition, we have $3c_0 \leq r < b$. Then the second term in (2.2.3) satisfies

$$(\frac{2}{3}r - c_0)(1 + k(r-b)^2) \geq (\frac{2}{3}r - \frac{r}{3})(1 + k(r-b)^2) \geq \frac{r}{3}.$$

So to balance the first term of (2.2.3), we just need

$$k(\sigma^2 + 2\kappa + 2b\beta)r \leq \frac{r}{3} \quad (\Leftrightarrow \quad k \leq \frac{1}{3(\sigma^2 + 2\kappa + 2b\beta)}). \tag{2.2.5}$$

One can check that, with $c_0$ and $k$ satisfying (2.2.4) and (2.2.5), $\mathcal{A}\varphi \geq 0$ on $[3c_0, \infty)$.

Secondly, let us consider the case when $r$ is small, i.e. when $r < 3c_0$. In this case,

$$(\frac{2}{3}r - c_0)(1 + k(r-b)^2) \geq -c_0(1 + kb^2) \geq -2c_0,$$

if

$$kb^2 \leq 1 \quad (\Leftrightarrow \quad k \leq \frac{1}{b^2}). \tag{2.2.6}$$

Now we will depend on the first term in (2.2.3) to make $\mathcal{A}\varphi \geq 0$. Note that when

$$k(\sigma^2 + 2\kappa + 2b\beta)r \leq \frac{1}{2}k\kappa b \quad (\Leftrightarrow \quad r \leq \frac{\kappa b}{2(\sigma^2 + 2\kappa + 2b\beta)}), \tag{2.2.7}$$

with (2.2.3) we have

$$\mathcal{A}\varphi \geq \frac{3}{2}k\kappa b + \Big((\frac{2}{3}r - c_0)(1 + k(r-b)^2\Big) \geq \frac{3}{2}k\kappa b - 2c_0 \geq 0,$$

if

$$c_0 \leq \frac{3k\kappa b}{4}. \tag{2.2.8}$$

Note that we are discussing the case when $r < 3c_0$, so to make (2.2.7) hold, we need

$$3c_0 \le \frac{\kappa b}{2(\sigma^2 + 2\kappa + 2b\beta)},$$

which is true, because of (2.2.8), if

$$k \le \frac{2}{9(\sigma^2 + 2\kappa + 2b\beta)}. \tag{2.2.9}$$

One can check that, with $c_0$ and $k$ satisfying (2.2.6), (2.2.8) and (2.2.9), $\mathcal{A}\varphi \ge 0$ on $(0, 3c_0)$.

In summary, by defining

$$k = \min\left\{\frac{1}{b^2}, \frac{2}{9(\sigma^2 + 2\kappa + 2b\beta)}\right\}, \quad c_0 = \min\left\{\frac{b}{4}, \frac{3k\kappa b}{4}\right\}, \tag{2.2.10}$$

we have (2.2.4), (2.2.5), (2.2.6), (2.2.8) and (2.2.9) hold and $\mathcal{A}\varphi \ge 0$ with

$$\varphi(r) = 1 + k(r-b)^2 \mathbf{1}_{\{r<b\}},\ r > 0. \tag{2.2.11}$$

Then since $\varphi \ge 1$,

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\varphi = \mathcal{A}\varphi + \left(\frac{1}{3}r + c_0\right)\varphi \ge \frac{1}{3}r + c_0 \ge \min\{\frac{1}{3}, c_0\}(1+r).$$

Note that in the previous discussion, we take $b$ be an arbitrary positive constant. Now we can pick

$$b = 1.$$

Then $c_0 \le \frac{b}{4} = \frac{1}{4} < \frac{1}{3}$, so

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\varphi \ge c_0(1+r).$$

Now, let

$$\phi = \frac{|||F|||}{c_0}\varphi.$$

Then

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\phi = \frac{|||F|||}{c_0}\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\varphi \geq \frac{|||F|||}{c_0}c_0(1+r)$$
$$\geq \frac{|F|}{1+r}(1+r) = |F| \geq F.$$

$$\phi(r) = \frac{|||F|||}{c_0}\varphi(r) \geq 0 \text{ in } (\epsilon, \frac{1}{\epsilon}).$$

And if we assume

$$\epsilon < 1, \tag{2.2.12}$$

then $\frac{1}{\epsilon} > 1 = b$ and $\frac{\partial \phi}{\partial \vec{n}}|_{r=\frac{1}{\epsilon}} = 0$. Also $\frac{\partial \phi}{\partial \vec{n}}|_{r=-\epsilon} \geq 0$ since $\epsilon \leq 1 = b$. So $\phi$ is a super-solution of Problem (2.1.1). One can check that $-\phi$ is a sub-solution of Problem (2.1.1). So by the comparison principle,

$$|u^\epsilon| \leq \phi = \frac{|||F|||}{c_0}\varphi \leq \frac{2}{c_0}|||F||| = C_{\kappa,\beta,\sigma}|||F|||,$$

since $\varphi \leq 1 + kb^2 \leq 1 + 1 = 2$ by (2.2.10). □

## 2.3 A PRIOR BOUND OF $\frac{\partial u^\epsilon}{\partial t}$

**Lemma 2.** *Under the assumptions of Theorem 1,*

$$\left\|\left\|\left\|\frac{\partial u^\epsilon}{\partial T}\right\|\right\|\right\|_\epsilon \leq C_{\kappa,\beta,\sigma}(|||F||| + \left\|\left\|\left\|\frac{\partial F}{\partial T}\right\|\right\|\right\|), \tag{2.3.1}$$

$$\left\|\frac{\partial u^\epsilon}{\partial T}\right\|_{L^\infty((0,\infty)\times(0,\infty))} \leq C_{\kappa,\beta,\sigma}\left(\|F(\cdot,0)\|_{L^\infty((0,\infty))} + \left\|\left\|\left\|\frac{\partial F}{\partial T}\right\|\right\|\right\|\right). \tag{2.3.2}$$

*Proof.* Taking derivative with respect to $T$ from both sides of the first and third equation of (2.1.1), we have

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u^\epsilon_T = \frac{\partial F}{\partial T} \text{ in } (\epsilon, \epsilon^{-1}) \times (0,\infty)$$
$$\frac{\partial}{\partial r} u^\epsilon_T = 0 \text{ on } \{\epsilon, \epsilon^{-1}\} \times (0,\infty)$$

By the first equation of (2.1.1), we have

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u^\epsilon(\cdot,0) = F(\cdot,0) \text{ in } (\epsilon, \epsilon^{-1}).$$

Then by the second equation of (2.1.1), $(\mathcal{L}+c)u^\epsilon(\cdot,0) = 0$ in $(\epsilon,\epsilon^{-1})$. So

$$u^\epsilon_T(\cdot,0) = F(\cdot,0) \text{ in } (\epsilon,\epsilon^{-1}).$$

Put them together we have

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u^\epsilon_T = \frac{\partial F}{\partial T} & \text{in } (\epsilon,\epsilon^{-1}) \times (0,\infty), \\ u^\epsilon_T(\cdot,0) = F(\cdot,0) & \text{in } (\epsilon,\epsilon^{-1}), \\ \frac{\partial}{\partial r} u^\epsilon_T = 0 & \text{on } \{\epsilon,\epsilon^{-1}\} \times (0,\infty). \end{cases}$$

Then $u^\epsilon_T = w_1 + w_2$ where $w_1$ and $w_2$ are respectively unique solutions of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) w_1 = \frac{\partial F}{\partial T} & \text{in } (\epsilon,\epsilon^{-1}) \times (0,\infty), \\ w_1(\cdot,0) = 0 & \text{in } (\epsilon,\frac{1}{\epsilon}), \\ \frac{\partial}{\partial r} w_1 = 0 & \text{on } \{\epsilon,\epsilon^{-1}\} \times (0,\infty), \end{cases} \tag{2.3.3}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) w_2 = 0 & \text{in } (\epsilon, \epsilon^{-1}) \times (0, \infty), \\ w_2(\cdot, 0) = F(\cdot, 0) & \text{in } (\epsilon, \frac{1}{\epsilon}), \\ \frac{\partial}{\partial r} w_2 = 0 & \text{on } \{\epsilon, \epsilon^{-1}\} \times (0, \infty). \end{cases} \tag{2.3.4}$$

Apply Lemma 1 to Problem (2.3.3), we have

$$|||w_1|||_\epsilon \le \|w_1\|_{L^\infty((\epsilon,\epsilon^{-1})\times(0,\infty))} \le C_{\kappa,\beta,\sigma} \left|\left|\left| \frac{\partial F}{\partial T} \right|\right|\right|; \tag{2.3.5}$$

by the same proof as Lemma 3.4 in [13], one can show that for $w_2$ of Problem (2.3.4),

$$|||w_2|||_\epsilon \le C_{\kappa,\beta,\sigma} |||F|||. \tag{2.3.6}$$

Combining (2.3.5) and (2.3.6), we have

$$\left|\left|\left| \frac{\partial u^\epsilon}{\partial T} \right|\right|\right|_\epsilon = |||w_1 + w_2|||_\epsilon \le C_{\kappa,\beta,\sigma}(|||F||| + |||F_T|||).$$

So (2.3.1) holds true.

Now, if $\|F(\cdot,0)\|_{L^\infty((0,\infty))} = \infty$, then (1.5.6) holds automatically. Otherwise, we have $\|F(\cdot,0)\|_{L^\infty((0,\infty))} < \infty$. Let

$$\psi = \|F(\cdot,0)\|_{L^\infty((0,\infty))}\varphi,$$

where $\varphi$ is defined in (2.2.11) in the proof of Lemma 1. Then

$$\begin{aligned} &\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\psi \ge \mathcal{A}\psi = \|F(\cdot,0)\|_{L^\infty((0,\infty))}\mathcal{A}\varphi \ge 0 \text{ in } (\epsilon, \epsilon^{-1}) \times (0,\infty), \\ &\psi(\cdot,0) = \psi(\cdot) = \|F(\cdot,0)\|_{L^\infty((0,\infty))}\varphi \ge \|F(\cdot,0)\|_{L^\infty((0,\infty))} \ge F \text{ in } (\epsilon, \epsilon^{-1}), \\ &\frac{\partial}{\partial \vec{n}}\psi = \|F(\cdot,0)\|_{L^\infty((0,\infty))}\frac{\partial}{\partial \vec{n}}\varphi \ge 0 \text{ on } \{\epsilon, \frac{1}{\epsilon}\} \times (0,\infty). \end{aligned}$$

So $\psi$ is a super-solution to Problem (2.3.4).One can easily check that $-\psi$ is a sub-solution to Problem (2.3.4). Then by the comparison principle,

$$\|w_1\|_{L^\infty((\epsilon,\epsilon^{-1})\times(0,\infty))} \le \psi = \|F(\cdot,0)\|_{L^\infty((0,\infty))}\varphi \le 2\|F(\cdot,0)\|_{L^\infty((0,\infty))}. \tag{2.3.7}$$

Combine (2.3.5) and (2.3.7) we have

$$\begin{aligned}\|\frac{\partial u^\epsilon}{\partial T}\|_{L^\infty((\epsilon,\epsilon^{-1})\times(0,\infty))} &\le \|w_1\|_{L^\infty((\epsilon,\epsilon^{-1})\times(0,\infty))} + \|w_2\|_{L^\infty((\epsilon,\epsilon^{-1})\times(0,\infty))} \\ &\le C_{\kappa,\beta,\sigma} \left|\!\left|\!\left| \frac{\partial F}{\partial T} \right|\!\right|\!\right| + 2\|F(\cdot,0)\|_{L^\infty((0,\infty))} \\ &= C_{\kappa,\beta,\sigma} \left( \left|\!\left|\!\left| \frac{\partial F}{\partial T} \right|\!\right|\!\right| + \|F(\cdot,0)\|_{L^\infty((0,\infty))} \right).\end{aligned}$$

So (2.3.2) holds true. □

## 2.4 A PRIOR BOUND OF $\frac{\partial u^\epsilon}{\partial r}$

**Lemma 3.** *Under the conditions of Theorem 1,*

$$\left\| \frac{\partial}{\partial r} u^\epsilon \right\|_{L^\infty\left((\epsilon,\frac{1}{\epsilon})\times(0,\infty)\right)} \le C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}} (|\!|\!| F |\!|\!| + \left|\!\left|\!\left| \frac{\partial F}{\partial T} \right|\!\right|\!\right|), \tag{2.4.1}$$

*where $C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}}$ is a constant depending only on $\kappa$, $\beta$, $\sigma$, $\ell$ and $\|c\|_{L^\infty([0,\ell])}$.*

*Proof.* We can rewrite the Problem (2.1.1) as

$$\begin{cases} (\mathcal{L}+c)\, u^\epsilon = F - \frac{\partial u^\epsilon}{\partial T} & \text{in } (\epsilon,\frac{1}{\epsilon})\times(0,\infty), \\ u^\epsilon(\cdot,0) = 0 & \text{in } (\epsilon,\frac{1}{\epsilon}), \\ \frac{\partial u^\epsilon}{\partial r}(\epsilon,\cdot) = 0,\ \frac{\partial u^\epsilon}{\partial r}(\frac{1}{\epsilon},\cdot) = 0 & \text{in } (0,\infty). \end{cases}$$

Now, fix an arbitrary $T \in (0,\infty)$. Let

$$\varphi(\cdot) := u^\epsilon(\cdot,T), \quad h(\cdot) := F(\cdot,T) - \frac{\partial u^\epsilon}{\partial T}(\cdot,T), \quad \text{in } (0,\infty), \tag{2.4.2}$$

then $\varphi(\cdot)$ and $h(\cdot)$ are just functions of $r \in (0,\infty)$ and $\varphi(\cdot) = u^\epsilon(\cdot,T)$ satisfies ODE problem

$$\begin{cases} (\mathcal{L}+c)\,\varphi = h & \text{in } (\epsilon,\frac{1}{\epsilon}), \\ \varphi'(\epsilon) = \varphi'\left(\frac{1}{\epsilon}\right) = 0. \end{cases} \tag{2.4.3}$$

This Problem (2.4.3) is a standard second order linear ODE problem, so the existence and uniqueness of its solution $\varphi$ is guaranteed by the standard ODE theory ([12], for example). And with (2.4.2), we can study the $L^\infty$ boundedness of $\frac{\partial u^\epsilon}{\partial r}$ through studying the $L^\infty$ boundedness of $\varphi'$. We will finish the proof in following six steps.

**(1) Write the explicit expression of the solution of Problem (2.4.3).**

Note that in $(\epsilon, \epsilon^{-1})$,

$$
\begin{aligned}
& (\mathcal{L}+c)\,\varphi = h \\
\Leftrightarrow\quad & -\frac{\sigma^2 r}{2}\frac{\partial^2\varphi}{\partial r^2} - (\kappa-\beta r)\frac{\partial\varphi}{\partial r} + (r+c)\,\varphi = h \\
\Leftrightarrow\quad & \frac{\partial^2\varphi}{\partial r^2} + \frac{\kappa-\beta r}{\frac{\sigma^2 r}{2}}\frac{\partial\varphi}{\partial r} + \frac{r+c}{-\frac{\sigma^2 r}{2}}\varphi = \frac{h}{-\frac{\sigma^2 r}{2}} \\
\Leftrightarrow\quad & \frac{\partial^2\varphi}{\partial r^2} + \left(\frac{2\kappa}{\sigma^2 r} - \frac{2\beta}{\sigma^2}\right)\frac{\partial\varphi}{\partial r} - \frac{2}{\sigma^2}\left(1+\frac{c(r)}{r}\right)\varphi = -\frac{2h(r)}{\sigma^2 r} \\
\Leftrightarrow\quad & \frac{\partial^2\varphi}{\partial r^2} + P(r)\frac{\partial\varphi}{\partial r} + Q(r)\varphi = R(r),
\end{aligned}
$$

where

$$P(r) := \frac{2\kappa}{\sigma^2 r} - \frac{2\beta}{\sigma^2},\ \ Q(r) := -\frac{2}{\sigma^2}\left(1+\frac{c(r)}{r}\right),\ \ R(r) := -\frac{2h(r)}{\sigma^2 r}. \tag{2.4.4}$$

So Problem (2.6) is equivalent to

$$
\begin{cases}
\frac{\partial^2\varphi}{\partial r^2} + P\frac{\partial\varphi}{\partial r} + Q\varphi = R \quad \text{in } (\epsilon, \frac{1}{\epsilon}), \\
\varphi'(\epsilon) = \varphi'\left(\frac{1}{\epsilon}\right) = 0,
\end{cases} \tag{2.4.5}
$$

where $P$, $Q$ and $R$ are as defined in (2.4.4). The corresponding homogeneous equation of the first equation in (2.4.5) is

$$\frac{\partial^2\varphi}{\partial r^2} + P\frac{\partial\varphi}{\partial r} + Q\varphi = 0 \text{ in } (\epsilon, \frac{1}{\epsilon}). \tag{2.4.6}$$

By [12], equation (2.4.6) has two linear independent solutions $\varphi_2$ and $\varphi_1$, and we can pick $\varphi_2$ so that

$$\varphi_2(\epsilon) = 1,\ \ \varphi_2'(\epsilon) = 0.$$

By equation (4.32) on page 165 of [12], the Wronskian $W$ of $\varphi_1$ and $\varphi_2$ on $(\epsilon, \epsilon^{-1})$ is given by

$$
\begin{aligned}
W &= Ce^{-\int P(r)dr} = Ce^{-\int \frac{2\kappa}{\sigma^2 r} - \frac{2\beta}{\sigma^2} dr} \\
&= Ce^{-\frac{2\kappa}{\sigma^2}\ln r + \frac{2\beta}{\sigma^2} r} = Cr^{-\frac{2\kappa}{\sigma^2}} e^{\frac{2\beta}{\sigma^2} r} \\
&= Cr^{-\nu} e^{\mu r},
\end{aligned} \tag{2.4.7}
$$

where $C$ is an arbitrary constant and

$$
\nu := \frac{2\kappa}{\sigma^2}, \ \mu := \frac{2\beta}{\sigma^2}.
$$

And with this notation, $P(r) = \frac{\nu}{r} - \mu$. We can take $C = 1$ and then the Wronskian of $\varphi_1$ and $\varphi_2$ on $(\epsilon, \epsilon^{-1})$ is

$$
W = r^{-\nu} e^{\mu r}.
$$

But by the definition of Wronskian (ref: equation (4.26) on page 164 of [12]),

$$
W = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \varphi_1 \varphi_2' - \varphi_2 \varphi_1',
$$

so if we take $\varphi_1$ with $\varphi_1'\left(\frac{1}{\epsilon}\right) = 0$, because of the version of Wronskian we use, we have

$$
\varphi_1\left(\frac{1}{\epsilon}\right) = \frac{W\left(\frac{1}{\epsilon}\right)}{\varphi_2'\left(\frac{1}{\epsilon}\right)}.
$$

Now, with these information of the corresponding homogeneous equation (2.4.6), by Theorem 4.28 on page 285 of [12], a solution of the first equation in (2.4.5) on $(\epsilon, \epsilon^{-1})$ is given by

$$
\varphi(r) = v_1(r)\varphi_1(r) + v_2(r)\varphi_2(r), \tag{2.4.8}
$$

where

$$
v_1(r) = -\int \varphi_2(r) \frac{R(r)}{W(r)} dr, \quad v_2(r) = \int \varphi_1(r) \frac{R(r)}{W(r)} dr.
$$

Then

$$
\begin{aligned}
\varphi' &= v_1'\varphi_1 + v_1\varphi_1' + v_2'\varphi_2 + v_2\varphi_2' \\
&= -\varphi_2\frac{R}{W}\varphi_1 + v_1\varphi_1' + \varphi_1\frac{R}{W}\varphi_2 + v_2\varphi_2' \\
&= v_1\varphi_1' + v_2\varphi_2'.
\end{aligned} \tag{2.4.9}
$$

To make $\varphi$ in (2.4.8) be a solution of Problem (2.4.5), we need

$$
\begin{aligned}
0 =& \varphi'(\epsilon) = v_1(\epsilon)\varphi_1'(\epsilon) + v_2(\epsilon)\varphi_2'(\epsilon) = v_1(\epsilon)\varphi_1'(\epsilon) + v_2(\epsilon)\cdot 0 = v_1(\epsilon)\varphi_1'(\epsilon), \\
0 =& \varphi'\left(\epsilon^{-1}\right) = v_1\left(\epsilon^{-1}\right)\varphi_1'\left(\epsilon^{-1}\right) + v_2\left(\epsilon^{-1}\right)\varphi_2'\left(\epsilon^{-1}\right) = v_2\left(\epsilon^{-1}\right)\varphi_2'\left(\epsilon^{-1}\right),
\end{aligned}
$$

which is true if $v_1(\epsilon) = 0$ and $v_2\left(\frac{1}{\epsilon}\right) = 0$. So we take

$$
v_1(r) = -\int_\epsilon^r \varphi_2(s)\frac{R(s)}{W(s)}ds, \quad v_2(r) = \int_{\frac{1}{\epsilon}}^r \varphi_1(s)\frac{R(s)}{W(s)}ds. \tag{2.4.10}
$$

In summary, the solution of Problem (2.4.3), which is equivalent with Problem (2.4.5), has the form (2.4.8):

$$
\begin{aligned}
\varphi(r) &= v_1(r)\varphi_1(r) + v_2(r)\varphi_2(r) \\
&= \varphi_1(r)\int_\varepsilon^r \varphi_2(s)\frac{-R(s)}{W(s)}ds + \varphi_2(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{-R(s)}{W(s)}ds,
\end{aligned} \tag{2.4.11}
$$

where $v_1$ and $v_2$ are as in (2.4.10), $\varphi_2$ and $\varphi_1$ are linear independent solutions of the corresponding homogeneous equation which satisfy

$$
\varphi_2(\epsilon) = 1,\ \varphi_2'(\epsilon) = 0,\ \varphi_1\left(\frac{1}{\epsilon}\right) = \frac{W\left(\frac{1}{\epsilon}\right)}{\varphi_2'\left(\frac{1}{\epsilon}\right)},\ \varphi_1'\left(\frac{1}{\epsilon}\right) = 0,\ W = r^{-\nu}e^{\mu r}. \tag{2.4.12}
$$

By (2.4.9), (2.4.10),

$$
\begin{aligned}
\varphi'(r) &= v_1(r)\varphi_1'(r) + v_2(r)\varphi_2'(r) \\
&= \varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{-R(s)}{W(s)}ds + \varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{-R(s)}{W(s)}ds.
\end{aligned} \tag{2.4.13}
$$

**(2) Determine signs of $\varphi_1$, $\varphi_2$, $\varphi_1'$ and $\varphi_2'$ on $(\epsilon, \epsilon^{-1})$.**

To find the bound for $\varphi'$, we need to know the signs of $\varphi_1$, $\varphi_2$, $\varphi_1'$ and $\varphi_2'$ on $(\epsilon, \epsilon^{-1})$. The idea to prove that $\varphi_2 > 0$ is to try to "solve" (2.4.6) formally using $W$. By (2.4.7), we know

$$W' = -PW. \tag{2.4.14}$$

So with (2.4.6) we have for $\varphi_i$, $i = 1, 2$ that

$$\begin{aligned} 0 &= \varphi_i'' + P\varphi_i' + Q\varphi_i = \varphi_i'' - \frac{W'}{W}\varphi_i' + Q\varphi_i \\ &= \frac{W\varphi_i'' - W'\varphi_i' + WQ\varphi_i}{W}. \end{aligned}$$

The format $W\varphi_i'' - W'\varphi_i'$ in the above equation remind me the quotient rule for taking derivatives, so divide both sides of the above equation by $W$, we have

$$0 = \frac{W\varphi_i'' - W'\varphi_i' + WQ\varphi_i}{W^2} = \left(\frac{\varphi_i'}{W}\right)' + Q\frac{\varphi_i}{W}.$$

So

$$\left(\frac{\varphi_i'(r)}{W(r)}\right)' = -Q(r)\frac{\varphi_i(r)}{W(r)}.$$

Integrate both sides w.r.t $r$ and multiply both sides by $W$, we have

$$\varphi_i'(r) = W(r)\int -Q(r)\frac{\varphi_i(r)}{W(r)}dr, \;\; i = 1, 2. \tag{2.4.15}$$

But $\varphi_2'(\epsilon) = 0$ by (2.4.12),

$$\varphi_2'(r) = W(r)\int_\epsilon^r -Q(s)\frac{\varphi_2(s)}{W(s)}ds, \tag{2.4.16}$$

Integrate w.r.t. $r$ again and noting that $\varphi_2(\epsilon) = 1$ by (2.4.12), we have

$$\varphi_2(r) = \int_\epsilon^r W(\rho)\int_\epsilon^\rho -Q(s)\frac{\varphi_2(s)}{W(s)}dsd\rho + 1.$$

Changing the integration order to be first w.r.t. $\rho$ then w.r.t. $s$, we have

$$\begin{aligned} \varphi_2(r) &= \int_\epsilon^r \int_\rho^r W(\rho)(-Q(s))\frac{\varphi_2(s)}{W(s)}d\rho ds + 1 \\ &= \int_\epsilon^r \varphi_2(s)\left(\frac{-Q(s)}{W(s)}\int_\rho^r W(\rho)d\rho\right)ds + 1. \end{aligned} \tag{2.4.17}$$

Note that $\varphi_2(\epsilon) = 1 > 0$ and that

$$\frac{-Q(s)}{W(s)} \int_\rho^r W(\rho)d\rho = \frac{2}{\sigma^2}\left(1 + \frac{c(r)}{r}\right)\frac{1}{s^{-\nu}e^{\mu s}} \int_\rho^r \rho^{-\nu}e^{\mu\rho}d\rho > 0,\ \epsilon \le s \le r, \tag{2.4.18}$$

so

$$\varphi_2(r) > 0 \text{ in } [\epsilon, \frac{1}{\epsilon}].$$

In fact, because $\varphi_2(\epsilon) = 1 > 0$ and the continuity of $\varphi_2$ implied by (2.4.17), $\varphi_2$ is at least positive in a small interval near $\epsilon$. Let $r_0 = \max\{r > \epsilon | \varphi_2 \text{ is positive on } [\epsilon, r_0]\}$. Assume for contradiction that $\varphi_2(r)$ is not always positive on $[\epsilon, \frac{1}{\epsilon}]$, then $r_0 < \frac{1}{\epsilon}$ and $\varphi_2(r_0) = 0$ because of the continuity of $\varphi_2$. But $\varphi_2$ is positive on $[\epsilon, r_0]$, by (2.4.17) and (2.4.18) we have $\varphi_2(r_0) = \int_\epsilon^{r_0} \varphi_2(s)\left(\frac{-Q(s)}{W(s)}\int_\rho^r W(\rho)d\rho\right)ds + 1 > 0 + 1 = 1$, which contradict with that $\varphi_2(r_0) = 0$. So (2.4.13) is true.

Now, note that $-Q(r) = \frac{2}{\sigma^2}\left(1 + \frac{c(r)}{r}\right) > 0$, $W > 0$, so by (2.4.13) and (2.4.16), we have

$$\varphi_2'(r) > 0 \text{ in } (\epsilon, \frac{1}{\epsilon}).$$

Note that $W = \varphi_1\varphi_2' - \varphi_2\varphi_1'$, and by (2.4.13) we know $\varphi_2(r) > 0$ in $[\epsilon, \frac{1}{\epsilon}]$. So

$$\left(\frac{\varphi_1}{\varphi_2}\right)' = \frac{\varphi_1'\varphi_2 - \varphi_1\varphi_2'}{\varphi_2^2} = \frac{-W}{\varphi_2^2}.$$

But by (2.4.12) we have $\varphi_1\left(\frac{1}{\epsilon}\right) = \frac{W\left(\frac{1}{\epsilon}\right)}{\varphi_2'\left(\frac{1}{\epsilon}\right)}$, so integrate both sides of above equation we have

$$\varphi_1(r) = \varphi_2(r)\int_r^{\frac{1}{\epsilon}} \frac{W(s)}{\varphi_2^2(s)}ds + \varphi_2(r)\frac{W\left(\frac{1}{\epsilon}\right)}{\varphi_2'\left(\frac{1}{\epsilon}\right)\varphi_2\left(\frac{1}{\epsilon}\right)} > 0.$$

Note that

$$\varphi_1'\left(\frac{1}{\epsilon}\right) = \varphi_2'\left(\frac{1}{\epsilon}\right)\int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon}} \frac{W(s)}{\varphi_2^2(s)}ds - \varphi_2\left(\frac{1}{\epsilon}\right)\frac{W\left(\frac{1}{\epsilon}\right)}{\varphi_2^2\left(\frac{1}{\epsilon}\right)} + \varphi_2'\left(\frac{1}{\epsilon}\right)\frac{W\left(\frac{1}{\epsilon}\right)}{\varphi_2'\left(\frac{1}{\epsilon}\right)\varphi_2\left(\frac{1}{\epsilon}\right)} = 0$$

is consistent with $\varphi_1'\left(\frac{1}{\epsilon}\right) = 0$ in (2.4.12). With this fact and (2.4.15), we have

$$\varphi_1'(r) = W(r)\int_{\frac{1}{\epsilon}}^r -Q(r)\frac{\varphi_1(r)}{W(r)}dr = -W(r)\int_r^{\frac{1}{\epsilon}} -Q(r)\frac{\varphi_1(r)}{W(r)}dr < 0.$$

In summary, we have

$$\varphi_1 > 0 \text{ and } \varphi_2 > 0 \text{ in } [\epsilon, \frac{1}{\epsilon}],\ \varphi_1' < 0 \text{ and } \varphi_2' > 0 \ \text{ in } (\epsilon, \frac{1}{\epsilon}). \tag{2.4.19}$$

**(3) Bound $|\varphi'(r)|$ by $M_1(r)|||h||| + M_2(r)|||h|||$.**

By (2.4.13) and (2.4.4),

$$\begin{aligned}\varphi'(r) &= \varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{-R(s)}{W(s)}ds + \varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{-R(s)}{W(s)}ds \\ &= \varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2h(s)}{\sigma^2 sW(s)}ds + \varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2h(s)}{\sigma^2 sW(s)}ds.\end{aligned}$$

Then with (2.4.19) we have

$$\begin{aligned}|\varphi'(r)| &= \left|\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2h(s)}{\sigma^2 sW(s)}ds + \varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2h(s)}{\sigma^2 sW(s)}ds\right| \\ &\le \left|\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2h(s)}{\sigma^2 sW(s)}ds\right| + \left|\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2h(s)}{\sigma^2 sW(s)}ds\right| \\ &\le |\varphi_1'(r)|\int_\epsilon^r \left|\varphi_2(s)\frac{2h(s)}{\sigma^2 sW(s)}\right| ds + |\varphi_2'(r)|\int_r^{\frac{1}{\epsilon}} \left|\varphi_1(s)\frac{2h(s)}{\sigma^2 sW(s)}\right| ds \\ &= -\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2|h(s)|}{\sigma^2 sW(s)}ds + \varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2|h(s)|}{\sigma^2 sW(s)}ds.\end{aligned}$$

Since we want to bound $|\varphi'(r)|$ by some constant times $|||h|||_\epsilon = \sup_{\epsilon\le r\le\frac{1}{\epsilon}}\left|\frac{h(r)}{1+r}\right|$, we multiply and then divide by $(1+r)$ in the last line of the above equation:

$$\begin{aligned}|\varphi'(r)| &\le -\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2|h(s)|}{\sigma^2 sW(s)}ds + \varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2|h(s)|}{\sigma^2 sW(s)}ds \\ &= -\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2(1+s)}{\sigma^2 sW(s)}\left|\frac{h(s)}{1+s}\right| ds + \varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2(1+s)}{\sigma^2 sW(s)}\left|\frac{h(s)}{1+s}\right| ds \\ &\le \left(-\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds\right)|||h|||_\epsilon + \left(\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds\right)|||h|||_\epsilon.\end{aligned} \tag{2.4.20}$$

Now our task is to show that

$$M_1(r) := -\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds \text{ and } M_2(r) := \left(\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds\right)$$

are bounded by some constants depending only on $\kappa$, $\beta$, $\sigma$, $\ell$ and $\|c\|_{L^\infty([0,\ell])}$.

**(4) Bound $M_1(r)$ by studying the lower bound of $\varphi_2'/\varphi_2$.**

Note that

$$M_1(r) = -\varphi_1'(r)\int_\epsilon^r \varphi_2(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds = \int_\epsilon^r -\varphi_1'(r)\varphi_2(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds.$$

The term $-\varphi_1'(r)\varphi_2(s)$ has not been written as explicit function, and it's not convenient to deal with two linear independent solutions $\varphi_1$ and $\varphi_2$ at the same time. So we want to use the Wronskian $W = \varphi_1\varphi_2' - \varphi_2\varphi_1'$ to get rid of $-\varphi_1'(r)$. In fact, with (2.4.19) we have

$$-\varphi_1' = \frac{W - \varphi_1\varphi_2'}{\varphi_2} \leq \frac{W}{\varphi_2},$$

so

$$\begin{aligned} M_1(r) &= \int_\epsilon^r -\varphi_1'(r)\varphi_2(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds \leq \int_\epsilon^r \frac{W(r)}{\varphi_2(r)}\varphi_2(s)\frac{2(1+s)}{\sigma^2 sW(s)}ds \\ &= \frac{2}{\sigma^2}\int_\epsilon^r \frac{\varphi_2(s)}{\varphi_2(r)}\frac{W(r)}{W(s)}\frac{1+s}{s}ds. \end{aligned}$$

Since $\varphi_2' > 0$, $\epsilon \leq s \leq r$, we have $\varphi_2(s) \leq \varphi_2(r)$. So we firstly want to try

$$\begin{aligned} M_1(r) &\leq \frac{2}{\sigma^2}\int_\epsilon^r \frac{\varphi_2(s)}{\varphi_2(r)}\frac{W(r)}{W(s)}\frac{1+s}{s}ds \leq \frac{2}{\sigma^2}\int_\epsilon^r \frac{W(r)}{W(s)}\frac{1+s}{s}ds \\ &= \frac{2}{\sigma^2}\int_\epsilon^r \frac{r^{-\nu}e^{\mu r}}{s^{-\nu}e^{\mu s}}\left(\frac{1}{s}+1\right)ds \\ &= \frac{2}{\sigma^2}\int_\epsilon^r \frac{s^{\nu-1}}{r^\nu}e^{\mu(r-s)}ds + \frac{2}{\sigma^2}\int_\epsilon^r \left(\frac{s}{r}\right)^\nu e^{\mu(r-s)}ds, \end{aligned}$$

where the first term is hard to integrate and the second term

$$\frac{2}{\sigma^2}\int_\epsilon^r \left(\frac{s}{r}\right)^\nu e^{\mu(r-s)}ds \leq \frac{2}{\sigma^2}\int_\epsilon^r e^{\mu(r-s)}ds = \frac{2}{-\mu\sigma^2}(1-e^{\mu(r-\epsilon)}) \leq \frac{2}{\mu\sigma^2}e^{\mu(r-\epsilon)}$$

cannot have a bound that does not depend on $\epsilon$. So it's not enough to directly use $\frac{\varphi_2(s)}{\varphi_2(r)} < 1$ on $\epsilon \leq s \leq r$, but to have sharper upper bound for $\frac{\varphi_2(s)}{\varphi_2(r)}$. In particular, if we have

$$\frac{\varphi_2(s)}{\varphi_2(r)} \leq C_{\kappa,\beta,\sigma}e^{\vartheta(s-r)} \text{ for } \epsilon \leq s \leq r, \tag{2.4.21}$$

where $\vartheta > 0$ satisfies $\vartheta > \mu$ and $C_{\kappa,\beta,\sigma}$ is some constant only depending on $\kappa, \beta, \sigma$, then

$$\begin{aligned}
M_1(r) &\leq \frac{2}{\sigma^2}\int_\epsilon^r \frac{\varphi_2(s)}{\varphi_2(r)}\frac{W(r)}{W(s)}\frac{1+s}{s}ds \leq C_{\kappa,\beta,\sigma}\int_\epsilon^r e^{\vartheta(s-r)}\frac{W(r)}{W(s)}\frac{1+s}{s}ds \\
&= C_{\kappa,\beta,\sigma}\int_\epsilon^r e^{\vartheta(s-r)}\frac{r^{-\nu}e^{\mu r}}{s^{-\nu}e^{\mu s}}\left(\frac{1}{s}+1\right)ds \\
&= C_{\kappa,\beta,\sigma}\int_\epsilon^r \frac{s^{\nu-1}}{r^\nu}e^{-(\vartheta-\mu)(r-s)}ds + C_{\kappa,\beta,\sigma}\int_\epsilon^r \left(\frac{s}{r}\right)^\nu e^{-(\vartheta-\mu)(r-s)}ds \\
&\leq C_{\kappa,\beta,\sigma}\int_\epsilon^r \frac{s^{\nu-1}}{r^\nu}ds + C_{\kappa,\beta,\sigma}\int_\epsilon^r e^{-(\vartheta-\mu)(r-s)}ds \\
&= C_{\kappa,\beta,\sigma}\frac{1}{\nu}\left(1-\left(\frac{\epsilon}{r}\right)^\nu\right) + C_{\kappa,\beta,\sigma}\frac{1}{\vartheta-\mu}\left(1-e^{-(\vartheta-\mu)(r-\varepsilon)}\right) \\
&\leq C_{\kappa,\beta,\sigma}\frac{1}{\nu} + C_{\kappa,\beta,\sigma}\frac{1}{\vartheta-\mu}. \qquad (2.4.22)
\end{aligned}$$

Because of (2.4.21) and (2.4.22), our task is to find a bound for $\frac{\varphi_2(s)}{\varphi_2(r)}e^{\vartheta(r-s)}$ on $\epsilon \leq s \leq r$. Note that by the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\log\frac{\varphi_2(s)}{\varphi_2(r)}e^{\vartheta(r-s)} &= (\log\varphi_2(s) - \vartheta s) - (\log\varphi_2(r) - \vartheta r) \\
&= \int_r^s \frac{d}{d\rho}(\log\varphi_2(\rho) - \vartheta\rho)\,d\rho \\
&= \int_s^r \frac{d}{d\rho}(\vartheta\rho - \log\varphi_2(\rho))\,d\rho.
\end{aligned}$$

So

$$\log\frac{\varphi_2(s)}{\varphi_2(r)}e^{\vartheta(r-s)} = \int_s^r \frac{d}{d\rho}(\vartheta\rho - \log\varphi_2(\rho))\,d\rho = \int_s^r \vartheta - \frac{\varphi_2'(\rho)}{\varphi_2(\rho)}d\rho$$

If we can show that $\frac{\varphi_2'(\rho)}{\varphi_2(\rho)} \geq \underline{\psi}(\rho)$ for some $\underline{\psi}(\rho)$ on $\epsilon \leq s \leq \rho \leq r$, then

$$\begin{aligned}
\log\frac{\varphi_2(s)}{\varphi_2(r)}e^{\vartheta(r-s)} &= \int_s^r \vartheta - \frac{\varphi_2'(\rho)}{\varphi_2(\rho)}d\rho \leq \int_s^r \vartheta - \underline{\psi}(\rho)d\rho \\
&\leq \int_\epsilon^\infty \left(\vartheta - \underline{\psi}(\rho)\right)^+ d\rho. \qquad (2.4.23)
\end{aligned}$$

To make $\int_\epsilon^\infty \left(\vartheta - \underline{\psi}(\rho)\right)^+ d\rho$ finite, we want

$$\underline{\psi}(\rho) > \vartheta > \mu \text{ for large value of } \rho.$$

On the other hand, since $\varphi_2$ is a solution of (2.4.6),

$$\varphi_2'' + P\varphi_2' + Q\varphi_2 = 0 \text{ in } (\epsilon, \epsilon^{-1}).$$

so $\psi := \frac{\varphi_2'}{\varphi_2}$ satisfies

$$\begin{aligned}\psi' = \left(\frac{\varphi_2'}{\varphi_2}\right)' &= \frac{\varphi_2''\varphi_2 - \varphi_2'^2}{\varphi_2^2} = \frac{(-P\varphi_2' - Q\varphi_2)\varphi_2 - \varphi_2'^2}{\varphi_2^2} \\ &= \frac{-P\varphi_2' - Q\varphi_2}{\varphi_2} - \left(\frac{\varphi_2'}{\varphi_2}\right)^2 \\ &= -P\psi - \psi^2 - Q.\end{aligned}$$

Moreover, $\psi(\epsilon) = \frac{\varphi_2'(\epsilon)}{\varphi_2(\epsilon)} = \frac{0}{1} = 0$. So if we denote $\mathcal{F}[\psi] := \psi' + P\psi + \psi^2 + Q$, then $\psi$ is the solution of

$$\begin{cases} \mathcal{F}\psi = \psi' + P\psi + \psi^2 + Q = 0 & \text{in } (\epsilon, \epsilon^{-1}) \\ \psi(\epsilon) = 0 \end{cases} \tag{2.4.24}$$

And we could take $\underline{\psi}$ be a sub-solution of Problem (2.4.24). Note that $\psi = \frac{\varphi_2'}{\varphi_2} \geq 0$. So we want $\underline{\psi}$ satisfying

$$\begin{aligned}&0 \geq \underline{\psi}' + P\underline{\psi} + \underline{\psi}^2 + Q = \underline{\psi}' + \left(\frac{\nu}{r} - \mu\right)\underline{\psi} + \underline{\psi}^2 - \frac{2}{\sigma^2}\left(1 + \frac{c(r)}{r}\right), \\ &\underline{\psi}(\epsilon) = 0,\ \underline{\psi} \geq 0, \\ &\underline{\psi}(r) > \vartheta \text{ for large value of } r.\end{aligned}$$

It suffices to take $\underline{\psi}$ satisfying

$$\underline{\psi}' + \left(\underline{\psi} - \mu + \frac{\nu}{r}\right)\underline{\psi} - \frac{2}{\sigma^2} \leq 0 \tag{2.4.25}$$

$$\underline{\psi}(\epsilon) = 0,\ \underline{\psi} \geq 0, \tag{2.4.26}$$

$$\underline{\psi}(r) > \vartheta \text{ for large value of } r. \tag{2.4.27}$$

Because of (2.4.25) and (2.4.27), we want to take $\underline{\psi} = a - \frac{b}{r}$ with $a > \vartheta$. This function has good properties that it approaches to $a$ for large $r$ and it's zero at $r = \frac{b}{a}$. But to satisfy (2.4.26), we would like it satisfies that $\underline{\psi} = 0$ for $\epsilon \le r \le \frac{b}{a}$. So we want to try

$$\underline{\psi} = \left(a - \frac{b}{r}\right)^+ \text{ with } a > \vartheta \text{ and } b > a\epsilon.$$

With such $\underline{\psi}$, on $\epsilon \le r < \frac{b}{a}$, $\underline{\psi} = 0$, so

$$\mathcal{F}\underline{\psi} \le \underline{\psi}' + \left(\underline{\psi} - \mu + \frac{\nu}{r}\right)\underline{\psi} - \frac{2}{\sigma^2} < 0 \text{ on } \epsilon \le r < \frac{b}{a};$$

and on $\frac{b}{a} < r < \frac{1}{\epsilon}$, $\underline{\psi} = a - \frac{b}{r}$, so $\underline{\psi}' = \frac{b}{r^2}$, so

$$\begin{aligned}
\mathcal{F}\underline{\psi} &\le \underline{\psi}' + \left(\underline{\psi} - \mu + \frac{\nu}{r}\right)\underline{\psi} - \frac{2}{\sigma^2} \\
&= \frac{b}{r^2} + \left((a - \frac{b}{r}) - \mu + \frac{\nu}{r}\right)(a - \frac{b}{r}) - \frac{2}{\sigma^2} \\
&= \frac{b}{r^2} + a^2 - \frac{2ab}{r} + \frac{b^2}{r^2} - \mu a + \frac{\mu b}{r} + \frac{a\nu}{r} - \frac{b\nu}{r^2} - \frac{2}{\sigma^2} \\
&= a^2 - \mu a - \frac{2}{\sigma^2} + \frac{1}{r^2}\left[b - 2abr + b^2 + \mu br + a\nu r - b\nu\right] \\
&= \left(a^2 - \mu a - \frac{2}{\sigma^2}\right) + \frac{1}{r^2}\left[b + b^2 - b\nu - (2ab - \mu b - a\nu)r\right].
\end{aligned}$$

To make $\mathcal{F}\underline{\psi} \le 0$ and $a > \vartheta,\ b > a\epsilon$, we want

$$\begin{cases}
a > \vartheta > \mu,\ b > a\epsilon, \\
a^2 - \mu a - \frac{2}{\sigma^2} \le 0, \\
2ab - \mu b - a\nu \ge 0, \\
b + b^2 - b\nu - (2ab - \mu b - a\nu)\frac{b}{a} \le 0,
\end{cases}$$

which is equivalent with

$$\begin{cases}
a > \vartheta > \mu,\ b > a\epsilon, \\
\frac{1}{2}\left(\mu - \sqrt{\mu^2 + \frac{8}{\sigma^2}}\right) \le a \le \frac{1}{2}\left(\mu + \sqrt{\mu^2 + \frac{8}{\sigma^2}}\right), \\
2a - \mu \ge 0,\ b \ge \frac{a\nu}{2a-\mu}, \\
a - \mu \ge 0,\ b \ge \frac{a}{a-\mu}.
\end{cases}$$

So we take

$$a = \frac{1}{2}\left(\mu + \sqrt{\mu^2 + \frac{8}{\sigma^2}}\right) > \mu, \tag{2.4.28}$$

$$\vartheta = \frac{1}{3}a + \frac{2}{3}\mu, \mu < \vartheta < a, \tag{2.4.29}$$

$$b = \frac{a\nu}{2a - \mu} + \frac{a}{a - \mu} > 0, \tag{2.4.30}$$

$$\epsilon < \frac{b}{a}. \tag{2.4.31}$$

With these choice of parameters, $\underline{\psi} = \left(a - \frac{b}{r}\right)^+$ is a sub-solution of Problem (2.4.24). So

$$\frac{\varphi_2'(\rho)}{\varphi_2(\rho)} = \psi(\rho) \geq \underline{\psi}(\rho) = \left(a - \frac{b}{\rho}\right)^+ \text{ on } \epsilon \leq s \leq \rho \leq r. \tag{2.4.32}$$

Then by (2.4.23),

$$\begin{aligned}
\log \frac{\varphi_2(s)}{\varphi_2(r)} e^{\vartheta(r-s)} &= \int_s^r \vartheta - \frac{\varphi_2'(\rho)}{\varphi_2(\rho)} d\rho \leq \int_s^r \vartheta - \underline{\psi}(\rho) d\rho \\
&\leq \int_\epsilon^\infty \left(\vartheta - \underline{\psi}(\rho)\right)^+ d\rho \leq \int_\epsilon^\infty \left(\vartheta - \left(a - \frac{b}{\rho}\right)^+\right)^+ d\rho \\
&= \int_\epsilon^{\frac{b}{a}} \left(\vartheta - \left(a - \frac{b}{\rho}\right)^+\right)^+ d\rho + \int_{\frac{b}{a}}^\infty \left(\vartheta - \left(a - \frac{b}{\rho}\right)^+\right)^+ d\rho \\
&= \vartheta\left(\frac{b}{a} - \epsilon\right) + \int_{\frac{b}{a}}^\infty \left(\vartheta - a + \frac{b}{\rho}\right)^+ d\rho \\
&= \vartheta\left(\frac{b}{a} - \epsilon\right) + \int_{\frac{b}{a}}^{\frac{b}{a-\vartheta}} \left(\vartheta - a + \frac{b}{\rho}\right) d\rho \\
&= \vartheta\left(\frac{b}{a} - \epsilon\right) + (\vartheta - a)\left(\frac{b}{a - \vartheta} - \frac{b}{a}\right) + b\left(\log \frac{b}{a - \vartheta} - \log \frac{b}{a}\right) \\
&\leq \vartheta \frac{b}{a} + (a - \vartheta)\left(\frac{b}{a} - \frac{b}{a - \vartheta}\right) + b\left(\log \frac{b}{a - \vartheta} - \log \frac{b}{a}\right) = b \log \frac{a}{a - \vartheta}.
\end{aligned}$$

So

$$\frac{\varphi_2(s)}{\varphi_2(r)} e^{\vartheta(r-s)} \leq \exp\left(b \log \frac{a}{a - \vartheta}\right) = \left(\frac{a}{a - \vartheta}\right)^b, \; \epsilon \leq s \leq \rho \leq r. \tag{2.4.33}$$

Note that, by (2.4.28) - (2.4.31), $\left(\frac{a}{a-\vartheta}\right)^b$ only depends on $\kappa, \beta, \sigma$, so (2.4.21) holds true. Then

$$M_1(r) \leq C_{\kappa,\beta,\sigma} \tag{2.4.34}$$

follows from (2.4.22).

**(5) Bound $M_2(r)$ by studying the upper bound of $\varphi_2'/\varphi_2$.**

When $\epsilon \le \ell \le r \le \frac{1}{\epsilon}$ or $\ell \le \epsilon \le r \le \frac{1}{\epsilon}$, since $0 < \varphi_1\varphi_2' < W$ and $\varphi_1$ is decreasing,

$$\begin{aligned}
M_2(r) =& \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{1+\frac{1}{s}}{W(s)}ds \\
\le& \frac{2}{\sigma^2}\frac{W(r)}{\varphi_1(r)}\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\left(1+\frac{1}{s}\right)\frac{1}{W(s)}ds \\
\le& \frac{2}{\sigma^2}\int_r^{\frac{1}{\epsilon}}\left(1+\frac{1}{s}\right)\frac{W(r)}{W(s)}ds \le \frac{2}{\sigma^2}\left(1+\frac{1}{r}\right)\int_r^{\frac{1}{\epsilon}}\frac{r^{-\nu}e^{\mu r}}{s^{-\nu}e^{\mu s}}ds \\
\le& \frac{2}{\sigma^2}\left(1+\frac{1}{\ell}\right)\int_r^{\frac{1}{\epsilon}}\left(\frac{s}{r}\right)^{\nu} e^{-\mu(s-r)}ds
\end{aligned}$$

Using $\rho = s - r$ to change variable, we have

$$\begin{aligned}
&\int_r^{\frac{1}{\epsilon}}\left(\frac{s}{r}\right)^{\nu} e^{-\mu(s-r)}ds = \int_0^{\frac{1}{\epsilon}-r}\left(\frac{\rho+r}{r}\right)^{\nu} e^{-\mu\rho}d\rho \\
\le& \int_0^{\frac{1}{\epsilon}-r}\left(1+\frac{\rho}{\ell}\right)^{\nu} e^{-\mu\rho}d\rho \le \int_0^{\infty}\left(1+\frac{\rho}{\ell}\right)^{\nu} e^{-\mu\rho}d\rho,
\end{aligned}$$

and $\int_0^\infty \left(1+\frac{\rho}{\ell}\right)^{\nu} e^{-\mu\rho}d\rho$ is finite, depending only on $\kappa, \beta, \sigma, \ell$. So

$$M_2(r) \le C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}} \text{ for } \epsilon \le \ell \le r \le \frac{1}{\epsilon} \text{ or } \ell \le \epsilon \le r \le \frac{1}{\epsilon}. \tag{2.4.35}$$

When $\ell > \epsilon$ and $\epsilon < r < \ell$,

$$M_2(r) = \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{1+\frac{1}{s}}{W(s)}ds.$$

Now, unlike the previous case, we don't have a lower bound $\ell$ for $r$. So the point to deal with $\frac{1}{r}$. Note that $P(r) = \frac{\nu}{r} - \mu$ has the $\frac{1}{r}$ component. We can also use $W$ for help, in particular, $W' = -PW$. So

$$W' = -PW = \left(\mu - \frac{\nu}{r}\right)W = \mu W - \nu W\frac{1}{r}.$$

So

$$\frac{1}{r} = \frac{\mu W - W'}{\nu W}.$$

So

$$M_2(r) = \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{1+\frac{1}{s}}{W(s)}ds = \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{1+\frac{\mu W(s)-W'(s)}{\nu W(s)}}{W(s)}ds$$
$$=\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{\nu W(s)+\mu W(s)-W'(s)}{\nu W(s)^2}ds$$
$$=\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \frac{\varphi_1(s)}{W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \frac{\varphi_1(s)\mu}{\nu W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)\frac{-W'(s)}{\nu W(s)^2}ds$$
$$=\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \frac{\varphi_1(s)}{W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \frac{\varphi_1(s)\mu}{\nu W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \varphi_1(s)d\left(\frac{1}{W(s)}\right)$$
$$=\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \frac{\varphi_1(s)}{W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \frac{\varphi_1(s)\mu}{\nu W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\frac{\varphi_1(s)}{W(s)}\bigg|_r^{\frac{1}{\epsilon}}-\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\frac{\varphi_1'(s)}{W(s)}ds$$
$$=\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \left(1+\frac{\mu}{\nu}\right)\frac{\varphi_1(s)}{W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\frac{\varphi_1(s)}{W(s)}\bigg|_r^{\frac{1}{\epsilon}}-\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\frac{\varphi_1'(s)}{W(s)}ds$$

But $\varphi_1\left(\frac{1}{\epsilon}\right)=\frac{W(\frac{1}{\epsilon})}{\varphi_2'(\frac{1}{\epsilon})}$ by (2.4.12),

$$\frac{2}{\sigma^2}\varphi_2'(r)\frac{\varphi_1(s)}{W(s)}\bigg|_r^{\frac{1}{\epsilon}} = \frac{2}{\sigma^2}\varphi_2'(r)\frac{\varphi_1(\frac{1}{\epsilon})}{W(\frac{1}{\epsilon})}-\frac{2}{\sigma^2}\varphi_2'(r)\frac{\varphi_1(r)}{W(r)} \le \frac{2}{\sigma^2}\varphi_2'(r)\frac{\varphi_1(\frac{1}{\epsilon})}{W(\frac{1}{\epsilon})}$$
$$=\frac{2}{\sigma^2}\varphi_2'(r)\frac{W\left(\frac{1}{\epsilon}\right)}{W(\frac{1}{\epsilon})\varphi_2'\left(\frac{1}{\epsilon}\right)}=\frac{2}{\sigma^2}\frac{\varphi_2'(r)}{\varphi_2'\left(\frac{1}{\epsilon}\right)}.$$

So

$$M_2(r) = \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \left(1+\frac{\mu}{\nu}\right)\frac{\varphi_1(s)}{W(s)}ds+\frac{2}{\sigma^2}\varphi_2'(r)\frac{\varphi_1(s)}{W(s)}\bigg|_r^{\frac{1}{\epsilon}}-\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\frac{\varphi_1'(s)}{W(s)}ds$$
$$\le \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \left(1+\frac{\mu}{\nu}\right)\frac{\varphi_1(s)}{W(s)}ds+\frac{2}{\sigma^2}\frac{\varphi_2'(r)}{\varphi_2'\left(\frac{1}{\epsilon}\right)}-\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\frac{\varphi_1'(s)}{W(s)}ds \qquad (2.4.36)$$

Then let us do it term by term.

It's a little hard to directly deal with $\varphi_2'(r)\frac{\varphi_1(s)}{W(s)}$, since $0<\varphi_1(s)\varphi_2'(s)<W(s)$ can only give us $\varphi_2'(r)\frac{\varphi_1(s)}{W(s)}<\frac{\varphi_2'(r)}{\varphi_2'(s)}$, which is a ratio of $\varphi_2'$. To overcome this difficulty, we try to firstly use $\frac{\varphi_1'}{W}$ instead. In fact,

$$\left(\frac{\varphi_1'}{W}\right)'=\frac{\varphi_1''W-\varphi_1'W'}{W^2}=\frac{W(-P\varphi_1'-Q\varphi_1)-\varphi_1'W'}{W^2}$$
$$=\frac{-PW\varphi_1'-QW\varphi_1-\varphi_1'W'}{W^2}=\frac{W'\varphi_1'-QW\varphi_1-\varphi_1'W'}{W^2}$$
$$=\frac{-QW\varphi_1}{W^2}=\frac{-Q\varphi_1}{W}\ge\frac{2}{\sigma^2}\frac{\varphi_1}{W}.$$

So

$$\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\left(1+\frac{\mu}{\nu}\right)\frac{\varphi_1(s)}{W(s)}ds \le \varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\left(1+\frac{\mu}{\nu}\right)\left(\frac{\varphi_1'(s)}{W(s)}\right)'ds$$
$$=\varphi_2'(r)\left(1+\frac{\mu}{\nu}\right)\frac{\varphi_1'(s)}{W(s)}\bigg|_r^{\frac{1}{\epsilon}} = \varphi_2'(r)\left(1+\frac{\mu}{\nu}\right)\left(\frac{\varphi_1'(\frac{1}{\epsilon})}{W(\frac{1}{\epsilon})}-\frac{\varphi_1'(r)}{W(r)}\right).$$

But by (2.4.12), $\varphi_1'\left(\frac{1}{\epsilon}\right)=0$. Also, $0<-\varphi_1'\varphi_2<W$. So

$$\begin{aligned}&\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\left(1+\frac{\mu}{\nu}\right)\frac{\varphi_1(s)}{W(s)}ds\\ \le&\varphi_2'(r)\left(1+\frac{\mu}{\nu}\right)\left(\frac{\varphi_1'(\frac{1}{\epsilon})}{W(\frac{1}{\epsilon})}-\frac{\varphi_1'(r)}{W(r)}\right)=\varphi_2'(r)\left(1+\frac{\mu}{\nu}\right)\frac{-\varphi_1'(r)}{W(r)}\\ \le&\left(1+\frac{\mu}{\nu}\right)\frac{\varphi_2'(r)}{\varphi_2(r)}.\end{aligned} \tag{2.4.37}$$

For the second term in (2.4.36), since $\varphi_2$ is increasing, $\varphi_2(r)\le\varphi_2(\frac{1}{\epsilon})$, so

$$\frac{2}{\sigma^2}\frac{\varphi_2'(r)}{\varphi_2'\left(\frac{1}{\epsilon}\right)}=\frac{2}{\sigma^2}\frac{\varphi_2'(r)\varphi_2(r)}{\varphi_2(r)\varphi_2'\left(\frac{1}{\epsilon}\right)}\le\frac{2}{\sigma^2}\frac{\varphi_2'(r)\varphi_2(\frac{1}{\epsilon})}{\varphi_2(r)\varphi_2'\left(\frac{1}{\epsilon}\right)}=\frac{2}{\sigma^2}\frac{\varphi_2'(r)}{\varphi_2(r)}\left(\frac{\varphi_2'\left(\frac{1}{\epsilon}\right)}{\varphi_2(\frac{1}{\epsilon})}\right)^{-1}.$$

By (2.4.32),

$$\frac{\varphi_2'\left(\frac{1}{\epsilon}\right)}{\varphi_2(\frac{1}{\epsilon})}\ge\left(a-\frac{b}{\frac{1}{\epsilon}}\right)^+=(a-b\epsilon)^+=a-b\epsilon>\frac{1}{3}a,$$

if we assume that

$$\epsilon<\frac{2a}{3b}. \tag{2.4.38}$$

Then

$$\frac{2}{\sigma^2}\frac{\varphi_2'(r)}{\varphi_2'\left(\frac{1}{\epsilon}\right)}\le\frac{2}{\sigma^2}\frac{\varphi_2'(r)}{\varphi_2(r)}\left(\frac{\varphi_2'\left(\frac{1}{\epsilon}\right)}{\varphi_2(\frac{1}{\epsilon})}\right)^{-1}\le\frac{6}{a\sigma^2}\frac{\varphi_2'(r)}{\varphi_2(r)}. \tag{2.4.39}$$

For the third term in (2.4.36), since $0<-\varphi_1'\varphi_2<W$,

$$\begin{aligned}&-\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\frac{\varphi_1'(s)}{W(s)}ds=\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\frac{-\varphi_1'(s)}{W(s)}ds\\ \le&\frac{2}{\sigma^2}\int_r^{\frac{1}{\epsilon}}\frac{\varphi_2'(r)}{\varphi_2(s)}ds=\frac{2}{\sigma^2}\int_r^{\frac{1}{\epsilon}}\frac{\varphi_2'(r)}{\varphi_2(r)}\frac{\varphi_2(r)}{\varphi_2(s)}ds.\end{aligned}$$

Note that here $s > r$, so by (2.4.33), $\frac{\varphi_2(r)}{\varphi_2(s)} \le e^{-\vartheta(s-r)} \left(\frac{a}{a-\vartheta}\right)^b$, therefore

$$\begin{aligned}
&-\frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}} \frac{\varphi_1'(s)}{W(s)}ds \le \frac{2}{\sigma^2}\int_r^{\frac{1}{\epsilon}} \frac{\varphi_2'(r)}{\varphi_2(r)}\frac{\varphi_2(r)}{\varphi_2(s)}ds \\
\le&\frac{2}{\sigma^2}\int_r^{\frac{1}{\epsilon}} \frac{\varphi_2'(r)}{\varphi_2(r)}e^{-\vartheta(s-r)}\left(\frac{a}{a-\vartheta}\right)^b ds \\
=&C_{\kappa,\beta,\sigma}\frac{\varphi_2'(r)}{\varphi_2(r)}\int_r^{\frac{1}{\epsilon}} e^{-\vartheta(s-r)}ds = C_{\kappa,\beta,\sigma}\frac{\varphi_2'(r)}{\varphi_2(r)}\frac{1}{-\vartheta}e^{-\vartheta(s-r)}\Big|_r^{\frac{1}{\epsilon}} \\
=&C_{\kappa,\beta,\sigma}\frac{\varphi_2'(r)}{\varphi_2(r)}\frac{1}{-\vartheta}(e^{-\vartheta(\frac{1}{\epsilon}-r)} - e^{-\vartheta(r-r)}) \\
\le&C_{\kappa,\beta,\sigma}\frac{\varphi_2'(r)}{\varphi_2(r)}\frac{1}{\vartheta} = C_{\kappa,\beta,\sigma}\frac{\varphi_2'(r)}{\varphi_2(r)} \qquad (2.4.40)
\end{aligned}$$

Combine (2.4.36), (2.4.37), (2.4.39) and (2.4.40), we have that for the case of $\epsilon < r < \ell$,

$$\begin{aligned}
M_2(r) &\le \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\left(1+\frac{\mu}{\nu}\right)\frac{\varphi_1(s)}{W(s)}ds + \frac{2}{\sigma^2}\frac{\varphi_2'(r)}{\varphi_2'\left(\frac{1}{\epsilon}\right)} - \frac{2}{\sigma^2}\varphi_2'(r)\int_r^{\frac{1}{\epsilon}}\frac{\varphi_1'(s)}{W(s)}ds \\
&\le \left(1+\frac{\mu}{\nu}\right)\frac{\varphi_2'(r)}{\varphi_2(r)} + \frac{6}{a\sigma^2}\frac{\varphi_2'(r)}{\varphi_2(r)} + C_{\kappa,\beta,\sigma}\frac{\varphi_2'(r)}{\varphi_2(r)} \\
&= C_{\kappa,\beta,\sigma}\frac{\varphi_2'(r)}{\varphi_2(r)}. \qquad (2.4.41)
\end{aligned}$$

Now we just need to find the bound for $\frac{\varphi_2'(r)}{\varphi_2(r)}$ on $\epsilon < r < \ell$. By (2.4.24), $\psi(r) = \frac{\varphi_2'(r)}{\varphi_2(r)}$ satisfies

$$\begin{cases} \psi' + P\psi + \psi^2 + Q = 0 \quad \text{in } (\epsilon, \epsilon^{-1}) \\ \psi(\epsilon) = 0 \end{cases}$$

Note that $W' = -PW$, so

$$\begin{aligned}
\left(\frac{\psi}{W}\right)' &= \frac{\psi'W - \psi W'}{W^2} = \frac{W(-P\psi - \psi^2 - Q) - \psi W'}{W^2} \\
&= \frac{-PW\psi - W\psi^2 - QW - \psi W'}{W^2} = \frac{W'\psi - W\psi^2 - QW - \psi W'}{W^2} \\
&= \frac{-W\psi^2 - QW}{W^2} = \frac{-\psi^2 - Q}{W} \le \frac{-Q}{W}.
\end{aligned}$$

So for $r > \epsilon$, integrate both sides from $\epsilon$ to $r$ we have

$$\frac{\psi(r)}{W(r)} - \frac{\psi(\epsilon)}{W(\epsilon)} \le \int_\epsilon^r \frac{-Q(s)}{W(s)}ds.$$

Note that $\psi(\epsilon) = 0$, so

$$\begin{aligned}
\psi(r) \leq & W(r) \int_\epsilon^r \frac{-Q(s)}{W(s)} ds = W(r) \int_\epsilon^r \frac{\frac{2}{\sigma^2}\left(1 + \frac{c(s)}{s}\right)}{W(s)} ds \\
= & r^{-\nu} e^{\mu r} \int_\epsilon^r \frac{\frac{2}{\sigma^2}\left(1 + \frac{c(s)}{s}\right)}{s^{-\nu} e^{\mu s}} ds = \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \int_\epsilon^r \frac{s + c(s)}{s^{1-\nu} e^{\mu s}} ds \\
= & \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \int_\epsilon^r \frac{1}{s^{-\nu} e^{\mu s}} ds + \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \int_\epsilon^r \frac{c(s)}{s^{1-\nu} e^{\mu s}} ds \\
\leq & \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \int_\epsilon^r s^\nu ds + \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \|c\|_{L^\infty([0,r])} \int_\epsilon^r s^{\nu-1} ds \\
= & \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \frac{1}{\nu+1} s^{\nu+1}|_\epsilon^r + \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \|c\|_{L^\infty([0,r])} \frac{1}{\nu} s^\nu|_\epsilon^r \\
\leq & \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \frac{1}{\nu+1} r^{\nu+1} + \frac{2}{\sigma^2} r^{-\nu} e^{\mu r} \|c\|_{L^\infty([0,r])} \frac{1}{\nu} r^\nu \\
= & \frac{2}{\sigma^2} e^{\mu r} \frac{r}{\nu+1} + \frac{2}{\sigma^2} e^{\mu r} \frac{\|c\|_{L^\infty([0,r])}}{\nu}
\end{aligned} \tag{2.4.42}$$

By (2.4.41) and (2.4.42), for $\epsilon < r < \ell$,

$$\begin{aligned}
M_2(r) &\leq C_{\kappa,\beta,\sigma} \frac{\varphi_2'(r)}{\varphi_2(r)} \leq C_{\kappa,\beta,\sigma} \left( \frac{2}{\sigma^2} e^{\mu r} \frac{r}{\nu+1} + \frac{2}{\sigma^2} e^{\mu r} \frac{\|c\|_{L^\infty([0,r])}}{\nu} \right) \\
&\leq C_{\kappa,\beta,\sigma} \left( \frac{2}{\sigma^2} e^{\mu \ell} \frac{\ell}{\nu+1} + \frac{2}{\sigma^2} e^{\mu \ell} \frac{\|c\|_{L^\infty([0,\ell])}}{\nu} \right) \\
&= C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}}.
\end{aligned} \tag{2.4.43}$$

By (2.4.35) and (2.4.43), we have

$$M_2(r) \leq C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}}. \tag{2.4.44}$$

**(6) Bound $\frac{\partial u^\epsilon}{\partial r}$.**

By (2.4.20), (2.4.34) and (2.4.44), we have

$$\begin{aligned}
|\varphi'(r)| &\leq M_1(r) |||h|||_\epsilon + M_2(r) |||h|||_\epsilon \\
&\leq C_{\kappa,\beta,\sigma} |||h|||_\epsilon + C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}} |||h|||_\epsilon \\
&= C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}} |||h|||_\epsilon.
\end{aligned} \tag{2.4.45}$$

By (2.4.2) and (2.3.1),

$$|||h|||_\epsilon \leq |||F||| + \left|\left|\left| \frac{\partial u^\epsilon}{\partial T} \right|\right|\right|_\epsilon = C_{\kappa,\beta,\sigma}(|||F||| + |||F_T|||).$$

So with (2.4.45) we have

$$|\varphi'(r)| \le C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}}(\|\!|F|\!\|+\|\!|F_T|\!\|) \text{ for all } \epsilon \le r \le \frac{1}{\epsilon}.$$

But $\varphi(\cdot) := u^\epsilon(\cdot, T)$ and that $T > 0$ is arbitrary, so

$$\left\|\frac{\partial}{\partial r}u^\epsilon\right\|_{L^\infty\left((\epsilon,\frac{1}{\epsilon})\times(0,\infty)\right)} \le C_{\kappa,\beta,\sigma,\ell,\|c\|_{L^\infty([0,\ell])}}(\|\!|F|\!\| + \left\|\!\left|\frac{\partial F}{\partial T}\right|\!\right\|).$$

The proof is done. □

## 2.5 WELL-POSEDNESS

**Lemma 4** (Existence)**.** *Under the assumptions of Theorem 1, Problem* (1.5.1) *has a solution* $u$*. The solution* $u$ *satisfies* (1.5.3)*,* (1.5.4)*,* (1.5.5) *and* (1.5.6)*.*

*Proof.* For any $\delta > 0$ and any $0 < \epsilon \le \frac{\delta}{2}$ satisfying (2.1.2), for any $p > 1$, by the $L^p$ interior estimate for parabolic equation (2.1.1) (reference: [24] page 212), there exists a constant $C_{\delta,p} > 0$ which only depends on $\delta$ and $p$ s.t.

$$\begin{aligned}\|u^\epsilon\|_{W_p^{2,1}\left([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}]\right)} &\le C_{\delta,p}\left(\|F\|_{L^p\left([\frac{\delta}{2},\frac{2}{\delta}]\times[0,\frac{4}{\delta^2}]\right)} + \|u^\epsilon\|_{L^\infty\left([\frac{\delta}{2},\frac{2}{\delta}]\times[0,\frac{4}{\delta^2}]\right)}\right)\\ &\le C_{\delta,p,\kappa,\beta,\sigma}(\|\!|F|\!\| + \|\!|F_T|\!\|),\end{aligned}$$

where the second line follows from Lemma 1.5.3 and the assumption (2.0.1). Note that $C_{\delta,p,\kappa,\beta,\sigma}(\|\!|F|\!\| + \|\!|F_T|\!\|)$ is irrelevant w.r.t. $\epsilon$, and $W_p^{2,1}\left([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}]\right)$ is compact embedded in $C^{1,\frac{1}{2}}\left([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}]\right)$ (reference: [24] page 20). So there is a sequence of $\{\delta_m\}_{m=1}^\infty$ with $\lim_{m\to\infty}\delta_m = 0$ and for each $m$ there is a sequence of $\{\epsilon_n^m\}_{n=1}^\infty$ with $0 < \epsilon_n \le \frac{\delta_m}{2}$ and each $\epsilon_n^m$ satisfies (2.1.2) s.t $\lim_{n\to\infty}\epsilon_n^m = 0$ and the bounded sequence $\{u^{\epsilon_n^m}\}_{n=1}^\infty$ in $W_p^{2,1}\left([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}]\right)$ converge in $C^{1,\frac{1}{2}}\left([\delta,\frac{1}{\delta}]\times[0,\frac{1}{\delta^2}]\right)$. Then the diagonal sequence $\{u^{\epsilon_n^n}\}$ converge to some $u \in C^{1,\frac{1}{2}}((0,\infty)\times[0,\infty))$, which satisfies the first two equations in Problem (1.5.1). By Lemma 3, we have the bound of $\frac{\partial u^\epsilon}{\partial r}$ which does not depend on $\epsilon$. So $\frac{\partial u}{\partial r} \in L^\infty((0,\infty)\times(0,\infty))$. Therefore $u$ is a solution to Problem (1.5.1). Moreover, since the bounds in Lemma 1, Lemma 3 and Lemma 2 all do not depend on $\epsilon$, taking the limit we will have (1.5.3), (1.5.4), (1.5.5) and (1.5.6) hold. □

**Lemma 5** (Uniqueness)**.** *Under the assumptions of Theorem 1, Problem (1.5.1) has at most one solution.*

*Proof.* Assume that there exists two solutions $u_1$, $u_2$ to Problem (1.5.1). Let $w = u_1 - u_2$. Then

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) w = 0 & \text{in } (0,\infty)\times(0,\infty), \\ w(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial w}{\partial r} \in L^\infty((0,\infty)\times(0,\infty)). \end{cases} \tag{2.5.1}$$

We want to show that $w = 0$. To prove this, we want to study the norm of a function $\phi$ which satisfies $\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\phi \leq 0$. The idea comes from the trick used in the "energy method" section in [10] page 41.

For any function $\phi(r,T)$ s.t. $\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\phi \leq 0$,

$$\begin{aligned}
0 &\geq \int_{r_a}^{r_b} \phi^+(r,T)\left[\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\phi(r,T)\right] dr \\
&= \int_{r_a}^{r_b} \phi^+(r,T)\left(\frac{\partial \phi(r,T)}{\partial T} - \frac{\sigma^2 r}{2}\frac{\partial^2 \phi(r,T)}{\partial r^2} - (\kappa - \beta r)\frac{\partial \phi(r,T)}{\partial r} + (r+c)\phi(r,T)\right) dr \\
&= \int_{r_a}^{r_b} \phi^+(r,T)\left(\frac{\partial \phi^+(r,T)}{\partial T} - \frac{\sigma^2 r}{2}\frac{\partial^2 \phi^+(r,T)}{\partial r^2} - (\kappa - \beta r)\frac{\partial \phi^+(r,T)}{\partial r} + (r+c)\phi^+(r,T)\right) dr \\
&\geq \int_{r_a}^{r_b} \phi^+(r,T)\left(\frac{\partial \phi^+(r,T)}{\partial T} - \frac{\sigma^2 r}{2}\frac{\partial^2 \phi^+(r,T)}{\partial r^2} - (\kappa - \beta r)\frac{\partial \phi^+(r,T)}{\partial r}\right) dr.
\end{aligned}$$

But

$$\int_{r_a}^{r_b} \phi^+(r,T)\frac{\partial \phi^+(r,T)}{\partial T} dr = \int_{r_a}^{r_b} \frac{1}{2}\frac{\partial}{\partial T}\left(\phi^+(r,T)\right)^2 dr,$$

and by integration by parts we have

$$\begin{aligned}
&\int_{r_a}^{r_b} \phi^+(r,T)\left(-\frac{\sigma^2 r}{2}\frac{\partial^2 \phi^+(r,T)}{\partial r^2}\right) dr = \int_{r_a}^{r_b} -\frac{\sigma^2 r}{2}\phi^+(r,T)\frac{\partial}{\partial r}\left(\frac{\partial \phi^+(r,T)}{\partial r}\right) dr \\
&= -\frac{\sigma^2 r}{2}\phi^+(r,T)\frac{\partial \phi^+(r,T)}{\partial r}\bigg|_{r_a}^{r_b} + \int_{r_a}^{r_b} \frac{\partial \phi^+(r,T)}{\partial r}\frac{\partial}{\partial r}\left(\frac{\sigma^2 r}{2}\phi^+(r,T)\right) dr \\
&= -\frac{\sigma^2 r}{2}\phi^+(r,T)\frac{\partial \phi^+(r,T)}{\partial r}\bigg|_{r_a}^{r_b} + \int_{r_a}^{r_b} \frac{\partial \phi^+(r,T)}{\partial r}\frac{\sigma^2}{2}\phi^+(r,T)dr + \int_{r_a}^{r_b} \frac{\sigma^2 r}{2}\left(\frac{\partial \phi^+(r,T)}{\partial r}\right)^2 dr,
\end{aligned}$$

so

$$0 \geq \int_{r_a}^{r_b} \frac{1}{2}\frac{\partial}{\partial T}\left(\phi^+(r,T)\right)^2 dr - \frac{\sigma^2 r}{2}\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r}\bigg|_{r_a}^{r_b} + \int_{r_a}^{r_b}\frac{\partial\phi^+(r,T)}{\partial r}\frac{\sigma^2}{2}\phi^+(r,T)dr$$

$$+ \int_{r_a}^{r_b}\frac{\sigma^2 r}{2}\left(\frac{\partial\phi^+(r,T)}{\partial r}\right)^2 dr - \int_{r_a}^{r_b}(\kappa-\beta r)\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r}dr$$

$$\geq \int_{r_a}^{r_b} \frac{1}{2}\frac{\partial}{\partial T}\left(\phi^+(r,T)\right)^2 dr - \int_{r_a}^{r_b}\left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r}dr$$

$$+ \int_{r_a}^{r_b}\frac{\sigma^2 r}{2}\left(\frac{\partial\phi^+(r,T)}{\partial r}\right)^2 dr - \frac{\sigma^2 r}{2}\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r}\bigg|_{r_a}^{r_b}.$$

Apply the formula $\mathfrak{ab} \leq \frac{\mathfrak{a}^2}{4\mathfrak{c}} + \mathfrak{c}\mathfrak{b}^2$ with $\mathfrak{a} = \left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)\phi^+(r,T)$, $\mathfrak{b} = \frac{\partial\phi^+(r,T)}{\partial r}$, $\mathfrak{c} = \frac{\sigma^2 r}{2}$, we have $\left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r} \leq \frac{\left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)^2\phi^+(r,T)^2}{2\sigma^2 r} + \frac{\sigma^2 r}{2}\left(\frac{\partial\phi^+(r,T)}{\partial r}\right)^2$, so

$$0 \geq \int_{r_a}^{r_b} \frac{1}{2}\frac{\partial}{\partial T}\left(\phi^+(r,T)\right)^2 dr - \int_{r_a}^{r_b}\frac{\left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)^2\phi^+(r,T)^2}{2\sigma^2 r}dr$$

$$- \frac{\sigma^2 r}{2}\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r}\bigg|_{r_a}^{r_b}.$$

Then

$$\frac{\partial}{\partial T}\int_{r_a}^{r_b}\left(\phi^+(r,T)\right)^2 dr \leq \int_{r_a}^{r_b}\frac{\left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)^2\phi^+(r,T)^2}{\sigma^2 r}dr + \sigma^2 r\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r}\bigg|_{r_a}^{r_b}$$

$$\leq \left(\sup_{r_a\leq r\leq r_b}\frac{\left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)^2}{\sigma^2 r}\right)\int_{r_a}^{r_b}(\phi^+(r,T))^2 dr + \sigma^2 r\phi^+(r,T)\frac{\partial\phi^+(r,T)}{\partial r}\bigg|_{r_a}^{r_b}.$$

If we know

$$\frac{\partial\phi^+(r,T)}{\partial r}\bigg|_{r=r_b} \leq 0 \text{ and } \frac{\partial\phi^+(r,T)}{\partial r}\bigg|_{r=r_a} \geq 0, \tag{2.5.2}$$

we will have

$$\frac{\partial}{\partial T}\int_{r_a}^{r_b}\left(\phi^+(r,T)\right)^2 dr \leq \left(\sup_{r_a\leq r\leq r_b}\frac{\left(-\frac{\sigma^2}{2}+\kappa-\beta r\right)^2}{\sigma^2 r}\right)\int_{r_a}^{r_b}(\phi^+(r,T))^2 dr;$$

and then by the Gronwall's inequality we will have

$$\int_{r_a}^{r_b}\left(\phi^+(r,T)\right)^2 dr = 0,$$

which will imply that

$$\phi(r, T) = 0. \tag{2.5.3}$$

But we cannot directly use this method for $\phi = w$, since $w_r \in L^\infty$ and we cannot have (2.5.2). We need to use some auxiliary positive function $\psi(r)$ with $\mathcal{L}\psi = 0$ s.t. $\lim_{r\downarrow 0} \psi = -\infty$ and $\lim_{r\uparrow\infty} \psi = +\infty$ and apply this method for $\phi = w - \alpha\psi$ with some $\alpha > 0$.

Let $\psi(r)$ be the one in Lemma 3.1 of [13] with $a^* = b^* = 0$, which satisfies

$$\mathcal{L}\psi = 0,\ \psi > 0 \text{ in } (0,\infty),\ \lim_{r\downarrow 0} \psi = -\infty,\ \lim_{r\uparrow\infty} \psi = \infty.$$

Then for any $\alpha > 0$, $\phi(r, T) := w(r, T) - \alpha\psi(r)$ satisfies

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\phi = \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)(w - \alpha\psi) = -\alpha c\psi \leq 0.$$

And by (2.5.3) we have

$$\phi(r, T) = 0. \tag{2.5.4}$$

Now, assume for contradiction that there exists some $(r_0, t_0)$ s.t. $w(r_0, t_0) \neq 0$. Without loss of generality, assume that $w(r_0, t_0) > 0$. For the case when $w(r_0, t_0) < 0$, consider $-w = u_2 - u_1$. Pick $\alpha = \frac{w(r_0,t_0)}{3\psi(r_0)} > 0$, then

$$\phi(r_0, t_0) = w(r_0, t_0) - \alpha\psi(r_0) > 0.$$

This contradicts with (2.5.4) and therefore our assumption is false. That is, $w(r, T) = 0$ on $(0, \infty) \times (0, \infty)$. So we have the uniqueness of the solution for (1.5.1).

□

## 3.0 EXPECTED VALUES OF CDSS

In this chapter we shall apply the result of the previous section to study the mathematical problems for the CDSs. Namely, we establish the well-posedness of the problem.

### 3.1 CDS TRANSFERABLE BY SELLER AT MOST ONE TIME

In this section we prove Theorem 2, which studies the following problem (i.e. (1.3.14)):

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_2\right) v = \Lambda - q & \text{in } (0,\infty)\times(0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) w = \Lambda - q - \Lambda_1 v^+ & \text{in } (0,\infty)\times(0,\infty), \\ u(\cdot,0) = 0,\ v(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} u \in L^\infty((0,\infty)\times(0,\infty)), \\ \frac{\partial}{\partial r} v \in L^\infty((0,\infty)\times(0,\infty)), \end{cases}$$

where $\mathcal{L}$ is as in (1.3.7), $\kappa$, $\beta$, $\sigma$ and $q$ are positive constants and $\Lambda(\cdot)$, $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ are given non-negative functions on $(0,\infty)$. Theorem 2 assumes that (i.e. (1.5.7), (1.5.8)):

$$|||\Lambda||| < \infty, \quad |||\Lambda_1||| < \infty, \quad \Lambda, \Lambda_1, \Lambda_2 \in L^\infty([0,\ell])\ \forall\, \ell > 0,$$

$$\|\Lambda\|_{L^\infty((0,\infty))} < \infty \text{ or } \|\Lambda_1\|_{L^\infty((0,\infty))} < \infty.$$

And the conclusion is that Problem (1.3.14) admits a unique solution.

*Proof of Theorem 2.* Since $\Lambda - q$ does not depend on $T$,

$$\left|\left|\left|\frac{\partial(\Lambda - q)}{\partial T}\right|\right|\right| = |||0||| = 0 < \infty.$$

By assumption (1.5.7),

$$\begin{aligned}|||\Lambda - q||| &= \sup_{r>0,T>0}\left|\frac{\Lambda(r) - q}{1+r}\right| = \sup_{r>0}\left|\frac{\Lambda(r) - q}{1+r}\right| \\ &\le \sup_{r>0}\left|\frac{\Lambda(r)}{1+r}\right| + q\sup_{r>0}\left|\frac{1}{1+r}\right| \\ &\le |||\Lambda||| + q < \infty.\end{aligned}$$

By assumptions for Problem (1.3.14), $\Lambda + \Lambda_2 > 0$, and by assumption (1.5.7),

$$\Lambda + \Lambda_2 \in L^\infty([0,\ell])\ \forall\, \ell > 0.$$

Therefore, condition 2.0.1 in the Theorem 1 below is satisfied with $c = \Lambda + \Lambda_2$ and $F = \Lambda - q$. Apply Theorem 1 with $c = \Lambda + \Lambda_2$ and $F = \Lambda - q$, we know the following Problem (3.1.1)

$$\begin{cases}\left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_2\right) v = \Lambda - q & \text{in } (0,\infty)\times(0,\infty), \\ v(\cdot, 0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} v \in L^\infty((0,\infty)\times(0,\infty)), \end{cases} \tag{3.1.1}$$

has a unique solution $v$ and $v$ satisfies

$$\|v\|_{L^\infty((0,\infty)\times(0,\infty))} \le C_{\kappa,\beta,\sigma}|||\Lambda - q||| < \infty, \tag{3.1.2}$$

$$\left|\left|\left|\frac{\partial v}{\partial T}\right|\right|\right| \le C_{\kappa,\beta,\sigma}|||\Lambda - q||| < \infty, \tag{3.1.3}$$

$$\left\|\frac{\partial v}{\partial T}\right\|_{L^\infty((0,\infty)\times(0,\infty))} \le C_{\kappa,\beta,\sigma}\,\|\Lambda - q\|_{L^\infty((0,\infty))} < \infty \text{ if } \|\Lambda\|_{L^\infty((0,\infty))} < \infty. \tag{3.1.4}$$

Now let's consider $u$. If $\|\Lambda_1\|_{L^\infty((0,\infty))} < \infty$ in (1.5.8) is true,

$$\left|\left|\left|\frac{\partial}{\partial T}(\Lambda - q - \Lambda_1 v^+)\right|\right|\right| = \left|\left|\left|\Lambda_1\frac{\partial v^+}{\partial T}\right|\right|\right| \le \|\Lambda_1\|_{L^\infty((0,\infty))}\left|\left|\left|\frac{\partial v^+}{\partial T}\right|\right|\right| < \infty;$$

otherwise $\|\Lambda\|_{L^\infty((0,\infty))} < \infty$ in (1.5.8) is true, and with (3.1.4) we have

$$\left|\left|\left|\frac{\partial}{\partial T}(\Lambda - q - \Lambda_1 v^+)\right|\right|\right| = \left|\left|\left|\Lambda_1\frac{\partial v^+}{\partial T}\right|\right|\right| \le |||\Lambda_1|||\left\|\frac{\partial v^+}{\partial T}\right\|_{L^\infty((0,\infty)\times(0,\infty))} < \infty.$$

In either case, we have

$$\left|\left|\left|\frac{\partial}{\partial T}(\Lambda - q - \Lambda_1 v^+)\right|\right|\right| \leq \infty. \tag{3.1.5}$$

By (1.5.7) and (3.1.2) we have

$$\left|\left|\left|\Lambda - q - \Lambda_1 v^+\right|\right|\right| \leq |||\Lambda||| + q + |||\Lambda_1||| \|v\|_{L^\infty((0,\infty)\times(0,\infty))} < \infty. \tag{3.1.6}$$

By assumptions for Problem (1.3.14), $\Lambda + \Lambda_1 + \Lambda_2 > 0$, and by assumption (1.5.7),

$$\Lambda + \Lambda_1 + \Lambda_2 \in L^\infty([0, \ell])\ \forall\, \ell > 0.$$

Therefore, with (3.1.5) and (3.1.6), condition 2.0.1 in the Theorem 1 below is satisfied with $c = \Lambda + \Lambda_1 + \Lambda_2$ and $F = \Lambda - q - \Lambda_1 v^+$. Apply Theorem 1 with $c = \Lambda + \Lambda_1 + \Lambda_2$ and $F = \Lambda - q - \Lambda_1 v^+$, we know the following Problem (3.1.7)

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u = \Lambda - q - \Lambda_1 v^+ & \text{in } (0,\infty) \times (0,\infty), \\ u(\cdot, 0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} u \in L^\infty((0,\infty) \times (0,\infty)), \end{cases} \tag{3.1.7}$$

has a unique solution $u$.

Therefore, Problem (1.3.14) has a unique solution $[v, u]$. □

## 3.2 CDS TRANSFERABLE BY SELLER AND BUYER AT MOST ONE TIME

In this section we prove Theorem 3, which studies the following problem (i.e. (1.3.20)):

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u_4 = \Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+ & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1\right) v_4 = \Lambda - q - \Lambda_1 u_1^- & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_2\right) w_4 = q - \Lambda - \Lambda_2 u_1^+ & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) u_1 = \Lambda - q & \text{in } (0,\infty)^2, \\ u_1(\cdot,0) = w_4(\cdot,0) = v_4(\cdot,0) = u_4(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial u_1}{\partial r}, \frac{\partial w_4}{\partial r}, \frac{\partial v_4}{\partial r}, \frac{\partial u_4}{\partial r} \in L^\infty((0,\infty)\times(0,\infty)), \end{cases}$$

where $\mathcal{L}$ is the operator defined as (1.3.7), $\kappa$, $\beta$, $\sigma$ and $q$ are positive constants, $\Lambda$, $\Lambda_1$ and $\Lambda_2$ are non-negative functions of $r$ s.t. (i.e. (1.5.9))

$$\|\Lambda\|_{L^\infty((0,\infty))} < \infty, \|\Lambda_1\|_{L^\infty((0,\infty))} < \infty, \|\Lambda_2\|_{L^\infty((0,\infty))} < \infty.$$

The conclusion of Theorem 3 is that there is a unique solution to Problem (1.3.20).

*Proof of Theorem 3.* With (1.5.9), apply Theorem 1 with $c = \Lambda$ and $F = \Lambda - q$, we know that

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) u_1 = \Lambda - q & \text{in } (0,\infty)\times(0,\infty), \\ u_1(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} u_1 \in L^\infty((0,\infty)\times(0,\infty)), \end{cases}$$

admit a unique solution $u_1$ and $u_1$ satisfies

$$\|u_1\|_{L^\infty((0,\infty)\times(0,\infty))} \leq C_{\kappa,\beta,\sigma} \|\Lambda - q\|_{L^\infty((0,\infty))} < \infty, \tag{3.2.1}$$

$$\left\|\frac{\partial u_1}{\partial T}\right\|_{L^\infty((0,\infty)\times(0,\infty))} \leq C_{\kappa,\beta,\sigma} \|\Lambda - q\|_{L^\infty((0,\infty))} < \infty. \tag{3.2.2}$$

By (1.5.9) and (3.2.1),

$$\begin{aligned}&\|q-\Lambda-\Lambda_2 u_1^+\|_{L^\infty((0,\infty)^2)}\\ \le& q+\|\Lambda\|_{L^\infty((0,\infty)^2)}+\|\Lambda_2\|_{L^\infty((0,\infty)^2)}\|u_1\|_{L^\infty((0,\infty)^2)}<\infty;\end{aligned}$$

by (1.5.9) and (3.2.2),

$$\left\|\frac{\partial(q-\Lambda-\Lambda_2 u_1^+)}{\partial T}\right\|=\left\|\Lambda_2\frac{\partial u_1^+}{\partial T}\right\|\le\|\Lambda_2\|_{L^\infty((0,\infty))}\left\|\frac{\partial u_1^+}{\partial T}\right\|_{L^\infty((0,\infty))}<\infty.$$

So apply Theorem 1 with $c=\Lambda+\Lambda_2$ and $F=q-\Lambda-\Lambda_2 u_1^+$, we know that

$$\begin{cases}\left(\frac{\partial}{\partial T}+\mathcal{L}+\Lambda+\Lambda_2\right)w_4=q-\Lambda-\Lambda_2 u_1^+ & \text{in } (0,\infty)\times(0,\infty),\\ w_4(\cdot,0)=0 & \text{in } (0,\infty),\\ \frac{\partial}{\partial r}w_4\in L^\infty((0,\infty)\times(0,\infty)),\end{cases}$$

admit a unique solution $w_4$ and $w_4$ satisfies

$$\|w_4\|_{L^\infty((0,\infty)\times(0,\infty))}\le C_{\kappa,\beta,\sigma}\left\|q-\Lambda-\Lambda_2 u_1^+\right\|_{L^\infty((0,\infty))}<\infty, \tag{3.2.3}$$

$$\left\|\frac{\partial w_4}{\partial T}\right\|_{L^\infty((0,\infty)\times(0,\infty))}\le C_{\kappa,\beta,\sigma}(\left\|q-\Lambda-\Lambda_2 u_1^+\right\|_{L^\infty((0,\infty))}+\left\|\frac{\partial u_1}{\partial T}\right\|_{L^\infty((0,\infty))})<\infty. \tag{3.2.4}$$

Similarly, apply Theorem 1 with $c=\Lambda+\Lambda_1$ and $F=\Lambda-q-\Lambda_1 u_1^-$, we know that

$$\begin{cases}\left(\frac{\partial}{\partial T}+\mathcal{L}+\Lambda+\Lambda_1\right)v_4=\Lambda-q-\Lambda_1 u_1^- & \text{in } (0,\infty)\times(0,\infty),\\ v_4(\cdot,0)=0 & \text{in } (0,\infty),\\ \frac{\partial}{\partial r}v_4\in L^\infty((0,\infty)\times(0,\infty)),\end{cases}$$

admit a unique solution $v_4$ and $v_4$ satisfies

$$\|v_4\|_{L^\infty((0,\infty)\times(0,\infty))}\le C_{\kappa,\beta,\sigma}\left\|\Lambda-q-\Lambda_1 u_1^-\right\|_{L^\infty((0,\infty))}<\infty, \tag{3.2.5}$$

$$\left\|\frac{\partial v_4}{\partial T}\right\|_{L^\infty((0,\infty)\times(0,\infty))}\le C_{\kappa,\beta,\sigma}(\left\|\Lambda-q-\Lambda_1 u_1^-\right\|_{L^\infty((0,\infty))}+\left\|\frac{\partial u_1}{\partial T}\right\|_{L^\infty((0,\infty))})<\infty. \tag{3.2.6}$$

By (1.5.9), (3.2.3) and (3.2.5),

$$\begin{aligned}&\left\|\Lambda-q+\Lambda_2 v_4^+-\Lambda_1 w_4^+\right\|_{L^\infty((0,\infty))}\\ \le&\|\Lambda\|_{L^\infty((0,\infty))}+q+\|\Lambda_2\|_{L^\infty((0,\infty))}\|v_4\|_{L^\infty((0,\infty))}+\|\Lambda_1\|_{L^\infty((0,\infty))}\|w_4\|_{L^\infty((0,\infty))}<\infty;\end{aligned}$$

by (1.5.9), (3.2.4) and (3.2.6),

$$\left\|\left|\frac{\partial(\Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+)}{\partial T}\right|\right\| = \left\|\left|\Lambda_2 \frac{\partial v_4^+}{\partial T} - \Lambda_1 \frac{\partial w_4^+}{\partial T}\right|\right\|$$
$$\leq \|\Lambda_2\|_{L^\infty((0,\infty))} \left\|\frac{\partial v_4^+}{\partial T}\right\|_{L^\infty((0,\infty))} + \|\Lambda_1\|_{L^\infty((0,\infty))} \left\|\frac{\partial w_4^+}{\partial T}\right\|_{L^\infty((0,\infty))} < \infty.$$

So apply Theorem 1 with $c = \Lambda + \Lambda_1 + \Lambda_2$ and $F = \Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+$, we know that

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u_4 = \Lambda - q + \Lambda_2 v_4^+ - \Lambda_1 w_4^+ & \text{in } (0,\infty) \times (0,\infty), \\ u_4(\cdot, 0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} u_4 \in L^\infty((0,\infty) \times (0,\infty)), \end{cases}$$

admit a unique solution $u_4$.

Therefore, Problem (1.3.20) has a unique solution $[u_1, w_4, v_4, u_4]$. □

## 3.3 CDS TRANSFERABLE BY SELLER AND BUYER ANY NUMBER OF TIMES

In this section we prove Theorem 4, which studies the well-posedness of Problem (1.3.21):

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) v = \Lambda - q + R_2 \Lambda_2 v^+ - \Lambda_1 w^+ & \text{in } (0,\infty)^2, \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) w = q - \Lambda - \Lambda_2 v^+ + R_1 \Lambda_1 w^+ & \text{in } (0,\infty)^2, \\ v(\cdot, 0) = w(\cdot, 0) = 0 & \text{in } (0,\infty), \\ \frac{\partial v}{\partial r},\ \frac{\partial w}{\partial r} \in L^\infty((0,\infty)^2), \end{cases}$$

where $\mathcal{L}$ is the operator defined as (1.3.7), $\kappa$, $\beta$, $\sigma$ and $q$ are positive constants, $\Lambda$, $\Lambda_1$ and $\Lambda_2$ are given non-negative functions satisfying

$$|||\Lambda||| < \infty, \quad \Lambda \in L^\infty([0,\ell])\ \forall\, \ell > 0,$$
$$\|\Lambda_1\|_{L^\infty((0,\infty))} < \infty, \qquad \|\Lambda_2\|_{L^\infty((0,\infty))} < \infty.$$

*Proof of Theorem 4.* Denote $c := \Lambda + \Lambda_1 + \Lambda_2$, $u_1 = v$, $u_2 = w$. We want to study

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u_1 = \Lambda - q + R_2\Lambda_2 u_1^+ - \Lambda_1 u_2^+ & \text{in } \Omega \times (0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) u_2 = q - \Lambda - \Lambda_2 u_1^+ + R_1\Lambda_1 u_2^+ & \text{in } \Omega \times (0,\infty), \\ u_1(\cdot,0) = u_2(\cdot,0) = 0 & \text{on } \bar{\Omega}. \end{cases} \tag{3.3.1}$$

Denote $\mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $F(\mathbf{u}) := (\Lambda - q)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} R_2\Lambda_2 & -\Lambda_1 \\ -\Lambda_2 & R_1\Lambda_1 \end{bmatrix}\mathbf{u}^+$, then

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) \mathbf{u} = F(\mathbf{u}) & \text{in } \Omega \times (0,\infty), \\ \mathbf{u}(\cdot,0) = \mathbf{0} & \text{on } \bar{\Omega}. \end{cases}$$

Let $\tilde{X} = C^{1,\frac{1}{2}}((0,\infty)\times[0,\infty))$ and $X := \tilde{X} \times \tilde{X}$. Define a map $\mathcal{T} : X \to X$ by $\mathbf{v} \mapsto \mathbf{u}$ where $\mathbf{u}$ is the solution of

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + c\right) \mathbf{u} = F(\mathbf{v}) & \text{in } \Omega \times (0,\infty), \\ \mathbf{u}(\cdot,0) = \mathbf{0} & \text{on } \bar{\Omega}. \end{cases} \tag{3.3.2}$$

The map $\mathcal{T}$ is well-defined because of the uniqueness and existence of the solution for Problem (3.3.3), which follows from Theorem 1. Note that $\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)\mathcal{T}(\mathbf{u}) = F(\mathbf{u})$, so

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c\right)(\mathcal{T}(\mathbf{u}_1) - \mathcal{T}(\mathbf{u}_2)) = F(\mathbf{u}_1) - F(\mathbf{u}_2). \tag{3.3.3}$$

Let $\mathbf{w}(r,T) = e^{-KT}(\mathcal{T}(\mathbf{u}_1) - \mathcal{T}(\mathbf{u}_2))$. Then

$$\left(\frac{\partial}{\partial T} + \mathcal{L} + c + K\right)\mathbf{w} = e^{-KT}(F(\mathbf{u}_1) - F(\mathbf{u}_2)).$$

Then

$$|\mathbf{w}| \le \frac{1}{K}\left|\!\left|\!\left| e^{-KT}(F(\mathbf{u}_1) - F(\mathbf{u}_2)) \right|\!\right|\!\right|,$$

where $|||M||| := \left\| \frac{M}{1+\frac{r}{K}} \right\|_\infty$. So

$$\|e^{-KT}(\mathcal{T}(\mathbf{u}_1) - \mathcal{T}(\mathbf{u}_2))\|_\infty \le \frac{1}{K}\left|\!\left|\!\left| \begin{bmatrix} R_2\Lambda_2 & -\Lambda_1 \\ -\Lambda_2 & R_1\Lambda_1 \end{bmatrix} \right|\!\right|\!\right| \|e^{-KT}(\mathbf{u}_1 - \mathbf{u}_2))\|_\infty.$$

Define norm $\|\cdot\|_*$ on $X$ by $\|\mathbf{u}\|_* = \|e^{-KT}\mathbf{u}\|_\infty$, then $\mathcal{T}$ is a contraction on Banach space $X$ when $K$ is large enough. Then by the Hahn-Banach Fixed Point Theorem, there exists a unique $\mathbf{u}^* \in X$ s.t. $\mathcal{T}(\mathbf{u}^*) = \mathbf{u}^*$. Such $u^*$ is the unique solution of problem (3.3.1). □

## 3.4 RELATION BETWEEN THE STRUCTURE MODEL AND THE INTENSITY MODEL

In this section we prove Theorem 5, which studies the relation between Problem (i.e. (1.5.13))

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right) v^* = \Lambda - q & \text{in } (0,R)\times(0,\infty), \\ v^* = 0 & \text{in } [R,\infty)\times(0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1\right) u^* = \Lambda - q - \Lambda_1 v^{*+} & \text{in } (0,R)\times(0,\infty), \\ u^* = 0 & \text{in } [R,\infty)\times(0,\infty), \\ v^*(\cdot,0) = 0,\ u^*(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} v^* \in L^\infty((0,\infty)^2),\ \frac{\partial}{\partial r} u^* \in L^\infty((0,\infty)^2), \end{cases}$$

and Problem (i.e. (1.5.15))

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \gamma H(r-R)\right) v^\gamma = \Lambda - q & \text{in } (0,\infty)\times(0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \gamma H(r-R)\right) u^\gamma = \Lambda - q - \Lambda_1 (v^\gamma)^- & \text{in } (0,\infty)\times(0,\infty), \\ v^\gamma(\cdot,0) = 0,\ u^\gamma(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} v^\gamma \in L^\infty((0,\infty)\times(0,\infty)), \\ \frac{\partial}{\partial r} u^\gamma \in L^\infty((0,\infty)\times(0,\infty)), \end{cases}$$

where $\kappa$, $\beta$, $\sigma$, $q$ and $R$ are positive constants, $\mathcal{L}$ is as in (1.3.7), $\Lambda$, $\Lambda_1$ and $\Lambda_2$ satisfy (i.e. (1.5.16), (1.5.17))

$$\begin{aligned} &|||\Lambda_1||| < \infty,\ |||\Lambda_2||| < \infty, \quad \Lambda_1, \Lambda_2 \in L^\infty([0,\ell])\ \forall\, \ell > 0, \\ &\|\Lambda\|_{L^\infty((0,\infty))} < \infty. \end{aligned}$$

Theorem 5 denotes by $[v^*, u^*]$ the solution of the structure model (1.5.13) and denotes by $[v^\gamma, u^\gamma]$ the solution of the intensity model (1.5.15) (for each $\gamma > 0$) and concludes that (i.e. (1.5.18))

$$\begin{aligned} &\lim_{\gamma\to\infty} v^\gamma = v^* \text{ locally uniformly in } (0,\infty)\times[0,\infty), \\ &\lim_{\gamma\to\infty} u^\gamma = u^* \text{ locally uniformly in } (0,\infty)\times[0,\infty). \end{aligned}$$

*Proof of Theorem 5.* Let us prove this theorem in two steps.

**(1) Prove (1.5.18).**

Apply Theorem (1) with $c = \Lambda + \gamma H(r-R)$ and $F = \Lambda$, we know that Problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \gamma H(r-R)\right) v_1^\gamma = K\Lambda & \text{in } (0,\infty)^2, \\ v_1^\gamma(\cdot, 0) = 0 & \text{in } (0,\infty), \\ \partial_r v_1^\gamma \in L^\infty((0,\infty)^2), \end{cases}$$

admits a unique solution $v_1^\gamma$; apply Theorem (1) with $c = \Lambda + \gamma H(r-R)$ and $F = q$, we know that Problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \gamma H(r-R)\right) w_1^\gamma = q & \text{in } (0,\infty)^2, \\ w_1^\gamma(\cdot, 0) = 0 & \text{in } (0,\infty), \\ \partial_r w_1^\gamma \in L^\infty((0,\infty)^2), \end{cases}$$

admits a unique solution $w_1^\gamma$. By the maximum principle, we know that

$$v_1^\gamma \geq 0 \text{ and } w_1^\gamma \geq 0, \ \forall \ \gamma > 0.$$

Moreover, for any $0 < \gamma < \hat{\gamma}$,

$$\begin{aligned} &\left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \gamma H(r-R)\right)\left(v_1^\gamma - v_1^{\hat{\gamma}}\right) = \Lambda - \Lambda + (\hat{\gamma} - \gamma) H(r-R) v_1^{\hat{\gamma}} \geq 0, \\ &\left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \gamma H(r-R)\right)\left(w_1^\gamma - w_1^{\hat{\gamma}}\right) = q - q + (\hat{\gamma} - \gamma) H(r-R) w_1^{\hat{\gamma}} \geq 0, \end{aligned}$$

so by the maximum principle again, we have

$$v_1^\gamma \geq v_1^{\hat{\gamma}}, \ w_1^\gamma \geq w_1^{\hat{\gamma}}.$$

So $v_1^\gamma$ and $w_1^\gamma$ are decreasing w.r.t. of $\gamma$. Since they are decreasing and have lower bound zero, by the monotone convergence theorem,

$$\lim_{\gamma \to \infty} v_1^\gamma = v_1^\infty \text{ pointwisely}, \ \lim_{\gamma \to \infty} w_1^\gamma = w_1^\infty \text{ pointwisely},$$

for some functions $v_1^\infty$ and $w_1^\infty$. Note that by linearity, $v^\gamma = v_1^\gamma - w_1^\gamma$. Define $v^\infty := v_1^\infty - w_1^\infty$, then

$$\lim_{\gamma\to\infty} v^\gamma = v^\infty \text{ pointwisely on } (0,\infty)\times[0,\infty).$$

Now we need to show that $v^\infty = v^*$. Apply Theorem (1) with $c = \mathcal{L} + \Lambda + \gamma H(r-R)$, $F = \Lambda - q$ and $\ell = R$, and noting that

$$\|c\|_{L^\infty([0,\ell])} = \|\Lambda + \gamma H(r-R)\|_{L^\infty([0,R])} = \|\Lambda\|_{L^\infty([0,R])},$$

we have

$$\begin{aligned} &\|v^\gamma\|_{L^\infty((0,\infty)^2)} \le C_{\kappa,\beta,\sigma}|\!|\!|\Lambda - q|\!|\!| < \infty, \\ &\left\|\frac{\partial v^\gamma}{\partial r}\right\|_{L^\infty((0,\infty)^2)} \le C_{\kappa,\beta,\sigma,R,\|\Lambda\|_{L^\infty([0,R])}}|\!|\!|\Lambda - q|\!|\!| < \infty, \\ &\left\|\frac{\partial v^\gamma}{\partial T}\right\|_{L^\infty((0,\infty)^2)} \le C_{\kappa,\beta,\sigma}\|\Lambda - q\|_{L^\infty((0,\infty))} < \infty. \end{aligned} \tag{3.4.1}$$

So

$$\lim_{\gamma\to\infty} v^\gamma = v^\infty \text{ locally uniformly on } (0,\infty)\times[0,\infty), \tag{3.4.2}$$

and we can pass the limit $\gamma\to\infty$ in the first, third and fourth equation of (1.5.15) and have

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda\right)v^\infty = \Lambda - q & \text{in } (0,R)\times(0,\infty), \\ v^\infty(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial v^\infty}{\partial r} \in L^\infty((0,\infty)\times(0,\infty)) \end{cases} \tag{3.4.3}$$

By similar proof as for Lemma 5.1 and Lemma 5.2 in [13],

$$|v^\gamma(R+r,T)| \le \frac{m(r)}{\gamma},\ \forall\ r>0, T\ge 0,$$

for some continuous bounded functions $m(\cdot)$. Let $\gamma\to\infty$ we have

$$v^\infty = 0 \text{ in } (R,\infty)\times[0,\infty).$$

But by passing the limit as $\gamma \to \infty$ in (3.4.1), we know that $u_1^\gamma$ is Lipschitz continuous and therefore

$$v^\infty = 0 \text{ in } [R, \infty) \times [0, \infty). \tag{3.4.4}$$

By (3.4.3), (3.4.4) and the uniqueness of the solution $v^*$, we have $v^\infty = v^*$. Then with (3.4.2) we know

$$\lim_{\gamma \to \infty} v^\gamma = v^* \text{ locally uniformly on } (0, \infty) \times [0, \infty),$$

This finishes the proof of (1.5.18).

**(2) Prove (1.5.19).** Apply Theorem (1) with $c = \Lambda + \Lambda_1 + \gamma H(r - R)$ and $F = \Lambda$, we know that Problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \gamma H(r-R)\right) v_2^\gamma = \Lambda & \text{in } (0,\infty)^2, \\ v_2^\gamma(\cdot, 0) = 0 & \text{in } (0, \infty), \\ \partial_r v_2^\gamma \in L^\infty((0,\infty)^2), \end{cases}$$

admits a unique solution $v_2^\gamma$; apply Theorem (1) with $c = \Lambda + \Lambda_1 + \gamma H(r - R)$ and $F = q + \Lambda_1 (u_1^\gamma)^- \Lambda_1$, we know that Problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \gamma H(r-R)\right) w_2^\gamma = q + \Lambda_1 (v^\gamma)^+ & \text{in } (0,\infty)^2, \\ w_2^\gamma(\cdot, 0) = 0 & \text{in } (0, \infty), \\ \partial_r w_2^\gamma \in L^\infty((0,\infty)^2), \end{cases} \tag{3.4.5}$$

admits a unique solution $w_2^\gamma$; apply Theorem (1) with $c = \Lambda + \Lambda_1 + \gamma H(r - R)$ and $F = q + \Lambda_1 (v^*)^+ \Lambda_1$, we know that Problem

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \gamma H(r-R)\right) \bar{w}_2^\gamma = q + \Lambda_1 (v^*)^+ & \text{in } (0,\infty)^2, \\ \bar{w}_2^\gamma(\cdot, 0) = 0 & \text{in } (0, \infty), \\ \partial_r \bar{w}_2^\gamma \in L^\infty((0,\infty)^2), \end{cases} \tag{3.4.6}$$

admits a unique solution $\bar{w}_2^\gamma$. By similar method as in step (1), we can show that $v_2^\gamma$ and $\bar{w}_2^\gamma$ are decreasing as $\gamma \to \infty$ and bounded below by zero. So by the monotone convergence theorem, there exist functions $v_2^\infty$ and $w_2^\infty$ s.t.

$$\lim_{\gamma\to\infty} v_2^\gamma = v_2^\infty \text{ pointwisely}, \;\; \lim_{\gamma\to\infty} \bar{w}_2^\gamma = w_2^\infty \text{ pointwisely}. \tag{3.4.7}$$

By (3.4.5) and (3.4.6),

$$\big(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \gamma H(r-R)\big)(w_2^\gamma - \bar{w}_2^\gamma) = \Lambda_1((v^\gamma)^+ - (v^*)^+) \text{ in } (0,\infty)^2,$$

By Theorem 1 with $c = \Lambda + \Lambda_1 + \gamma H(r-R)$ and $F = \Lambda_1((v^\gamma)^+ - (v^*)^+)$, we have

$$\|w_2^\gamma - \bar{w}_2^\gamma\|_{L^\infty((0,\infty)^2)} \le C_{\kappa,\beta,\sigma} \big|\big|\big|\Lambda_1((v^\gamma)^+ - (v^*)^+)\big|\big|\big| \le C_{\kappa,\beta,\sigma} |||\Lambda_1||| \|(v^\gamma)^+ - (v^*)^+\|_{L^\infty((0,\infty)^2)}.$$

So by (1.5.18), which we have proved in part (1), and (3.4.7),

$$\lim_{\gamma\to\infty} v_2^\gamma = v_2^\infty \text{ pointwisely}, \;\; \lim_{\gamma\to\infty} w_2^\gamma = w_2^\infty \text{ pointwisely}.$$

Note that by linearity, $u^\gamma = v_2^\gamma - w_2^\gamma$. Define $u^\infty := v_2^\infty - w_2^\infty$, then

$$\lim_{\gamma\to\infty} u^\gamma = u^\infty \text{ pointwisely on } (0,\infty)\times[0,\infty).$$

Then by similar discussion as in Step (1), we can show that the convergence is locally uniformly and that $u^\infty = u^*$, i.e. (1.5.19) holds. □

## 4.0 NUMERICAL COMPUTATION

We want to compute the price $q^*$ of the CDS as defined in Section 1.4. We will only study the CDS transferable by the seller at most one time. The other cases can be studied in a similar way.

By Subsection 1.3.6, if the compensation $K = 1$, the expected value $u$, from the buyer's point of view, of a CDS transferable by the seller at most one time, satisfies the PDE system

$$\begin{cases} \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_2\right) v = \Lambda - q & \text{in } (0,\infty) \times (0,\infty), \\ \left(\frac{\partial}{\partial T} + \mathcal{L} + \Lambda + \Lambda_1 + \Lambda_2\right) u = \Lambda - q - \Lambda_1 v^+ & \text{in } (0,\infty) \times (0,\infty), \\ u(\cdot,0) = 0,\ v(\cdot,0) = 0 & \text{in } (0,\infty), \\ \frac{\partial}{\partial r} u \in L^\infty((0,\infty) \times (0,\infty)), & \\ \frac{\partial}{\partial r} v \in L^\infty((0,\infty) \times (0,\infty)), & \end{cases} \tag{4.0.1}$$

where $\mathcal{L}$ is the operator defined as in (1.3.7):

$$\mathcal{L} = -\frac{\sigma^2 r}{2}\frac{\partial^2}{\partial r^2} - (\kappa - \beta r)\frac{\partial}{\partial r} + r,$$

with $\kappa$, $\beta$, $\sigma$ and $q$ being positive constants, $\Lambda$, $\Lambda_1$ and $\Lambda_2$ are given non-negative functions of $r$. By Theorem 2, if $\Lambda$, $\Lambda_1$ and $\Lambda_2$ satisfies

$$|||\Lambda||| < \infty, \quad |||\Lambda_1||| < \infty, \quad \Lambda, \Lambda_1, \Lambda_2 \in L^\infty([0,\ell])\ \forall\, \ell > 0, \tag{4.0.2}$$

$$\|\Lambda\|_{L^\infty((0,\infty))} < \infty \text{ or } \|\Lambda_1\|_{L^\infty((0,\infty))} < \infty. \tag{4.0.3}$$

Then Problem (4.0.1) has a unique solution.

Now we want to solve (4.0.1) numerically and find $q^*$ for given $T$ and $r$.

## 4.1 FINITE DIFFERENCE METHOD

We will use the finite difference method (ref: [20]) to find the numerical solution of Problem (4.0.1).

Let $N$ be the number of grid intervals in the spatial variable space for the interest rate $r$. Let $r_i,\ i = 0, 1, 2, \cdots, N$ be the interest rate grid. Denote

$$v_i(T) := u(r_i, T),\ u_i(T) := v(r, T).$$

Then by the finite difference method,

$$\frac{\partial v(r_i,T)}{\partial r} \approx \frac{v_{i+1}-v_i}{r_{i+1}-r_i},\ \frac{\partial u(r_i,T)}{\partial r} \approx \frac{u_{i+1}-u_i}{r_{i+1}-r_i},$$
$$\frac{\partial^2 v(r_i,T)}{\partial r^2} \approx \frac{v_{i+1}-2v_i+v_{i-1}}{(r_{i+1}-r_i)^2},\ \frac{\partial^2 u(r_i,T)}{\partial r^2} \approx \frac{u_{i+1}-2u_i+u_{i-1}}{(r_{i+1}-r_i)^2}.$$

Then

$$\begin{cases} v_i'(T) = \frac{\sigma^2 r_i}{2}\frac{v_{i+1}-2v_i+v_{i-1}}{(r_{i+1}-r_i)^2} + (\kappa - \beta r_i)\frac{v_{i+1}-v_i}{r_{i+1}-r_i} - (r_i + \Lambda(r_i) + \Lambda_2(r_i))\, v_i \\ \qquad\qquad +\Lambda(r_i) - q, \quad i = 2, 3, \cdots, N-1, \\ u_i'(T) = \frac{\sigma^2 r_i}{2}\frac{u_{i+1}-2u_i+u_{i-1}}{(r_{i+1}-r_i)^2} + (\kappa - \beta r_i)\frac{u_{i+1}-u_i}{r_{i+1}-r_i} - (r_i + \Lambda(r_i) + \Lambda_1(r_i) + \Lambda_2(r_i))\, v_i \\ \qquad\qquad +\Lambda(r_i) - q - \Lambda_1(r_i) v_i^+, \quad i = 2, 3, \cdots, N-1, \\ v_i(0) = u_i(0) = 0, \quad i = 0, 1, 2, \cdots, N, \\ v_0(T) = v_1(T),\ v_{N-1}(T) = v_N(T),\ u_0(T) = u_1(T),\ u_{N-1}(T) = u_N(T). \end{cases} \tag{4.1.1}$$

The last condition in (4.1.1) imposes an artificial condition that the numerical derivative is zero at the boundary. It is motivated by the last condition is (2.1.1).

We can use the ode45 function in MATLAB to solve this system of ordinary differential equations (4.1.1). The codes are in Appendix A.2.

## 4.2 RESULTS AND DISCUSSIONS

In the following, we will use $\kappa = 0.01$, $\beta = 0.2$, $\sigma = 0.02$ as an example, which is similar to the calibration of interest rates in [5]. We will do the computation for the interest rate on $[0, 1]$ with $N = 1000$. The codes are in Appendix A.2.

### 4.2.1 Case 1: No counterparty risk, constant intensity for credit event

Let us start from a very simple case: when there is no counterparty risk and the intensity for the underlying credit event is a constant. For example, we set

$$\Lambda_1 = \Lambda_2 = 0 \text{ and } \Lambda = 0.1.$$

Here $\Lambda = 0.1$ means the underlying entity averagely will default once every ten years. So we guess that the corresponding fair price $q^*$, as defined in Section 1.4, should approximately be 0.1. Our numerical computation confirms this guess. We consider the value of the CDS when $r = 0.053$ and $T = 1$.

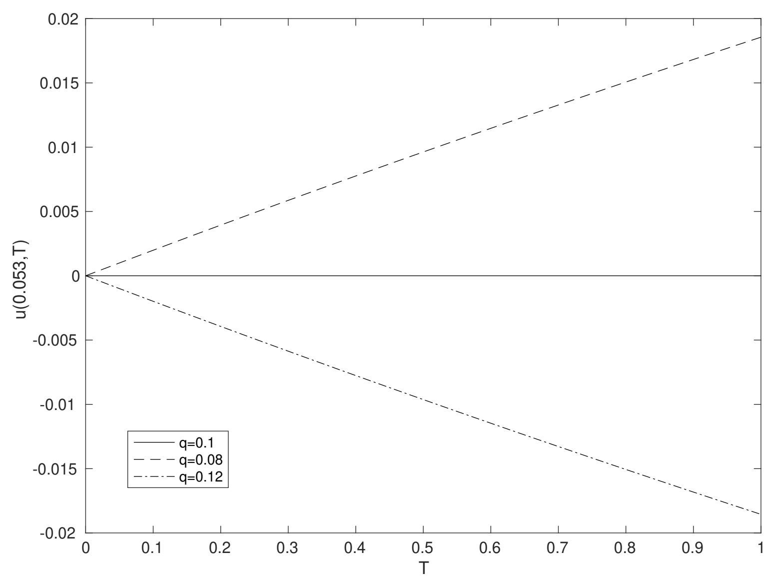


Figure 2: Case 1: CDS without counterparty risks ($r = 0.053$)

Using $q = 0.1$, we see from Figure 2 (the solid line) that $u(0.053, 1) = 0$. So the fair price is $q^* = 0.1$. For comparison, if we use $q = 0.08 < 0.1$, we see from Figure 2 (the dashed line) that the value of the CDS $u(0.053, 1)$ is positive. So with lower $q$, the seller is in the money. But if we use $q = 0.12 > 0.1$, we see from Figure 2 (the dash-dot line) that the value of the CDS $u(0.053, 1)$ is negative. So with higher $q$, the buyer is in the money.

#### 4.2.2 Case 2: Transferable CDS by the seller at most one time

Now we consider a more complicated case, transferable CDS by the seller at most one time with

$$\Lambda = \left(\frac{1}{3}\sqrt{r} + 0.1\right) \wedge 1, \;\; \Lambda_1 = \frac{1}{4}\sqrt{r} + 0.05, \;\; \Lambda_2 = 0.05(1 - \sqrt{r})^+.$$

Then $\Lambda$, $\Lambda_1$ and $\Lambda_2$ defined as above satisfy conditions (4.0.2) and (4.0.3). So by Theorem 2, the existence and uniqueness of the solution is guaranteed. So we can use numerical methods to approximate the solution. Coding with MATLAB, we have in Figure 3 the graph of the CDS value $u$ w.r.t. the expiry $T$ and interest rate $r$ (with $q$=0.1765).

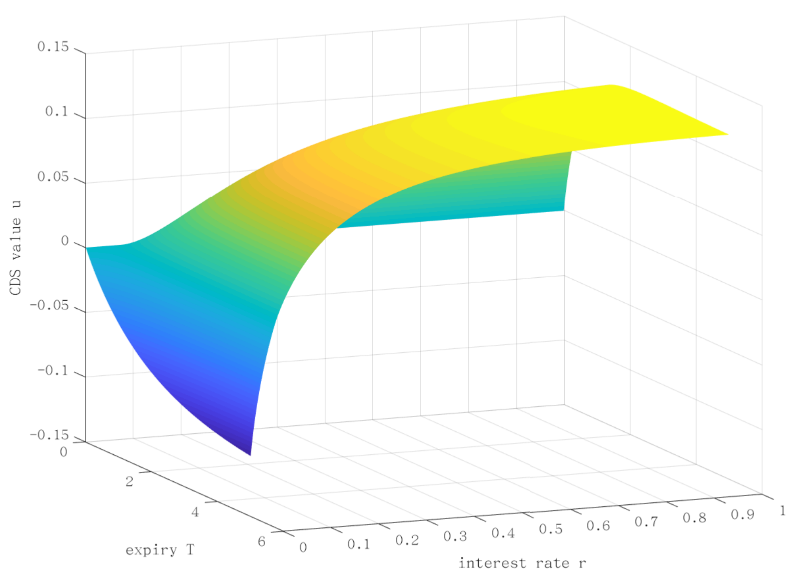


Figure 3: Value of the CDS transferable by the seller ($q$=0.1765)

Now let us consider the fair price $q^*$, as defined in Section 1.4, when $r = 0.053$. To see this, we plot the value $u(0.053, T)$ for different $q$ (Figure 4).

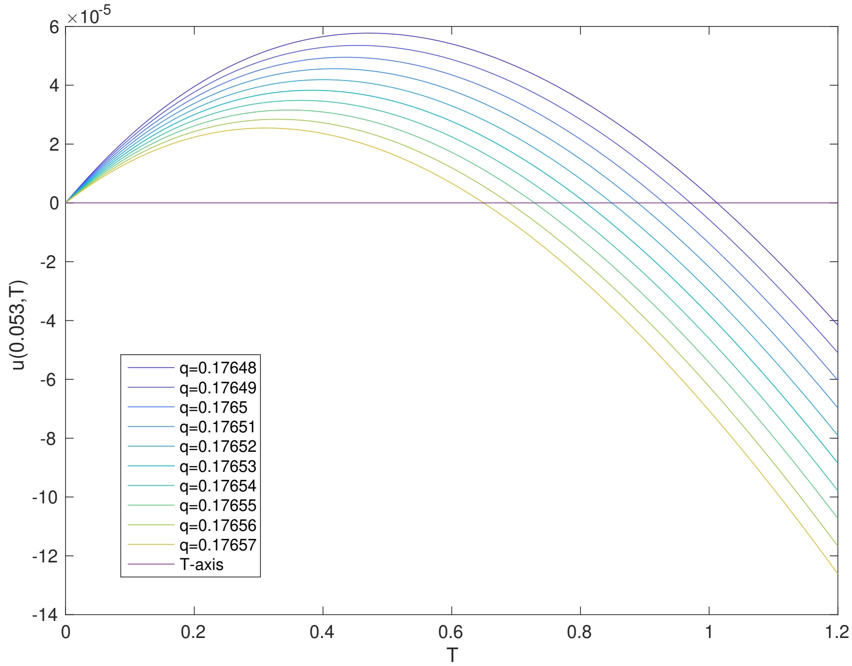


Figure 4: CDS value for different $q$ (Case 2)

For different value of $q$, by reading the $T$ value so that $u(0.053, T) = 0$ from Figure 4, we can have the fair price for different $T$, as in Table 1.

| | | | | | |
|---|---|---|---|---|---|
| T | 0.6482 | 0.6903 | 0.7271 | 0.7677 | 0.8098 |
| $q^*$ | 0.17657 | 0.17656 | 0.17655 | 0.17654 | 0.17653 |
| T | 0.8506 | 0.8892 | 0.9292 | 0.9717 | 1.011 |
| $q^*$ | 0.17652 | 0.17651 | 0.1765 | 0.17649 | 0.17648 |

Table 1: Fair price when $r = 0.053$ (Case 2)

From Table 1, we see that the fair price of the CDS is 0.17649 when $T = 0.9717$ and is 0.17648 when $T = 1.011$ with $r = 0.053$. To see the fair price when $T = 1$ and $r = 0.053$, we plot the CDS value $u(0.053, 1, q)$ for different $q$ in the region $[0.17648, 0.17649]$ (Figure 5). Then we can read from Figure 5 that $q* = 0.176483$ when $T = 1$ and $r = 0.053$.

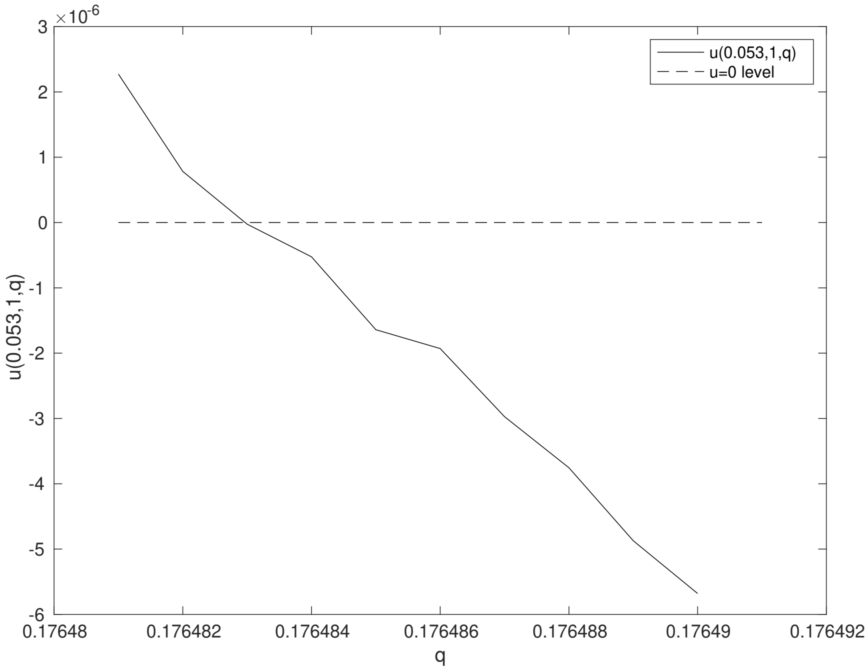


Figure 5: CDS value for different $q \in [0.17648, 0.17649]$ (Case 2)

## 5.0 CONCLUSIONS

A CDS is a financial contract between two parties who exchange cash flows based on the occurrence of an underlying credit event, typically, a default of an underlying entity. Under a CDS contract, the seller will pay a lump sum of compensation to the buyer at the time of the credit event, if it happens before the expiry; in return, from the date the contact was signed and until the occurrence of the credit event or the expiry, whichever earlier, the buyer agrees to pay a continuous premium to the seller.

Besides the credit risk of the underlying credit event, there is another kind of risk involving in a CDS contract, which is called the counterparty risk. The counterparty risk is the risk of defaults of the buyer and/or the seller of the CDS contract. Regarding default as credit event, we can describe the defaults by either an intensity model or a structure model. The intensity model describes credit events by first arrival times of Poisson processes, whereas the structure model describes a credit event by the crossing of a boundary on a state space of state variables.

We call a contract *nontransferable* if the contract automatically ends at the default time of one party. When one party, say the seller, defaults, the CDS contract may be in favor of the seller and therefore he would like to sell the contract to a third party. Hence, we use the word *transferable* to designate a CDS which allows a party to sell his contract to a third party at time of his default. In this dissertation, we considered three types of transferability: (1) The CDS can be sold by the original seller to a third party at time of his default, and the new contract becomes non-transferable; (2) The CDS can be sold by the original seller or buyer to a third party at time of his default, and the new contract becomes non-transferable; (3) The CDS can be sold by the original seller or buyer to a third party at time of his default, and the new contract is still transferable any number of times.

We modeled these three types of transferable CDSs by systems of partial differential equations and studied the corresponding well-posedness problems. We overcame three main difficulties. Firstly, we don't assume a specific expression of the intensity functions, for example linear functions, but only assume some boundedness or local boundedness properties for the intensity functions; secondly, since we need to deal with PDE systems, one component of the solution may depend on other components of the solution, and in particular, the function $F$ in (1.5.1) is a function of both $r$ and $T$, not just $r$; thirdly, for the CDS that is transferable any number of times, the two components of the solution to the corresponding PDE system depends on each other and cannot be solved one by one.

The boundedness or local boundedness assumption over the the intensity function is much more practical in reality than the assumption that the intensity functions are linear in the interest rate. In reality, we usually cannot guarantee that the frequency of credit events and defaults be linear function of the interest rate, but the frequency of credit event and default times are usually bounded, or at least bounded when the interest rate is not too big. The study of well-posedness of the PDE systems corresponding to the CDS problems are important, because it gives the theoretical support for the numerical computation of the value of the CDS using the PDE.

We also studied the link between the structure model and the intensity model. So even though Theorem 2 need the frequency of defaults to be somewhat bounded to have the well-posedness of the solution, we can still deal with the case when the frequency of defaults tends to infinity. In fact, with the structure model, the default will happen immediately when the interest rate attains some level; this corresponds to the default frequency equaling to infinity when the interest rate attains that level. Theorem 5 provides theoretical proof for this insight.

Theorem 1 not only is important for the proof of all other theorems in this dissertation, but also can be used to give the boundedness of the solutions and their derivatives w.r.t. the interest rate and expiry, which is useful in practice. As long as the frequency of credit event and defaults have some boundedness property, which is the usual case, the value of the CDS is bounded and changes w.r.t. the interest rate and expiry smoothly. In other words, if the interest rate and the expiry do not have a big jump, the value of the CDS won't change dramatically. But if the interest rate changes dramatically in a short period, the corresponding CDS value will also change with big jumps.

Finally in Chapter 4 we discussed the numerical solution and gave two examples about how to calculate the fair price of a CDS as defined in Section 1.4.

# APPENDIX

## A.1 DERIVATIONS OF EXPECTATIONS IN SECTION 1.3

### A.1.1 Derivation of (1.3.5)

By (1.2.1),

$$p_{10} := K\ \mathbb{E}\left[e^{-\int_0^\tau r_t dt}\mathbf{1}_{\{\tau\leq T\}} - \int_0^{\tau\wedge T} qe^{-\int_0^t r_\theta d\theta}dt\middle|\mathscr{F}_0, \tau > 0\right]. \tag{A.1.1}$$

By (1.3.1),

$$\begin{aligned}\mathbb{E}\left[e^{-\int_0^\tau r_t dt}\mathbf{1}_{\{\tau<T\}}\middle|\mathscr{F}_0, \tau>0\right] =&\mathbb{E}\left[\int_0^T \mathbb{P}\left\{\tau\in[s,s+ds)|\mathscr{F}_0,\tau>0\right\}e^{-\int_0^s r_t dt}\middle| \begin{matrix}\mathscr{F}_0,\\ \tau>0\end{matrix}\right]\\ =&\mathbb{E}\left[\int_0^T \lambda_s e^{-\int_0^s(r_t+\lambda_t)dt}ds\middle|\mathscr{F}_0\right];\end{aligned} \tag{A.1.2}$$

by (1.3.1) again and applying integration by parts, we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^{\tau\wedge T} qe^{-\int_0^t r_\theta d\theta}dt\middle|\mathscr{F}_0,\tau>0\right]\\ =&\mathbb{E}\left[\int_0^\infty\int_0^{s\wedge T} qe^{-\int_0^t r_\theta d\theta}dt\ \mathbb{P}\left\{\tau\in[s,s+ds)|\mathscr{F}_s,\tau>0\right\}\middle| \begin{matrix}\mathscr{F}_0,\\ \tau>0\end{matrix}\right]\\ =&\mathbb{E}\left[\int_0^\infty\int_0^{s\wedge T} qe^{-\int_0^t r_\theta d\theta}dt\ d\left(1-e^{-\int_0^s(\lambda_\theta)d\theta}\right)\middle|\mathscr{F}_0\right]\\ =&\mathbb{E}\left[\int_0^\infty\left(e^{-\int_0^t(\lambda_\theta)d\theta}\right)d\left(\int_0^{s\wedge T} qe^{-\int_0^s r_\theta d\theta}ds\right)dt\middle|\mathscr{F}_0\right]\\ =&\mathbb{E}\left[\int_0^T qe^{-\int_0^s(r_\theta+\lambda_\theta)d\theta}dt\middle|\mathscr{F}_0\right].\end{aligned} \tag{A.1.3}$$

Combine (A.1.1), (A.1.2) and (A.1.3), we have

$$p_{10} = K\,\mathbb{E}\left[\int_0^T e^{-\int_0^s (r_\theta+\lambda_\theta)d\theta}\,(\lambda_s - q)\,ds\middle|\mathscr{F}_0\right].$$

which is (1.3.5).

### A.1.2 Derivation of (1.3.9)

By (1.2.3),

$$w_{3t} = \mathbb{E}\left[\int_t^{\tau\wedge\tau_2\wedge T} e^{-\int_t^s r_\theta d\theta} q ds - e^{-\int_t^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_2\wedge T\}}\middle|\mathscr{F}_t, \tau_2\wedge\tau > t\right]. \tag{A.1.4}$$

Note that with (1.3.2),

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_2\wedge T\}}\middle|\mathscr{F}_t, \tau_2\wedge\tau > t\right]\\
=&\mathbb{E}\left[\int_t^T \mathbb{P}\left\{\tau\in[s,s+ds), s<\tau_2|\mathscr{F}_s, \tau_2\wedge\tau>t\right\}e^{-\int_t^s r_\theta d\theta}\middle|\begin{array}{c}\mathscr{F}_t,\\ \tau_2\wedge\tau>t\end{array}\right]\\
=&\mathbb{E}\left[\int_t^T \lambda_s e^{-\int_t^s (r_\theta+\lambda_\theta+\lambda_{2\theta})d\theta}ds\middle|\mathscr{F}_t\right];
\end{aligned} \tag{A.1.5}$$

with (1.3.1) and (1.3.2), by applying integration by parts, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_t^{\tau\wedge\tau_2\wedge T} q e^{-\int_t^s r_\theta d\theta}ds\middle|\mathscr{F}_t, \tau_2\wedge\tau>t\right]\\
=&\mathbb{E}\left[\int_t^\infty\int_t^{s'\wedge T} q e^{-\int_t^s r_\theta d\theta}ds\;\mathbb{P}\left\{\tau\wedge\tau_2\in[s', s'+ds')|\mathscr{F}_s, \tau_2\wedge\tau>t\right\}\middle|\begin{array}{c}\mathscr{F}_t,\\ \tau_2\wedge\tau>t\end{array}\right]\\
=&\mathbb{E}\left[\int_t^\infty\int_t^{s'\wedge T} q e^{-\int_t^s r_\theta d\theta}ds\; d\left(1-e^{-\int_t^{s'}(\lambda_\theta+\lambda_{2\theta})d\theta}\right)\middle|\mathscr{F}_t\right]\\
=&\mathbb{E}\left[\int_t^\infty\left(e^{-\int_t^{s'}(\lambda_\theta+\lambda_{2\theta})d\theta}\right)d\left(\int_t^{s'\wedge T} q e^{-\int_t^s r_\theta d\theta}ds\right)\middle|\mathscr{F}_t\right]\\
=&\mathbb{E}\left[\int_t^T q e^{-\int_t^s (r_\theta+\lambda_\theta+\lambda_{2\theta})d\theta}ds\middle|\mathscr{F}_t\right].
\end{aligned} \tag{A.1.6}$$

Combine (A.1.4), (A.1.5) and (A.1.6), we have

$$w_{3t} = \mathbb{E}\left[\int_t^T (\lambda_s - q)e^{-\int_t^s (r_\theta+\lambda_\theta+\lambda_{2\theta})d\theta}ds\middle|\mathscr{F}_t\right],$$

which is (1.3.9).

### A.1.3 Derivation of (1.3.11)

By (1.2.4),

$$
\begin{aligned}
p_{30} =& K\ \mathbb{E}\left[e^{-\int_0^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_1\wedge\tau_2\wedge T\}} - q\int_0^{\tau\wedge\tau_1\wedge\tau_2\wedge T} e^{-\int_0^t r_\theta d\theta}\,dt\right.\\
&\left.-\mathbf{1}_{\{\tau_1<\tau\wedge\tau_2\wedge T\}}e^{-\int_0^{\tau_1} r_\theta d\theta}w_{3\tau_1}^{+}\middle|\mathscr{F}_0, \tau\wedge\tau_1\wedge\tau_2>0\right]. \qquad \text{(A.1.7)}
\end{aligned}
$$

By (1.3.2),

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{\tau_1<\tau\wedge\tau_2\wedge T\}}e^{-\int_0^{\tau_1} r_\theta d\theta}w_3^{+}(r_{\tau_1}, T-\tau_1)\middle|\mathscr{F}_0, \tau\wedge\tau_1\wedge\tau_2>0\right]\\
=&\mathbb{E}\left[\int_0^T \mathbb{P}\{\tau_1\in[s,s+ds), s<\tau\wedge\tau_2|\mathscr{F}_s, \tau\wedge\tau_1\wedge\tau_2>0\}\,e^{-\int_0^s r_\theta d\theta}w_3^{+}(r_s, T-s)\middle|\begin{matrix}\mathscr{F}_0,\\ \tau\wedge\tau_1\wedge\tau_2>0\end{matrix}\right]\\
=&\mathbb{E}\left[\int_0^T \lambda_{1s}e^{-\int_0^s(\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta})d\theta}e^{-\int_0^s r_\theta d\theta}w_3^{+}(r_s, T-s)ds\middle|\mathscr{F}_0\right]\\
=&\mathbb{E}\left[\int_0^T w_3^{+}(r_s, T-s)\lambda_{1s}e^{-\int_0^s(r_\theta+\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta})d\theta}ds\middle|\mathscr{F}_0\right]. \qquad \text{(A.1.8)}
\end{aligned}
$$

And similar to (A.1.5) and (A.1.6), we have

$$
\mathbb{E}\left[e^{-\int_0^\tau r_\theta d\theta}\mathbf{1}_{\{\tau<\tau_1\wedge\tau_2\wedge T\}}\middle|\begin{matrix}\mathscr{F}_0,\\ \tau\wedge\tau_1\wedge\tau_2>0\end{matrix}\right] = \mathbb{E}\left[\int_0^T \lambda_s e^{-\int_0^s(r_\theta+\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta})d\theta}ds\middle|\mathscr{F}_0\right], \qquad \text{(A.1.9)}
$$

$$
\mathbb{E}\left[\int_0^{\tau\wedge\tau_1\wedge\tau_2\wedge T} qe^{-\int_0^s r_\theta d\theta}ds\middle|\begin{matrix}\mathscr{F}_0,\\ \tau\wedge\tau_1\wedge\tau_2>0\end{matrix}\right] = \mathbb{E}\left[\int_0^T qe^{-\int_0^s(r_\theta+\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta})d\theta}ds\middle|\mathscr{F}_0\right]. \qquad \text{(A.1.10)}
$$

Combine (A.1.7), (A.1.8), (A.1.9) and (A.1.10), we have

$$
p_{30} = K\ \mathbb{E}\left[\int_0^T e^{-\int_0^s(r_\theta+\lambda_\theta+\lambda_{1\theta}+\lambda_{2\theta})d\theta}\left(\lambda_s-q-\lambda_{1s}w_3^{+}(r_s, T-s)\right)ds\middle|\mathscr{F}_0\right],
$$

which is (1.3.11).

## A.2 MATLAB CODES

### A.2.1 Applying the finite difference method

File name: **odesys.m**

Purpose: Using the finite difference method to solve the PDE system.

```
function dudt = odesys(t, u, N, dr, sigma, kappa, beta, q, lbdt, lbd1t, lbd2t)
% u1=v, u2=u
% u: (2N-2)*1
% N: number of intervals for r
% dr: interval length for r

du1dt = zeros(N-1,1);
du2dt = zeros(N-1,1);
u1 = u(1:N-1);
u2 = u(N:2*N-2);

for i=2:N-2
    ri = i * dr;
    du1dt(i) = sigma^2*ri*(u1(i+1)-2*u1(i)+u1(i-1))/(2*dr^2)...
        + (kappa - beta*ri)*(u1(i+1)-u1(i))/dr...
        - (ri+lbd(ri, lbdt)+lbd2(ri,lbd2t))*(u1(i))+lbd(ri, lbdt)-q;

    du2dt(i) = sigma^2*ri*(u2(i+1)-2*u2(i)+u2(i-1))/(2*dr^2)...
        + (kappa - beta*ri)*(u2(i+1)-u2(i))/dr...
        - (ri+lbd(ri, lbdt)+lbd1(ri,lbd1t)+lbd2(ri,lbd2t))*(u2(i))+lbd(ri, lbdt)-q...
        - lbd1(ri,lbd1t) * max(0, u1(i));
```

```
end

% u1(0)=u1(1), u2(0)=u2(1)
% i=1,
ri = dr;
du1dt(1) = sigma^2*(u1(2)-u1(1))/(2*ri)...
     + (kappa - beta*ri)*(u1(2)-u1(1))/ri...
     - (ri+lbd(ri, lbdt)+lbd2(ri,lbd2t))*(u1(1))+lbd(ri, lbdt)-q;

du2dt(1) = sigma^2*(u2(2)-u2(1))/(2*ri)...
     + (kappa - beta*ri)*(u2(2)-u2(1))/ri...
     - (ri+lbd(ri, lbdt)+lbd1(ri,lbd1t)+lbd2(ri,lbd2t))*(u2(1))+lbd
        (ri, lbdt)-q...
     - lbd1(ri,lbd1t) * max(0, u1(1));

% u1(N-1)=u1(N), u2(N-1)=u2(N)
% i=N-1,
ri=(N-1) * dr;
du1dt(N-1) = sigma^2*ri*(-u1(N-1)+u1(N-2))/(2*dr^2)...
     - (ri+lbd(ri, lbdt)+lbd2(ri,lbd2t))*(u1(N-1))+lbd(ri, lbdt)-q;

du2dt(N-1) = sigma^2*ri*(-u2(N-1)+u2(N-2))/(2*dr^2)...
     - (ri+lbd(ri, lbdt)+lbd1(ri,lbd1t)+lbd2(ri,lbd2t))*(u2(N-1))+
        lbd(ri, lbdt)-q...
     - lbd1(ri,lbd1t) * max(0, u1(N-1));

dudt = [du1dt;du2dt];
```

### A.2.2 Defining intensity functions

**(1) lbd.m** (defining the function $\Lambda(\cdot)$)

```
function lambda = lbd(r, lbdt)
if lbdt == 1
    lambda = 0.1;
elseif lbdt == 2
    lambda = 0.1+0.5*r;
elseif lbdt == 3
    lambda = min(sqrt(r)/3+0.1,1);
elseif lbdt == 4
    lambda = r^2/3+0.1;
end
```

**(2) lbd1.m** (defining the function $\Lambda_1(\cdot)$)

```
function lambda1 = lbd1(r, lbd1t)
if lbd1t == 1
    lambda1=0;

elseif lbd1t == 2
    if r>0.04
        lambda1=0.5;
    else
        lambda1=0;
    end

elseif lbd1t == 3
    lambda1 = sqrt(r)/4+0.05;
elseif lbd1t == 4
    lambda1 = r^2/4+0.05;
```

```
end
```

**(3) lbd2.m** (defining the function $\Lambda_2(\cdot)$)

```
function lambda2 = lbd2(r, lbd2t)
if lbd2t == 1
    lambda2 = 0;
elseif lbd2t == 2
    if r<0.06
        lambda2=0.3;
    else
        lambda2=0;
    end
elseif lbd2t == 3
    lambda2 = 0.05*max((1-sqrt(r)),0);
elseif lbd2t == 4
    lambda2 = 0.05*(1-r^2);
end
```

### A.2.3 Studying Case 1

```
clear
clc
N=1000; dr=0.001;
sigma=0.02; kappa=0.01; beta=0.25;
lbdt=1; lbd1t=1; lbd2t=1;
tspan=[0 1];


q=0.1;
u0 = zeros(2*N-2,1);
```

```
[t,u]=ode45(@(t,u) odesys(t,u, N, dr, sigma, kappa, beta, q, lbdt,
    lbd1t, lbd2t), tspan, u0);
plot(t,u(:,1052),'k', 'DisplayName', ['q=' num2str(q)])
hold on

q=0.08;
u0 = zeros(2*N-2,1);
[t,u]=ode45(@(t,u) odesys(t,u, N, dr, sigma, kappa, beta, q, lbdt,
    lbd1t, lbd2t), tspan, u0);
plot(t,u(:,1052), '--k', 'DisplayName', ['q=' num2str(q)])

q=0.12;
u0 = zeros(2*N-2,1);
[t,u]=ode45(@(t,u) odesys(t,u, N, dr, sigma, kappa, beta, q, lbdt,
    lbd1t, lbd2t), tspan, u0);
plot(t,u(:,1052), '-.k', 'DisplayName', ['q=' num2str(q)])

xlabel('T')
ylabel('u(0.053,T)')
hold off
legend show
```

### A.2.4 Studying Case 2

```
clear
clc
N=1000; dr=0.001;
sigma=0.02; kappa=0.01; beta=0.2;
q=0.1765; lbdt=3; lbd1t=3; lbd2t=3;
tspan=[0 5];
```

```
u0 = zeros(2*N-2,1);
[t,u]=ode45(@(t,u) odesys(t,u, N, dr, sigma, kappa, beta, q, lbdt,
    lbd1t, lbd2t), tspan, u0);
[T, R]=meshgrid(t',linspace(1*dr,(N-1)*dr,N-1));
surf(T', R',u(:,N:2*N-2));
%colormap('gray');
xlabel('expiry T')
ylabel('interest rate r')
zlabel('CDS value u')
shading interp
```

#### A.2.5 CDS value for different $q$ (Case 2)

```
clear
clc

colormap parula;
cmap=colormap;

for j = 1:10
N=1000; dr=0.001;
sigma=0.02; kappa=0.01; beta=0.2;
q=0.17647+j*0.00001; lbdt=3; lbd1t=3; lbd2t=3;
tspan=[0 1.2];
u0 = zeros(2*N-2,1);
[t,u]=ode45(@(t,u) odesys(t,u, N, dr, sigma, kappa, beta, q, lbdt,
    lbd1t, lbd2t), tspan, u0);
Plot_color=cmap(j*5,:);
plot(t,u(:,1052),'DisplayName',['q=' num2str(q)], 'Color',
    Plot_color);
```

```
xlabel('T')
ylabel('u(0.053,T)')
hold on
end
sizet=size(t);
plot(t, zeros(1,sizet(1)),'DisplayName','T-axis')
hold off
legend show
```

### A.2.6 CDS value for different $q \in [0.17648, 0.17649]$ (Case 2)

```
clear
clc

for j = 1:10
N=1000; dr=0.001;
sigma=0.02; kappa=0.01; beta=0.2;
q=0.17648+j*0.000001;
lbdt=3; lbd1t=3; lbd2t=3;
tspan=[0 1];
u0 = zeros(2*N-2,1);
[t,u]=ode45(@(t,u) odesys(t,u, N, dr, sigma, kappa, beta, q, lbdt,
    lbd1t, lbd2t), tspan, u0);
Q(j)=q;
U(j)=u(433,1052);
end
plot(Q, U, 'k', 'DisplayName','u(0.053,1,q)')
xlabel('q');
ylabel('u(0.053,1,q)')
hold on
```

```
plot([Q,0.176491],zeros(1,11), '--k','DisplayName','u=0 level')
hold off
legend show
```

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