RATIONAL ZETA SERIES FOR $\zeta(2N)$ AND $\zeta(2N+1)$

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I will begin by using the cotangent function to find rational zeta series with $\zeta(2n)$ in terms of $\zeta(2k+1)$ and $\beta(2k)$, the Dirichlet beta function. I then develop a certain family of generalized rational zeta series using the generalized Clausen function $\mathrm{Cl}_m(x)$ and use those results to discover a second family of generalized rational zeta series. As a special case of my results from Theorem 3.1, I prove a conjecture given in 2012 by F.M.S. Lima. Later, I use the same analysis but for the digamma function $\psi(x)$ and negapolygammas $\psi^{(-m)}(x)$. With these, I extract the same two families of generalized rational zeta series with $\zeta(2n+1)$ on the numerator rather than $\zeta(2n)$. Afterwards, I look into the applications of these rational zeta series and how they are related to other special functions such as the multiple zeta function.

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1.0 INTRODUCTION

In 1734, Leonard Euler proved an amazing result, now known as the celebrated Euler series:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

where $\zeta(s)$ is the Riemann zeta function, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \;\; \Re(s) > 1.$$

Later, Euler gave the formula

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}, \;\; k \in \mathbb{N}_0,$$

where B_n are the Bernoulli numbers, defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \;\; |z| < 2\pi.$$

These Bernoulli numbers also arise in certain power series, namely of the cotangent function. The odd arguments of $\zeta(s)$ are the interesting ones as they do not have a closed form, though many mathematicians have studied them in detail. In 1979, Roger Apéry [3] proved that $\zeta(3)$ is irrational using the fast converging series

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

It is still unknown whether $\zeta(5)$ is irrational however, it is known that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ is irrational (see [19]). I will begin by investigating $\int_0^{\pi z} x^p \cot(x)\, dx$ for $|z| < 1$, which is studied in [15]. I then compute this same integral using the power series for the cotangent and obtain a rational zeta series representation with $\zeta(2n)$ on the numerator for $z = 1/2$ and $z = 1/4$. Afterwords, I work on a generalized rational zeta series with $\zeta(2n)$ on the numerator and an arbitrary number of monomials on the denominator using the generalized Clausen function $\mathrm{Cl}_m(z)$. As a special case, I immediately prove a conjecture given in 2012 by F.M.S. Lima. After doing this, I go back to the cotangent function and discover a separate class of generalized $\zeta(2n)$ series. Lastly, I perform the same analysis but for the digamma function $\psi(x)$ instead of $\cot(x)$ and instead of $\mathrm{Cl}_m(x)$, I use the negapolygammas $\psi^{(-m)}(x)$. Using the $\zeta(2n)$ sums from earlier, I extract the same rational zeta series representations with $\zeta(2n+1)$ on the numerator for $z = 1/2$ and $z = 1/4$. As applications, some of these rational zeta series have been connected to deeper functions, such as the multiple zeta function.

2.0 RATIONAL $\zeta(2N)$ SERIES

In this paper, we will use the Taylor series for $\cot(x)$, given by

$$\cot(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} = -2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} x^{2n-1}, \ |x| < \pi. \tag{2.1}$$

Another main function needed is the Clausen function (or Clausen's integral),

$$\mathrm{Cl}_2(\theta) := \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^2} = -\int_0^{\theta} \log\left|2\sin\left(\frac{\phi}{2}\right)\right| d\phi, \ |\theta| < 2\pi. \tag{2.2}$$

The Clausen function also has a power series representation which will be used later in the paper. It is given as

$$\frac{\mathrm{Cl}_2(\theta)}{\theta} = 1 - \log|\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)} \left(\frac{\theta}{2\pi}\right)^{2n}, \ |\theta| < 2\pi. \tag{2.3}$$

There are also higher order Clausen-type function defined as

$$\mathrm{Cl}_{2m}(\theta) := \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^{2m}}, \quad \mathrm{Cl}_{2m+1}(\theta) := \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^{2m+1}}. \tag{2.4}$$

The Clausen function is widely studied and has many applications in mathematics and mathematical physics ([5], [6], [9], [12], [14], [15], [18]). We will also discuss the Dirichlet beta function

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \ \Re(s) > 0.$$

When $s = 2$,

$$\beta(2) = G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

is known as Catalan's constant. Using this and the Riemann zeta function, we find

$$\mathrm{Cl}_{2m}(\pi) = 0, \quad \mathrm{Cl}_{2m+1}(\pi) = -\frac{(4^m - 1)\zeta(2m+1)}{4^m}, \tag{2.5}$$

and

$$\mathrm{Cl}_{2m}(\pi/2) = \beta(2m), \quad \mathrm{Cl}_{2m+1}(\pi/2) = -\frac{(4^m - 1)\zeta(2m+1)}{2^{4m+1}}. \tag{2.6}$$

Also, from (4) we can see

$$\frac{d}{d\theta}\,\mathrm{Cl}_{2m}(\theta) = \mathrm{Cl}_{2m-1}(\theta), \quad \frac{d}{d\theta}\,\mathrm{Cl}_{2m+1}(\theta) = -\,\mathrm{Cl}_{2m}(\theta), \tag{2.7}$$

$$\int_0^\theta \mathrm{Cl}_{2m}(x)\,dx = \zeta(2m+1) - \mathrm{Cl}_{2m+1}(\theta), \quad \int_0^\theta \mathrm{Cl}_{2m-1}(x)\,dx = \mathrm{Cl}_{2m}(\theta). \tag{2.8}$$

Using (4) and (7), we find

$$\mathrm{Cl}_1(\theta) = -\log\left|2\sin\left(\frac{\theta}{2}\right)\right|, \;\; |\theta| < 2\pi. \tag{2.9}$$

Writing others out,

$$\mathrm{Cl}_3(z) = \zeta(3) - \int_0^z \mathrm{Cl}_2(t)\,dt,$$

$$\mathrm{Cl}_4(z) = \int_0^z \mathrm{Cl}_3(x)\,dx = z\zeta(3) - \int_0^z \int_0^x \mathrm{Cl}_2(t)\,dt\,dx = z\zeta(3) - \int_0^z (z-t)\,\mathrm{Cl}_2(t)\,dt,$$

$$\mathrm{Cl}_5(z) = \zeta(5) - \int_0^z \mathrm{Cl}_4(x)\,dx = \zeta(5) - \frac{1}{2}z^2\zeta(3) + \frac{1}{2}\int_0^z (z - t)^2\,\mathrm{Cl}_2(t)\,dt,$$

and by induction, for $m \geq 3$,

$$\mathrm{Cl}_m(z) = (-1)^{\lfloor \frac{m-1}{2} \rfloor} \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^k z^{m-2k-1}}{(m-2k-1)!}\zeta(2k+1) + \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-3)!}\int_0^z (z-t)^{m-3}\,\mathrm{Cl}_2(t)\,dt. \quad (2.10)$$

2.1 USING COTANGENT FUNCTION

Theorem 2.1.1. *For $p \in \mathbb{N}$ and $|z| < 1$,*

$$\int_0^{\pi z} x^p \cot(x)\,dx = (\pi z)^p \sum_{k=0}^{p} \frac{p!(-1)^{\lfloor \frac{k+3}{2} \rfloor}}{(p-k)!(2\pi z)^k}\,\mathrm{Cl}_{k+1}(2\pi z) + \delta^{\frac{p}{2}}_{\lfloor \frac{p}{2} \rfloor}\frac{p!(-1)^{\frac{p}{2}}}{2^p}\zeta(p+1), \quad (2.11)$$

where δ^k_j is the Kronecker delta function.

Proof. Let $f(z)$ be the left hand side of the equation and let $g(z)$ be the right hand side. Note that $f'(z) = \pi^{p+1} z^p \cot(\pi z)$. Using (2.7) and (2.9),

$$g'(z) = \frac{1}{2^p}\sum_{k=0}^{p} \frac{p!(-1)^{\lfloor \frac{k+3}{2} \rfloor}(p-k)(2\pi)^{p-k} z^{p-k-1}}{(p-k)!}\,\mathrm{Cl}_{k+1}(2\pi z) + \frac{(\pi z)^p 2\pi \cos(\pi z)}{2\sin(\pi z)} + \frac{1}{2^p}\sum_{k=1}^{p} \frac{p!(-1)^{\lfloor \frac{k+3}{2} \rfloor}(2\pi z)^{p-k}}{(p-k)!}\Big\{(-1)^{k+1} 2\pi\,\mathrm{Cl}_k(2\pi z)\Big\}$$

$$= \pi^{p+1} z^p \cot(\pi z) + (\pi z)^p \sum_{k=1}^{p} \frac{p!\,\mathrm{Cl}_k(2\pi z)}{(p-k)! z^{k+1}(2\pi)^{k-1}}\Big((-1)^{\lfloor \frac{k+2}{2} \rfloor} + (-1)^{\lfloor \frac{k+3}{2} \rfloor}(-1)^k\Big).$$

So indeed, $g'(z) = \pi^{p+1} z^p \cot(\pi z) = f'(z)$. Clearly, $f(0) = 0$. For $g(z)$, note that all terms in the sum are zero except when $k = p$. So we have

$$g(0) = \frac{1}{2^p} p! (-1)^{\lfloor \frac{p+3}{2} \rfloor} \operatorname{Cl}_{p+1}(0) + \delta^{\frac{p}{2}}_{\lfloor \frac{p}{2} \rfloor} \frac{p!(-1)^{\frac{p}{2}}}{2^p} \zeta(p+1) = \frac{p!}{2^p} \zeta(p+1) \delta^{\frac{p}{2}}_{\lfloor \frac{p}{2} \rfloor} \left((-1)^{\lfloor \frac{p+3}{2} \rfloor} + (-1)^{\frac{p}{2}} \right),$$

since from (2.4), $\operatorname{Cl}_{p+1}(0) = \delta^{\frac{p}{2}}_{\lfloor \frac{p}{2} \rfloor} \zeta(p+1)$. So we see $g(0) = 0$. Since $f(0) = g(0)$ and $f'(z) = g'(z)$, $f(z) = g(z)$. □

Using (2.5), (2.6), and (2.9), setting $z = 1/2$ and $z = 1/4$, we find

$$\int_0^{\pi/2} x^p \cot(x)\, dx = \left(\frac{\pi}{2}\right)^p \left(\log 2 + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \frac{p!(-1)^k (4^k - 1)}{(p-2k)!(2\pi)^{2k}} \zeta(2k+1) \right) + \delta^{\frac{p}{2}}_{\lfloor \frac{p}{2} \rfloor} \frac{p!(-1)^{\frac{p}{2}} \zeta(p+1)}{2^p}, \tag{2.12}$$

and

$$\int_0^{\pi/4} x^p \cot(x)\, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^p \left(\log 2 + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \frac{p!(-1)^k (4^k - 1)}{(p-2k)!(2\pi)^{2k}} \zeta(2k+1) \right.$$
$$\left. - \sum_{k=1}^{\lfloor \frac{p+1}{2} \rfloor} \frac{p!(-4)^k \beta(2k)}{(p+1-2k)!\pi^{2k-1}} \right) + \delta^{\frac{p}{2}}_{\lfloor \frac{p}{2} \rfloor} \frac{p!(-1)^{\frac{p}{2}}}{2^p} \zeta(p+1). \tag{2.13}$$

We can also integrate $x^p \cot(x)$ using (2.1) and Fubini's theorem. Doing so, we obtain

$$\int_0^{\pi z} x^p \cot(x)\, dx = -2 \int_0^{\pi z} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} x^{2n-1+p}\, dx = -2(\pi z)^p \sum_{n=0}^{\infty} \frac{\zeta(2n) z^{2n}}{2n+p},$$

and using (2.11),

$$-2 \sum_{n=0}^{\infty} \frac{\zeta(2n) z^{2n}}{2n+p} = \sum_{k=0}^{p} \frac{p!(-1)^{\lfloor \frac{k+3}{2} \rfloor}}{(p-k)!(2\pi z)^k} \operatorname{Cl}_{k+1}(2\pi z) + \delta^{\frac{p}{2}}_{\lfloor \frac{p}{2} \rfloor} \frac{p!(-1)^{\frac{p}{2}}}{(2\pi z)^p} \zeta(p+1). \tag{2.14}$$

So for $z = 1/2$ and $z = 1/4$, we have

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+p)4^n} = \log 2 + \sum_{k=1}^{\lfloor\frac{p}{2}\rfloor}\frac{p!(-1)^k(4^k-1)\zeta(2k+1)}{(p-2k)!(2\pi)^{2k}} + \delta_{\lfloor\frac{p}{2}\rfloor}^{\frac{p}{2}}\frac{p!(-1)^{\frac{p}{2}}\zeta(p+1)}{\pi^p}, \quad (2.15)$$

and

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+p)16^n} = \frac{1}{2}\log 2 + \frac{1}{2}\sum_{k=1}^{\lfloor\frac{p}{2}\rfloor}\frac{p!(-1)^k(4^k-1)\zeta(2k+1)}{(p-2k)!(2\pi)^{2k}}$$
$$-\frac{\pi}{2}\sum_{k=1}^{\lfloor\frac{p+1}{2}\rfloor}\frac{p!(-4)^k\beta(2k)}{(p+1-2k)!\pi^{2k}} + \delta_{\lfloor\frac{p}{2}\rfloor}^{\frac{p}{2}}\frac{p!(-1)^{\frac{p}{2}}2^p\zeta(p+1)}{\pi^p}. \quad (2.16)$$

Plugging in $p = 1$ into both of the equations yields two nice series representations for $\log 2$,

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n} = \log 2, \quad (2.17)$$

and

$$-\frac{4G}{\pi} - 4\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)16^n} = \log 2. \quad (2.18)$$

Other series from (2.15) and (2.16) are

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(n+1)4^n} = -\log 2 + \frac{7\zeta(3)}{2\pi^2},$$
$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+3)4^n} = -\frac{1}{2}\log 2 + \frac{9\zeta(3)}{4\pi^2},$$
$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(n+2)4^n} = -\log 2 + \frac{9\zeta(3)}{\pi^2} - \frac{93\zeta(5)}{2\pi^4},$$
$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(n+1)16^n} = -\frac{1}{2}\log 2 + \frac{35\zeta(3)}{4\pi^2} - \frac{4G}{\pi},$$

and

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+3)16^n}=-\frac{1}{4}\log 2+\frac{9\zeta(3)}{8\pi^2}-\frac{3G}{\pi}+\frac{24\beta(4)}{\pi^3}.$$

Using a few of these relations, one can find

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+3)4^n}=-\frac{9\zeta(3)}{8\pi^2}$$

and

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(n+1)4^n}=-\frac{7\zeta(3)}{2\pi^2}.$$

Subtracting linear combinations of (2.15) and (2.16), one finds

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+p)4^n}\left(1-\frac{2}{4^n}\right)=\delta^{\frac{p}{2}}_{\lfloor\frac{p}{2}\rfloor}\frac{p!(-1)^{\frac{p}{2}}(2^{p+1}-1)}{2\pi^p}\zeta(p+1)-\frac{\pi}{2}\sum_{k=1}^{\lfloor\frac{p+1}{2}\rfloor}\frac{p!(-4)^k\beta(2k)}{(p+1-2k)!\pi^{2k}}. \quad (2.19)$$

For $p=1$, we have a nice formula for G given as

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n}\left(1-\frac{2}{4^n}\right)=\frac{2G}{\pi}. \quad (2.20)$$

Formula (2.19) focuses more on the Dirichlet beta function rather than the zeta function. Other formulas include

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2)4^n}\left(1-\frac{2}{4^n}\right)=-\frac{7\zeta(3)}{\pi^2}+\frac{4G}{\pi}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+3)4^n}\left(1-\frac{2}{4^n}\right)=\frac{6G}{\pi}-\frac{48\beta(4)}{\pi^3}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+4)4^n}\left(1-\frac{2}{4^n}\right)=\frac{372\zeta(5)}{\pi^4}+\frac{8G}{\pi}-\frac{192\beta(4)}{\pi^3}$$

Using some of these relations, one can find

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)(2n+3)4^n}\left(1-\frac{2}{4^n}\right)=\frac{7\zeta(3)}{\pi^2}-\frac{24\beta(4)}{\pi^3}.$$

Later in the paper, we will find an easier way of obtaining such rational zeta series with more monomials on the denominator.

2.2 USING THE GENERALIZED CLAUSEN FUNCTION

Theorem 2.2.1. *For* $p\in\mathbb{N}_0$, $m\in\mathbb{N}$ *and* $|z|<1$,

$$\int_0^{2\pi z} x^p \operatorname{Cl}_m(x)\,dx = -\sum_{k=m+1}^{m+p+1}\frac{(2\pi z)^{p+m+1-k}p!(-1)^{\lfloor\frac{m}{2}\rfloor}(-1)^{\lfloor\frac{k}{2}\rfloor}}{(p+m+1-k)!}\operatorname{Cl}_k(2\pi z)$$
$$+\,\delta^{\frac{p+m}{2}}_{\lfloor\frac{p+m}{2}\rfloor}(-1)^{\lfloor\frac{m}{2}\rfloor}p!(-1)^{\frac{p+m}{2}}\zeta(p+m+1). \quad (2.21)$$

Proof. Similar to the proof of (13), we will call the left hand side $f(z)$ and the right hand side $g(z)$. Note $f(0)=0$ and $f'(z)=(2\pi)^{p+1}z^p\operatorname{Cl}_m(2\pi z)$. Using (7),

$$g'(z) = -\sum_{k=m+1}^{m+p}\frac{p!(2\pi)^{p+m+1-k}z^{p+m-k}(-1)^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{k}{2}\rfloor}}{(p+m-k)!}\operatorname{Cl}_k(2\pi z)$$
$$-\sum_{k=m+1}^{m+p+1}\frac{p!(2\pi z)^{p+m+1-k}(-1)^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{k}{2}\rfloor}}{(p+m+1-k)!}\Big\{(-1)^k 2\pi\operatorname{Cl}_{k-1}(2\pi z)\Big\}$$

$$= -(2\pi)^{p+1}z^p(-1)^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{m+1}{2}\rfloor}(-1)^{m+1}\operatorname{Cl}_m(2\pi z)$$
$$-\,2\pi\sum_{k=m+1}^{m+p}\frac{p!(2\pi z)^{p+m-k}(-1)^{\lfloor\frac{m}{2}\rfloor}}{(p+m-k)!}\operatorname{Cl}_k(2\pi z)\Big\{(-1)^{\lfloor\frac{k+1}{2}\rfloor}(-1)^{k+1}+(-1)^{\lfloor\frac{k}{2}\rfloor}\Big\}$$

$$= (2\pi)^{p+1}z^p\operatorname{Cl}_m(2\pi z).$$

So indeed $f'(z) = g'(z)$. Plugging in $z = 0$, the sum in $g(z)$ only has the last term remaining. So $g(0)$ simplifies to

$$g(0) = -p!(-1)^{\lfloor \frac{m}{2} \rfloor}(-1)^{\lfloor \frac{m+p+1}{2} \rfloor} \operatorname{Cl}_{m+p+1}(0) + \delta_{\lfloor \frac{p+m}{2} \rfloor}^{\frac{p+m}{2}}(-1)^{\lfloor \frac{m}{2} \rfloor} p!(-1)^{\frac{p+m}{2}} \zeta(p+m+1)$$

and since $\operatorname{Cl}_{m+p+1}(0) = \delta_{\lfloor \frac{p+m}{2} \rfloor}^{\frac{p+m}{2}} \zeta(p+m+1)$, we see $g(0) = 0$. Since $f(0) = g(0)$ and $f'(z) = g'(z)$, $f(z) = g(z)$. □

We can see if $p = 0$, we recover (2.8). Now letting $z = 1/2$ and $z = 1/4$, we find

$$\int_0^{\pi} x^p \operatorname{Cl}_m(x)\, dx = (-1)^{\lfloor \frac{m}{2} \rfloor} p! \pi^{p+m} \sum_{k=\lfloor \frac{m+1}{2} \rfloor}^{\lfloor \frac{p+m}{2} \rfloor} \frac{(-1)^k (4^k - 1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}} + \delta_{\lfloor \frac{p+m}{2} \rfloor}^{\frac{p+m}{2}}(-1)^{\lfloor \frac{m}{2} \rfloor} p!(-1)^{\frac{p+m}{2}} \zeta(p+m+1), \quad (2.22)$$

and

$$\int_0^{\pi/2} x^p \operatorname{Cl}_m(x)\, dx = (-1)^{\lfloor \frac{m}{2} \rfloor} p! \left(\frac{\pi}{2}\right)^{p+m} \left(\frac{1}{2} \sum_{k=\lfloor \frac{m+1}{2} \rfloor}^{\lfloor \frac{p+m}{2} \rfloor} \frac{(-1)^k (4^k - 1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}} - \frac{\pi}{2} \sum_{k=\lfloor \frac{m+2}{2} \rfloor}^{\lfloor \frac{p+m+1}{2} \rfloor} \frac{(-1)^k 4^k \beta(2k)}{(p+m+1-2k)!\pi^{2k}} \right) + \delta_{\lfloor \frac{p+m}{2} \rfloor}^{\frac{p+m}{2}}(-1)^{\lfloor \frac{m}{2} \rfloor} p!(-1)^{\frac{p+m}{2}} \zeta(p+m+1). \quad (2.23)$$

Now, we will integrate the left hand side of (2.21) using (2.3) and (2.10), giving us

$$\int_0^{2\pi z} x^p \operatorname{Cl}_m(x)\, dx = \int_0^{2\pi z} x^p \left(\sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor + k} x^{m-2k-1}}{(m-2k-1)!} \zeta(2k+1) + \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-3)!} \int_0^x (x-t)^{m-3} \operatorname{Cl}_2(t)\, dt \right) dx$$

$$= \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor + k}(2\pi z)^{m+p-2k}}{(m-1-2k)!(m+p-2k)}\zeta(2k+1) + \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-3)!}\int_0^{2\pi z}\int_0^x x^p (x-t)^{m-3}\,\mathrm{Cl}_2(t)\,dt\,dx$$

$$= \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor + k}(2\pi z)^{m+p-2k}}{(m-1-2k)!(m+p-2k)}\zeta(2k+1) + \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-3)!}\int_0^{2\pi z}\int_0^x x^p (x-t)^{m-3}\left(t - t\log t + \sum_{n=1}^{\infty}\frac{\zeta(2n)t^{2n+1}}{n(2n+1)(2\pi)^{2n}}\right)dt\,dx$$

$$= \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor + k}(2\pi z)^{m+p-2k}}{(m-1-2k)!(m+p-2k)}\zeta(2k+1) + \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-3)!}\int_0^{2\pi z} x^p\left(\frac{x^{m-1}(H_{m-1} - \log x)}{(m-1)(m-2)} + \sum_{n=1}^{\infty}\frac{2\zeta(2n)\Gamma(m-2)x^{m+2n-1}}{2n(2n+1)(2n+2)\dots(2n+m-1)(2\pi)^{2n}}\right)dx,$$

where H_k is the k-th harmonic number and $H_0 := 0$. Integrating again and simplifying, we arrive at

$$\int_0^{2\pi z} x^p\,\mathrm{Cl}_m(x)\,dx = \frac{(2\pi z)^{p+m}(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-1)!}\left(\frac{H_{m-1} - \log(2\pi z)}{(p+m)} + \frac{1}{(p+m)^2} + \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor}\frac{(m-1)!(-1)^k\zeta(2k+1)}{(m-1-2k)!(p+m-2k)(2\pi z)^{2k}} + \sum_{n=1}^{\infty}\frac{(m-1)!\zeta(2n)z^{2n}}{n(2n+1)\dots(2n+m-1)(2n+p+m)}\right). \tag{2.24}$$

Using (2.21), we can rearrange this and find

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)z^{2n}}{n(2n+1)\dots(2n+m-1)(2n+m+p)} = \frac{\log(2\pi z) - H_{m-1}}{(m-1)!(p+m)} + \sum_{k=m}^{m+p}\frac{(-1)^m p!(-1)^{\lfloor \frac{k+1}{2} \rfloor}}{(p+m-k)!(2\pi z)^k}\,\mathrm{Cl}_{k+1}(2\pi z) + \delta_{\lfloor \frac{p+m}{2} \rfloor}^{\frac{p+m}{2}}\frac{(-1)^{m+1}p!(-1)^{\frac{p+m}{2}}}{(2\pi z)^{p+m}}\zeta(p+m+1) - \frac{1}{(m-1)!(p+m)^2} - \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor}\frac{(-1)^k\zeta(2k+1)}{(m-1-2k)!(m+p-2k)(2\pi z)^{2k}}. \tag{2.25}$$

Corollary 2.2.2. *Conjecture from Lima, 2012 [13]. For N odd,*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+0)\dots(2n+N)2^{2n}} = \frac{1}{2}\left(\frac{\log \pi - H_N}{N!} + \sum_{k=1}^{(N-1)/2} (-1)^{k+1} \frac{\zeta(2k+1)}{(N-2k)!\pi^{2k}}\right),$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+0)\dots(2n+N)4^{2n}} = \frac{1}{2}\left(\frac{\log(\pi/2) - H_N}{N!} - (-1)^{\frac{N+1}{2}}\left(\frac{2}{\pi}\right)^N \beta(N+1) - \sum_{k=1}^{(N-1)/2} (-1)^k \left(\frac{2}{\pi}\right)^{2k} \frac{\zeta(2k+1)}{(N-2k)!}\right).$$

Proof. Note when $p = 0$ in (2.25), we have the very special formula

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)z^{2n}}{n(2n+1)\dots(2n+m)} = \frac{(-1)^m(-1)^{\lfloor\frac{m+1}{2}\rfloor}}{(2\pi z)^m} \operatorname{Cl}_{m+1}(2\pi z) - \delta^{\frac{m}{2}}_{\lfloor\frac{m}{2}\rfloor} \frac{(-1)^{\frac{3m}{2}}}{(2\pi z)^m}\zeta(m+1) - \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor} \frac{(-1)^k\zeta(2k+1)}{(m-2k)!(2\pi z)^{2k}} + \frac{\log(2\pi z) - H_m}{m!}. \quad (2.26)$$

Letting $z = 1/2$ and $z = 1/4$, we see that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)\dots(2n+m)4^n} = \delta^{\frac{m}{2}}_{\lfloor\frac{m}{2}\rfloor} \frac{(-1)^{\frac{3m+2}{2}}(2^{m+1}-1)\zeta(m+1)}{(2\pi)^m} - \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor} \frac{(-1)^k\zeta(2k+1)}{(m-2k)!\pi^{2k}} + \frac{\log \pi - H_m}{m!}, \quad (2.27)$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)\dots(2n+m)16^n}=\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(-1)^{\frac{3m+1}{2}}2^m\beta(m+1)}{\pi^m}$$
$$-\frac{1}{2}\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(-1)^{\frac{3m}{2}}(2^{2m+1}+2^m-1)\zeta(m+1)}{(2\pi)^m}-\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-4)^k\zeta(2k+1)}{(m-2k)!\pi^{2k}}+\frac{\log(\pi/2)-H_m}{m!},\tag{2.28}$$

Dividing both sides of each equation by 2 yields two stronger statements than the conjecture since $m\in\mathbb{N}$. In any case, the conjecture is proven. □

For general p, we have

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)\dots(2n+m-1)(2n+m+p)4^n}=\frac{\log\pi-H_{m-1}}{(m-1)!(p+m)}$$
$$+(-1)^{m+1}p!\left(\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{(-1)^{\frac{p+m}{2}}}{\pi^{p+m}}\zeta(p+m+1)+\sum_{k=\lfloor\frac{m+1}{2}\rfloor}^{\lfloor\frac{p+m}{2}\rfloor}\frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}}\right)$$
$$-\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-1)^k\zeta(2k+1)}{(m-1-2k)!(p+m-2k)\pi^{2k}}-\frac{1}{(m-1)!(p+m)^2},\tag{2.29}$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)\dots(2n+m-1)(2n+m+p)16^n}=\frac{\log(\pi/2)-H_{m-1}}{(m-1)!(p+m)}-\frac{1}{(m-1)!(p+m)^2}$$
$$+\frac{\pi}{2}\sum_{k=\lfloor\frac{m+2}{2}\rfloor}^{\lfloor\frac{p+m+1}{2}\rfloor}\frac{p!(-1)^{m+k}4^k\beta(2k)}{(p+m+1-2k)!\pi^{2k}}+\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{(-1)^{m+1}p!(-1)^{\frac{p+m}{2}}2^{p+m}}{\pi^{p+m}}\zeta(p+m+1)$$
$$-\frac{p!(-1)^m}{2}\sum_{k=\lfloor\frac{m+1}{2}\rfloor}^{\lfloor\frac{p+m}{2}\rfloor}\frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}}-\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-1)^k4^k\zeta(2k+1)}{(m-1-2k)!(p+m-2k)\pi^{2k}}.\tag{2.30}$$

Remark. When $m=1$ and $p=0$, one has

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)4^n} = \log \pi - 1,$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)16^n} = \frac{2G}{\pi} - 1 + \log\left(\frac{\pi}{2}\right),$$

the first of which is a famous series (see [16]) and both are immediate from (3) with $\theta = \pi$ and $\theta = \pi/2$, respectively. Manipulating these and equations (19) and (20), one can show

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)4^n}\left(1 - \frac{1}{4^n}\right)\left(\frac{1}{n} + 4\right) = \frac{2G}{\pi}$$

Below we compute other sums for certain m and p.

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+3)4^n} = -\frac{3\zeta(3)}{2\pi^2} + \frac{1}{3}\log \pi - \frac{1}{9}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(n+1)4^n} = \frac{7\zeta(3)}{2\pi^2} - \log \pi - \frac{3}{2}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(n+1)(2n+3)4^n} = \frac{2\zeta(3)}{\pi^2} + \frac{1}{3}\log \pi - \frac{11}{18}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(n+1)(n+2)4^n} = \frac{2\zeta(3)}{\pi^2} + \frac{31\zeta(5)}{4\pi^4} + \frac{1}{2}\log \pi - \frac{7}{8}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)\dots(2n+5)4^n} = \frac{\zeta(3)}{6\pi^2} - \frac{\zeta(5)}{\pi^4} + \frac{1}{120}\log \pi - \frac{137}{7200}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)16^n} = -\frac{35\zeta(3)}{4\pi^2} + \frac{4G}{\pi} + \log\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(n+1)16^n} = \frac{35\zeta(3)}{4\pi^2} + \log\left(\frac{\pi}{2}\right) - \frac{3}{2}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)16^n} = \frac{3\zeta(3)}{8\pi^2} + \frac{8\beta(4)}{\pi^3} + \frac{1}{3}\log\left(\frac{\pi}{2}\right) - \frac{4}{9}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)\dots(2n+4)16^n} = \frac{2\zeta(3)}{\pi^2} - \frac{527\zeta(5)}{32\pi^4} + \frac{1}{24}\log\left(\frac{\pi}{2}\right) - \frac{25}{288}$$

2.3 REVISITING THE COTANGENT FUNCTION

Using the cotangent function again, we will investigate the double integral

$$\int_0^{\pi z} \int_0^x x^p (x-t)^m t \cot(t)\, dt\, dx.$$

Using the binomial theorem, (13), and a change of variables among other things,

$$\int_0^{\pi z} \int_0^x x^p (x-t)^m t \cot(t)\, dt\, dx = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_0^{\pi z} x^{p+m-j} \int_0^x t^{j+1} \cot(t)\, dt\, dx$$

$$= \sum_{j=0}^{m} \sum_{k=0}^{j+1} \binom{m}{j} \frac{(-1)^j (j+1)! (-1)^{\lfloor \frac{k+3}{2} \rfloor}}{(j+1-k)! 2^k} \int_0^{\pi z} x^{p+m+1-k}\, \mathrm{Cl}_{k+1}(2x)\, dx$$
$$+ \sum_{j=0}^{m} \delta^{\frac{j+1}{2}}_{\lfloor \frac{j+1}{2} \rfloor} \frac{(-1)^j (j+1)! (-1)^{\frac{j+1}{2}}}{2^{j+1}} \binom{m}{j} \zeta(j+2) \int_0^{\pi z} x^{p+m-j}\, dx$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^{j+1}}{2^{m+p+2}} \int_0^{2\pi z} u^{p+m+1}\, \mathrm{Cl}_1(u)\, du$$
$$+ \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{m}{j} \frac{(-1)^j (j+1)! (-1)^{\lfloor \frac{k}{2} \rfloor}}{(j-k)! 2^{p+m+2}} \int_0^{2\pi z} u^{p+m-k}\, \mathrm{Cl}_{k+2}(u)\, du$$
$$- \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(2k)(-1)^k m! (\pi z)^{p+m+2-2k}}{2^{2k} (m+1-2k)! (p+m+2-2k)} \zeta(2k+1)$$

$$= -\frac{\delta_m^0}{2^{p+2}} \int_0^{2\pi z} u^{p+1} \operatorname{Cl}_1(u)\, du - (1-\delta_m^0) \frac{(-1)^{\lfloor \frac{m}{2} \rfloor} m!}{2^{p+m+2}} \int_0^{2\pi z} u^{p+1} \operatorname{Cl}_{m+1}(u)\, du$$
$$+ \frac{(m+1)!(-1)^{\lfloor \frac{m+1}{2} \rfloor}}{2^{p+m+2}} \int_0^{2\pi z} u^{p} \operatorname{Cl}_{m+2}(u)\, du + \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(2k)(-1)^{k+1} m! (\pi z)^{p+m+2-2k} \zeta(2k+1)}{2^{2k}(m+1-2k)!(p+m+2-2k)}.$$

where we've evaluated the sum on j and k. The terms with $\delta_{m,0}$ cancel each other. We can use (21) to evaluate the other integrals. Rearranging and simplifying, we will arrive at

$$\int_0^{\pi z} \int_0^x x^p (x-t)^m t \cot(t)\, dt\, dx = (p+m+2) \Bigg(2(-1)^m (\pi z)^{m+p+3} p! m! *$$
$$\sum_{k=m+3}^{p+m+3} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor} \operatorname{Cl}_k(2\pi z)}{(p+m+3-k)!(2\pi z)^k} + \delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} \frac{p!(-1)^{\frac{p+m}{2}} (-1)^m m! \zeta(p+m+3)}{2^{m+p+2}} \Bigg)$$
$$- \frac{(-1)^{\lfloor \frac{m+1}{2} \rfloor} m! (\pi z)^{p+1} \operatorname{Cl}_{m+2}(2\pi z)}{2^{m+1}} + \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(2k)(-1)^{k+1} m! (\pi z)^{p+m+2-2k} \zeta(2k+1)}{2^{2k}(m+1-2k)!(p+m+2-2k)}. \quad (2.31)$$

Another way to evaluate this double integral is to use (2.1) and Fubini's theorem, similar to the first section. Doing so, we see

$$\int_0^{\pi z} \int_0^x x^p (x-t)^m t \cot(t)\, dt\, dx = -2 \int_0^{\pi z} x^p \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \int_0^x (x-t)^m t^{2n}\, dt\, dx$$
$$= -2 \sum_{n=0}^{\infty} \frac{\zeta(2n) m! \Gamma(2n+1)}{\pi^{2n} \Gamma(m+2n+2)} \int_0^{\pi z} x^{2n+m+p+1}\, dx$$
$$= -2 \sum_{n=0}^{\infty} \frac{\zeta(2n) m! (\pi z)^{m+p+2} z^{2n}}{(2n+1)\dots(2n+m+1)(2n+m+p+2)}.$$

Putting this and (2.31) together, we find

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)z^{2n}}{(2n+1)\dots(2n+m+1)(2n+m+p+2)}=(p+m+2)\Bigg((-1)^{m+1}p!*$$
$$\sum_{k=m+3}^{p+m+3}\frac{\pi z(-1)^{\lfloor\frac{k}{2}\rfloor}\operatorname{Cl}_k(2\pi z)}{(p+m+3-k)!(2\pi z)^k}-\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{p!(-1)^{\frac{p+m}{2}}(-1)^m\zeta(p+m+3)}{2(2\pi z)^{m+p+2}}\Bigg)$$
$$+\frac{(-1)^{\lfloor\frac{m+1}{2}\rfloor}\operatorname{Cl}_{m+2}(2\pi z)}{2(2\pi z)^{m+1}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)}{(2\pi z)^{2k}(m+1-2k)!(m+p+2-2k)}. \quad (2.32)$$

When $p=0$, this formula simplifies to

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)z^{2n}}{(2n+1)\dots(2n+m+2)}=\frac{(-1)^{\lfloor\frac{m+1}{2}\rfloor}\operatorname{Cl}_{m+2}(2\pi z)}{2(2\pi z)^{m+1}}$$
$$+\frac{(-1)^{\lfloor\frac{m}{2}\rfloor}(m+2)\Big(\operatorname{Cl}_{m+3}(2\pi z)-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\zeta(m+3)\Big)}{2(2\pi z)^{m+2}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)}{(2\pi z)^{2k}(m+2-2k)!}. \quad (2.33)$$

Setting $z=1/2$ and $z=1/4$, we obtain

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)\dots(2n+m+2)4^n}=-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{2(2\pi)^{m+1}}$$
$$-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(-1)^{\frac{m}{2}}(2^{m+3}-1)(m+2)\zeta(m+3)}{2(2\pi)^{m+2}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)}{(2\pi z)^{2k}(m+2-2k)!}, \quad (2.34)$$

and

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)\dots(2n+m+2)16^n}=$$
$$(-1)^{\lfloor\frac{m}{2}\rfloor}(m+2)\Bigg(\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{2^{m+1}\beta(m+3)}{\pi^{m+2}}-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(2^{2m+5}+2^{m+2}-1)\zeta(m+3)}{4(2\pi)^{m+2}}\Bigg)$$
$$+(-1)^{\lfloor\frac{m+1}{2}\rfloor}\Bigg(\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{2^m\beta(m+2)}{\pi^{m+1}}-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(2^{m+1}-1)\zeta(m+2)}{4(2\pi)^{m+1}}\Bigg)+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-4)^k\zeta(2k+1)}{\pi^{2k}(m+2-2k)!}.$$
$$(2.35)$$

For general p,

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)\dots(2n+m+1)(2n+m+p+2)4^n}=$$
$$\frac{(-1)^m p!(m+p+2)}{2}\left(\sum_{k=\lfloor\frac{m+3}{2}\rfloor}^{\lfloor\frac{p+m+2}{2}\rfloor}\frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m+2-2k)!(2\pi)^{2k}}-\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+3)}{\pi^{m+p+2}}\right)$$
$$-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{2(2\pi)^{m+1}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)}{\pi^{2k}(m+1-2k)!(m+p+2-2k)},\quad(2.36)$$

and

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)\dots(2n+m+1)(2n+m+p+2)16^n}=$$
$$\frac{(-1)^m p!(m+p+2)}{4}\left(\sum_{k=\lfloor\frac{m+3}{2}\rfloor}^{\lfloor\frac{p+m+2}{2}\rfloor}\frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m+2-2k)!(2\pi)^{2k}}-\sum_{k=\lfloor\frac{m+4}{2}\rfloor}^{\lfloor\frac{p+m+3}{2}\rfloor}\frac{(-1)^k\beta(2k)4^k}{(p+m+3-2k)!\pi^{2k-1}}\right.$$
$$\left.-\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+3)2^{p+m+3}}{\pi^{m+p+2}}\right)+\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(-1)^{\lfloor\frac{m+1}{2}\rfloor}2^m\beta(m+2)}{\pi^{m+1}}$$
$$-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{4(2\pi)^{m+1}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)4^k}{\pi^{2k}(m+1-2k)!(m+p+2-2k)}.\quad(2.37)$$

Remark. For $m=0$ and $p=0$, we have

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(n+1)4^n}=-\frac{7\zeta(3)}{2\pi^2},$$

and

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(n+1)16^n}=\frac{2G}{\pi}-\frac{35\zeta(3)}{4\pi^2},$$

the first of which was rediscovered by Ewell (see [11]). Similar to (2.20), we can use these sums and obtain

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(n+1)4^n}\left(\frac{5}{2}-\frac{1}{4^n}\right)=-\frac{2G}{\pi} \tag{2.38}$$

Below we compute other sums for certain m and p.

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+3)4^n}=-\frac{9\zeta(3)}{8\pi^2}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(n+2)4^n}=-\frac{3\zeta(3)}{\pi^2}+\frac{31\zeta(5)}{2\pi^4}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)(2n+3)4^n}=-\frac{5\zeta(3)}{8\pi^2}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(n+1)(n+2)4^n}=-\frac{\zeta(3)}{2\pi^2}-\frac{31\zeta(5)}{2\pi^4}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)\dots(2n+5)4^n}=-\frac{\zeta(3)}{6\pi^2}+\frac{49\zeta(5)}{32\pi^4}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+3)16^n}=-\frac{9\zeta(3)}{16\pi^2}+\frac{G}{\pi}-\frac{12\beta(4)}{\pi^3}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)(2n+3)16^n}=-\frac{61\zeta(3)}{16\pi^2}+\frac{12\beta(4)}{\pi^3}$$

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)\dots(2n+4)16^n}=-\frac{2\zeta(3)}{\pi^2}+\frac{527\zeta(5)}{16\pi^4}-\frac{4\beta(4)}{\pi^3}$$

Now, we will use these results to establish the same families of general rational zeta series but for $\zeta(2n+1)$.

3.0 RATIONAL $\zeta(2N+1)$ SERIES

The polygamma function is given by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\Big(\log\Gamma(z)\Big),\ \ n \in \mathbb{N}_0.$$

This paper will discuss when $n = 0$. For $n = 0$, $\psi^{(0)}(z) = \psi(z)$ is called the digamma function. It has been shown (see [2]) that there is a closed form of $\int_0^z x^n \psi(x)\,dx$ in terms of sums involving Harmonic numbers, Bernoulli numbers and Bernoulli polynomials. There are also definitions for negative order polygamma functions, called negapolygammas, given by

$$\psi^{(-1)}(z) = \log\Gamma(z),$$

$$\psi^{(-2)}(z) = \int_0^z \log\Gamma(x)\,dx,$$

$$\psi^{(-3)}(z) = \int_0^z \int_0^x \log\Gamma(t)\,dt\,dx = \int_0^z \int_t^z \log\Gamma(t)\,dx\,dt = \int_0^z (z-t)\log\Gamma(t)\,dt,$$

and by induction, for $n \geq 2$,

$$\psi^{(-n)}(z) = \frac{1}{(n-2)!}\int_0^z (z-t)^{n-2}\log\Gamma(t)\,dt. \tag{3.1}$$

Note the Taylor series for $\log\Gamma(z)$ is

$$\log\Gamma(z) = -\log z - \gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} z^k, \quad |z| < 1, \tag{3.2}$$

where γ is the Euler-Mascheroni constant. This series can be obtained using the Taylor series for the Hurwitz zeta function, defined by

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \quad a \in \mathbb{R} \backslash -\mathbb{N}, \quad \Re(s) > 1.$$

The negapolygammas are related to the derivative of the Hurwitz zeta function with respect to the first variable (see [2]).

3.1 USING DIGAMMA FUNCTION

Theorem 3.1.1. *For $p \in \mathbb{N}$ and $|z| < 1$,*

$$\int_0^z x^p \psi(x)\, dx = \sum_{k=0}^{p} \frac{p!(-1)^k \psi^{(-k-1)}(z)}{(p-k)!} z^{p-k}. \tag{3.3}$$

Proof. Let $f(z)$ be the left hand side and $g(z)$ be the right hand side. It is clear that $f(0) = g(0) = 0$ and $f'(z) = z^p \psi(z)$. Taking the derivative of $g(z)$, we find

$$\begin{aligned} g'(z) &= \sum_{k=0}^{p} \frac{p!(-1)^k \psi^{(-k)}(z)}{(p-k)!} z^{p-k} + \sum_{k=0}^{p-1} \frac{p!(-1)^k \psi^{(-k-1)}(z)}{(p-k-1)!} z^{p-k-1} \\ &= z^p \psi(z) + \sum_{k=1}^{p} \frac{p!(-1)^k \psi^{(-k)}(z)}{(p-k)!} z^{p-k} + \sum_{k=0}^{p-1} \frac{p!(-1)^k \psi^{(-k-1)}(z)}{(p-k-1)!} z^{p-k-1}. \end{aligned}$$

Reindexing the first sum, one can see the two sums cancel each other out. So, $g'(z) = f'(z)$ for all z. Since they are equal at $z = 0$, then $f(z) = g(z)$. □

We can also compute this integral using the derivative of (3.2). So we will have

$$\int_0^z x^p\psi(x)\,dx = \int_0^z x^p\Big(-\gamma - \frac{1}{x} + \sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1}\Big)\,dx$$

$$= -\frac{\gamma}{p+1}z^{p+1} - \frac{z^p}{p} + \sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)}{k+p}z^{k+p}.$$

Setting these two results equal to each other, one has

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)}{k+p}z^k = \frac{1}{p} + \frac{\gamma z}{p+1} + \sum_{k=0}^{p}\frac{p!(-1)^k\psi^{(-k-1)}(z)}{(p-k)!z^k}. \tag{3.4}$$

Now we can split this up as follows:

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^k}{k+p} = \sum_{n=1}^{\infty}\frac{\zeta(2n)z^{2n}}{2n+p} - \sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n+1}}{2n+p+1}.$$

Using (2.14) and rearranging, we see

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n}}{2n+p+1} = -\frac{1}{2zp} - \frac{\gamma}{p+1} + \sum_{k=0}^{p}\frac{p!(-1)^{\lfloor\frac{k+1}{2}\rfloor}\pi}{(p-k)!(2\pi z)^{k+1}}\,\mathrm{Cl}_{k+1}(2\pi z)$$
$$- \delta^{\frac{p}{2}}_{\lfloor\frac{p}{2}\rfloor}\frac{p!(-1)^{\frac{p}{2}}\pi}{(2\pi z)^{p+1}}\zeta(p+1) - \sum_{k=0}^{p}\frac{p!(-1)^k\psi^{(-k-1)}(z)}{(p-k)!z^{k+1}}. \tag{3.5}$$

For $z = 1/2$ and $z = 1/4$, we find

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+p+1)4^n} = -\frac{1}{p} - \log 2 - \frac{\gamma}{p+1} - \sum_{k=1}^{\lfloor\frac{p}{2}\rfloor}\frac{p!(-1)^k(4^k-1)\zeta(2k+1)}{(p-2k)!(2\pi)^{2k}}$$
$$- \delta^{\frac{p}{2}}_{\lfloor\frac{p}{2}\rfloor}\frac{p!(-1)^{\frac{p}{2}}\zeta(p+1)}{\pi^p} - 2\sum_{k=0}^{p}\frac{p!(-2)^k\psi^{(-k-1)}(1/2)}{(p-k)!}, \tag{3.6}$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+p+1)16^n}=-\frac{2}{p}-\log 2-\frac{\gamma}{p+1}-\sum_{k=1}^{\lfloor\frac{p}{2}\rfloor}\frac{p!(-1)^k(4^k-1)\zeta(2k+1)}{(p-2k)!(2\pi)^{2k}}$$
$$+\pi\sum_{k=1}^{\lfloor\frac{p+1}{2}\rfloor}\frac{p!(-4)^k\beta(2k)}{(p+1-2k)!\pi^{2k}}-\delta_{\lfloor\frac{p}{2}\rfloor}^{\frac{p}{2}}\frac{p!(-1)^{\frac{p}{2}}2^{p+1}\zeta(p+1)}{\pi^p}-4\sum_{k=0}^{p}\frac{p!(-4)^k\psi^{(-k-1)}(1/4)}{(p-k)!}. \quad (3.7)$$

Below we compute a few examples for specific p. Note that A is the Glaisher-Kinkelin constant, defined by $\log A=\frac{1}{12}-\zeta'(-1)=-\zeta(-1)-\zeta'(-1)$.

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(n+1)4^n}=-2-\gamma+12\log A-\frac{1}{3}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+3)4^n}=-\frac{1}{2}-\frac{\gamma}{3}+4\log A-\frac{1}{3}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+5)4^n}=-\frac{199}{180}-\frac{\gamma}{5}+8\log A-56\zeta'(-3)-\frac{3}{5}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+2)16^n}=-2-\frac{\gamma}{2}+18\log A+\log 2\pi^2-4\log\Gamma\Big(\frac{1}{4}\Big)$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+3)16^n}=-64\zeta'\Big(-2,\frac{1}{4}\Big)+\frac{1}{2}+4\log A+\log 2\pi^2-\frac{\gamma}{3}+\frac{3\zeta(3)}{2\pi^2}-4\log\Gamma\Big(\frac{1}{4}\Big)$$

3.2 USING NEGAPOLYGAMMA FUNCTIONS

Theorem 3.2.1. *For $p\in\mathbb{N}_0$, $m\in\mathbb{N}$ and $|z|<1$,*

$$\int_0^z x^p\psi^{(-m)}(x)\,dx=\sum_{k=0}^{p}\frac{p!(-1)^k\psi^{(-k-m-1)}(z)}{(p-k)!}z^{p-k}. \quad (3.8)$$

Proof. The proof is exactly the same as the proof of (3.3) as there was no dependence on $\psi(x) = \psi^{(0)}(x)$. ☐

Now, let us compute this integral using (3.1) and (3.2). Doing so, we see

$$\int_0^z x^p \psi^{(-m)}(x)\,dx = \int_0^z \frac{x^p}{(m-2)!}\int_0^x (x-t)^{m-2}\left(-\gamma t - \log t + \sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)}{k}t^k\right)dt\,dx$$

$$= \int_0^z \frac{-x^p}{(m-2)!}\left(\frac{\gamma x^m}{m(m-1)} + \frac{x^{m-1}(\log x - H_{m-1})}{m-1} - \sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)\Gamma(m-1)x^{m+k-1}}{k(k+1)\dots(k+m-1)}\right)dx$$

$$= -\frac{\gamma z^{m+p+1}}{m!(m+p+1)} + \frac{z^{m+p}(H_{m-1} - \log z)}{(m-1)!(m+p)} + \frac{z^{m+p}}{(m-1)!(m+p)^2} + \sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^{m+p+k}}{k(k+1)\dots(k+m-1)(k+m+p)}.$$

Setting this result equal to (3.8) and simplifying, one has the nice result

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^k}{k(k+1)\dots(k+m-1)(k+m+p)} = \frac{\gamma z}{m!(m+p+1)} + \frac{\log z - H_{m-1}}{(m-1)!(m+p)} - \frac{1}{(m-1)!(m+p)^2} + \sum_{k=0}^{p}\frac{p!(-1)^k\psi^{(-k-m-1)}(z)}{(p-k)!z^{m+k}}. \quad (3.9)$$

In the special case of $p = 0$, we have

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^k}{k(k+1)\dots(k+m)} = \frac{\gamma z}{(m+1)!} + \frac{\log z - H_m}{m!} + \frac{\psi^{(-m-1)}(z)}{z^m}. \quad (3.10)$$

Again, we can split (3.9) into

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^k}{k(k+1)\dots(k+m-1)(k+m+p)} = \sum_{n=1}^{\infty}\frac{\zeta(2n)z^{2n}}{2n(2n+1)\dots(2n+m-1)(2n+m+p)} - \sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n+1}}{(2n+1)\dots(2n+m)(2n+1+m+p)},$$

and the first sum has been computed earlier. Using (2.21), we find

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n}}{(2n+1)\dots(2n+m)(2n+m+1+p)} = -\sum_{k=0}^{p}\frac{p!(-1)^k\psi^{(-k-m-1)}(z)}{(p-k)!z^{m+k+1}}$$
$$+\frac{(-1)^m p!}{2z}\left(\sum_{k=m}^{m+p}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(p+m-k)!(2\pi z)^k}\mathrm{Cl}_{k+1}(2\pi z)-\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+1)}{(2\pi z)^{p+m}}\right)$$
$$-\frac{1}{2z}\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-1)^k\zeta(2k+1)}{(m-1-2k)!(m+p-2k)(2\pi z)^{2k}}+\frac{\log(2\pi/z)+H_{m-1}}{2z(m-1)!(p+m)}$$
$$+\frac{1}{2z(m-1)!(p+m)^2}-\frac{\gamma}{m!(p+m+1)}. \quad (3.11)$$

If $p=0$, we have the nice representation

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n}}{(2n+1)\dots(2n+m+1)} = -\frac{\psi^{(-m-1)}(z)}{z^{m+1}}-\frac{1}{2z}\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-1)^k\zeta(2k+1)}{(m-2k)!(2\pi z)^{2k}}$$
$$+\frac{(-1)^{\lfloor\frac{m}{2}\rfloor}(\mathrm{Cl}_{m+1}(2\pi z)-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\zeta(m+1))}{2z(2\pi z)^m}+\frac{\log(2\pi/z)+H_m}{2zm!}-\frac{\gamma}{(m+1)!}. \quad (3.12)$$

and for $z=1/2$ and $z=1/4$, we have

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)\dots(2n+m+1)4^n} = \frac{\log 4\pi+H_m}{m!}-\frac{\gamma}{(m+1)!}$$
$$-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(-1)^{\frac{m}{2}}(2^{m+1}-1)\zeta(m+1)}{(2\pi)^m}-\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-1)^k\zeta(2k+1)}{(m-2k)!\pi^{2k}}-2^{m+1}\psi^{(-m-1)}(1/2), \quad (3.13)$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)\dots(2n+m+1)16^n}=\frac{2\log 8\pi+2H_m}{m!}-\frac{\gamma}{(m+1)!}$$
$$-\delta^{\frac{m+1}{2}}_{\lfloor\frac{m+1}{2}\rfloor}\frac{(-1)^{\frac{m+1}{2}}2^{m+1}\beta(m+1)}{\pi^m}-\delta^{\frac{m}{2}}_{\lfloor\frac{m}{2}\rfloor}\frac{(-1)^{\frac{m}{2}}(2^{2m+1}+2^m-1)\zeta(m+1)}{(2\pi)^m}$$
$$-\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{2(-4)^k\zeta(2k+1)}{(m-2k)!\pi^{2k}}-4^{m+1}\psi^{(-m-1)}(1/4). \quad (3.14)$$

For general p,

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)\dots(2n+m)(2n+1+m+p)4^n}=\frac{\log 4\pi+H_{m-1}}{(m-1)!(m+p)}$$
$$+\frac{1}{(m-1)!(m+p)^2}-\frac{\gamma}{m!(m+p+1)}-(-1)^m p!\sum_{k=\lfloor\frac{m+1}{2}\rfloor}^{\lfloor\frac{p+m}{2}\rfloor}\frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}}$$
$$+\delta^{\frac{p+m}{2}}_{\lfloor\frac{p+m}{2}\rfloor}\frac{(-1)^{m+1}p!(-1)^{\frac{p+m}{2}}}{\pi^{p+m}}\zeta(p+m+1)-2^{m+1}\sum_{k=0}^{p}\frac{p!(-2)^k\psi^{(-k-m-1)}(1/2)}{(p-k)!}$$
$$-\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{(-1)^k\zeta(2k+1)}{(m-1-2k)!(m+p-2k)\pi^{2k}}, \quad (3.15)$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)\dots(2n+m)(2n+1+m+p)16^n}=\frac{2}{(m-1)!}\left(\frac{\log 8\pi+H_{m-1}}{(m+p)}\right.$$
$$\left.+\frac{1}{(m+p)^2}\right)+\sum_{k=\lfloor\frac{m+2}{2}\rfloor}^{\lfloor\frac{p+m+1}{2}\rfloor}\frac{p!(-1)^{m+k}4^k\beta(2k)}{(p+m+1-2k)!\pi^{2k-1}}-\sum_{k=\lfloor\frac{m+1}{2}\rfloor}^{\lfloor\frac{p+m}{2}\rfloor}\frac{p!(-1)^{k+m}(4^k-1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}}$$
$$-\delta^{\frac{p+m}{2}}_{\lfloor\frac{p+m}{2}\rfloor}\frac{p!(-1)^m(-1)^{\frac{p+m}{2}}2^{p+m+1}\zeta(p+m+1)}{\pi^{p+m}}-\frac{\gamma}{m!(m+p+1)}$$
$$-\sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{2(-4)^k\zeta(2k+1)}{(m-1-2k)!(m+p-2k)\pi^{2k}}-4^{m+1}\sum_{k=0}^{p}\frac{p!(-4)^k\psi^{(-k-m-1)}(1/4)}{(p-k)!}. \quad (3.16)$$

Below we compute some sums for certain m and p.

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)(n+1)4^n}=-12\log A+2-\gamma+\frac{7}{3}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)(2n+3)4^n}=-2\log A+\frac{1}{4}-\frac{\gamma}{3}+\frac{2}{3}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)(n+2)4^n}=-4\log A+20\zeta'(-3)+\frac{19}{36}-\frac{\gamma}{2}-\frac{89}{90}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)(n+1)(2n+3)4^n}=-8\log A+\frac{3}{2}-\frac{\gamma}{3}+\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)(n+1)(n+2)4^n}=-8\log A-20\zeta'(-3)+\frac{53}{36}-\frac{\gamma}{2}+\frac{121}{90}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)\dots(2n+5)4^n}=-\frac{2}{3}\log A+\frac{8}{3}\zeta'(-3)+\frac{551}{4320}-\frac{\gamma}{120}+\frac{1}{24}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)(n+1)16^n}=-36\log A+4-\gamma+8\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)(2n+3)16^n}=32\zeta'\Big(-2,\frac{1}{4}\Big)-2\log A-\frac{3}{4\pi^2}\zeta(3)-\frac{1}{4}-\frac{\gamma}{3}+2\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+1)\dots(2n+4)16^n}=-8\log A+45\zeta'(-3)+\frac{187}{144}-\frac{\gamma}{24}+\frac{41}{60}\log 2$$

To conclude, we will revisit the digamma function and find another general $\zeta(2n+1)$ series.

3.3 REVISITING THE DIGAMMA FUNCTION

Consider the double integral

$$\int_0^z\int_0^x x^p(x-t)^m t\psi(t)\,dt\,dx.$$

Using the binomial theorem and (3.3) among other things, we find

$$\int_0^z \int_0^x x^p(x-t)^m t\psi(t)\,dt\,dx = \sum_{j=0}^{m}(-1)^j \binom{m}{j} \int_0^z x^{p+m-j} \int_0^x t^{j+1}\psi(t)\,dt\,dx$$

$$= \sum_{j=0}^{m}\sum_{k=0}^{j+1} \binom{m}{j} \frac{(-1)^j (j+1)!(-1)^k}{(j+1-k)!} \int_0^z x^{p+m+1-k}\psi^{(-k-1)}(x)\,dx$$

$$= \sum_{j=0}^{m} \binom{m}{j} (-1)^j \int_0^z x^{p+m+1}\psi^{(-1)}(x)\,dx$$

$$+ \sum_{k=0}^{m}\sum_{j=k}^{m} \binom{m}{j} \frac{(-1)^j (j+1)!(-1)^{k+1}}{(j-k)!} \int_0^z x^{p+m-k}\psi^{(-k-2)}(x)\,dx$$

$$= \delta_m^0 \int_0^z x^{p+1}\psi^{(-1)}(x)\,dx + (1-\delta_m^0)m! \int_0^z x^{p+1}\psi^{(-m-1)}(x)\,dx - (m+1)! \int_0^z x^p\psi^{(-m-2)}(x)\,dx$$

Simplifying and using (3.8), we find

$$\int_0^z \int_0^x x^p(x-t)^m t\psi(t)\,dt\,dx = m!z^p\left(z\psi^{(-m-2)}(z) - (m+p+2)\sum_{k=0}^{p} \frac{p!(-1)^k\psi^{(-k-m-3)}(z)}{(p-k)!z^k}\right). \tag{3.17}$$

Now we will evaluate the same integral using the power series for $\psi(x)$ and Fubini's theorem once again. Doing so, we see

$$\int_0^z \int_0^x x^p(x-t)^m t\psi(t)\,dt\,dx = -\int_0^z x^p \int_0^x (x-t)^m \left(t\gamma + 1 - \sum_{k=2}^{\infty}(-1)^k\zeta(k)t^k\right) dt\,dx$$

$$= -\int_0^z x^p \left(\frac{x^{m+2}\gamma}{(m+1)(m+2)} + \frac{x^{m+1}}{m+1} - \sum_{k=2}^{\infty} \frac{(-1)^k\zeta(k)m!x^{k+m+1}}{(k+1)\dots(k+m+1)}\right) dx$$

$$= \frac{-z^{p+m+3}\gamma}{(m+1)(m+2)(m+p+3)} + \frac{-z^{p+m+2}}{(m+1)(m+p+2)}$$

$$+ \sum_{k=2}^{\infty} \frac{(-1)^k\zeta(k)m!z^{p+m+2+k}}{(k+1)\dots(k+m+1)(k+m+p+2)}$$

Using this result and (3.17), we arrive at

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^k}{(k+1)\dots(k+m+1)(k+m+p+2)}=\frac{\gamma z}{(m+2)!(p+m+3)}$$
$$+\frac{1}{(m+1)!(p+m+2)}+\frac{\psi^{(-m-2)}(z)}{z^{m+1}}-(m+p+2)\sum_{k=0}^{p}\frac{p!(-1)^k\psi^{(-k-m-3)}(z)}{(p-k)!z^{k+m+2}}. \quad (3.18)$$

Note when $p=0$,

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^k}{(k+1)\dots(k+m+2)}=\frac{\gamma z+m+3}{(m+3)!}+\frac{\psi^{(-m-2)}(z)}{z^{m+1}}-\frac{(m+2)\psi^{(-m-3)}(z)}{z^{m+2}}. \quad (3.19)$$

As before, we will split the sum as

$$\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)z^k}{(k+1)\dots(k+m+1)(k+m+p+2)}=\sum_{n=1}^{\infty}\frac{\zeta(2n)z^{2n}}{(2n+1)\dots(2n+m+1)(2n+m+p+2)}$$
$$-\sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n+1}}{(2n+2)\dots(2n+m+2)(2n+m+p+3)}$$

and using (2.32), we have

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n}}{(2n+2)\dots(2n+m+2)(2n+m+p+3)}=(p+m+2)\Bigg((-1)^{m+1}p!*$$
$$\sum_{k=m+3}^{p+m+3}\frac{\pi(-1)^{\lfloor\frac{k}{2}\rfloor}\operatorname{Cl}_k(2\pi z)}{(p+m+3-k)!(2\pi z)^k}-\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{p!(-1)^{\frac{p+m}{2}}(-1)^m\zeta(p+m+3)}{2z(2\pi z)^{m+p+2}}$$
$$+\sum_{k=0}^{p}\frac{p!(-1)^k\psi^{(-k-m-3)}(z)}{(p-k)!z^{k+m+3}}\Bigg)-\frac{\gamma}{(m+2)!(p+m+3)}-\frac{1}{2z(m+1)!(p+m+2)}$$
$$+\frac{(-1)^{\lfloor\frac{m+1}{2}\rfloor}\operatorname{Cl}_{m+2}(2\pi z)}{2z(2\pi z)^{m+1}}-\frac{\psi^{(-m-2)}(z)}{z^{m+2}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)}{z(2\pi z)^{2k}(m+1-2k)!(m+p+2-2k)}. \quad (3.20)$$

If $p=0$, this simplifies to

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)z^{2n}}{(2n+2)\dots(2n+m+3)}=\frac{(-1)^{\lfloor\frac{m+1}{2}\rfloor}\operatorname{Cl}_{m+2}(2\pi z)}{2z(2\pi z)^{m+1}}$$
$$+\frac{(-1)^{\lfloor\frac{m}{2}\rfloor}\left(\operatorname{Cl}_{m+3}(2\pi z)-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\zeta(m+3)\right)(m+2)}{2z(2\pi z)^{m+2}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)}{z(2\pi z)^{2k}(m+2-2k)!}$$
$$-\frac{(2\gamma z+m+3)}{2z(m+3)!}-\frac{z\psi^{(-m-2)}(z)-(m+2)\psi^{(-m-3)}(z)}{z^{m+3}}. \quad (3.21)$$

For $z=1/2$ and $z=1/4$, we see

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+2)\dots(2n+m+3)4^n}=-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{(2\pi)^{m+1}}$$
$$-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(-1)^{\frac{m}{2}}(2^{m+3}-1)(m+2)\zeta(m+3)}{(2\pi)^{m+2}}+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{2k(-1)^k\zeta(2k+1)}{\pi^{2k}(m+2-2k)!}$$
$$-2^{m+2}\psi^{(-m-2)}(1/2)+(m+2)2^{m+3}\psi^{(-m-3)}(1/2)-\frac{\gamma}{(m+3)!}-\frac{1}{(m+2)!}, \quad (3.22)$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+2)\dots(2n+m+3)16^n}=(m+2)\Bigg(4^{m+3}\psi^{(-m-3)}(1/4)$$
$$+(-1)^{\lfloor\frac{m}{2}\rfloor}\left(\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{2^{m+3}\beta(m+3)}{\pi^{m+2}}-\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(2^{2m+5}+2^{m+2}-1)\zeta(m+3)}{(2\pi)^{m+2}}\right)\Bigg)$$
$$+(-1)^{\lfloor\frac{m+1}{2}\rfloor}\left(\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{2^{m+2}\beta(m+2)}{\pi^{m+1}}-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(2^{m+1}-1)\zeta(m+2)}{(2\pi)^{m+1}}\right)$$
$$-4^{m+2}\psi^{(-m-2)}(1/4)+4\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-4)^k\zeta(2k+1)}{\pi^{2k}(m+2-2k)!}-\frac{\gamma}{(m+3)!}-\frac{2}{(m+2)!}. \quad (3.23)$$

For general p,

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+2)\dots(2n+m+2)(2n+m+p+3)4^n}=-2^{m+2}\psi^{(-m-2)}(1/2)$$
$$+(-1)^m p!(m+p+2)\left(\sum_{k=\lfloor\frac{m+3}{2}\rfloor}^{\lfloor\frac{p+m+2}{2}\rfloor}\frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m+2-2k)!(2\pi)^{2k}}-\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+3)}{\pi^{m+p+2}}\right.$$
$$\left.+8(-2)^m\sum_{k=0}^{p}\frac{(-2)^k\psi^{(-k-m-3)}(1/2)}{(p-k)!}\right)-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{(2\pi)^{m+1}}$$
$$+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{2k(-1)^k\zeta(2k+1)}{\pi^{2k}(m+1-2k)!(m+p+2-2k)}-\frac{1}{(m+2)!}\left(\frac{\gamma}{p+m+3}+\frac{m+2}{p+m+2}\right),\quad (3.24)$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(2n+2)\dots(2n+m+2)(2n+m+p+3)16^n}=-4^{m+2}\psi^{(-m-2)}(1/4)$$
$$+(-1)^m p!(m+p+2)\left(\sum_{k=\lfloor\frac{m+3}{2}\rfloor}^{\lfloor\frac{p+m+2}{2}\rfloor}\frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m+2-2k)!(2\pi)^{2k}}-\sum_{k=\lfloor\frac{m+4}{2}\rfloor}^{\lfloor\frac{p+m+3}{2}\rfloor}\frac{(-1)^k\beta(2k)4^k}{(p+m+3-2k)!\pi^{2k-1}}\right.$$
$$\left.+64(-4)^m\sum_{k=0}^{p}\frac{(-4)^k\psi^{(-k-m-3)}(1/4)}{(p-k)!}-\delta_{\lfloor\frac{p+m}{2}\rfloor}^{\frac{p+m}{2}}\frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+3)2^{p+m+3}}{\pi^{m+p+2}}\right)$$
$$+\delta_{\lfloor\frac{m}{2}\rfloor}^{\frac{m}{2}}\frac{(-1)^{\lfloor\frac{m+1}{2}\rfloor}2^{m+2}\beta(m+2)}{\pi^{m+1}}-\delta_{\lfloor\frac{m+1}{2}\rfloor}^{\frac{m+1}{2}}\frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{(2\pi)^{m+1}}$$
$$+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{k(-1)^k\zeta(2k+1)4^{k+1}}{\pi^{2k}(m+1-2k)!(m+p+2-2k)}-\frac{1}{(m+2)!}\left(\frac{\gamma}{p+m+3}+\frac{2(m+2)}{p+m+2}\right).\quad (3.25)$$

Below we compute some sums for certain m and p.

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(n+1)(2n+3)4^n}=4\log A-1-\frac{\gamma}{3}+\frac{1}{3}\log 2$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{(n+1)(n+2)4^n}=60\zeta'(-3)-\frac{5}{12}-\frac{\gamma}{2}+\frac{19}{30}\log 2$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(n+1)(2n+5)4^n} = -\frac{4}{3}\log A + \frac{112\zeta'(-3)}{3} + \frac{19}{270} - \frac{\gamma}{5} + \frac{13}{45}\log 2$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(n+1)(2n+3)(n+2)4^n} = 8\log A - 60\zeta'(-3) - \frac{19}{12} - \frac{\gamma}{6} + \frac{1}{30}\log 2$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(2n+2)(2n+3)(2n+5)4^n} = \frac{4}{3}\log A - \frac{28\zeta'(-3)}{3} - \frac{289}{1080} - \frac{\gamma}{30} + \frac{1}{90}\log 2$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(2n+2)\dots(2n+5)4^n} = \frac{2}{3}\log A - \frac{17\zeta'(-3)}{3} - \frac{277}{2160} - \frac{\gamma}{120} - \frac{1}{360}\log 2$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(n+1)(2n+3)16^n} = 128\zeta'\Big(-2, \frac{1}{4}\Big) + 28\log A - \frac{3\zeta(3)}{\pi^2} - 5 - \frac{\gamma}{3}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(n+1)(n+2)16^n} = 384\zeta'\Big(-2, \frac{1}{4}\Big) + 24\log A + 540\zeta'(-3) - \frac{9\zeta(3)}{\pi^2} - \frac{41}{12} - \frac{\gamma}{2} + \frac{1}{5}\log 2$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(2n+2)\dots(2n+4)16^n} = -32\zeta'\Big(-2, \frac{1}{4}\Big) + 8\log A - 135\zeta'(-3) + \frac{3\zeta(3)}{4\pi^2} - \frac{79}{48} - \frac{\gamma}{24} - \frac{1}{20}\log 2$$

4.0 CONCLUSION

In this paper, I discussed multiple approaches for obtaining rational zeta series with $\zeta(2n)$ and $\zeta(2n+1)$. I first investigated the cotangent function, knowing its Taylor series can be written in terms of $\zeta(2n)$. Later, the generalized Clausen functions act like generalized cotangent functions and using them, one can get an entire family of rational zeta series. Lastly, using the cotangent function again, I extracted a second family of generalized rational zeta series. Using these three formulas, one can get the exact same series but with $\zeta(2n+1)$ on the numerator and I did exactly that, taking advantage of the digamma function and negapolygammas.

A few extensions are to consider

$$\int_0^{\pi z} \int_0^x x^p (x-t)^q t \, \mathrm{Cl}_m(t) \, dt \, dx$$

$$\int_0^z x^p \left(1 - \frac{x}{z}\right)^q \cot(x) \, dx$$

$$\int_0^z x^p \left(1 - \frac{x}{z}\right)^q \mathrm{Cl}_m(x) \, dx$$

I have looked into the first two slightly and a problem may arise in the first, one might get a double sum out of this expression. Nonetheless, it could be something to look into. The second integral is nice but involves hypergeometric functions and could be hard to evaluate for general z. As an application, the first section of Chapter 2 has been found useful in classifying multiple zeta functions. Don Zagier defines the multiple zeta function by

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

in ([17]) (usually, it is defined with a decreasing sequence of n_i rather than increasing). The weight of a multiple zeta value (MZV) is $s = s_1 + s_2 + \ldots + s_k$. Let ξ_s as the $\mathbb{Q}$-vector space spanned by all MZV of weight s. Denoting d_s as the number of MZV with weight s whose arguments are 2's and 3's, Michael Hoffman conjectured that these special MZV span ξ_s, that is $\dim(\xi_s) = d_s$. It has been proven so far that $\dim(\xi_s) \le d_s$. Don Zagier worked with a special type of these MZV, namely $H(a,b) := \zeta(\{2\}^a, 3, \{2\}^b)$. Denoting

$$H(n) = \zeta(\{2\}^n) = \zeta(2, 2, \ldots, 2) = \frac{\pi^{2n}}{(2n+1)!}$$

Zagier proved that

$$H(a,b) = 2\sum_{r=1}^{a+b+1} (-1)^r \left(\binom{2r}{2a+2} - \left(1 - \frac{1}{2^{2r}}\right) \binom{2r}{2b+1} \right) \zeta(2r+1) H(a+b+1-r) =: \hat{H}(a,b)$$

However, the right hand side of the equation is actually a generalized rational zeta series from this paper. In fact, I am able to show

$$\int_0^{\pi/2} x^{2a+2} \left(1 - \frac{2x}{\pi}\right)^{2b+1} \cot(x)\, dx = \frac{(2a+2)!(2b+1)!}{2^{2a+3}\pi^{2b}} \hat{H}(a,b) \tag{4.1}$$

and using rational zeta series,

$$\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2a+2)(2n+2a+3)\cdots(2n+2a+2b+3)2^{2n}} = -\frac{(2a+2)!}{4\pi^{2a+2b+2}} \hat{H}(a,b). \tag{4.2}$$

Cezar Lupu was able to show for $b = 0$,

$$H(a,0) = \frac{1}{(2a+2)!} \sum_{n=0}^{\infty} \frac{-4\pi^{2a+2}\zeta(2n)}{(2n+2a+2)(2n+2a+3)2^{2n}}$$

and using (4.2), he proved $H(a,0) = \hat{H}(a,0)$ independent of Zagier's formula. One would expect there is an extension of Lupu's work for $b \neq 0$, which he and I plan to work on. Also with this result, one may ask if other rational zeta series are related to different multiple L-functions such as the multiple eta or multiple beta function. I am currently looking into analgous formulas for the multiple beta function.

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