OPTIMAL INVESTMENT STRATEGIES FOR MINIMIZING THE PROBABILITY OF RUIN

by

Chigozie Okechukwu Nwoke

B.Sc. Mathematics, University of Nigeria, 2013

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This thesis was presented

by

Chigozie Okechukwu Nwoke

It was defended on

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and approved by

John M. Chadam, Professor (Mathematics)

Juan J. Manfredi, Professor (Mathematics)

Song Yao, Associate Professor (Mathematics)

Thesis Advisor: John M. Chadam, Professor (Mathematics)

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Chigozie Okechukwu Nwoke, M.S.

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Probability of Ruin measures the likelihood for an investor's wealth (given consumption and randomness of risky asset) to go below a pre-assigned level. We consider an investor with a wealth dependent consumption rate and two investment choices: a risk-free asset and a risky asset. We obtain optimal investment strategies using a probability approach and stochastic control approach for finite and infinite time horizons. We extend this analysis to address the problem of finding the optimal investment strategy that simultaneously maximizes terminal wealth while avoiding ruin.

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1.0 INTRODUCTION

For an investor with an initial wealth $X(0) = x$ and two investment options - a risky asset (stock, mutual fund, etc.) whose price follows a geometric Brownian motion and a money market with a risk-free rate of return r - we seek the optimal investment strategy that solves two problems: (1) minimize the probability of ruin, where ruin occurs when the wealth falls below a, $0 < a < x$, causing the investor to go into bankruptcy, and (2) the optimal investment strategy which maximizes wealth while minimizing ruin at terminal time T. We study both problems in the context of the Wealth Dependent Consumption (WDC) Model.

1.1 The Wealth Dependent Consumption (WDC) Model

In this model we consider one risky asset whose price at time t, S_t, follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \,, \tag{1.1}$$

where μ and σ are positive constants and $\{W_t : t > 0\}$ is a standard Brownian motion. The other investment option is a riskless asset (money market or bond) whose price at time t, B_t, follows

$$dB_t = rB_t dt \tag{1.2}$$

where $r > 0$ and $\mu > r$.

We are interested in finding the investment strategy, $\pi(t)$, which denotes the proportion of wealth invested in the risky asset at time t, with $\pi(t) \geq 0$ that optimizes the financial objectives listed above. We assume the investor consumes a proportion of the wealth at the rate

$$c(t) = \gamma(t)X(t); \quad 0 < r < \gamma < \mu \tag{1.3}$$

The wealth process, $X(t)$, then follows

$$dX(t) = \pi(t)X(t)\frac{dS_t}{S_t} + (1-\pi(t))X(t)\frac{dB_t}{B_t} - \gamma(t)X(t)$$

Substituting (1.1) and (1.2) we obtain

$$\begin{aligned} dX(t) &= [(r-\gamma) + (\mu - r)\pi]X(t)dt + \sigma\pi X(t)dW(t) \\ X(0) &= x \end{aligned} \tag{1.4}$$

In the case where $\gamma < r$, which means the interest rate of the money market is greater than the consumption, so ruin would never occur if all the wealth is immediately invested in the money market. On the otherhand, if $\gamma > \mu$ then there is a high probability of ruin or bankruptcy since the investor is consuming at a higher rate than the maximum possible growth rate of the wealth. Thus, we take $0 < r < \gamma < \mu$ in this study.

For a preassigned level a, with $0 < a < x = X(0)$, ruin is said to occur at time τ_a defined by

$$\tau_a = \inf \{t \geqslant 0, X(t) \leq a\} \tag{1.5}$$

In this thesis we study the mathematical problem of finding the strategy (allocation among risky asset and money market), π^*, for which

$$\inf_{\pi} \Pr(\tau_a^{\pi} \leq T) = \Pr\left(\tau_a^{\pi^*} \leq T\right) \tag{1.6}$$

In section 2 we study this problem in the finite horizon setting ($T < \infty$) using a probability approach based on the first crossing of a given level by Brownian motion. We also study the infinite horizon limit, $T \to \infty$. Section 3 begins by posing this problem of minimizing the probability of ruin as an optimal stopping time problem in a stochastic control theory setting. Financial considerations lead to a wider class of possible problems than in the probability approach of section 2. We conclude section 3 with the solutions of two of these problems in the infinite horizon setting, showing the equivalence of the corresponding result in the probability setting of section 2. Section 4 addresses the problem of maximizing the expected terminal wealth, $\mathrm{E}[X^{\pi}(T)]$, at a finite horizon, $T < \infty$, while simultaneously avoiding ruin by minimizing $\Pr(\tau_a \leq T)$. This is carried out using the probability approach. In the limit of large T, we obtain the same optimal strategy, π^*, of sections 2 and 3. that simply minimizes the probability of ruin.

2.0 MINIMIZING THE PROBABILITY OF RUIN (MPR)

2.1 Probability Approach for MPR

The Probability Approach for MPR is based on the study of the probability of the first crossing time of Brownian motion through a given boundary. The Brownian motion mentioned in section 1.1 is defined as the Wiener process W_t which satisfies:

1. $W_0 = 0$
2. W_t is almost surely continuous
3. W_t has independent increments
4. $W_t - W_s \sim N(0, t-s)$ for $0 \leq s \leq t$ where $N(0, t-s)$ is the normal distribution with mean 0, and variance $t - s$.

Brownian motion with drift μ and volatility (standard deviation) σ can then be written in the form

$$X(t) = X(0) + \mu t + \sigma W(t) \tag{2.1}$$

where $W(t)$ is a standard Brownian motion. This implies that the process $X(t)$ follows

$$dX(t) = \mu dt + \sigma dW(t) \tag{2.2}$$

Let $\{W_t\}$ be a Brownian motion. For $B < 0$ define the first time W_t reaches the point B as

$$\tau = \inf \{t > 0 : W_t \leq B\} \tag{2.3}$$

With $0 \leq s \leq t$, the transition density $p(s, t; x, y)$ for Brownian motion is given by [1, p. 108]

$$p(s, t; x, y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} \tag{2.4}$$

which implies that

$$\Pr(a \leq W(t) \leq b | W(s) = x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-(y-x)^2/2(t-s)} dy \tag{2.5}$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} 1_{[a,b]}(y) e^{-(y-x)^2/2(t-s)} dy \tag{2.6}$$

This transition pdf (2.4) satisfies the heat equation [1, p. 108]

$$p_t(s,t,x,y) = \frac{1}{2} p_{yy}(s,t,x,y) \tag{2.7}$$

2.1.1 First Crossing Time for Brownian Motion without Drift

For a Brownian motion $W(t)$ ($\mu = 0$, $\sigma = 1$), $W(0) = 0$ for $s = 0$, so that

$$p(0,t;0,y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}$$

Notice that

$$\begin{aligned} &\lim_{t \to 0^+} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} f(y) dy \quad \text{take } x = \frac{y}{\sqrt{t}} \\ &= \lim_{t \to 0^+} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} f(x\sqrt{t}) dx \\ &= f(0) \quad \text{(by the Dominated Convergence Theorem)} \end{aligned} \tag{2.8}$$

i.e. $\lim_{t \to 0^+} p(0,t,0,y) = \delta_0(y)$ in the sense of distributions.

To treat the first crossing problem, we must eliminate sample paths that cross the barrier, $y = B$, and subsequently return to the acceptable region. Thus we seek a solution $u(x,t)$ of (2.7) that has $u(B,t) = 0$ and $\lim_{t \to 0} u(x,t) = \delta_0(x)$, where $B < 0$ is the level we want $W(t)$ to stay above.

Taking

$$u(y,t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-y^2/2t} - e^{-(y-2B)^2/2t} \right) \tag{2.9}$$

we notice that $u(y,t)$ is a solution to the problem

$$su_t = \frac{1}{2} u_{yy}; \quad B < y < \infty,\ t > 0 \tag{2.10a}$$

$$u(B,t) = 0 \quad \text{and} \quad \lim_{y \to \infty} u(y,t) = 0,\ t > 0 \tag{2.10b}$$

$$u(y,0) = \delta_0(y) - \delta_{2B}(y) \tag{2.10c}$$

with $\delta_{2B}(y)$ being outside the region of interest.

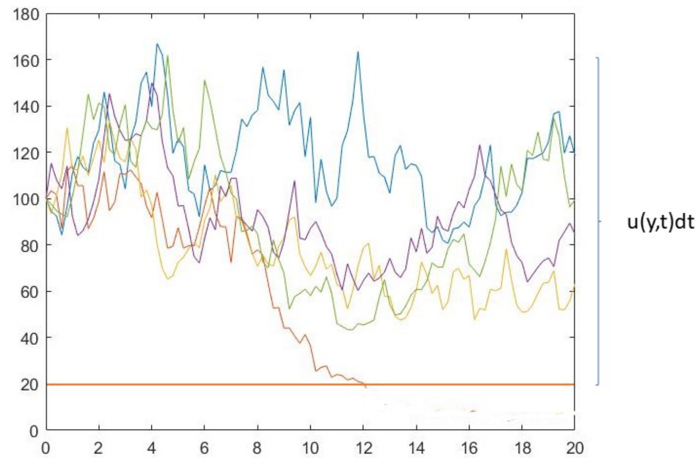

Figure 1: Brownian Motion sample paths above $a = 20$.

Counting only paths above B (i.e. ignoring paths that hit the absorbing boundary)

$$\Pr(y \le W(t) \le y + dy) = u(y,t)dy \tag{2.11}$$

which implies

$$\Pr(\tau > t) = \int_B^\infty u(y,t)dt = \frac{1}{\sqrt{2\pi t}} \int_B^\infty \left(e^{-y^2/2t} - e^{-(y-2B)^2/2t} \right) dy \tag{2.12}$$

Substitute $z = y/\sqrt{t}$ in the first integral and $z = (y-2B)/\sqrt{t}$ in the second then set $x = -z$, to obtain

$$\begin{aligned} \Pr(\tau > t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-B/\sqrt{t}} e^{-x^2/2}dx - \int_{-\infty}^{B/\sqrt{t}} e^{-x^2/2}dx \\ &= \Phi(-B/\sqrt{t}) - \Phi(B/\sqrt{t}) \end{aligned} \tag{2.13}$$

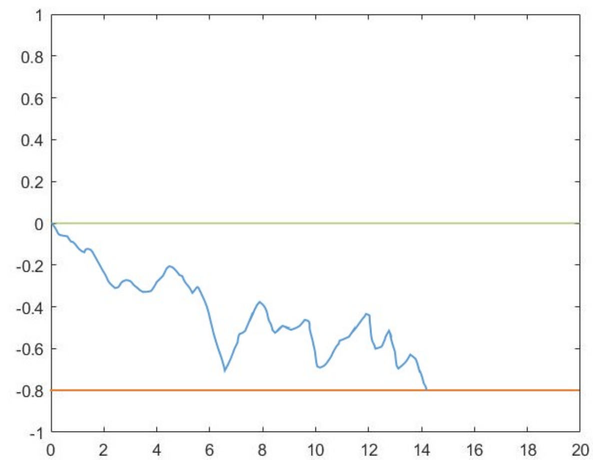

Figure 2: Brownian Motion sample path above $B = -0.8$.

where $\Phi(z)$ is the standard normal distribution function

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy \,.$$

Thus the CDF for the first crossing time τ for a Brownian motion is [1, p. 113]

$$\Pr(\tau \leq t) = 1 - \Phi(-B/\sqrt{t}) + \Phi(B/\sqrt{t}) \tag{2.14}$$

By direct calculation one has that the pdf for τ [1, p. 116]

$$f_\tau(t) = \frac{d}{dt} \Pr(\tau \leq t) = \frac{(-B)}{\sqrt{2\pi} t^{3/2}} e^{-B^2/2t} \tag{2.15}$$

and the expected value of τ

$$\begin{aligned} \mathbb{E}[\tau] &= \int_0^\infty t f_\tau(t) dt \\ &= \infty \end{aligned} \tag{2.16}$$

2.1.2 First Crossing Time for Brownian Motion with Drift

Let $W(t)$ be standard Brownian motion and consider

$$X(t) = \mu t + \sigma W(t) > b \quad (b < 0) \tag{2.17}$$

For the Brownian motion, this is equivalent to

$$W(t) > \frac{b}{\sigma} - \frac{\mu t}{\sigma} = B + At \tag{2.18}$$

where $B = \frac{b}{\sigma}(< 0)$ and $A = -\frac{\mu}{\sigma}$ (arbitrary sign)

Following the ideas from section 2.1.1, define

$$\tau = \inf\{t > 0 : X(t) \leq b\} = \inf\{t > 0 : W(t) \leq B + At\} \tag{2.19}$$

Notice that

$$u(y,t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-y^2/2t} - e^{-2AB} e^{-(y-2B)^2/2t} \right) \tag{2.20}$$

solves the PDE problem analogous to (2.7) and that

$$u(B + At, t) = 0 \,,$$

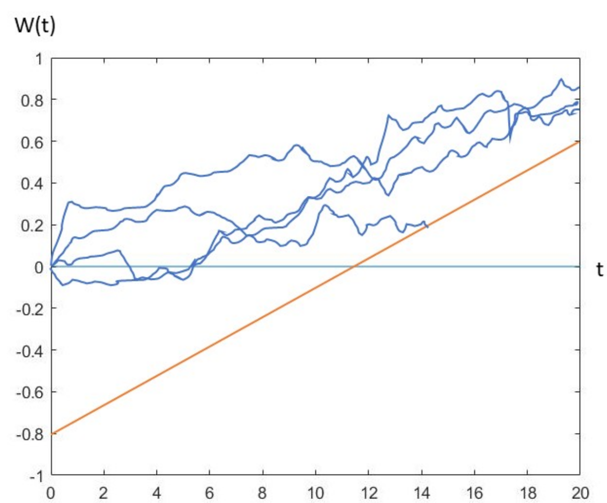

Figure 3: Brownian Motion with Drift. $A > 0, B = -0.8$.

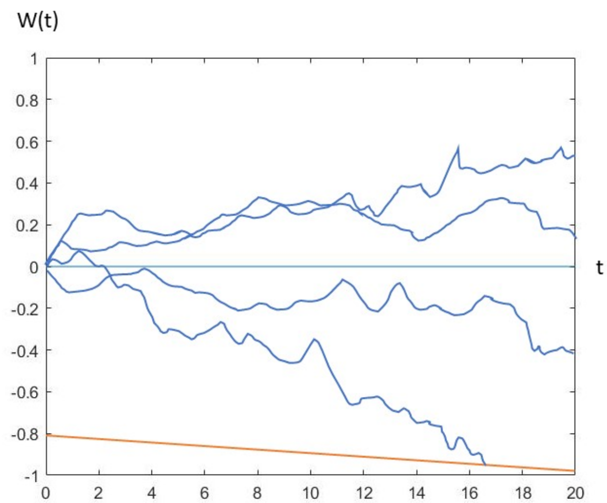

Figure 4: Brownian Motion with Drift. $A < 0, B = -0.8$.

Moreover, in the region of interest above the absorbing boundary $B + At$

$$u(y, 0) = \delta_0(y) - e^{-AB}\delta_{2B}(y) = \delta_0(y) \tag{2.21}$$

Thus,

$$\begin{aligned} \Pr(\tau > t) &= \int_{B+At}^{\infty} u(y,t)\, dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{B+At}^{\infty} \left[e^{-y^2/2t} - e^{-2AB} e^{-(y-2B)^2/2t} \right] dy \end{aligned} \tag{2.22}$$

Again, in the first integral set $z = y/\sqrt{t}$ and let $z = (y - 2B)/\sqrt{t}$ in the second, then with $x = -z$ in both cases one obtains

$$\begin{aligned} \Pr(\tau > t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\left(\frac{B}{\sqrt{t}} + A\sqrt{t}\right)} e^{-x^2/2} dx + \frac{1}{\sqrt{2\pi}} e^{-2AB} \int_{-\infty}^{\frac{B}{\sqrt{t}} - A\sqrt{t}} e^{-x^2/2} dx \\ &= \Phi\left(-\frac{B}{\sqrt{t}} - A\sqrt{t}\right) - e^{-2AB}\Phi\left(\frac{B}{\sqrt{t}} - A\sqrt{t}\right) \end{aligned} \tag{2.23}$$

The CDF for the first crossing time τ for Brownian motion with drift (2.17) is

$$\Pr(\tau \le t) = 1 - \Phi\left(\frac{-B}{\sqrt{t}} - A\sqrt{t}\right) + e^{-2AB}\Phi\left(\frac{B}{\sqrt{t}} - A\sqrt{t}\right) \tag{2.24}$$

The pdf for τ is

$$\begin{aligned} \frac{d}{dt}\Pr(\tau \le t) &= \frac{-1}{\sqrt{2\pi}} e^{\frac{-(-B/\sqrt{t} - A\sqrt{t})^2}{2}} \frac{d}{dt}\left(\frac{-B}{\sqrt{t}} - A\sqrt{t}\right) \\ &+ \frac{1}{\sqrt{2\pi}} e^{-2AB}\left(e^{\frac{-(B/\sqrt{t} - A\sqrt{t})^2}{2}}\right) \frac{d}{dt}\left(\frac{B}{\sqrt{t}} - A\sqrt{t}\right) \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-(B+At)^2}{2t}} \left(\frac{-B}{t^{3/2}}\right) \end{aligned} \tag{2.25}$$

Notice that all these formulas reduce to the corresponding ones in the previous section 2.1.1 by taking $A = 0$.

2.1.3 Application to the WDC Model

Recall that the WDC model (eqn (1.4))

$$dX(t) = [(r-\gamma) + (\mu - r)\pi]X(t)dt + \sigma\pi X(t)dW(t)$$
$$X(0) = x$$

is a geometric Brownian Motion with drift which has the solution

$$X(t) = x\exp\left\{\left[(r-\gamma) + (\mu - r)\pi - \frac{\sigma^2\pi^2}{2}\right]t + \sigma\pi W(t)\right\} \tag{2.26}$$

To avoid ruin we require that $X(t) > a$ for some pre-assigned $a < x = X(0)$. Using (2.26) this is equivalent to

$$W(t) > \frac{1}{\sigma\pi}\ln\left(\frac{a}{x}\right) - \frac{1}{\sigma\pi}\left[(r-\gamma) + (\mu - r)\pi - \frac{\sigma^2\pi^2}{2}\right]t$$

Thus, for

$$\tau = \inf\{s \geq 0 | X(s) \leq a\}$$

the probability of ruin before time t is obtained from the previous section 2.1.2 as

$$\Pr(\tau \leq t) = 1 - \Phi\left(\frac{-B}{\sqrt{t}} - A\sqrt{t}\right) + e^{-2AB}\Phi\left(\frac{B}{\sqrt{t}} - A\sqrt{t}\right)$$

where $B = \frac{1}{\sigma\pi}\ln\left(\frac{a}{x}\right)$ (< 0 since $a < x$) and $A = \frac{-1}{\sigma\pi}\left[(r-\gamma) + (\mu - r)\pi - \frac{\sigma^2\pi^2}{2}\right]$ (A has arbitrary sign)

The fixed-mix optimal investment strategy, π^*, for the finite time horizon, T, is then obtained from the first order optimality condition $\frac{\partial}{\partial\pi}\Pr(\tau \leq T : \pi) = 0$;
that is

$$\frac{\partial}{\partial\pi}\left[1 - \Phi\left(\frac{-B}{\sqrt{T}} - A\sqrt{T}\right) + e^{-2AB}\Phi\left(\frac{B}{\sqrt{T}} - A\sqrt{T}\right)\right] = 0 \tag{2.27}$$

Note that

$$\frac{\partial B}{\partial\pi} = -\frac{1}{\sigma}\ln\left(\frac{a}{x}\right)\cdot\frac{1}{\pi^2} \tag{2.28}$$

$$\frac{\partial A}{\partial\pi} = \frac{1}{\sigma\pi^2}(r-\gamma) + \frac{\sigma}{2} = \frac{(r-\gamma) + \frac{\sigma^2\pi^2}{2}}{\sigma\pi^2} \tag{2.29}$$

$$
\begin{aligned}
\frac{\partial AB}{\partial \pi} &= -\frac{1}{\sigma\pi}\left[(r-\gamma)+(\mu-r)\pi-\frac{\sigma^2\pi^2}{2}\right]\left[-\frac{1}{\sigma\pi^2}\ln\left(\frac{a}{x}\right)\right]+\frac{1}{\sigma\pi}\ln\left(\frac{a}{x}\right)\left[\frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2}\right] \\
&= \frac{-\ln\left(\frac{a}{x}\right)}{\sigma^2\pi^3}\left[-(\mu-r)\pi+2(\gamma-r)\right]
\end{aligned}
\tag{2.30}
$$

Hence, multiplying both sides by $\sqrt{2\pi}$ we obtain

$$
\begin{aligned}
\frac{\partial}{\partial \pi}\Pr(\tau \le T : \pi) = &-\left\{\frac{1}{\sqrt{T}}\left(\frac{1}{\sigma\pi^2}\ln\left(\frac{a}{x}\right)\right)-\sqrt{T}\left(\frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2}\right)\right\}e^{\frac{-\left(-\frac{B}{\sqrt{T}}-A\sqrt{T}\right)^2}{2}} \\
&+\frac{2\ln\left(\frac{a}{x}\right)}{\sigma^2\pi^3}\left[-(\mu-r)\pi+2(\gamma-r)\right]e^{-2AB}\int_{-\infty}^{\frac{\beta}{\sqrt{T}}-A\sqrt{T}} e^{-y^2/2}dy \\
&+e^{-2AB}\left\{\frac{1}{\sqrt{T}}\left(-\frac{1}{\sigma\pi^2}\ln\left(\frac{a}{x}\right)\right)-\sqrt{T}\left(\frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2}\right)\right\}e^{\frac{-\left(\frac{B}{\sqrt{T}}-A\sqrt{T}\right)^2}{2}}
\end{aligned}
$$

Recall that

$$
\begin{aligned}
e^{-2AB}\cdot e^{\frac{-\left(\frac{B}{\sqrt{T}}-A\sqrt{T}\right)^2}{2}} &= e^{-2AB}\cdot e^{\frac{-\left(\frac{B^2}{T}-2AB+A^2T\right)}{2}} \\
&= e^{\frac{-\left(B^2-2ABT+A^2T^2+4ABT\right)^2}{2T}} \\
&= e^{\frac{-\left(\frac{B}{\sqrt{T}}+A\sqrt{T}\right)^2}{2}}
\end{aligned}
$$

Thus the optimality condition (2.27) becomes

$$
\begin{aligned}
\frac{-2}{\sqrt{T}}\left(\frac{1}{\sigma\pi^2}\ln\left(\frac{a}{x}\right)\right)e^{-\frac{\left(\frac{B}{\sqrt{T}}+A\sqrt{T}\right)^2}{2}}+&\frac{2\ln\left(\frac{a}{x}\right)}{\sigma^2\pi^3}\left[-(\mu-r)\pi+2(\gamma-r)\right] \\
&\cdot e^{-2AB}\int_{-\infty}^{\frac{B}{\sqrt{T}}-A\sqrt{T}} e^{-y^2/2}dy = 0
\end{aligned}
$$

By dividing both sides by $\frac{2\ln\left(\frac{a}{x}\right)}{\sigma\pi^2}$ we obtain

$$
\pi\left[(\mu-r)e^{-2AB}\int_{-\infty}^{\frac{B}{\sqrt{T}}-A\sqrt{T}} e^{-y^2/2}dy+\frac{\sigma}{\sqrt{T}}e^{-\left(\frac{B}{\sqrt{T}}+A\sqrt{T}\right)^2}\right] = 2(\gamma-r)e^{-2AB}\int_{-\infty}^{\frac{B}{\sqrt{T}}-A\sqrt{T}} e^{-y^2/2}dy
$$

so that $\pi^* = \pi^*(T)$ is a solution of

$$\pi^* = \frac{2(\gamma - r)e^{-2AB}\int_{-\infty}^{\frac{B}{\sqrt{T}} - A\sqrt{T}} e^{-y^2/2}dy}{(\mu - r)e^{-2AB}\int_{-\infty}^{\frac{B}{\sqrt{T}} - A\sqrt{T}} e^{-y^2/2}dy + \frac{\sigma}{\sqrt{T}}e^{-\left(\frac{B}{\sqrt{T}} + A\sqrt{T}\right)^2}}$$

where A, B both depend on π^*

For the infinite time horizon problem (i.e., $T \to \infty$) with $A < 0$ we obtain

$$\pi^* = \frac{2(\gamma - r)e^{-2AB} \cdot 1}{(\mu - r)e^{-2AB} \cdot 1 + 0}$$
$$\Rightarrow \pi^* = \frac{2(\gamma - r)}{\mu - r}.$$

On the other hand when $A > 0$, the situation is more complicated. Write

$$\pi = \frac{2(\gamma - r)}{(\mu - r) + f_\pi(T)} \tag{2.31}$$

where

$$\begin{aligned} f_\pi(T) &= \frac{\frac{\sigma}{\sqrt{T}}\, e^{\frac{-\left(\frac{B}{\sqrt{T}} + A\sqrt{T}\right)^2}{2}}}{e^{-2AB} \cdot \int_{-\infty}^{\frac{B}{\sqrt{T}} - A\sqrt{T}} e^{-y^2/2}dy} \\ &= \frac{\frac{\sigma}{\sqrt{T}}\, e^{\frac{-\left(\frac{B}{\sqrt{T}} + A\sqrt{T}\right)^2}{2}}}{e^{-2AB}\, e^{\frac{-\left(\frac{B}{\sqrt{T}} - A\sqrt{T}\right)^2}{2}}\, e^{\frac{\left(\frac{B}{\sqrt{T}} - A\sqrt{T}\right)^2}{2}} \cdot \int_{-\infty}^{\frac{B}{\sqrt{T}} - A\sqrt{T}} e^{-y^2/2}dy} \\ &= \frac{\sigma}{\sqrt{T}\, e^{\frac{\left(\frac{B}{\sqrt{T}} - A\sqrt{T}\right)^2}{2}} \cdot \int_{-\infty}^{\frac{B}{\sqrt{T}} - A\sqrt{T}} e^{-y^2/2}dy} \end{aligned} \tag{2.32}$$

Using

$$e^{-2AB}\, e^{\frac{-\left(\frac{B}{\sqrt{T}} - A\sqrt{T}\right)^2}{2}} = e^{\frac{-\left(\frac{B}{\sqrt{T}} + A\sqrt{T}\right)^2}{2}}$$

and letting $z = -y/\sqrt{2}$, then

$$f_\pi(T) = \frac{\sigma}{\sqrt{2}\sqrt{T}} \left(\frac{1}{e^{\frac{\left(A\sqrt{T} - \frac{B}{\sqrt{T}}\right)^2}{2}} \cdot \int_{\frac{(A\sqrt{T} - B/\sqrt{T})}{\sqrt{2}}}^{\infty} e^{-z^2/2}dz} \right)$$

For $x \geq 0$ one has the following estimate for the error function [2, p. 298]

$$\frac{1}{x+\sqrt{x^2+2}} < e^{x^2}\int_x^\infty e^{-t^2}dt \leq \frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}} \tag{2.33}$$

So, if $x = \frac{\left(A\sqrt{T}-\frac{B}{\sqrt{T}}\right)}{\sqrt{2}} \geq 0$

$$\begin{aligned}\frac{\left(A\sqrt{T}-\frac{B}{\sqrt{T}}\right)}{\sqrt{2}} + \sqrt{\frac{\left(A\sqrt{T}-\frac{B}{\sqrt{T}}\right)^2}{2}+\frac{4}{\pi}} &\leq \frac{\sqrt{2}\sqrt{T}}{\sigma}f_\pi(T) \\ &< \frac{\left(A\sqrt{T}-\frac{B}{\sqrt{T}}\right)}{\sqrt{2}} + \sqrt{\frac{\left(A\sqrt{T}-\frac{B}{\sqrt{T}}\right)^2}{2}+\frac{2}{\sqrt{T}}}\end{aligned}$$

$$\Leftrightarrow \frac{\left(A-\frac{B}{T}\right)}{\sqrt{2}} + \sqrt{\frac{\left(A-\frac{B}{T}\right)^2}{2}+\frac{4}{\pi T}} \leq \frac{\sqrt{2}}{\sigma}f_\pi(T) < \frac{\left(A-\frac{B}{T}\right)}{\sqrt{2}} + \sqrt{\frac{\left(A-\frac{B}{T}\right)^2}{2}+\frac{2}{T}}$$

$$\Leftrightarrow \frac{\left(A-\frac{B}{T}\right)}{\sqrt{2}}\left[1+\sqrt{1+\frac{8}{\left(A-\frac{B}{T}\right)\pi T}}\right] \leq \frac{\sqrt{2}}{\sigma}f_\pi(T) < \frac{\left(A-\frac{B}{T}\right)}{\sqrt{2}}\left[1+\sqrt{1+\frac{4}{T\left(A-\frac{B}{T}\right)}}\right] \tag{2.34}$$

Thus as $T \to \infty$

$$\frac{A}{\sqrt{2}}\cdot 2 \leq \frac{\sqrt{2}}{\sigma}f_\pi(\infty) < \frac{A}{\sqrt{2}}\cdot 2$$

$$\Rightarrow \quad f_\pi(\infty) = \sigma A \tag{2.35}$$

and, from (2.31),

$$\pi = \frac{2(\gamma - r)}{(\mu - r) - \frac{1}{\pi}\left[(r-\gamma)+(\mu-r)\pi - \frac{\sigma^2\pi^2}{2}\right]}$$

Solving one obtains

$$\pi_f^*(\infty) = \left[\frac{2(\gamma-r)}{\sigma^2}\right]^{1/2} \tag{2.36}$$

We verify the consistency of signs of A with range of π in Appendix 1

Hence,

$$\pi_f^*(\infty) = \begin{cases} \frac{2(\gamma-r)}{\mu-r}, & A<0 \text{ i.e., } \frac{\mu-r}{\sigma} > \sqrt{2}(\gamma-r)^{1/2} \\ \left[\frac{2(\gamma-r)}{\sigma^2}\right]^{1/2}, & A>0 \text{ i.e., } \frac{\mu-r}{\sigma} < \sqrt{2}(\gamma-r)^{1/2}\end{cases} \tag{2.37}$$

2.2 Stochastic Control Approach for MPR

2.2.1 WDC Model with Infinite Time Horizon

Frequently optimal stopping problems in the infinite time horizon setting are more amenable to analysis since they admit closed form solutions. This is the case for the MPR problem. In this subsection we shall obtain this solution using stochastic control techniques and compare it to the solution obtained in the last subsection using probabilistic methods.

Once again we begin with the wealth process. (eqns (1.4) and (2.26))

$$dX(t) = [(r-\gamma) + (\mu - r)\pi]X(t)dt + \sigma\pi X(t)dW(t)$$
$$X(0) = x$$

with solution

$$X(t) = x\exp\left\{\left[(r-\gamma) + (\mu - r)\pi - \frac{\sigma^2\pi^2}{2}\right]t + \sigma\pi W(t)\right\}$$

Defining the stopping time τ_a as the first time that the wealth process falls below the ruin level a $(< x = X(0))$

$$\tau_a = \inf_t\{t \geq 0; X(t) \leq a\}$$

then the probability of ruin before $T < \infty$ (a finite time horizon) is $\Pr(\tau_a \leq T)$.

To merge the Stochastic Control Theory (SCT) approach with this Optimal Stopping Time (OST) problem we first recall the associated Probability/ PDE approach of section 2.1.2. By analogy,

$$\Pr(\tau_a \leq T) = \int_a^\infty u(y,T)dy$$

where in this geometric Brownian motion setting $u(y,T)$ is the solution of the analog of eqns (2.10) a)-c):

$$u_t = [(r-\gamma) + (\mu - r)\pi]yu_y + \frac{\sigma^2\pi^2}{2}y^2u_{yy} \qquad a < y < \infty,\ t > 0$$

$$u(a,t) = 0 \ \text{ and } \ \lim_{t\to\infty} u(y,t) = 0,\ t > \infty$$

$$u(y,0) = \delta_x(y)$$

First notice that $u(y,t) = u^\pi(y,t)$ above depends on the control π. Minimizing the probability of ruin can then be written in the SCT/OST setting as follows:

$$\inf_{\pi\in\mathcal{A}} \Pr(\tau_a \le T) = \inf_{\pi\in\mathcal{A}} \int_a^\infty u^\pi(y,t)\,dy = \inf_{\pi\in\mathcal{A}} \mathrm{E}_{0,x}\left[v\left(X^\pi(\tau_a)\right)\right] \tag{2.38}$$

where $\mathrm{E}_{0,x}$ is the conditional expectation, given that $X^\pi(0) = x$, v is some function which satisfies the second equality, and $\mathcal{A}$ is the set of admissible controls $\pi \ge 0$. The associated value function then is

$$V(x,t) = \inf_{\pi\in\mathcal{J}} \inf_{\pi\in\mathcal{A}} \mathrm{E}_{t,x}\left[v\left(X^\pi(\tau_a)\right)\right] \tag{2.39}$$

where $\mathcal{J}$ is the set of admissible stopping times in the time interval $[t,T]$ and $\mathrm{E}_{t,x}$ is the conditional expectation given that $X^\pi(t) = x$.

Standard SCT, using the Dynamic Programming Principle, says that the solution to this optimization problem is obtained as a solution to the Quasi-Variational Inequality (QVI) [3]

$$\min\left\{V_t + \inf_{\pi\in\mathcal{A}} \mathcal{L}^\pi V, v - V\right\} = 0, \quad y \in \mathbb{R} \text{ and } 0 < t < T \tag{2.40}$$

where $\mathcal{L}^\pi V = [(r-\gamma) + (\mu - r)\pi] y V_y + \frac{\sigma^2\pi^2}{2} y^2 V_{yy}$ is the generator of the process $X^\pi(t)$. The approach to solving this QVI is to split the region $(y,t) \in \mathbb{R} \times [0,T]$ into a stopping region

$$S = \{(y,t) \in \mathbb{R} \times [0,T], V(t,y) = v(y)\}$$

and its complement, S^c, the continuation region where

$$V_t + \inf_{\pi\in\mathcal{A}} \left\{[(r-\gamma) + (\mu - r)\pi] y V_y + \frac{\sigma^2\pi^2}{2} y^2 V_{yy}\right\} = 0 \tag{2.41}$$

For the MPR problem, the boundaries for the stopping region are clear: $x = a$ and $x \to \infty$. Moreover, using the "initial" data $X^\pi(t) = x$ which corresponds to $u^\pi(x,t) = \delta_x(y)$. The boundary condition for the value function at $x = a$ is

$$V(a,t) = \int_a^\infty \delta_a(y)\,dy = 1 \tag{2.42}$$

This corresponds to choosing $v(x)$ so that it assigns to the boundary $x = a$ the value $v(a) = 1$ so that $\mathrm{E}_{t,a}\left[v\left(X^{\pi}\left(\tau_a\right)\right)\right] = 1$. The resulting problem is to solve the Dynamic Programming Equation (DPE) - HJB equation

$$V_t + \inf_{\pi \in \mathcal{A}} \left\{ [(r-\gamma) + (\mu - r)\pi] x V_x + \frac{\sigma^2 \pi^2}{2} x^2 V_{xx} \right\} = 0$$

in the continuation region, $S^c = \{(x,t) \in (a, \infty) \times [0, T]\}$ with the conditions on the boundary of the stopping region, $x = a$ and $x \to \infty$

$$V(a,t) = 1 \quad \text{and} \quad \lim_{x \to \infty} V(x,t) \quad \text{(TBD)}$$

along with the terminal condition

$$V(x,T) = 1 \quad a < x < \infty$$

Notice that the value function (and hence the probability of ruin, $V(x,0)$) depends on the parameters r, μ, σ and γ as pointed out in the previous section.

Finally, in the infinite time horizon case, $T \to \infty$, one expects convergence to the associated stationary problem ($V_t = 0$):

$$\begin{gathered} \inf_{\pi \in \mathcal{A}} \left\{ [(r-\gamma) + (\mu - r)\pi] x V_x + \tfrac{\sigma^2 \pi^2}{2} x^2 V_{xx} \right\} = 0\,, \quad a < x < \infty \\ V(a) = 1 \quad \text{and} \quad \lim_{x \to \infty} V(x) \quad \text{(TBD)} \end{gathered} \tag{2.43}$$

Equivalently, the problem of maximizing $\Pr(\tau_a > T)$ as $T \to \infty$ is similar to (2.43) with inf replaced with sup and with boundary condition $V(a) = 0$.
When $\frac{\mu - r}{\sigma} > \sqrt{2}(\gamma - r)^{1/2}$, the solution to problem (2.43) is

$$\pi^* = \frac{2(\gamma - r)}{\mu - r} \tag{2.44}$$

and

$$V(x) = \frac{-1}{a^\beta}\left(x^\beta - a^\beta\right) = 1 - \left(\frac{x}{a}\right)^\beta \tag{2.45}$$

with $\beta = \frac{\rho}{r - \gamma} + 1 < 0$, $\rho = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$

$$\sup_\pi \left\{ [(r-\gamma) + (\mu - r)\pi] x V_x + \frac{\sigma^2 \pi^2 x^2}{2} V_{xx} \right\} = 0 \tag{2.46}$$

$$V(a)=0, \quad V(x)\to 1 \text{ as } x\to\infty$$

Take $\frac{\partial}{\partial \pi}=0$, we obtain

$$(\mu-r)xV_x+\sigma^2x^2\pi V_{xx}=0$$

which implies

$$\pi^*=\frac{-(\mu-r)xV_x}{\sigma^2x^2V_{xx}} \tag{2.47}$$

Substituting π^* into (2.46) one obtains

$$(r-\gamma)xV_x-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\frac{V_x^2}{V_{xx}}=0 \tag{2.48}$$

or equivalently (with $V_x>0$)

$$(r-\gamma)x-\rho\frac{V_x}{V_{xx}}=0 \tag{2.49}$$

where $\rho=\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2$.

A direct calculation shows that for β as above the general solution of (2.49) is

$$V(x)=k_1+k_2x^\beta \tag{2.50}$$

Matching the boundary condition at $x=a$

$$V(a)=k_1+k_2a^\beta=0$$

which implies

$$k_1=-k_2a^\beta \tag{2.51}$$

and hence

$$V(x)=k_2\left(x^\beta-a^\beta\right), \quad \beta=1+\frac{\rho}{r-\gamma} \tag{2.52}$$

To see that $\beta=1+\frac{\rho}{r-\gamma}<0$ we require

$$\frac{\rho}{\gamma-r}>1 \Leftrightarrow \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2>(\gamma-r) \Leftrightarrow \frac{\mu-r}{\sigma}>\sqrt{2}(\gamma-r)^{1/2}$$

So,

$$V(x)\to k_2\left(-a^\beta\right)=1 \text{ as } x\to\infty$$

by choosing

$$k_2 = \frac{-1}{a^\beta} \tag{2.53}$$

which leads to

$$V(x) = \frac{-1}{a^\beta}\left(x^\beta - a^\beta\right) = 1 - \left(\frac{x}{a}\right)^\beta$$

Finally, the optimal strategy is

$$\begin{aligned}
\pi^* &= \left(\frac{\mu - r}{\sigma^2}\right) \cdot \frac{x\left[-\beta\left(\frac{1}{a}\right)^\beta \cdot x^{\beta-1}\right]}{x^2 \cdot \beta(\beta-1)\left(\frac{1}{a}\right)^\beta \cdot x^{\beta-2}} \\
&= -\frac{(\mu - r)}{\sigma^2} \cdot \frac{1}{\beta - 1} = \frac{-\left(\frac{\mu-r}{\sigma^2}\right)}{\frac{\rho}{r-\gamma}} \\
&= (\gamma - r)\frac{(\mu - r)}{\sigma^2 \rho} = \frac{(\gamma - r)(\mu - r)}{\sigma^2 \cdot \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}} = \pi^* = \frac{2(\gamma - r)}{\mu - r}
\end{aligned}$$

2.2.2 WDC Model with Finite Time Horizon

Financial considerations in the case of a finite time horizon, $T < \infty$, allow for a wider class of MPR problems, leading to non-constant optimal strategies, π. For example, if at the time $t < T$, the wealth $X^\pi(t)$ exceeds $ae^{(\gamma-r)(T-t)}$ (which would be infinite in the previous infinite time horizon setting) then it could all be put in the money market ($\pi = 0$ in $[t, T]$) and, with probability 1, the resulting wealth process for $s \in [t, T]$,

$$X(s) = X^\pi(t) \cdot e^{-(\gamma-r)(t-s)} = ae^{(\gamma-r)(T-s)}$$

satisfies $X(s) \geq a$ for all $s \in [t, T]$. That is, this strategy guarantees that there is no ruin after t.

In the context of eqn (2.38) of the previous subsection, this problem results in the following mathematical formulation for $\Pr(\tau_a > T)$, in terms of the value function, $V(t, x)$, in the region $a < x < \hat{x}(t) = ae^{(\gamma-r)(T-t)}$, $0 < t < T$:

$$\begin{aligned}
&V_t + \sup_{\pi \in \mathcal{A}} \left\{[(r - \gamma) + (\mu - r)\pi]xV_x + \frac{\sigma^2\pi^2x^2}{2}V_{xx}\right\} = 0 \\
&V(a, t) = 0, \quad V(\hat{x}(t), t) = 1
\end{aligned} \tag{2.54}$$

while in the region $x > ae^{(\gamma-r)(T-t)},\ 0 < t < T$, where the wealth is invested in the money market (i.e., $\pi = 0$),

$$\begin{aligned} &V_t + (r-\gamma)xV_x = 0 \\ &V(\hat{x}(t), t) = 1 \end{aligned} \tag{2.55}$$

Thus, there is regime switching at the time-dependent boundary $x = \hat{x}(t)$. Preliminary analysis suggest that it is difficult to find an explicit solution to this problem. This will be discussed further in the last subsection, 4.2.

One might consider replacing the time dependent boundary $x = \hat{x}(t) = ae^{(\gamma-r)(T-t)}$ with $x = b = a^{(\gamma-r)T} > \hat{x}(t)$ to obtain the modified problem in $a < x < b$, $0 < t < T$:

$$\begin{aligned} &V_t + \sup_{\pi\in\mathcal{A}}\left\{[(r-\gamma)+(\mu-r)\pi]xV_x + \frac{\sigma^2\pi^2x^2}{2}V_{xx}\right\} = 0 \\ &V(a,t) = 0, \quad V(b,t) = 1 \quad 0 < t < T \\ &V(x,T) = 1 \end{aligned} \tag{2.56}$$

This problem clearly provides a sub-optimal solution for $\Pr(\tau_a > T)$ since keeping part of the wealth in the risky asset when $\hat{x}(t) < X^{\pi}(t) < b$ could lead to ruin before T due to the randomness of the Brownian motion. In spite of the sub-optimal implication, this problem also does not have a closed form solution due to the presence of the terminal condition $V(x,T) = 1$.

By taking $T = \infty$ (the infinite horizon approximation) one finally arrives at a problem that has a closed form solution for the problem of maximizing $\Pr(\tau_b < \tau_a)$:

$$\begin{aligned} &\sup_{\pi\in\mathcal{A}}\left\{[(r-\gamma)+(\mu-r)\pi]xV_x + \frac{\sigma^2\pi^2x^2}{2}V_{xx}\right\} = 0 \\ &V(a) = 0, \quad \text{and} \quad V(b) = 1 \end{aligned} \tag{2.57}$$

Clearly this leads to a further reduction in the optimality since one is choosing a strategy π that must work not only for $0 < t < T$ but also for $T < t < \infty$.

Once again, as in the previous subsection, first order optimality leads to the optimal strategy (see eqn (2.47))

$$\pi^* = -\frac{(\mu-r)V_x}{\sigma^2 xV_{xx}}$$

which in turn results in the HJB equation (see eqns (2.48), (2.49))

$$(\gamma - r)x - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{V'}{V''} = 0$$

with general solution (see eqn (2.50))

$$V(x) = k_1 + k_2 x^\beta$$

with $\beta = 1 + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 \frac{1}{(r-\gamma)}$.
Matching the boundary conditions leads to

$$0 = V(a) = k_1 + k_2 a^\beta \quad \text{and}$$
$$1 = V(b) = k_1 + k_2 b^\beta$$

Solving, one obtains $k_1 = \frac{-a^\beta}{b^\beta - a^\beta}$ and $k_2 = \frac{1}{b^\beta - a^\beta}$ which (provided that $\beta \neq 0$) leads to

$$V(x) = \frac{x^\beta - a^\beta}{b^\beta - a^\beta} = \left[\left(\frac{x}{a}\right)^\beta - 1\right]\left[\left(\frac{b}{a}\right) - 1\right]^{-1} \quad \text{and}$$
$$\pi^* = \frac{2(\gamma - r)}{\mu - r}$$

Moreover, if $\beta < 0$ then the above solution agrees with that of the previous subsection by taking $b \to \infty$. Recall however that this is a finite time horizon - it assumes that the investor puts his wealth into the money market the instant it exceeds $b = ae^{(\gamma-r)T}$. For T time units the wealth remains above a but after π, it drops below. So this is solely a finite time horizon approximation to the original optimization set up involving the regime switching boundary $\hat{x}(t) = ae^{(\gamma-r)(T-t)}$.

Browne in [4], considered a fixed consumption model with c as a fixed consumption rate per unit time and obtained the strategy

$$\pi^*(x) = \frac{2r}{\mu - r}\left(\frac{c}{r} - x\right) \tag{2.58}$$

and the value function

$$V(x : a, b) = \frac{(c - ra)^{\frac{\rho}{r}+1} - (c - rx)^{\frac{\rho}{r}+1}}{(c - ra)^{\frac{\rho}{r}+1} - (c - rb)^{\frac{\rho}{r}+1}} \tag{2.59}$$

3.0 MAXIMIZING TERMINAL WEALTH WHILE AVOIDING RUIN

In this chapter, we consider the problem of finding an optimal investment strategy which would simultaneously minimize the probability of ruin and generate the most wealth at $T < \infty$.

First notice that if the investor begins with a large initial wealth, say $X(0) \geq ae^{(\gamma - r)T}$, and invests it all in the money market, then $X(t) \geq ae^{(\gamma-r)(T-t)}$ for $0 \leq t \leq T$, so $X(T) \geq a$ which for $\pi = 0$ provides a strategy that ensures avoidance of ruin. However, it is certainly not optimal from the viewpoint of maximizing the wealth, $X(T)$, at terminal time T.

To combine the two optimizations, we seek a strategy, π, which provides

$$\begin{aligned} &\sup_{\pi} \{\mathrm{E}\left[X(t)|\tau \leq T\right] \cdot \Pr(\tau \leq T) + \mathrm{E}\left[X(T)|\tau > T\right] \cdot \Pr(\tau > T)\} \\ &= \sup_{\pi} \left\{ a\left[1 - \Pr(\tau > T)\right] + \frac{1}{\sqrt{2\widetilde{\pi} T}} \int_{B+AT}^{\infty} x e^{AT + \sigma \pi y} u(y,T) dy \cdot \Pr(\tau > T) \right\} \end{aligned} \tag{3.1}$$

where we have used $X^{\pi}(\tau_a) = a$, $X^{\pi}(t) = e^{A^{\pi}t + \sigma\pi W(t)}$ and $u(y,t)$ is the pdf for Brownian motion with drift in eqn (2.20). Continuing, the above can be written (dropping the subscript a and superscript π) using the explicit form of $u(y,t)$ in eqn (2.20)

$$\sup_{\pi} \left\{ a - a\Pr(\tau > T) + \frac{xe^{AT}}{\sqrt{2\widetilde{\pi} T}} \int_{B+AT}^{\infty} e^{\frac{-y^2 + 2\sigma\pi y T}{2T}} - e^{-2AB + \sigma\pi y} e^{\frac{-(y-2B)^2 + 2\sigma\pi y T}{2T}} \, dy \cdot \Pr(\tau > T) \right\}$$

To find the optimal strategy, π we consider the first order optimization condition

$$\frac{\partial}{\partial \pi} \left\{ a + \Pr(\tau > T) \left[-a + \frac{xe^{AT}}{\sqrt{2\widetilde{\pi} T}} \int_{B+AT}^{\infty} e^{\frac{-y^2 + 2\sigma\pi y T}{2T}} - e^{\frac{-4ABT - 2\sigma\pi y T - (y-2B)^2 + 2\sigma\pi y T}{2T}} \, dy \right] \right\} = 0 \tag{3.2}$$

Straightforward calculations give

$$
\begin{aligned}
&\frac{\partial}{\partial \pi}\left\{a+\Pr(\tau>T)\left[-a+\frac{xe^{AT}}{\sqrt{2\widetilde{\pi}T}}\int_{B+AT}^{\infty} e^{\frac{-y^2+2\sigma\pi yT}{2T}}-e^{\frac{-4ABT-2\sigma\pi yT-(y-2B)^2+2\sigma\pi yT}{2T}}\,dy\right]\right\}\\
&=\left\{\left\{\frac{1}{\sqrt{T}}\left(\frac{1}{\sigma\pi^2}\ln\left(\frac{a}{x}\right)\right)-\sqrt{T}\left(\frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2}\right)\right\}e^{-\frac{\left(-\frac{B}{\sqrt{T}}-A\sqrt{T}\right)^2}{2}}\right.\\
&-\frac{2\ln\left(\frac{a}{x}\right)}{\sigma^2\pi^3}[-(\mu-r)\pi+2(\gamma-r)]e^{-2AB}\int_{-\infty}^{\frac{B}{\sqrt{T}}-A\sqrt{T}} e^{-y^2/2}\,dy\\
&\left.-e^{-2AB}\left[\frac{1}{\sqrt{T}}\left(-\frac{1}{\sigma\pi^2}\ln\left(\frac{a}{x}\right)\right)-\sqrt{T}\left(\frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2}\right)\right]e^{-\frac{\left(\frac{B}{\sqrt{T}}-A\sqrt{T}\right)^2}{2}}\right\}\cdot\\
&\left[-a+\frac{xe^{AT}}{\sqrt{2\widetilde{\pi}T}}\int_{B+AT}^{\infty} e^{\frac{-y^2+2\sigma\pi yT}{2T}}-e^{\frac{-4ABT+4\sigma\pi yT-(y-2B)^2}{2T}}\,dy\right]\\
&+\Pr(\tau>T)\left\{\left[\frac{xT}{\sqrt{2\widetilde{\pi}T}}\left(\frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2}\right)e^{AT}\int_{B+AT}^{\infty} e^{-\frac{y^2+2\sigma\pi yT}{2T}}-e^{\frac{4\sigma\pi yT-4ABT-(y-2B)^2}{2T}}\,dy\right]\right.\\
&\left.+xe^{AT}\frac{\partial}{\partial\pi}\int_{B+AT}^{\infty}\frac{1}{\sqrt{2\widetilde{\pi}T}}e^{\frac{-y^2+2\sigma\pi yT}{2T}}-e^{\frac{4\sigma\pi yT-4ABT-(y-2B)^2}{2T}}\,dy\right\}
\end{aligned}
$$

By completing the squares

$$
\begin{aligned}
&\frac{\partial}{\partial\pi}\left\{\frac{1}{\sqrt{2\widetilde{\pi}T}}\int_{B+AT}^{\infty} e^{\frac{-y^2+2\sigma\pi yT}{2T}}-e^{\frac{4\sigma\pi yT-4ABT-y^2+4yB-4B^2}{2T}}\,dy\right\}\\
&=\frac{\partial}{\partial\pi}\left\{\frac{1}{\sqrt{2\widetilde{\pi}T}}\int_{B+AT}^{\infty} e^{\frac{\sigma^2\pi^2T}{2}}\cdot e^{-\frac{(y-\sigma\pi T)^2}{2T}}-e^{2\sigma^2\pi^2T+4\sigma\pi B-2AB}\cdot e^{-\frac{[y-(2\sigma\pi T+2B)]^2}{2T}}\,dy\right\}
\end{aligned}
$$

Then for the first part set $z=(y-\sigma\pi T)/\sqrt{T}$ and for the second part set $z=[y-(2\sigma\pi T+2B)]/\sqrt{T}$ and letting $x=-z$ we obtain the above

$$
=\frac{\partial}{\partial\pi}\left\{\frac{-1}{\sqrt{2\widetilde{\pi}}}\left[\int_{-\infty}^{-\left[\frac{B}{\sqrt{T}}+(A-\sigma\pi)\sqrt{T}\right]} e^{-x^2/2}\cdot e^{\frac{\sigma^2\pi^2T}{2}}dx-\int_{-\infty}^{\frac{B}{\sqrt{T}}-(A-2\sigma\pi)\sqrt{T}} e^{-x^2/2}\cdot e^{k+2\sigma^2\pi^2T}dx\right]\right\} \tag{3.3}
$$

where $k = 4\sigma\pi B - 2AB$. Carrying out the differentiation one obtains that

$$\begin{aligned}
&\frac{\partial}{\partial\pi}\left\{\frac{1}{\sqrt{2\tilde{\pi}T}}\int_{B+AT}^{\infty} e^{\frac{-y^2+2\sigma\pi yT}{2T}} - e^{\frac{4\sigma\pi yT-4ABT-y^2+4yB-4B^2}{2T}}\,dy\right\} = \\
&\left\{\frac{-1}{\sqrt{2\tilde{\pi}}}\left\{e^{\frac{\sigma^2\pi^2T}{2}}e^{-\frac{\left[\frac{B}{\sqrt{T}}+(A-\sigma\pi)\sqrt{T}\right]^2}{2}}\left[\frac{\ln\left(\frac{a}{x}\right)}{\sigma\pi^2\sqrt{T}}-\left(\frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2}\right)\sqrt{T}+\sigma\sqrt{T}\right]\right.\right. \\
&\left.+\,\sigma^2\pi T e^{\frac{\sigma^2\pi^2T}{2}}\int_{-\infty}^{-\left[\frac{B}{\sqrt{T}}+(A-\sigma\pi)\sqrt{T}\right]} e^{-x^2/2}dx\right\} \qquad (3.4) \\
&+\frac{1}{\sqrt{2\tilde{\pi}}}\left\{e^{k+2\sigma^2\pi^2T}\left[e^{-\frac{\left[\frac{B}{\sqrt{T}}-(A-2\sigma\pi)\sqrt{T}\right]^2}{2}}\cdot\left[\frac{\ln\left(\frac{a}{x}\right)}{\sigma\pi^2\sqrt{T}}+2\sigma\sqrt{T}-\sqrt{T}\frac{\partial A}{\partial\pi}\right]\right]\right. \\
&\left.\left.+\left(4\sigma B+4\sigma^2\pi T\right)e^{k+2\sigma^2\pi^2T}\int_{-\infty}^{\frac{B}{\sqrt{T}}-(A-2\sigma\pi)\sqrt{T}} e^{-x^2/2}dx\right\}\right\}
\end{aligned}$$

With

$$\begin{aligned}
P &= \frac{1}{\sigma\pi^2}\ln\left(\frac{a}{x}\right) = -\frac{\partial B}{\partial\pi} \\
Q &= \frac{(r-\gamma)+\frac{\sigma^2\pi^2}{2}}{\sigma\pi^2} = \frac{\partial A}{\partial\pi} \qquad (3.5) \\
R &= -\frac{2\ln\left(\frac{a}{x}\right)}{\sigma^2\pi^3}\left[-(\mu-r)\pi+2(\gamma-r)\right] = 2\cdot\frac{\partial AB}{\partial\pi}
\end{aligned}$$

one obtains the optimality condition to be

$$\begin{aligned}
&\left\{\left[\frac{2P}{\sqrt{T}}e^{-\frac{\left(\frac{B}{\sqrt{T}}+A\sqrt{T}\right)^2}{2}} + Re^{-2AB}\int_{-\infty}^{\frac{B}{\sqrt{T}}-A\sqrt{T}} e^{-y^2/2}dy\right]\right. \\
&\cdot\left[-a + \frac{xe^{AT}}{\sqrt{2\pi T}}\int_{B+AT}^{\infty} e^{-\frac{y^2}{2T}+\sigma\pi y} - e^{2\sigma\pi y-2AB-\frac{(y-2B)^2}{2T}}\,dy\right] \\
&+\Pr(\tau>T)\left\{\left[\frac{x}{\sqrt{2\widetilde{\pi}}}\sqrt{T}Qe^{AT}\int_{B+AT}^{\infty} e^{\frac{-y^2}{2T}+\sigma\pi y} - e^{2\sigma\pi y-2AB} - \frac{(y-2B)^2}{2T}\,dy\right]\right. \\
&+\frac{xe^{AT}}{\sqrt{2\widetilde{\pi}}}\left\{\left\{e^{\frac{\sigma^2\pi^2T}{2}}e^{-\frac{\left[\frac{B}{\sqrt{T}}+(A-\sigma\pi)\sqrt{T}\right]^2}{2}}\left[\frac{-P}{\sqrt{T}}+Q\sqrt{T}+\sigma\sqrt{T}\right]\right.\right. \\
&+\left.\sigma^2\pi Te^{\frac{\sigma^2\pi^2T}{2}}\int_{-\infty}^{-\left[\frac{B}{\sqrt{T}}+(A-\sigma\pi)\sqrt{T}\right]} e^{-x^2/2}dx\right\} \\
&+\left\{e^{k+2\sigma^2\pi^2T}\left[e^{-\frac{\left[\frac{B}{\sqrt{T}}-(A-2\sigma\pi)\sqrt{T}\right]^2}{2}}\left(\frac{-P}{\sqrt{T}}+Q\sqrt{T}+2\sigma\sqrt{T}\right)\right]\right. \\
&+\left.\left.\left.\left.\left(4\sigma B+4\sigma^2\pi T\right)e^{k+2\sigma^2\pi^2T}\int_{-\infty}^{\frac{B}{\sqrt{T}}-(A-2\sigma\pi)\sqrt{T}} e^{-x^2/2}dx\right\}\right\}\right\}\right\} \\
&=0
\end{aligned} \tag{3.6}$$

where $\Pr(\tau > T)$ in terms of π is explicitly given in eqn (2.23). Of course, for $T < \infty$ this equation must be solved numerically.

On the other hand, as $T \to \infty$, the estimates in [2] can be used as follows: for $A < 0$ and as $T \to \infty$, then, using $\Pr(\tau > T) = 0$ to estimate the second term,

$$Re^{-2AB}(-a) = 0$$

implying that

$$R = \frac{-2\ln\left(\frac{a}{x}\right)}{\sigma^2\pi^3}\left[-(\mu-r)\pi + 2(\gamma-r)\right] = 0$$

Thus,

$$\pi^* = \frac{2(\gamma-r)}{\mu-r} \tag{3.7}$$

Hence, the optimal investment strategy which would simultaneously minimize the probability of ruin and yield the maximum wealth for very large time horizons T, is to invest $\frac{2(\gamma-r)}{\mu-r}$ in the risky asset; i.e., precisely the same strategy as for MPR.

4.0 CONCLUSIONS

4.1 Summary of Results

This research examines two approaches for obtaining the optimal investment strategy for minimizing the probability of ruin. The first (probability approach) applies ideas from the first crossing time of geometric Brownian motion through a boundary to the wealth process at a fixed terminal time. In this setting we also derive the optimal investment strategy for simultaneously minimizing the probability of ruin and maximizing wealth at a fixed terminal time. The second (stochastic control approach) is best suited for the infinite time horizon. We derive the optimal investment strategy for minimizing the probability of ruin with an infinite terminal time and find that it agrees with the probability approach.

Using parameters in [5] and from [6] with $\mu = 0.12, \sigma = 0.12, r = 0.05, X(0) = 1$ and varying values of γ we obtain figures 5 and 6. Both figures show the value of the investors wealth when the optimal investment strategy, π^* is followed and the value when a significantly different strategy is followed for $T = 5$ years and $T = 50$ years respectively.

4.2 Future Work

We wish to adapt the stochastic control approach to minimizing the probability of ruin for a finite terminal time and to simultaneously include maximizing the wealth at the terminal time.

A second problem that arose in this analysis was that of minimizing the probability of ruin for a random terminal time, such as the death of the investor. Let e_λ denote the random terminal time with constant parameter $\lambda > 0$ (e.g., $e_\lambda = 1/\lambda$ if the random terminal time τ^* has an exponential distribution). Our objective is to find the optimal strategy π that would minimize the probability for the wealth $X(t)$ to get to the ruin level a before the random

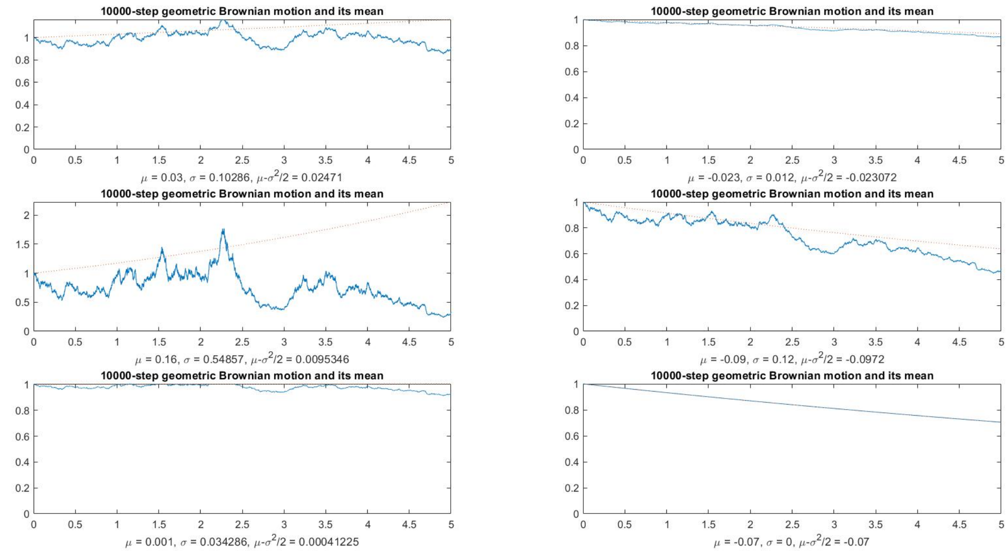

Figure 5: Wealth Process when terminal time is 5 years.

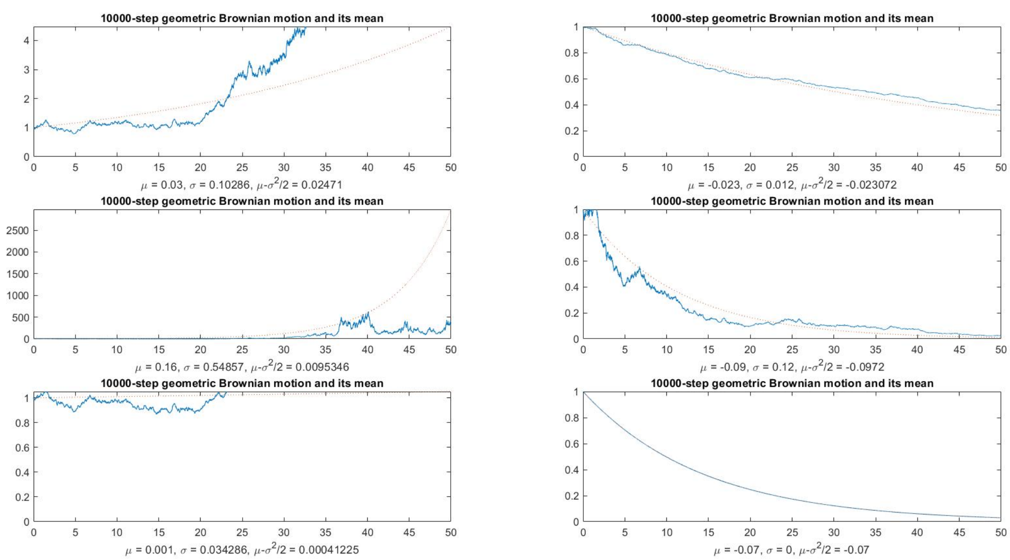

Figure 6: Wealth Process when terminal time is 50 years.

terminal time τ^*, i.e., to find

$$\inf_{\pi\in\mathcal{A}} \mathrm{P}_{0,x}\left(\tau^* < e_\lambda\right) = \inf_{\pi\in\mathcal{A}} \mathrm{E}_{0,x}\left(e^{-\lambda\tau^*}\right) \tag{4.1}$$

The value function

$$V(t,x) = \inf_{\pi\in\mathcal{A}} \mathrm{E}_{t,x}\left(e^{-\lambda\tau^*}\right) \tag{4.2}$$

Applying the DPP, one obtains the problem

$$\begin{aligned} &V_t + \inf_{\pi}\left\{[(r-\gamma)+(\mu-r)\pi]xV_x + \frac{\sigma^2\pi^2x^2}{2}V_{xx}\right\} = 0 \\ &V(a,t) = e^{-\lambda t} \quad \text{and} \quad V(\infty,t) = 0 \end{aligned} \tag{4.3}$$

With $\pi^* = -\left(\frac{\mu-r}{\sigma^2}\right)\frac{V_x}{xV_{xx}}$ one obtains, with $\rho = \left(\frac{\mu-r}{\sigma}\right)^2$

$$V_t - (\gamma-r)xV_x - \frac{\rho}{2}\frac{\left(V_x\right)^2}{V_{xx}} = 0 \tag{4.4}$$

Letting $V(t,x) = e^{-\lambda t}G(x)$, then

$$\lambda G + (\gamma-r)xG' + \frac{\rho}{2}\frac{\left(G'\right)^2}{G''} = 0 \tag{4.5}$$

We hope to use the methods in [5] to find the solutions of (4.5) to solve both of the above problems.

APPENDIX

Appendix 1

Here we provide a check for consistency of the signs of A with the range of π $(0 < \pi < 1)$

$$A = \frac{-1}{\sigma\pi}\left[-(\gamma - r) + (\mu - r)\pi - \frac{\sigma^2\pi^2}{2}\right] > 0$$

$$\Leftrightarrow \frac{\sigma^2\pi^2}{2} - (\mu - r)\pi + (\gamma - r) > 0$$

Taking

$$\alpha(\pi) = \pi^2 - \frac{2(\mu - r)}{\sigma^2}\pi + \frac{2(\gamma - r)}{\sigma^2} > 0$$

$$\alpha(0) = \frac{2(\gamma - r)}{\sigma^2} \geq 0$$

$$\alpha'(\pi) = 2\pi - \frac{2(\mu - r)}{\sigma^2} = 0 \Leftrightarrow \pi = \frac{\mu - r}{\sigma^2}$$

$\alpha''(\pi) = 2 > 0 \Rightarrow \frac{\mu - r}{\sigma^2}$ is a min

$\alpha(\pi) = 0$ when

$$\pi = \frac{\frac{2(\mu-r)}{\sigma^2} \pm \sqrt{\frac{4(\mu-r)^2}{\sigma^4} - \frac{8(\gamma-r)}{\sigma^2}}}{2}$$

$$\begin{aligned}\Rightarrow \pi &= \frac{\mu - r}{\sigma^2} \pm \sqrt{\frac{(\mu - r)^2}{\sigma^4} - \frac{2(\gamma - r)\sigma^2}{\sigma^4}} \\ &= \frac{\mu - r}{\sigma^2}\left[1 \pm \sqrt{1 - \frac{2(\gamma - r)\sigma^2}{(\mu - r)^2}}\right]\end{aligned}$$

So, if

$$\frac{2(\gamma - r)\sigma^2}{(\mu - r)^2} > 1 \Rightarrow \frac{\mu - r}{\sigma} < \sqrt{2}(\gamma - r)^{1/2}$$

then we have imaginary roots which implies that $\alpha(\pi) > 0 \quad \forall\, 0 \leq \pi < 1$ and $A > 0 \quad \forall\, 0 < \pi < 1$

If $\frac{2(\gamma-r)\sigma^2}{(\mu-r)^2} < 1$, then $\alpha(\pi)$ has a root at

$$0 < \frac{\mu - r}{\sigma^2}\left[1 - \sqrt{1 - \frac{2(\gamma - r)\sigma^2}{(\mu - r)^2}}\right] < 1$$

which is a contradiction since $\alpha(\pi) > 0$.

So,

$$\frac{2(\gamma - r)\sigma^2}{(\mu - r)^2} > 1 \quad \text{for } A > 0 \tag{.1}$$

Next,

$$A = \frac{-1}{\sigma\pi}\left[-(\gamma - r) + (\mu - r)\pi - \frac{\sigma^2\pi^2}{2}\right] < 0$$

$$\Leftrightarrow \frac{\sigma^2\pi^2}{2} - (\mu - r)\pi + (\gamma - r) < 0$$

$$\Leftrightarrow \pi^2 - \frac{2(\mu - r)\pi}{\sigma^2} + \frac{2(\gamma - r)}{\sigma^2} < 0$$

Taking

$$\beta(\pi) = \pi^2 - \frac{2(\mu - r)}{\sigma^2}\pi + \frac{2(\gamma - r)}{\sigma^2} \tag{.2}$$

$\beta(\pi)$ attains a maximum at π if π is a root of $\beta(\pi)$

So,

$$\beta(\pi) = 0 \tag{.3}$$

$$\Rightarrow \pi = \frac{\mu - r}{\sigma^2}\left[1 \pm \sqrt{1 - \frac{2(\gamma - r)\sigma^2}{(\mu - r)^2}}\right]$$

$$\Rightarrow \frac{2(\gamma - r)\sigma^2}{(\mu - r)^2} < 1 \quad \text{(otherwise we get a complex root)}$$

$$\Rightarrow \frac{(\mu - r)^2}{\sigma^2} > 2(\gamma - r)$$

$$\Rightarrow \frac{\mu - r}{\sigma} > \sqrt{2}(\gamma - r)^{1/2} \tag{.4}$$

So,

$$\frac{2(\gamma - r)\sigma^2}{(\mu - r)^2} < 1 \quad \text{for } A < 0 \tag{.5}$$

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