Polynomial equations and symmetries in numbers

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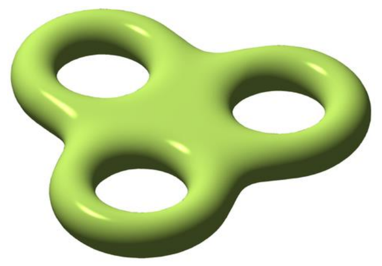
Motivation

- Since Diophantus in the 3rd century, humans have been seeking understanding how to solve polynomial equations like
  $$y^2 = x^7 + 45x^3 - x + 1$$
  with rational numbers.
- Today these efforts have developed into the mathematical field of "Diophantine geometry," reflecting the fact that we use techniques and intuition from geometry in order to attack these problems.
- For example, a landmark theorem of Faltings (1983) links the number of holes in the surface made of the complex number solutions (x,y) of such equations to the number of rational solutions: if there are at least two holes, then there are finitely many solutions.
- The particular equation above gives rise to a surface of complex number solutions with three holes (see figure), so it has finitely many solutions!



Project Description

- While Faltings's theorem is powerful, it gives no way to find this finite set of rational solutions.
- In order to find these solutions, around 2010, Minhyong Kim introduced a new suite of techniques into Diophantine geometry, known as the "non-abelian Chabauty method."
- Netan Dogra and I are producing a conceptual framework that incorporates and extends the existing ways of carrying out the computations used to implement the non-abelian Chabauty method.

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Finding **integer solutions to polynomial equations** is known to be difficult: there is provably no completely general algorithm to do it.

We are **contributing to a toolkit** that will enable us to find solutions in **new cases.**

Context

- Dogra has been involved in recent field-leading projects that carry out the computations of the non-abelian Chabauty method in the "quadratic" or "order 2" case.
- Extending the scope of the method requires that we work orders 3 and higher.
- At order 2, it is straightforward to identify the right computations, but it becomes difficult at order 3 to even identify what to compute, much less carry it out.
- My expertise in Galois representations, which has to do with symmetries among numbers, complements Dogra's as we develop a framework that incorporates all orders.

Project Deliverables

- In the next year, Dogra and I plan to complete a manuscript setting up this framework and demonstrating an application to finding solutions to a few equations of interest.
- I will organize a workshop at Pitt in May 2021 to bring together experts in the non-abelian Chabauty method and other new developments.
- The workshop will also be pitched at PhD students and junior researchers – both at Pitt and externally – who can benefit from the lectures.

Potential Impact

- My joint work with Dogra provides a useful framework to extend the applicability of the non-abelian Chabauty method.
- The workshop will hopefully spur on new developments linking Galois representations to Diophantine questions, as several ways of linking these will be featured.

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