KHOVANSKII-GRÖBNER BASES

by

Daniel James Ehrmann

B.S. in Mathematics, University of Tampa, 2014

Submitted to the Graduate Faculty of

the Dietrich School of Arts and Sciences in partial

fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2020

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Daniel James Ehrmann

It was defended on

November 20, 2019

and approved by

Kiumars Kaveh, PhD, Department of Mathematics, University of Pittsburgh

Thomas Hales, PhD, Department of Mathematics, University of Pittsburgh

Gregory Constaintine, PhD, Department of Mathematics, University of Pittsburgh

Christopher Manon, PhD, Department of Mathematics, University of Kentucky

Dissertation Director: Kiumars Kaveh, PhD, Department of Mathematics, University of Pittsburgh

KHOVANSKII-GRÖBNER BASES

Daniel James Ehrmann, PhD

University of Pittsburgh, 2020

In this thesis a natural generalization and further extension of Gröbner theory using Kaveh and Manon's Khovanskii basis theory [KM19] is constructed. Suppose A is a finitely generated domain equipped with a valuation $\mathfrak{v}$ with a finite Khovanskii basis. We develop algorithmic processes for computations regarding ideals in the algebra A. We introduce the notion of a Khovanskii-Gröbner basis for an ideal J in A and give an analogue of the Buchberger algorithm for it (accompanied by a Macaulay2 code). We then use Khovanskii-Gröbner bases to suggest an algorithm to solve a system of equations from A. Finally we suggest a notion of relative tropical variety for an ideal in A and sketch ideas to extend the tropical compactification theorem to this setting.

TABLE OF CONTENTS

PREFACE

Acknowledgments

There are so many people that have helped me get to where I am today. I first need to thank my advisor, Dr. Kiumars Kaveh, for his constant guidance. Dr. Kaveh, thank you for walking me through the problems I struggled to understand and challenging me to ensure I mastered the details of every proof. I greatly appreciate the endless hours, days, and months you invested in helping me improve my writing—a critical component in finishing this dissertation.

I am also very thankful to my committee, Dr. Thomas Hales, Dr. Christopher Manon, and Dr. Gregory Constantine, for their advice and patience. Your questions and proposed research directions led me to more impactful results. Thank you for being amazing instructors and teaching me the important fundamentals of the field; you gave me the passion to pursue the field of algebraic geometry.

Without the support of both my family and friends, my journey to and through graduate school would have been nearly impossible. First to note are my amazing parents, Mary and Tom; they have been nothing but supportive of everything I do. Thank you for noticing that I was skilled in mathematics and pushing me as far as you did. I also need to thank my older brother, Dr. Tom Ehrmann, for being an excellent role model and inspiring my life goals. Our competition in education helped me push through this program during difficult times. I am also grateful for my friends, Marcello Codianni, Youngmin Park, Abby Pekoske, Nathan Fulton, Ben Letson, who struggled with me and helped me relax when the pressure from school started to build. And I cannot forget Ed Robinson; thank you for being my biggest champion during my time in school.

Most importantly, I have to thank my wife, Sam Ehrmann, who is the reason that I chose to go to graduate school. You encouraged me to pursue this career and have been with me every step of the way, through the bad and the good. You have never left my side and always reminded me that earning my Ph. D. is what I wanted to do. Thank you for your non-stop support and listening to me explain my research every night (even though you didn't understand what I was saying). I would not be here without you and I want to thank you for never letting me give up and helping me achieve my dream.

1.0 INTRODUCTION

This thesis continues the work of Kaveh and Manon [KM19] which aims to use higher rank valuations to extend powerful and extremely useful methods of Gröbner basis theory for ideals in a polynomial ring to general algebras. When a general algebra has a finite Khovanskii basis with respect to a valuation, we introduce the notion of a Khovanskii-Gröbner basis for an ideal in that algebra. We will show how one can construct Khovanskii-Gröbner bases and use them to extend Gröbner basis theory to this more general context. The rest of this introduction will give a survey of the main results of this thesis. Proofs and unexplained notation will appear in later chapters.

Let $\mathbf{k}$ be a field. In the usual Gröbner theory for ideals in a polynomial ring $\mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1, \ldots, x_n]$, one begins by fixing a term order $\succ$ (Example 2.1.1) on the additive semigroup $\mathbb{Z}^n_{\geq 0}$, that is, a well-ordering on $\mathbb{Z}^n_{\geq 0}$ that respects addition. Let $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in \mathbf{k}[x_1, \ldots, x_n]$, where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ and $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$. One defines the initial monomial of f as:

$$in_{\succ}(f) = c_{\mathbf{b}} \mathbf{x}^{\mathbf{b}},$$

where $\mathbf{b} = \min\{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$ and the minimum is with respect to the term order $\succ$. A Gröbner basis for an ideal $I \subset \mathbf{k}[\mathbf{x}]$ is a finite collection of elements in I

whose initial monomials generate the initial ideal

$$in_{\succ}(I) = \langle in_{\succ}(f) \mid f \in I \rangle.$$

The Hilbert basis theorem (Theorem 2.1.5) asserts that we always have a finite Gröbner basis and the well-known Buchberger algorithm produces a Gröbner basis for I starting with a set of generators for it. Given a Gröbner basis for an ideal I, one can provide efficient algorithms to solve many computational problems concerning I, such as the ideal membership problem (i.e. deciding whether a given polynomial lies in I or not).

One attempt to expand Gröbner basis theory beyond polynomial rings was done by Robbiano-Sweedler [RS90] and Ollivier [Oll91] with their work on SAGBI bases. They were able to expand the algorithms and properties of Gröbner bases to certain subalgebras of polynomial rings, such as determining ideal membership, computing syzygies, and a Buchberger criterion and algorithm [Mil96]. Throughout this thesis, unless noted otherwise, we start with a finitely presented commutative algebra over $\mathbf{k}$ and domain A and equip it with a valuation map $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^s$ (Definition 2.3.1) where $\mathbb{Z}^s$ has a total order $\succ$. We say that $\mathfrak{v}$ has one-dimensional leaves if $\forall a$ $\dim(F_{\mathfrak{v}\succeq a}/F_{\mathfrak{v}\succ a})$ is at most 1-dimensional, where $F_{\mathfrak{v}\succeq a} = \{f \in A \mid \mathfrak{v}(f) \succeq a\} \cup \{0\}$. For most of the results in this manuscript we will assume that the valuation has one-dimensional leaves. If $\mathfrak{v}$ has one-dimensional leaves, the rank of the sublattice of Z^s generated by $S(A, \mathfrak{v}) = S$ is equal to $d = \dim(A)$, where

$$S = \{\mathfrak{v}(f) \mid 0 \neq f \in A\}.$$

The valuation $\mathfrak{v}$ will play the role of the exponent of the initial term of a polynomial. One defines the associated graded algebra $\mathrm{gr}_{\mathfrak{v}}(A)$ with respect to the valuation map by:

$$\mathrm{gr}_{\mathfrak{v}}(A) = \bigoplus_{a \in \mathbb{Z}^n} F_{\mathfrak{v} \succeq a} / F_{\mathfrak{v} \succ a}.$$

From the definition of a valuation map it follows that $\mathrm{gr}_{\mathfrak{v}}(A)$ is a domain. We can then consider the image $\bar{f}$ of an element $f \in A$ in $\mathrm{gr}_{\mathfrak{v}}(A)$ to be the analog of the initial monomial.

Definition 1.0.1. (Khovanskii Basis) Given a valuation map $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^s$ with the associated graded ring $\mathrm{gr}_{\mathfrak{v}}(A)$, we say that $\mathcal{B} \subset A$ is a *Khovanskii basis* for A if the image $\bar{\mathcal{B}}$ of $\mathcal{B}$ in $\mathrm{gr}_{\mathfrak{v}}(A)$ is a set of algebra generators for $\mathrm{gr}_{\mathfrak{v}}(A)$.

Throughout this thesis we assume $(A, \mathfrak{v})$ has a finite Khovanskii basis or equivalently $\mathrm{gr}_{\mathfrak{v}}(A)$ is a finitely generated algebra. The basics of Khovanskii basis theory have been developed by Kaveh and Manon [KM19] and give a natural generalization of SAGBI theory. In contrast to SAGBI theory Khovanskii bases can be utilized when working with affine varieties whose coordinate rings are not subalgebras of polynomial rings (e.g. if the variety is not unirational). The name was suggested by B. Sturmfels in honor of A. G. Khovanskii's contributions to combinatorial algebraic geometry and convex geometry and in particular, his central role in developing the theory of Newton-Okounkuv bodies where valuations on algebras play an important role.

Let $(A, \mathfrak{v})$ have a finite Khovanskii basis. The following is an extension of the notion of Gröbner basis for ideals in A.

Definition 1.0.2. (Khovanskii-Gröbner Basis) Let $J \subset A$ be an ideal. We say that $G = \{g_1, \ldots, g_m\} \subset J$ is a *Khovanskii-Gröbner basis* for J if

$$\langle \bar{G} \rangle = \langle \bar{g}_1, \ldots, \bar{g}_m \rangle = \langle \bar{J} \rangle = \langle \bar{j} \mid j \in J \rangle$$

where $\bar{j}$ is the image of j in $\mathrm{gr}_{\mathfrak{v}}(A)$.

The next definition is our extension of the notion of standard monomials for an ideal of a polynomial algebra.

Definition 1.0.3. (Adapted Basis) A set $\mathbb{B} \subset A$ is an *adapted basis* for $(A, \mathfrak{v})$ if $\bar{\mathbb{B}} = \{\bar{f} \mid f \in \mathbb{B}\}$ is a vector space basis for $\mathrm{gr}_{\mathfrak{v}}(A)$.

Remark 1.0.4. Since an adapted basis is a vector space basis, we can decompose the algebra as $A = \bigoplus_{a \in \mathbb{Z}^s} V_a$ where $V_a = \mathrm{span}\{b \in \mathbb{B} \mid \mathfrak{v}(b) = a\}$. We call $\bigoplus_{a \in \mathbb{Z}^s} V_a$ the *direct sum decomposition adapted to the valuation* $\mathfrak{v}$. Note that $V_a \cong F_{\mathfrak{v} \succeq a} / F_{\mathfrak{v} \succ a}$.

We introduce notions of minimal Khovanskii-Gröbner basis and reduced Khovanskii-Gröbner basis. However, unlike the usual Gröbner theory for polynomial rings, while every ideal $J \subset A$ has a reduced Khovanskii-Gröbner basis, it may not be unique. For the purposes of computation in this context we have the following analog of the division algorithm (Algorithm 3.0.2). This will give us the necessary and basic tools needed to compute a Khovanskii-Gröbner basis and to solve problems related to ideals in a general algebra with a finite Khovanskii basis.

Proposition 1.0.5. *(Division Algorithm 3.0.2) Let $f \in A = \bigoplus_a V_a$ and consider a set of elements $G = \{g_1, \ldots, g_m\} \subset A$. Then f can be written as a linear combination $f = c_1 g_1 + \cdots + c_m g_m + r$ where $c_i \in A$ and $r = \sum_a r_a$ with $r_a \in V_a$ such that $\bar{r}_a \notin \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$. In particular $r = 0$ implies that $f \in \langle G \rangle$.*

Similar to the division algorithm for polynomials (Algorithm 2.1.3), the generalized division algorithm does not always give a unique remainder. In the polynomial case we only get a unique remainder when dividing by a Gröbner basis. In this context we can replace a Gröbner basis with a Khovanskii-Gröbner basis, provided that the valuation has one-dimensional leaves. In this case $\mathrm{gr}_{\mathfrak{v}}(A)$ is a semigroup algebra $\mathbf{k}[S]$ where $S = \{\mathfrak{v}(f) \mid 0 \neq f \in A\}$ is the value semigroup of $(A, \mathfrak{v})$ (Definition 2.6.1). Recall that the *rank* of a valuation $\mathfrak{v}$ is the rank of the sublattice generated by S. A valuation is *full rank* if the image of A generates a full rank sublattice of $\mathbb{Z}^s$. One shows that if a valuation is full rank and the base field $\mathbf{k}$ is algebraically closed, then it has one-dimensional leaves.

Proposition 1.0.6. *(Proposition 3.0.6) If G is a Khovanskii-Gröbner basis and the valuation has one-dimensional leaves, then the remainder term r in the division algorithm (Algorithm 3.0.2) is unique. In particular, $f \in \langle G \rangle$ if and only if $r = 0$.*

We prove an analogue of the Buchberger criterion (Theorem 3.0.20) and the Buchberger algorithm (Algorithm 3.0.23) for Khovanskii-Gröbner bases. This theorem and algorithm use the notion of syzygies of a graded ring 3.0.19 in place of S-polynomials in the usual Gröbner theory, similar to how it is used in SAGBI theory. Moreover, we provide a Macaulay2 code for computation of Khovanskii-Gröbner bases when the algebra is given by means of generators and relations.

Gröbner bases can be used to give an efficient algorithm to find solutions of a system of polynomial equations. We propose an extension of this algorithm to the setting of an algebra A with a finite Khovanskii basis (Proposal 4.0.6). Given a smooth point $p \in \mathrm{Spec}(A) = X$ consider its local ring $\mathcal{O}_p$ and let $\mathfrak{m}_p$ denote the maximal ideal of $\mathcal{O}_p$. Since p is a smooth point, $\mathcal{O}_p$ is a regular local ring and we can write $\mathfrak{m}_p = \langle u_1, \ldots, u_d \rangle$ where $d = \dim X$. One calls $\{u_1, \ldots, u_d\}$ a system of parameters at p. With this system of parameters we can define a lowest term order valuation on A (Example 2.3.2). Suppose we have a finite Khovanskii basis with this valuation. We propose a procedure for solving a system of equations in A (Proposal 4.0.6). The next proposition is a key step in this regard.

Proposition 1.0.7. *(Elimination Ideal 4.0.2) With notation as above, consider the ideal $J \subset A$ with a finite Khovanskii-Gröbner basis G with respect to the valuation $\mathfrak{v}$. Then $G_i = G \cap \mathbf{k}[[u_{i+1}, \ldots, u_d]]$ is a Khovanskii-Gröbner basis for the i^{th} elimination ideal $J_i = J \cap \mathbf{k}[[u_{i+1}, \ldots, u_d]]$.*

Note that a Khovanskii basis gives an embedding of $X = \mathrm{Spec}(A)$ in an affine space $\mathbb{A}^n$. Suppose $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a finite Khovanskii basis for A with respect to the valuation above.

Proposal 1.0.8. (Proposal 4.0.6)

Input: A finite set $\Gamma \subset A$.

Output: Points on $\mathrm{Spec}(A) \subset \mathbb{A}^n$ that satisfy $f = 0$ for all $f \in \Gamma$.

Suppose $J \subset A \cong \mathbf{k}[x_1, \ldots, x_n]/I$ is an ideal, then we can use the usual Gröbner basis theory for the ideal $\tilde{J} = \pi^{-1}(J) + I \subset \mathbf{k}[x_1, \ldots, x_n]$, where $\pi : \mathbf{k}[x_1, \ldots, x_n] \to A$ is the natural homomorphism, to find solutions of the system Γ. We expect that the algorithm 4.0.6 is more efficient than using the classical Gröbner basis algorithm when $\dim(V(I)) = d \ll n$. This is because running time for finding a Gröbner basis is exponential in the number of variables in the polynomial ring.

Tropical geometry is an interface between algebraic geometry, combinatorics and convex geometry. The tropical variety of a subvariety of an algebraic torus is a polyhedral fan in $\mathbb{R}^n$ that encodes the asymptotic directions in the subvariety. Recall that the tropical variety $\mathrm{trop}(I)$ is the set of all $\omega \in \mathbb{Q}^n$ such that the corresponding initial ideal $in_\omega(I)$ (Definition 2.1.14) contains no monomials. The tropical variety $\mathrm{trop}(I)$ has a fan structure coming from the Gröbner fan of the homogenization of I. In particular, each cone $C \subset \mathrm{trop}(I)$ has an associated initial ideal $in_C(I)$ (Section 2.7). Following [KM19] we say that C is a *prime cone* if the corresponding initial ideal $in_C(I)$ is a prime ideal.

Many advanced tools in algebraic geometry, such as intersection theory, are tailored to varieties that are complete/compact. However, many varieties that one deals with are not compact. In such a case one first needs a compactification of the variety. A compactification of a variety Z is a complete variety $\tilde{Z}$ such that Z is a Zariski dense subset. One can use tropical varieties to find a "nice" compactification for a subvariety in an algebraic torus [Tev07].

In this thesis we introduce the notion of a relative tropical variety $\mathrm{trop}_C(J)$ of an ideal $J \subset A$ with respect to a prime cone C. We show that $\mathrm{trop}_C(J)$

encodes directions at infinity of the subvarieties Y defined by J with respect to prime divisors defined by C. For $\omega \in C$ one constructs a partial compactification $\tilde{X}_\omega$ of $X = \mathrm{Spec}(A)$. The variety $\tilde{X}_\omega$ consists of X with a divisor D_ω at infinity attached to it. $\tilde{X}_\omega$ is defined as Proj of a Rees algebra (see Section 5). Let $\tilde{Y}_\omega$ to be the closure of $Y = V(J)$ in $\tilde{X}_\omega$.

Theorem 1.0.9. *(Theorem 5.0.3) Let $\omega \in C$ and let $J \subset A$ be an ideal with $V(J) = Y \subset X$, and $\tilde{Y}_\omega$ its closure in $\tilde{X}_\omega$. Then $\tilde{Y}_\omega \setminus Y \neq \emptyset$ if and only if $\omega \in trop_C(J)$.*

This theorem tells us that given a subvariety Y of X, its relative tropical variety $\mathrm{trop}_C(J)$ gives us information about the closure of Y in the partial compactification of X defined using C. A complete generalization of tropical compactification theorem should say that if Σ is a fan whose support is $\mathrm{trop}_C(J)$, then $\mathrm{trop}_C(J)$, in a sense that can be made precise, gives the most efficient way to compactify $Y \subset X$ among the compactifications coming from C.

Remark 1.0.10. We expect that a complete generalization of the compactification theorem should follow from the above result as well as a generalization of the construction of a toric variety from a fan to associate a partial compactification of X to a fan in a prime cone C.

The development of the theory of Khovanskii bases and construction of valuations with finite Khovanskii bases from prime cones is the work of Kaveh and Manon [KM19]. The contribution of this thesis is developing the theory of Khovanskii-Gröbner bases. More precisely we give: (1) an analog of the

Buchberger criterion and algorithm with corresponding Macaulay2 code, the associated division algorithm and solution to the ideal membership problem with corresponding Macaulay2 code, (2) a proposal for finding solutions of a system of equations in an algebra A, and (3) the notion of relative tropical variety of an ideal $J \subset A$ with respect to a prime cone C.

Notation. Throughout this paper we use the following notation.

- $\mathbf{k}$, a field which we take to be our base field throughout the paper.
- $\mathbf{k}[\mathbf{x}]$, the polynomial ring over $\mathbf{k}$ associated to a finite set of indeterminates $\mathbf{x} = (x_1, \ldots, x_n)$.
- A, a finitely generated $\mathbf{k}$-algebra and domain with Krull dimension $d = \dim(A)$.
- $X = \operatorname{Spec}(A)$, the affine variety associated to A.
- $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^s$, a discrete valuation on A (Definition 2.3.1). We denote the corresponding associated graded algebra by $\operatorname{gr}_{\mathfrak{v}}(A)$.
- $\mathfrak{v}_\omega : A \setminus \{0\} \to \mathbb{Z}$, a valuation defined by a weight vector ω (Section 2.7).
- $S(A, \mathfrak{v})$, the value semigroup of $(A, \mathfrak{v})$ (Definition 2.6.1).
- $\mathcal{B} = \{b_1, \ldots, b_n\}$, a set of $\mathbf{k}$-algebra generators for A. We denote the kernel of $\mathbf{k}[x_1, \ldots, x_n] \to A$ by I.
- $\mathbb{B}$, a set of vector space generators for A.
- $GF(I)$, the Gröbner fan of a homogenous ideal I in $\mathbf{k}[x_1, \ldots, x_n]$.
- $\operatorname{trop}(I)$, the tropical variety of an ideal I in $\mathbf{k}[x_1, \ldots, x_n]$.
- $\mathbf{k}[V]$, the coordinate ring of an affine variety V.
- $\operatorname{Syz}(S)$, the set of syzygies on S (Definition 2.5.4).
- D, a divisor on a normal variety X.
- $\tilde{X}$, a compactification of a variety X.

- I_B, the kernel of the map $\Phi : \mathbf{k}[\mathbf{x}] \to \mathrm{gr}_{\mathfrak{v}}(A)$ which sends $x_i \to \bar{b}_i$.
- $\mathrm{Frac}(A)$, the field of fractions of A.
- $\mathrm{Proj}(A)$, the Proj construction of a positively graded ring A.
- $\mathbb{B}_{\omega,\geq -k} = \{b \in \mathbb{B} \mid \mathfrak{v}_\omega(b) \geq -k\}$.
- $\tilde{X}_\omega$, the partial compactification of X obtained by adding the divisor D_ω.
- $\mathrm{trop}_C(J)$, the tropical variety relative to a prime cone C of an ideal $J \subset A$ (Definition 5.0.1).
- $\mathcal{O}_p$, the local ring associated to a smooth point p in a variety.
- $\mathfrak{m}_p = \{u_1, \ldots, u_d\}$, the maximal ideal of a local ring $\mathcal{O}_p$.
- J_i, the i^{th} elimination ideal of an ideal J (Definition 4.0.1).

List of Assumptions. Throughout this paper we make the following assumption unless stated otherwise.

- The algebra A is a domain.
- The base field $\mathbf{k}$ is of characteristic 0.
- A has a finite Khovanskii basis with respect to a natural choice of a valuation $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^s$.
- The values of the valuation $\mathfrak{v}$ can be computed effectively.
- Arithmetic in $\mathbf{k}$ and A can be done effectively.
- Arithmetic in the graded ring $\mathrm{gr}_{\mathfrak{v}}(A)$ can be done effectively.
- Unless otherwise stated $\mathbb{Z}^n$ has the lexicographic term order.

2.0 PRELIMINARIES

This chapter is a review of the known literature that will be needed for the rest of the thesis.

2.1 GRÖBNER BASES

We start by introducing Gröbner basis theory for ideals in a polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$. The concept of a Gröbner basis was first introduced by Buchberger in 1965 [Buc06] and is an extremely powerful tool in the field of computational algebra and algebraic geometry. It provides efficient algorithms to solve many problems in commutative algebra and algebraic geometry such as the *ideal membership problem*:

Given an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$, and a polynomial $f \in \mathbf{k}[x_1, \ldots, x_n]$, determine if $f \in I$.

Before going into the details of Gröbner bases we first need to introduce the concept of a term order and an initial ideal. We define a *term order* on the group $\mathbb{Z}^n_{\geq 0}$ as a total order with the property that if $\mathbf{a} \succ \mathbf{b}$ then $\mathbf{a} + \mathbf{c} \succ \mathbf{b} + \mathbf{c}$ $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^n_{\geq 0}$. We assume that the term order is minimum well-ordered, i.e. every chain of elements has a minimum element.

Example 2.1.1. An example of a term order is the lexicographic order on $\mathbb{Z}^n_{\geq 0}$. Let $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = b_1, \cdots_n) \in \mathbb{Z}^n_{\geq 0}$. We say $\mathbf{a} \succ \mathbf{b}$ if $a_1 < b_1$ or $a_i = b_i \ \forall i < k$ and $a_k < b_k$.

Given a term order $\succ$ on $\mathbb{Z}^n_{\geq 0}$, we define the *initial term* of a polynomial $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in \mathbf{k}[x_1, \ldots, x_n]$, where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ as:

$$in_{\succ}(f) = c_{\mathbf{b}} \mathbf{x}^{\mathbf{b}},$$

where $\mathbf{b} = \min\{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$, the minimum is taken with respect to the term order $\succ$. For an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ we define the initial ideal of I as

$$in_{\succ}(I) = \langle in_{\succ}(f) \mid f \in I \rangle.$$

It is clear that $in_{\succ}(I)$ is generated by monomials and hence is a monomial ideal. Note that the initial terms of a set of generators of an ideal may not generate the initial ideal.

Example 2.1.2. Let $I = \langle x^3 - 2xy, x^2y + x - 2y^2 \rangle \subset \mathbf{k}[x, y]$ with the reverse lexicographic order. Notice that $x(x^2y + x - 2y^2) - y(x^3 - 2xy) = x^2 \in I$.
Thus, $in_{\succ}(x^2) = x^2 \in in_{\succ}(I)$ but,
$x^2 \notin \langle in_{\succ}(x^3 - 2xy), in_{\succ}(x^2y + x - 2y^2) \rangle = \langle x^3, x^2y \rangle$.

A set of generators for I is a Gröbner basis if their initial terms generate $in_{\succ}(I)$. The follow two results (Lemma 2.1.4 and Theorem 2.1.5) provides the ground work for finding a Gröbner basis. We will start by introducing the division algorithm for polynomials.

Algorithm 2.1.3. (Division Algorithm) Fix a term order $\succ$ on $\mathbb{Z}^n_{\geq 0}$ and let $\Gamma = \{f_1, \ldots, f_t\}$ be a set of polynomials in $\mathbf{k}[x_1, \ldots, x_n]$. Then for every $f \in \mathbf{k}[x_1, \ldots, x_n]$, we can write f as,

$$f = a_1 f_1 + \cdots + a_t f_t + r,$$

where $a_i, r \in \mathbf{k}[x_1, \ldots, x_n]$ and either $r = 0$ or none of the nonzero monomials in r are divisible by $in_{\succ}(f_1)$,…,$in_{\succ}(f_t)$.

Input: $f_1, \ldots, f_t, f$

Output: $a_1, \ldots, a_t, r$

```
Set a_1 := 0; ... ; a_t := 0; r := 0
p := f
WHILE p ≠ 0 DO
    i := 1
    divisionoccurred := false
    WHILE i ≤ t AND divisionoccurred = false DO
        IF in(f_i) divides in(p) THEN
            a_i := a_i + in(p)/in(f_i)
            p := p − (in(p)/in(f_i))f_i
            divisionoccurred := true
        ELSE
            i := i + 1
    IF divisionoccurred = false THEN
        r := r + in(p)
        p := p − in(p)
```

We will write $f \xrightarrow{\Gamma} r$ to mean that if we apply the division algorithm to f by the set of polynomials Γ we will get the remainder r. It is important to notice that if the set Γ is chosen arbitrarily then the remainder may not be unique and depends on the order in which f_i appear in Γ.

Lemma 2.1.4. *(Dickson's Lemma) Let $I \subset \mathbf{k}[x_1, \ldots, x_n]$ be a monomial ideal. Then there exists a finite set of monomials $\{x^{\alpha_1}, \ldots, x^{\alpha_j}\}$ that generate I.*

Proof: The proof can be found in [CLO15, p.71].

□

Theorem 2.1.5. *(Hilbert Basis Theorem) Every ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ has a finite generating set.*

Proof: Let $I = \{0\}$, then I is clearly finitely generated. We can assume I contains a nonzero element.

First consider $in_{\succ}(I)$. Since $in_{\succ}(I)$ is a monomial ideal, then by Dickson's Lemma (Lemma 2.1.4) we have that $in_{\succ}(I)$ is finitely generated, say
$in_{\succ}(I) = \langle in_{\succ}(g_1), \ldots, in_{\succ}(g_m) \rangle$ where $g_i \in I$.

We claim that $I = \langle g_1, \ldots, g_m \rangle$. Since $g_i \in I$ for all i, we can see that $\langle g_1, \ldots, g_m \rangle \subset I$. Let $f \in I$ and apply the division algorithm (Algorithm 2.1.3) to f to get:

$$f = a_1 g_1 + \cdots + a_m g_m + r$$

where no term of r is divisible by any $in_{\succ}(g_i)$. Since $r = f - a_1 g_1 - \cdots - a_m g_m \in I$. We have $in_{\succ}(r) \in in_{\succ}(I) = \langle in_{\succ}(g_1), \ldots, in_{\succ}(g_m) \rangle$, which implies that $in_{\succ}(r) = 0$. Thus $r = 0$ and $f \in \langle g_1, \ldots, g_m \rangle$. Therefore $I = \langle g_1, \ldots, g_m \rangle$.

$\square$

The above two results give us the existence and finiteness of a Gröbner basis, G, for an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$.

Definition 2.1.6. (Gröbner Basis) A set $G = \{g_1, \ldots, g_m\} \subset \mathbf{k}[x_1, \ldots, x_n]$ is a *Gröbner basis* for $I = \langle g_1, \ldots, g_m \rangle$ if $in_\succ(I) = \langle in_\succ(g_1), \ldots, in_\succ(g_m) \rangle$.

Given a Gröbner basis $G = \{g_1, \ldots, g_m\}$ for an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ the division algorithm (Algorithm 2.1.3) gives us a unique way to write $f \in \mathbf{k}[x_1, \ldots, x_n]$ as $f = g + r$ where $g \in I$ and no monomial of r is in $in_\succ(I)$. Thus we have that $f \in I$ if and only if the remainder $r = 0$, giving us a solution to the ideal membership problem.

We can compute a Gröbner basis using the well-known Buchberger criterion (Theorem 2.1.8) and Buchberger algorithm (Algorithm 2.1.9). We first introduce the S-polynomials, which play a crucial role in the Buchberger criterion.

Definition 2.1.7. (S-Polynomial) Let $f, g \in \mathbf{k}[x_1, \ldots, x_n]$ be nonzero polynomials.

(i) Let $\deg(f) = \mathbf{a}$ and $\deg(g) = \mathbf{b}$. For each i, let $c_i = \max(a_i, b_i)$ and let $\mathbf{c} = (c_1, \ldots, c_n)$. We call $x^{\mathbf{c}}$ the *least common multiple* of $in_\succ(f)$ and $in_\succ(g)$.

(ii) The *S-polynomial* of f and g is

$$S(f, g) = \frac{x^{\mathbf{c}}}{in_\succ(f)} f - \frac{x^{\mathbf{c}}}{in_\succ(g)} g$$

Theorem 2.1.8. *(Buchberger Criterion) Let I be a polynomial ideal. Then an ideal generating set $G = \{g_1, \ldots, g_m\}$ for I is a Gröbner basis if and only if for all (i, j), $i \neq j$, the remainder of $S(g_i, g_j)$ when divided by G is zero.*

Proof: The proof can be found in [CLO15, p.85].

$\square$

Algorithm 2.1.9. (Buchberger Algorithm)
Input: $\Gamma = \{f_1, \ldots, f_t\}$ where $I = \langle f_1, \ldots, f_t \rangle$.
Output: A Gröbner basis $G = \{g_1, \ldots, g_m\}$ for I.

(1) Set $\Gamma = G$.

(2) For each pair $\{p, q\} \subset G$ where $p \neq q$, do:

  (a) Compute $S(p, q)$.

  (b) Apply the division algorithm to $S(p, q)$ with G.

  (c) If we get a nonzero remainder S, then $G := G \cup \{S\}$.
  Go to step (2).

  (d) If all reminders are zero, give G as output.

Gröbner bases computed using the Buchberger algorithm (Algorithm 2.1.9) are often bigger than necessary. This leads us to the notions of minimal and reduced Gröbner bases.

Definition 2.1.10. (Minimal Gröbner Basis) For an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ a Gröbner basis G is *minimal* if $in_\succ(g) \notin \langle in_\succ(G \setminus \{g\})\rangle$.

Definition 2.1.11. (Reduced Gröbner Basis) A minimal Gröbner basis G for an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ is *reduced* if for all $g \in G$, no monomial of g, besides $in_\succ(g)$, lies in $in_\succ(I) = \langle in_\succ(G)\rangle$.

Remark 2.1.12. One can show that every ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ has a unique reduced Gröbner basis (see [CLO15, Proposition 7.6]).

We can also use Gröbner bases for finding solutions of a system of polynomial equations. We start by defining the i^{th} elimination ideal of an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ to be:

$$I_i = I \cap \mathbf{k}[x_{i+1}, \ldots, x_n]$$

noting that $I_0 = I$.

Theorem 2.1.13. *(The Elimination Theorem) Let $I \subset \mathbf{k}[x_1, \ldots, x_n]$ be an ideal and let G be a Gröbner basis for I with respect to a term order $\succ$ such that $x_1 \succ x_2 \succ \cdots \succ x_n$. Then for every $0 \leq i \leq n-1$, the set*

$$G_i = G \cap \mathbf{k}[x_{i+1}, \ldots, x_n]$$

is a Gröbner basis for the i^{th} elimination ideal I_i.

Proof: Fix $0 \leq i \leq n$. It suffices to show that $\langle in_{\succ}(G_i)\rangle = in_{\succ}(I_i)$.

Let $f \in I_i$, and note that this implies that $f \in I$. Since G is a Gröbner basis for I we see that $in_{\succ}(f)$ is divisible by some $in_{\succ}(g)$, $g \in G$. But $f \in I_i$ implies that $in_{\succ}(g) \in I_i$. If there exists a monomials in g that contained x_j for $j \leq i$, then because of the term order we see that this monomial should be $in_{\succ}(g)$, which would imply that $in_{\succ}(g) \notin I_i$, a contradiction. Hence, $g \in G_i$, and $\langle in_{\succ}(I_i)\rangle \subset \langle in_{\succ}(G_i)\rangle$.

□

Note that the $(n-1)^{st}$ elimination ideal I_{n-1} is that set of all polynomials in I in one variable x_n. The fact that G_{n-1} is a Gröbner basis for I_{n-1} implies that if we solve the polynomials in G_{n-1}, we get all the possible solutions for x_n. We continue this process to get all solutions for all the variables. Here we make the assumption that we have an efficient process for root finding in our algebraically closed field [FJ08].

It can be useful to replace a term order with a weight vector $\omega \in \mathbb{R}^n$. We define the initial form of a polynomial $f = \sum_{\alpha} c_\alpha x^\alpha \in \mathbf{k}[x_1, \ldots, x_n]$ with respect to $\omega \in \mathbb{R}^n$ as:

$$in_\omega(f) = \sum_{\omega \cdot \beta = \gamma} c_\beta x^\beta,$$

where $\gamma = \min\{\omega \cdot \alpha \mid c_\alpha \neq 0\}$ and the minimum is taken with respect to the usual ordering of numbers. This leads to the natural definition of the initial form.

Definition 2.1.14. (Initial Ideal) Given an ideal $I \in \mathbf{k}[\mathbf{x}]$ the *initial ideal* of I with respect to some weight vector $\omega \in \mathbb{R}^n$ is

$$in_\omega(I) = \langle in_\omega(f) \mid f \in I\rangle.$$

It is easy to see that the initial form of a polynomial may not be a monomial. However, when ω is in general position, we can see that the initial form is a monomial. When ω is in general position then the initial ideal $in_\omega(I)$ coincides with $in_\succ(I)$ for some term order $\succ$. This leads us to the concept of a Gröbner region.

Definition 2.1.15. (Gröbner Region) A *Gröbner region* of an ideal $I \subset k[\mathbf{x}]$, $GR(I)$, is the set of all $\omega \in \mathbb{R}^n$ such that $in_\succ(in_\omega(I)) = in_\succ(I)$ for some term order $\succ$.

Note that if I is homogenous then $GR(I) = \mathbb{R}^n$. Given an ideal I, we have an equivalence relation among the weights in the Gröbner region, $GR(I)$, we say that $\omega_1 \sim \omega_2$ if $in_{\omega_1}(I) = in_{\omega_2}(I)$. It can be shown that the equivalence classes of $\sim$ are relative interior of polyhedral cones and the closures of the equivalence classes form a fan (Definition 2.2.4), see [Stu96, Chapter 1].

2.2 TORIC VARIETIES

In this section we take $\mathbf{k} = \mathbb{C}$. For a comprehensive reference on toric varieties we refer the reader to [CLS10].

Definition 2.2.1. (Torus) The affine variety $(\mathbb{C}^*)^n$ is a group under component wise multiplication. A *torus* $\mathbb{T}$ is an affine variety isomorphic to $(\mathbb{C}^*)^n$.

Definition 2.2.2. (Toric Variety) A *toric variety* is an irreducible normal variety X containing a torus $\mathbb{T} \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of $\mathbb{T}$ on itself extends to an algebraic action of $\mathbb{T}$ on X.

Toric varieties are classified by polyhedral cones and fans. We recall these notions from polyhedral geometry.

Definition 2.2.3. (Convex Polyhedral Cone) A *convex polyhedral cone* in $\mathbb{R}^n$ is the cone generated by a finite set $S \subset \mathbb{R}^n$:

$$\sigma = \mathrm{Cone}(S) = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\} \subset \mathbb{R}^n.$$

σ is a *strongly* convex polyhedral cone if it does not contain a 1-dimensional subspace (i.e. a line through the origin).

Definition 2.2.4. (Fan) A *fan* Σ in $\mathbb{R}^n$ is a finite collection of cones $\sigma \subset \mathbb{R}^n$ such that:

(1) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

(2) For all $\sigma \in \Sigma$, each face of σ is also in Σ.

(3) For all $\sigma_1, \sigma_2 \in \Sigma$ with $\sigma_1 \cap \sigma_2 \neq \emptyset$ we have $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 and belongs to Σ.

For every cone $\sigma \in \mathbb{R}^n$ we define the dual cone of σ by:

$$\sigma^\vee = \{a \in \mathbb{R}^n \mid \langle a, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

To a cone σ we associate an affine toric variety U_σ defined by $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n])$. One can construct an abstract X_Σ from a fan Σ by gluing affine toric varieties U_σ [CLS10, Section 3.1]. The following theorem is the main result about classification of toric varieties (see [CLS10, Chapter 3]):

Theorem 2.2.5. *There is a one-to-one correspondence (in fact equivalence of categories) between fans and the isomorphism classes of toric varieties.*

2.3 TROPICAL GEOMETRY

Tropical geometry is an interface between algebraic geometry, combinatorics, and convex geometry and has many connections to other fields. Tropical geometry is the geometry over the tropical semiring $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, with the usual operation of addition replaced by minimum and the usual operation of multiplication replaced by addition. This turns polynomials into piecewise linear functions, and turns algebraic varieties into objects from polyhedral geometry. The notion of a valuation map is an important concept connected to tropical varieties. We warn the reader that in some of the tropical geometry literature, in the definitions, maximum convention is used instead of minimum convention in this thesis.

Definition 2.3.1. (Valuation Map) A function $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^s$, where $\mathbb{Z}^s$ is equipped with a term order $\succ$, is a *valuation map* over a field $\mathbf{k}$ if it satisfies the following:

(1) For all $0 \neq f, g \in A$ with $0 \neq f + g$ we have $\mathfrak{v}(f + g) \succeq \min\{\mathfrak{v}(f), \mathfrak{v}(g)\}$.

(2) For all $0 \neq f, g \in A$ we have $\mathfrak{v}(fg) = \mathfrak{v}(f) + \mathfrak{v}(g)$.

(3) For all $0 \neq f \in A$ and $0 \neq c \in \mathbf{k}$ we have $\mathfrak{v}(cf) = \mathfrak{v}(f)$.

One extends $\mathfrak{v} : A \to \mathbb{Z}^s \cup \{\infty\}$ by defining $\mathfrak{v}(0) = +\infty$.

An important example of a valuation map is the minimum term valuation.

Example 2.3.2. Consider the polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$. Fix a term order on $\mathbb{Z}^n$. We define $\mathfrak{v} : \mathbf{k}[x_1, \ldots, x_n] \setminus \{0\} \to \mathbb{Z}^n$ as follows. For $f \in \mathbf{k}[x_1, \ldots, x_n]$ let $\mathfrak{v}(f) = \mathbf{b} = \min\{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$. It is easy to check that this does give us a valuation map. Note that the above definition works for power series ring $\mathbf{k}[[x_1, \ldots, x_n]]$ as well. We can also similarly define a valuation on $\mathbf{k}[x_1, \ldots, x_n]$ using $\mathbf{b}' = \max\{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$ and $\mathfrak{v}(f) = -\mathbf{b}'$.

Fix a valuation $\mathfrak{v} : \mathbf{k} \setminus \{0\} \to \mathbb{Z}$. Let $\mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$ denote the ring of Laurent polynomials over $\mathbf{k}$. Given a function $f = \sum_{\mathbf{a} \in \mathbb{Z}^n} c_{\mathbf{a}} x^{\mathbf{a}} \subset \mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$, we define the tropical polynomial $\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ associated to f by:

$$\mathrm{trop}(f)(\mathbf{w}) = \min_{\mathbf{a} \in \mathbb{Z}^n} (\mathfrak{v}(c_{\mathbf{a}}) + \sum_{i=1}^{n} a_i w_i).$$

Note that the tropicalization of a Laurent polynomial f transforms it to a piecewise linear function $\mathrm{trop}(f)$.

Definition 2.3.3. (Tropical Hypersurface) The *tropical hypersurface* $\mathrm{trop}(V(f))$ is the set

$$\{\omega \in \mathbb{R}^n \mid \text{ the minimum in } \mathrm{trop}(f)(\omega) \text{ is achieved at least twice}\}.$$

The tropical hypersurface is the set of points in $\mathbb{R}^n$ where the piecewise linear function $\mathrm{trop}(f)$ fails to be linear. To compute a tropical variety, we now introduce the Fundamental Theorem of Tropical Algebraic Geometry.

Theorem 2.3.4. *(Fundamental Theorem of Tropical Algebraic Geometry) Let I be an ideal in $\mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$ and $Z = V(I)$ its variety. Then the following three subsets of $\mathbb{R}^n$ coincide:*

(1) The tropical variety $\operatorname{trop}(Z) = \bigcap_{f \in I} \operatorname{trop}(V(f))$.

(2) The closure in $\mathbb{R}^n$ of the set of all vectors $\mathbf{w} \in \mathbb{Q}^s$ with $in_{\mathbf{w}}(I) \neq \langle 1 \rangle$.

(3) The closure of the set of coordinatewise valuations of points in Z:

$$\mathfrak{v}(Z) = \{(\mathfrak{v}(u_1), \ldots, \mathfrak{v}(u_n)) \mid (u_1, \ldots, u_n) \in Z\}.$$

It is an important result that any tropical variety is the support of a fan. Suppose C is a cone contained in $\operatorname{trop}(Z)$ such that for any $\mathbf{w}_1, \mathbf{w}_2 \in C^o$, the relative interior of the cone C, then $in_{\mathbf{w}_1}(I) = in_{\mathbf{w}_2}(I)$. We denote this common initial ideal by $in_C(I)$. Kaveh and Manon introduced the notion of a prime cone [KM19, Section 4] which will be used throughout the remainder of this paper in correspondence with valuation maps.

Definition 2.3.5. (Prime Cone) Assume $A \cong \mathbf{k}[\mathbf{x}]/I$ and $\operatorname{trop}(V(I))$ be the tropical variety of I. Let $C \subset \operatorname{trop}(V(I))$ be a cone as above. We call C a *prime cone* if the corresponding initial ideal $in_C(I)$ is a prime ideal.

2.4 EFFECTIVE FIELD THEORY

This section will address when algebraic operations can be effectively computed by a computer. For a more detailed background in effective field theory we refer the reader to [FJ08]. The basic structures are the prime fields $\mathbb{Q}$, $\mathbb{F}_p$, finite algebraic extensions of these fields, the ring $\mathbb{Z}$ and the rings of polynomials over these rings. We consider elements of these rings as recognizable and we may perform computations with them effectively.

A primitive recursive function is a function that can be computed by a computer program whose loops are all "for" loops, which implies an upper bound on the number of iterations of every loop can be determined before entering the loop. We define the set Λ as the set of all formal quotients of polynomial words, where polynomial words are defined inductively. Consider a sequence of symbols $(\alpha_1, \alpha_2, \dots)$, then each element of $\mathbb{Z}$ and each symbol is a polynomial word. If t_1 and t_2 are polynomial words and $n \in \mathbb{Z}$, then nt_1, $(t_1 + t_2)$, and $(t_1 t_2)$ are polynomial words.

The main focus of this section will be presented rings and fields. A presented ring is a ring in which we can explicitly recognize the elements and in which we can explicitly carry out addition and multiplication, such as $\mathbb{Q}$ and its algebraic closer $\bar{\mathbb{Q}}$.

Definition 2.4.1. (Presented Ring) A ring is said to be *presented* if there exists an injective map $j : R \to \Lambda$ such that $j(R)$ is a primitive recursive subset of Λ and addition and multiplication are primitive recursive functions over R. In addition the primitive recursive construction of this data from basic functions should be "given explicitly" (in a naive way suitable for practical purposes).

It can be shown that computations of a presented ring can be effectively done with a computer. Now that we know which rings we can work with we need to understand what kind of functions we can effectively compute. The following is the main theorem of this section.

Theorem 2.4.2. *If R is a presented ring then,*

(a) The theory of algebraically closed fields of a given characteristic is primitive recursive.

(b) The theory of algebraically closed fields is primitive recursive.

A field $\mathbf{k}$ is presented if the characteristic of $\mathbf{k}$ is given explicitly, and the inverse function of $\mathbf{k}^*$ is a primitive recursive function, explicitly given in terms of basic functions. An integral domain R is presented in its quotient field $\mathbf{k}$ if R and $\mathbf{k}$ are both presented with R a primitive recursive subset of $\mathbf{k}$. Note that a presented ring is countable and that $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{F}_p$ can be presented. Let $\mathbf{K}$ be a field extension of a presented field $\mathbf{k}$. Then an element $x \in \mathbf{K}$ is presented over $\mathbf{k}$ if x is algebraic over $\mathbf{k}$.

Lemma 2.4.3. *If an element x is presented over a presented field $\mathbf{k}$, the field $\mathbf{k}(x)$ is also a presented field. An n-tuple $(x_1, x_2, \ldots, x_n)$ is said to be presented over $\mathbf{k}$ if x_i is presented over $\mathbf{k}(x_1, \ldots, x_{i-1})$ for all $i = 1, 2, \ldots, n$.*

An effective algorithm over a presented ring R is an explicitly given primitive recursive map $\lambda : A \to B$, where A and B are explicitly given primitive recursive subsets of $[R(\mathbf{x})]^a$ and $[R(\mathbf{x})]^b$, respectively.

Definition 2.4.4. (Splitting Algorithm) A presented field $\mathbf{k}$ is said to have a *splitting algorithm* if $\mathbf{k}$ has an effective algorithm for factoring each element of

$\mathbf{k}[\mathbf{x}]$ into a product of irreducible factors.

We should note that it follows from the above results that $\bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, is a presented field and has a splitting algorithm, as well as $\bar{\mathbb{Q}}[\mathbf{x}]$.

2.5 SAGBI BASES

Gröbner basis theory works in polynomial rings. Sweedler [RS90] and Ollivier [Oll91] extended the framework of Gröbner basis theory to subalgebras of polynomial rings. It is important to recognize that a subalgebra R may not be Noetherian. Hence, when working with SAGBI (Subalgebra Analog to Gröbner Bases for Ideals) bases we will always make the assumption that R is finitely generated $\mathbf{k}$-subalgebra in a polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$.

Let $R \subset \mathbf{k}[x_1, \ldots, x_n]$ be a subalgebra. The initial subalgebra $in_{\succ}(R)$ is the subalgebra of $\mathbf{k}[x_1, \ldots, x_n]$ generated by $in_{\succ}(f)$, $f \in R$.

Definition 2.5.1. (SAGBI Basis) A *SAGBI basis* for a finitely generated $\mathbf{k}$-algebra $R \subset \mathbf{k}[x_1, \ldots, x_n]$ is a subset C of R such that $in_{\succ}(R)$ is generated as a $\mathbf{k}$-algebra by the set of monomials $\{in_{\succ}(f) \mid f \in C\}$.

Remark 2.5.2. Even if R is finitely generated, it is still possible that $in_{\succ}(R)$ is not finitely generated, and hence R would not have a finite SAGBI basis. However, when $in_{\succ}(R)$ is finitely generated we can imitate all of Gröbner theory, including an analogous Buchberger criterion and Buchberger algorithm, for ideals in R.

The following example is due to Göbel [Gob95] giving us a finitely generated subalgebra that does not have a finite SAGBI basis with respect to a certain lexicographic order.

Example 2.5.3. Let $R = \mathbf{k}[x_1, x_2, x_3]^{A_3} \subset \mathbf{k}[x_1, x_2, x_3]$ the subalgebra which is invariant under the cyclic permutation $x_1 \to x_2, x_2 \to x_3, x_3 \to x_1$. We can see that R has four minimal generators and is hence finitely generated:

$$R = \mathbf{k}[x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3, (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]$$

If we consider $\succ$ to be the lexicographic term order with $x_3 \succ x_2 \succ x_1$. It can be shown that $in_{\succ}(R)$ is not finitely generated (see [Stu96, p.99]). Hence R does not have a finite SAGBI basis.

□

Next we introduce the notation of a syzygy. The use of syzygies will replace the S-polynomials in the Buchberger algorithm.

Definition 2.5.4. (Syzygy) A *syzygy* on $S = \{s_1, \ldots, s_m\} \subset R$ is a n-tuple $(a_{s_1}, \ldots, a_{s_m}) \in R^m$ such that $\sum_{i=1}^{m} a_{s_i} s_i = 0$. We denote the set of all syzygies on S by $\mathrm{Syz}(S)$.

To simplify notation we write $(a_{s_1}, \ldots, a_{s_m}) = (a_1, \ldots, a_m) \in R^m$ when the order of $S = \{s_1, \ldots, s_m\}$ is clear.

Theorem 2.5.5. *(Analogue of Buchberger Criterion) Let* $G = \{g_1, \ldots, g_m\}$ *with* $g_i \in R$*, and let* $\mathbf{H}$ *be a subset of* R^m *such that* $\{(in_\succ(h_1), \ldots, in_\succ(h_m)) \mid (h_1, \ldots, h_m) \in \mathbf{H}\}$ *generates the* R*-module* $Syz(\{in_\succ(g_1), \ldots, in_\succ(g_m)\})$*. Then* G *is a Gröbner basis for the ideal* $\langle G \rangle$ *with respect to the term order if and only if, for every* $\mathbf{h} = (h_1, \ldots, h_m) \in \mathbf{H}$*, the polynomial* $h_1g_1 + \cdots + h_mg_m \xrightarrow{G} 0$.

Proof: Proof in [Mil94, Theorem 4.9].

Algorithm 2.5.6. (Analogue of Buchberger Algorithms)
Input: A generating set S for an ideal $I \subset R$ with a term order $\succ$.
Output: A Gröbner basis G for I.

(1) Let $S = \{s_1, \ldots, s_m\}$ and $in_\succ(s_i) = \mathbf{x}^{\mathbf{c}_i}$.

(2) Compute a finite generating set $\mathbf{H}$ for $\mathrm{Syz}(\mathbf{x}^{\mathbf{c}_1}, \ldots, \mathbf{x}^{\mathbf{c}_m})$.

(3) Set $N := \emptyset$.

(4) For each $\mathbf{h} = (h_1, \ldots, h_m) \in \mathbf{H}$ do:

  (a) Find $k = (k_1, \ldots, k_m) \in R^m$ such that $in_\succ(k_i) = h_i$.

  (b) Compute $k_1s_1 + \cdots + k_ms_m \xrightarrow{S} \bar{k}$.

  (c) Set $N := N \cup \{\bar{k}\}$.

(5) If $N \neq \{0\}$ then set $S := S \cup N \setminus \{0\}$, and return to step 1.

(6) If $N = \{0\}$, print S.

One important example where we can apply the theory of SAGBI bases is the total coordinate ring of the Grassmanian, namely, the Plücker algebra. It was shown by Sturmfels [Stu93, Section 3.1] that the set of $r \times r$-minors of an $r \times n$-matrix of indeterminants is a SAGBI basis for the subalgebra they generate, with respect to any diagonal term order on $\mathbf{k}[x_{11}, \ldots, x_{rn}]$.

However, there are many examples where SAGBI theory cannot be applied, one of which being Example 2.5.3. For another example where SAGBI theory cannot be applied, consider an affine cubic curve $x^3 + ax + b = y^2$, and let $R = \mathbf{k}[x, y]/\langle x^3 + ax + b - y^2 \rangle = \mathbf{k}[E]$. Let us assume for a contradiction that R is a subalgebra of a polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$, then there exists an imbedding $Q : R \hookrightarrow \mathbf{k}[x_1, \ldots, x_n]$. By Lüroth's Theorem there exist $\mathbf{k}(E) \hookrightarrow \mathbf{k}(x_1, \ldots, x_n)$ which implies that $\mathbf{k}(E) \hookrightarrow \mathbf{k}(t)$, which is not possible because an elliptic curve is not a rational curve. Hence, R is not a subalgebra of a polynomial ring, and thus we cannot apply SAGBI basis theory.

Such examples show the need for a further generalization of Gröbner theory. It is because of this that Kaveh and Manon developed what is known as Khovanskii bases [KM19]. The notion of a Khovanskii basis generalizes the notion of a SAGBI basis and makes sense for any finitely generated algebra.

2.6 KHOVANSKII BASES

There are many motivations for the theory of Khovanskii bases: they are a natural generalization of SAGBI theory, they can be utilized when working with affine varieties whose coordinate rings are not a subalgebra of a polynomial ring, and there is a direct connection between Khovanskii bases and the theory of Newton-Okounkov bodies. Before we introduce Khovanskii bases we need to introduce some more terminology, starting by revisiting the definition of a valuation map (Definition 2.3.1).

For the remainder of this paper we will consider $\mathbb{Z}^s$ with a lexicographic term order. Given a valuation map $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^s$ (for some $s \in \mathbb{N}$), we construct a filtration $\mathcal{F}_{\mathfrak{v}} = (F_{\mathfrak{v}\succeq a})_{a\in\mathbb{Q}^s}$ on A with $F_{\mathfrak{v}\succeq a} = \{f \in A \mid \mathfrak{v}(f) \succeq a\} \bigcup \{0\}$ and $F_{\mathfrak{v}\succ a}$ defined in a similar way. With this filtration we construct the associated graded ring

$$\mathrm{gr}_{\mathfrak{v}}(A) = \bigoplus_{a\in\mathbb{Q}^s} F_{\mathfrak{v}\succeq a}/F_{\mathfrak{v}\succ a}.$$

For $0 \neq f \in A$ we let $\bar{f} \in \mathrm{gr}_{\mathfrak{v}}(A)$ be the image of f in $F_{\mathfrak{v}\succeq a}/F_{\mathfrak{v}\succ a}$ where $\mathfrak{v}(f) = a$.

Definition 2.6.1. (Value Semigroup) Given a valuation map $\mathfrak{v} : A\setminus\{0\} \to \mathbb{Z}^s$ we define the *value semigroup* of $(A, \mathfrak{v})$ by $S = S(A, \mathfrak{v}) = \{\mathfrak{v}(f) \mid 0 \neq f \in A\}$.

We say that the rational rank of the valuation $\mathfrak{v}$ is the rank of the sublattice of $\mathbb{Z}^s$ generated by S. A valuation is called *full rank* if the rational rank of the valuation is equal to the dimension of the algebra A.

We say that a valuation has one-dimensional leaves if for every $a \in \mathbb{Q}^s$ the quotient vector space $F_{\mathfrak{v}\succeq a}/F_{\mathfrak{v}\succ a}$ is at most 1-dimensional.

Theorem 2.6.2. *Let* $\mathbf{k}$ *be algebraically closed and assume that* $\mathfrak{v}$ *has full rank* $d = \mathit{dim}(A)$ *then* $(A, \mathfrak{v})$ *has one-dimensional leaves.*

Proof: The proof can be found in [KM19, Theorem 2.3].

The one-dimensional leaves property has the following important implication. Suppose $(A, \mathfrak{v})$ has one-dimensional leaves and let $L \subset A$ be a $\mathbf{k}$-vector subspace. Then $\dim_{\mathbf{k}}(L) = |\mathfrak{v}(L \setminus \{0\})|$. When $(A, \mathfrak{v})$ has one-dimensional leaves, the associated graded $\mathrm{gr}_{\mathfrak{v}}(A)$ is isomorphic to the semigroup algebra $\mathbf{k}[S]$. Thus we assume that for any pair $b_i \neq b_j \in \mathcal{B} \subset A$, a Khovanskii basis, we have $\mathfrak{v}(b_i) \neq \mathfrak{v}(b_j)$.

Let $X \subset \mathbb{P}^N$ be a d-dimensional projective variety with homogenous coordinate ring $A = \mathbf{k}[X]$. Let $\mathfrak{v}' : \mathbf{k}(X) \setminus \{0\} \to \mathbb{Z}^d$ be a valuation on the field of rational functions of X. We can extend $\mathfrak{v}'$ to $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z} \times \mathbb{Z}^d$ as follows. Fix a homogenous degree one element $0 \neq h \in \mathbf{k}[X]$. For $f \in \mathbf{k}[X]_m$ homogenous of degree m define:

$$\mathfrak{v}(f) = (-m, \mathfrak{v}'(\frac{f}{h^m})).$$

Example 2.6.3. Now let X be a projective curve in $\mathbb{P}^2$ and let $\mathfrak{v}'$ be the order of vanishing at a smooth point P.

Theorem 2.6.4. *(Riemann-Roch Theorem) Given a smooth projective curve X and D a divisor on it, we have:*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g,$$

where g is the genus of X and K is the canonical divisor of X. Also $\ell(D)$ denotes the dimension of the vector space $\{f \mid (f) + D > 0\}$.

Let $D = \mathrm{div}(h) = \sum_P a_P P$ be the divisor of h. We can compute the value semigroup of an elliptic curve using the Riemann-Roch theorem. We note that for an elliptic curve the genus $g = 1$, and if $\deg(D) \geq 2g - 1 = 1$, then $\ell(K - D) = 0$. Consider the divisor $nD - kP$ where D is an arbitrary divisor. Then we have $\ell(nD - kP) = 3n - k$ when $\deg(nD - kP) \geq 1$. Thus we can see that when $k = i < 3n$, $\ell = 3n - i$, if $k = 3n$ then $\ell = 1$. Hence, $S = \{(x, y) \mid 0 \leq y, 0 \leq x \leq 3y\}$. Note that when $P = 0$ then S is finitely generated and if P is a general point then S is not finitely generated [KM19, p. 9].

Definition 2.6.5. (Khovanskii Basis) Given a valuation map $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^s$ and the associated graded ring $\mathrm{gr}_{\mathfrak{v}}(A)$, we say that $\mathcal{B} \subset A$ is a *Khovanskii basis* for $(A, \mathfrak{v})$ if the image $\bar{\mathcal{B}}$ of $\mathcal{B}$ in $\mathrm{gr}_{\mathfrak{v}}(A)$ is a set of algebra generators for $\mathrm{gr}_{\mathfrak{v}}(A)$.

If $(A, \mathfrak{v})$ has one-dimensional leaves then $\mathcal{B}$ is a Khovanskii basis if and only if $\{\mathfrak{v}(b) \mid b \in \mathcal{B}\}$ generates $S = S(A, \mathfrak{v})$ as a semigroup. This follows from the fact that $\mathrm{gr}_{\mathfrak{v}}(A) \cong \mathbf{K}[S]$. We can represent the elements of A as a polynomial in the elements of $\mathcal{B}$ using a subduction algorithm (Algorithm 2.6.6).

Algorithm 2.6.6. (Subduction algorithm)
Input: A Khovanskii basis $\mathcal{B} \subset A$ and an element $0 \neq f \in A$.
Output: A polynomial expression for f in terms of a finite number of elements of $\mathcal{B}$.

(1) Since the image of $\mathcal{B}$ in $\mathrm{gr}_{\mathfrak{v}}(A)$ generates this algebra, we can find $b_1, \ldots, b_n \in \mathcal{B}$ and a polynomial $p(x_1, \ldots, x_n)$ such that $\bar{f} = p(\bar{b}_1, \ldots, \bar{b}_n)$. Thus we either have $f = p(b_1, \ldots, b_n)$ of $\mathfrak{v}(f - p(b_1, \ldots, b_n)) \succ \mathfrak{v}(f)$.

(2) If $f = p(b_1, \ldots, b_n)$ we are done. Otherwise replace f with $f - p(b_1, \ldots, b_n)$ and go to step (1).

The subduction algorithm (Algorithm 2.6.6) terminates after a finite number of steps if the value semigroup $S = S(A, \mathfrak{v})$ is maximum well-ordered. It is important to know how to compute a Khovanskii basis for our algebra A. For any finite set of generators $\mathcal{B} = \{b_1, \ldots, b_n\}$ let $\Phi : \mathbf{k}[\mathbf{x}] \to \mathrm{gr}_{\mathfrak{v}}(A)$ be the natural homomorphism that sends $x_i \to \bar{b}_i$. The following algorithm is from [KM19, Section 2].

Algorithm 2.6.7. (Computing a Khovanskii Basis)
Input: A finite set of $\mathbf{k}$-algebra generators $\{b_1, \ldots, b_n\}$ for A.
Output: A finite Khovanskii basis $\mathcal{B}$.

(1) Put $\mathcal{B} = \{b_1, \ldots, b_n\}$. Let $\bar{\mathcal{B}}$ be the image of $\mathcal{B}$ in $\mathrm{gr}_{\mathfrak{v}}(A)$.
Let $\Phi_{\mathcal{B}} : \mathbf{k}[\mathbf{x}] \to \mathrm{gr}_{\mathfrak{v}}(A)$ be the homomorphism defined by $x_i \mapsto \bar{b}_i$.

(2) Let I_B be the kernel of the homomorphism $\Phi_{\mathcal{B}}$. Let G be a finite set of generators for I_B.

(3) Take an element $g \in G$. Let $h \in A$ be the element obtained by plugging b_i for x_i in g, $i = 1, \ldots, n$. Let $\bar{h}$ denote the image of h in $\mathrm{gr}_{\mathfrak{v}}(A)$.

(4) Verify if $\bar{h}$ lies in the subalgebra generated by $\bar{\mathcal{B}}$.

(5) If this is the case, find a polynomial $p(x_1, \ldots, x_n)$ such that $\bar{h} = p(\bar{b}_1, \ldots, \bar{b}_n)$. This means that either $h = p(b_1, \ldots, b_n)$ or $\mathfrak{v}(h - p(b_1, \ldots, b_n)) \succ \mathfrak{v}(h)$. Put $h_1 = h - p(b_1, \ldots, b_n)$. If $h_1 = 0$ go to step (7). Otherwise, replace h with h_1 and go to step (4).

(6) If $\bar{h}$ does not lie in the subalgebra generated by $\bar{\mathcal{B}}$ then add h to $\mathcal{B}$.

(7) Repeat until there are no more generators left in G.

(8) If no elements were added to G, the set $\mathcal{B}$ is our desired finite Khovanskii basis. Otherwise go to step (1).

It is shown in [KM19, Corollary 2.19] that the algorithm for computing a Khovanskii basis (Algorithm 2.6.7) terminates in finite time if and only if A has a finite Khovanskii basis with the valuation $\mathfrak{v}$. We now introduce the vector space counterpart of a Khovanskii basis, namely, an adapted basis.

Definition 2.6.8. (Adapted Basis) A $\mathbf{k}$-vector space basis $\mathbb{B} \subset A$ is an *adapted basis* with respect to a decreasing algebra filtration $\mathcal{F} = \{F_a\}_{a \in Q^s}$, given by $\mathbf{k}$-vector subspaces, if $F_a \bigcap \mathbb{B}$ is a vector space basis for all $a \in \mathbb{Q}^s$.

We can write any element in A as a linear combination of elements in the adapted basis $\mathbb{B}$ using the following algorithm.

Algorithm 2.6.9. (Vector Space Subduction)
Input: A vector space basis $\bar{\mathbb{B}} \subset \mathrm{gr}_{\mathfrak{v}}(A)$, a lift $\mathbb{B} \subset A$ of $\bar{\mathbb{B}}$ and an element $f \in A$.
Output: An expression of f as a linear combination of the elements in $\mathbb{B}$.

(1) Compute $\mathfrak{v}(f) = a$ and take the equivalence class $\bar{f} \in F_{\mathfrak{v} \succeq a}/F_{\mathfrak{v} \succ a}$.
(2) Express $\bar{f}$ as a linear combination of elements in $\bar{\mathbb{B}}$, that is, $\bar{f} = \sum c_i \bar{b}_i$.
(3) If $f = \sum c_i b_i$ we are done. Otherwise replace f with $f - \sum c_i b_i \in F_{\mathfrak{v} \succ a}$ and go to (1).

Remark 2.6.10. Let $\bar{\mathbb{B}} \subset \mathrm{gr}_{\mathfrak{v}}(A)$ be a vector space basis. A lift $\mathbb{B} \subset A$ of $\bar{\mathbb{B}}$ is a vector space basis for A (and hence an adapted basis) if and only if the vector space subduction algorithm (Algorithm 2.6.9) terminates for all $f \in A$.

2.7 VALUATIONS FROM A PRIME CONE

Fix a total order $\succ$ on $\mathbb{Q}^s$. Given an $s \times n$ matrix $M \in \mathbb{Q}^{s\times n}$ we can define a partial order on the group $\mathbb{Q}^n$. Given $\alpha, \beta \in \mathbb{Q}^n$, we say that $\alpha \succ_M \beta$ if $M\alpha \succ M\beta$. Note that it is possible to have $\alpha \neq \beta$ and $M\alpha = M\beta$. Given this term order we define the notion of initial form of a polynomial $\tilde{f} = \sum c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[\mathbf{x}]$. Let $m(p) = \min\{M\alpha \mid c_\alpha \neq 0\}$ where the minimum is taken with respect to $\succ$. We define the initial term to be

$$in_M(f) = \sum_\beta c_\beta \mathbf{x}^\beta,$$

where the sum is over all β such that $M\beta = m$.

Recall that a quasivaluation $\mathfrak{v}$ is defined with the same axioms as a valuation except that $\mathfrak{v}(fg) \succeq \mathfrak{v}(f) + \mathfrak{v}(g)$. One can construct a quasivaluation $\tilde{\mathfrak{v}} : \mathbf{k}[\mathbf{x}] \setminus \{0\} \to \mathbb{Q}^s$ from a weight matrix $M \in \mathbb{Q}^{s\times n}$ [KM19, Section 3], by sending $\tilde{f} = \sum c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[\mathbf{x}]$ to

$$\tilde{\mathfrak{v}}_M(\tilde{f}) = \min\{M\alpha \mid c_\alpha \neq 0\},$$

where the min is with respect to the total order $\succ$ in $\mathbb{Q}^s$. Then we use $\pi : \mathbf{k}[\mathbf{x}] \to A$ and define the quasivaluation map $\mathfrak{v}_M : A \setminus \{0\} \to \mathbb{Q}^s$ by sending $f \in A$ to

$$\mathfrak{v}_M(f) = \pi_*(\tilde{\mathfrak{v}}_M)(f) = \max\{\tilde{\mathfrak{v}}_M(\tilde{f}) \mid \tilde{f} \in \mathbf{k}[\mathbf{x}], \pi(\tilde{f}) = f\}.$$

For an arbitrary weight matrix M, this defines a quasivaluation, however if the weight matrix is chosen appropriately then this construction will give us a valuation.

The above extends the theory of monomial weightings to weightings by $\mathbb{Q}^s$ where $\mathbb{Q}^s$ is equipped with a term order $\succ$. We then define a higher rank

Gröbner region $GR^s(I) \subset \mathbb{Q}^{s\times n}$ of an ideal $I \subset \mathbf{k}[\mathbf{x}]$ as follows.

Definition 2.7.1. (Gröbner region) A matrix $M \in \mathbb{Q}^{s\times n}$ is in the *Gröbner region* $GR^s(I)$ if there is some monomial order $>$ on $\mathbb{Q}^n$ such that the following holds:

$$in_{\succ}(in_M(I)) = in_{\succ}(I).$$

If $M \in GR^s(I)$ then the standard monomial basis for the cone C that contains M is an adapted basis with respect to $\mathfrak{v}_M$ [KM19, Proposition 3.3]. We also have that for an algebra $A \cong \mathbf{k}[\mathbf{x}]/I$ the associated graded algebra $\mathrm{gr}_{\mathfrak{v}}(A) \cong \mathbf{k}[\mathbf{x}]/in_M(I)$. Moreover, let $0 \neq f \in A$ and write $f = \Sigma c_\alpha b_\alpha$ as a linear combination of adapted basis elements $b_\alpha \in \mathbb{B}$, where b_α is the image of a standard monomial $\mathbf{x}^\alpha$. Then $\mathfrak{v}_M(f)$ can be computed by:

$$\mathfrak{v}_M(f) = \min\{M\alpha \mid c_\alpha \neq 0\}.$$

Finally, we will use a prime cone to construct a valuation map on an algebra $A \cong \mathbf{k}[\mathbf{x}]/I$. Given a prime cone $C \subset \mathrm{trop}(I)$ of dimension s. Let $\{u_1, \ldots, u_s\} \subset C$ be linearly independent vectors and let M be the matrix with the u_i as its rows. When M is constructed this way from a prime cone one shows that the quasivaluation is in fact a valuation, and has rank equal to $s = \mathrm{rank}(M)$ (see [KM19, Proposition 4.2]).

We conclude this section with the following example [KM19, p.8].

Example 2.7.2. Consider the algebra $A = \mathbf{k}[x,y,z]/\langle y^2z - x^3 + 7xz^2 - 2z^3\rangle$ the homogenous coordinate ring of the elliptic curve $E : y^2z - x^3 + 7xz^2 - 2z^3 = 0$. The tropical variety $\mathcal{T}$ of the affine cone over E, is the union of the three half-planes $\mathbb{Q}(1,1,1) + \mathbb{Q}_{\geq 0}(1,0,0), \mathbb{Q}(1,1,1) + \mathbb{Q}_{\geq 0}(0,1,0),$ and $\mathbb{Q}(1,1,1) +$

$\mathbb{Q}_{\geq 0}(-2,-3,0)$ with initial forms $zy^2 - 2z^3, x^3 + 7xz^2 - 2z^3$, and $y^2z - x^3$ respectively. The half-plane $\mathbb{Q}(1,1,1) + \mathbb{Q}_{\geq 0}(-2,-3,0)$ is the only prime cone and we can use it to construct a valuation with one dimensional leaves. We use the Gröbner Fan $GF(E)$ to compute an adapted basis for this valuation. We note that $\mathcal{T} \subset GF(E) = \mathbb{R}^3$, and we look at the codimension 1 cones of $GF(E)$ which contain the prime cone.

The codimension 1 cones $C_1 = \{(\omega_1, \omega_2, \omega_3) \mid 2\omega_2 + \omega_3 \geq \max\{3\omega_1, \omega_1 + 2\omega_3, 3\omega_3\}\}$ and $C_2 = \{(\omega_1, \omega_2, \omega_3) \mid 3\omega_1 \geq \max\{2\omega_2 + \omega_3, \omega_1 + 2\omega_3, 3\omega_3\}\}$, are the two cones that correspond to $in_\omega(E) = y^2z$ and $in_\omega(E) = -x^3$. We can choose either of these cones to give us a correct adapted basis. Consider the cone C_1 that corresponds to $in_\omega(E) = y^2z$. The adapted basis for E is then the set of standard monomials of $in_\omega(E)$, namely $\mathbb{B} = \{$monomials which do not lie in $in_\omega(E)\}$.

Let $\mathbb{B}$ be an adapted basis for A, and consider the map $\Psi : \mathbf{k}[\mathbf{x}] \to A$ that sends x^α to b_α. For $f = \sum c_\alpha b_\alpha \in A$ we have that $\mathfrak{v}(f) = \min\{M\alpha \mid c_\alpha \neq 0\}$ where M (constructed from the prime cone) is

$$M = \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & 0 \end{bmatrix}.$$

We then compute the following values of the valuation:

$$\mathfrak{v}(x^2 - y^2) = \min\left\{ \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\} =$$

$$\min\{(-2,-4),(-2,-6)\} = (-2,-6),$$

$$\mathfrak{v}(xy+yz) = \min\{ \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \} =$$

$$\min\{(-2,-5),(-2,-3)\} = (-2,-5),$$

$$\mathfrak{v}(z^2) = \min\{ \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \} = (-2,0).$$

Since the valuation comes from a prime cone, we have $\mathrm{gr}_{\mathfrak{v}}(A) \cong \mathbf{k}[x,y,z]/\langle y^2z - x^3 \rangle$ and the adapted basis $\mathbb{B} = \{x^{\alpha_1}z^{\beta_1}, x^{\alpha_2}yz^{\beta_2}, x^{\alpha_3}y^{\beta_3} \mid \alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}, \forall i\}$. We also use the computing Khovanskii basis algorithm (Algorithm 2.6.7) to see that a Khovanskii basis for our algebra A is $\mathcal{B} = \{x,y,z\}$.

$\square$

2.8 TROPICAL COMPACTIFICATIONS

Given an affine variety $Y \subset \mathbb{A}^n$ one can consider its closure $\tilde{Y} \subset \mathbb{P}^n$. It is a projective variety that contains Y as an open subset. For many purposes, such as doing intersection theory or resolution of singularities, $\tilde{Y}$ may not be the best compactification of Y to work with. When $Y \subset \mathbb{T} = (\mathbf{k}^*)^n$ is a subvariety of the algebraic torus one can construct suitable compactifications of Y by taking its closure in the toric varieties of T. Among such compactifications, there is a minimal one called the tropical compactification of Y and is constructed from a fan structure on the tropical variety $\mathrm{trop}(Y)$ of Y (note that trop(Y) does not have a unique fan structure). We record the following fact which we will refer to later. In section 5 we give a generalization of this in the context of Khovanskii bases and prime cones.

Theorem 2.8.1. *Fix a toric variety X_Σ with torus $\mathbb{T}$. Let Y be a subvariety of $\mathbb{T}$ and $\tilde{Y}$ its closure in X_Σ. For any $\sigma \in \Sigma$, we have $\tilde{Y} \cap \mathcal{O}_\sigma \neq \emptyset$ if and only if* trop(Y) *intersects the relative interior of the cone σ.*

Proof: The proof can be found in [MS15, p. 276].

□

This theorem tells us that given a subvariety $Y \subset \mathbb{T}$, its tropicalization trop(Y) gives us information about the closure of Y in any toric compactification of $\mathbb{T}$. The tropical compactification of Y is its closure $\tilde{Y}$ in a toric variety X_Σ where Σ is a fan whose support is trop(Y). It is a most economical way to compactify Y (see [Tev07] and [MS15, Section 6.3]).

3.0 KHOVANSKII-GRÖBNER BASES

When starting with a general algebra and domain Kaveh and Manon [KM19] give a criterion for it to have a finite Khovanskii basis and show how to compute it. We now wish to use Khovanskii bases for computations in such general algebras and more importantly develop a Gröbner theory for them. We start by constructing a division algorithm (Algorithm 3.0.2); using the division algorithm we construct what we call a Khovanskii-Gröbner basis for an ideal J in an algebra and domain A, along with a corresponding Buchberger criterion and Buchberger algorithm. Throughout this section we assume that the value semigroup is maximum well-ordered and that our algebra A and its associated graded $\mathrm{gr}_{\mathfrak{v}}(A)$ are finitely generated.

Remark 3.0.1. Since the adapted basis is a vector space basis, we can decompose our algebra $A = \bigoplus_a V_a$ where $V_a = \mathrm{span}\{b \in \mathbb{B} \mid \mathfrak{v}(b) = a\}$. We call $\bigoplus_a V_a$ the *direct sum decomposition adapted to the valuation* $\mathfrak{v}$. Note that $V_a \cong F_{\mathfrak{v}\succeq a}/F_{\mathfrak{v}\succ a} \subset \mathrm{gr}_{\mathfrak{v}}(A)$, and if we have a valuation with one dimensional leaves then V_a is at most 1-dimensional.

The following is a generalized division algorithm (Algorithm 3.0.2) that can be applied to a general algebra and domain A with a finite Khovanskii basis. Similar to the original division algorithm in a polynomial ring (Algorithm 2.1.3), the remainder may not be unique depending on the set of elements $G \subset A$.

Algorithm 3.0.2. (Division Algorithm)
Input: $f \in A$, a finite set $G = \{g_1, \ldots, g_m\} \subset A$, a valuation on A with a finite Khovanskii basis.
Output: $f = c_1 g_1 + \cdots + c_m g_m + r$ where $c_i \in A$ and $r = \sum_a r_a$ with $r_a \in V_a$ such that $\bar{r}_a \notin \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$, and $f \in \langle G \rangle$ if $r = 0$.

(1) If $f = 0$ set $r := 0$ and Terminate.
If $f \neq 0$, write $f = v_{a_1} + \cdots + v_{a_\alpha}$ where $0 \neq v_{a_i} \in V_{a_i}$ and $a_1 < \cdots < a_\alpha$.
Set $i = 1$.

(2) If $\bar{v}_{a_i} \in \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$,

(a) Write $\bar{v}_{a_i} = \sum_j \bar{d}_j \bar{h}_j$ with $\bar{h}_j \in \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$.

(b) For each j find $h_j \in \langle G \rangle$ such that the image of h_j in $\mathrm{gr}_{\mathfrak{v}}(A)$ is $\bar{h}_j$.

(c) Write $h_j = \sum_l k_{lj} g_l$.
Set $c_l := c_l + \sum_j d_j k_{lj}$.
$f := f - \sum_j d_j h_j$.
Go to Step (1).

(3) If $\bar{v}_{a_i} \notin \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$,
Set $i := i + 1$.
If $i > \alpha$ set $r := f$ and Terminate.
Else go to step (2).

Corollary 3.0.3. *The division algorithm (Algorithm 3.0.2) terminates in a finite number of steps if the term order on value semigroup $S(A, \mathfrak{v})$ is maximum well-ordered.*

Proof: Note that the function $f = v_{a_1} + v_{a_2} + \ldots$ only changes at step (2.c). If $i = 1$ at this change then we have $\bar{f} = \bar{v}_{a_1} = \sum \bar{c}_j \bar{h}_j$. Then by construction, for $h = f - \sum c_j h_j = v_{b_1} + v_{b_2} + \ldots$, we have that $\mathfrak{v}(f) < \mathfrak{v}(h)$, thus $\mathfrak{v}(v_{a_1}) < \mathfrak{v}(v_{b_1})$. Since the value semigroup is maximum well-ordered, then this chain of inequalities must stabilize in a finite number of steps and give us a v_{a_1} with maximal valuation. Note that in this algorithm $\bar{v}_{a_1}$ will not change once $i > 1$, thus when v_{a_1} stabilizes at its maximal valuation, we will never get $\bar{v}_{a_1} \in \langle \bar{g}_1, ..., \bar{g}_m \rangle$ again. Similarly each v_{a_i} will eventually stabilize at a maximal valuation at which time $\bar{v}_{a_i}$ will no longer change. If we consider v_{a_i} to be the stabilized element for all i, then since the value semigroup is maximum well-ordered, then the valuations of the set $\{f, f - v_{a_1}, f - v_{a_1} - v_{a_2}, \ldots\}$ must have a maximal element and obtain its maximal element in a finite number of steps. This maximal element must be 0, for if $g = v_{c_1} + v_{c_2} + \cdots \neq 0$ was the maximal element, then $g - v_{c_1}$ would be in the list with $\mathfrak{v}(g) < \mathfrak{v}(g - v_{c_1})$, a contradiction.

$\square$

Example 3.0.4. Using $A = \mathbf{k}[x, y, z]/\langle y^2 z - x^3 + 7xz^2 - 2z^3 \rangle$ with the same valuation map and adapted basis as in Example 2.7.2. Consider $f = 2x^3 - 7xz^2 + 2z^3 \in A$ and $G = \{xy + yz, z^2, -y^2 + x^2\}$.

Since the valuation map has one dimensional leaves we can write $v_{a_1} = 2x^3$, $v_{a_2} = -7xz^2$, and $v_{a_3} = 2z^3$. Then we see that $\bar{G} = \{-\bar{y}^2, \bar{x}\bar{y}, \bar{z}^2\}$.

Since $\bar{v}_{a_1} = 2\bar{x}^3 \notin \langle -\bar{y}^2, \bar{x}\bar{y}, \bar{z}^2 \rangle$ then v_{a_1} will not change and it is at its highest valuation, hence $\bar{v}_{a_1}$ will no longer change.

Then consider v_{a_2} and we have that $\bar{v}_{a_2} = -7\bar{x}\bar{z}^2 \in \langle -\bar{y}^2, \bar{x}\bar{y}, \bar{z}^2 \rangle$ and $\bar{v}_{a_2} = -7\bar{x}\bar{z}^2 = \bar{z}^2(-7\bar{x})$. Then choose $h = -7x(z^2)$ and set $c_2 := -7x$ and $f := 2x^3 + 2z^3$.

This makes $v_{a_2} := 2z^3$ and once again we have $\bar{v}_{a_2} = 2\bar{z}^3 \in \langle -\bar{y}^2, \bar{x}\bar{y}, \bar{z}^2 \rangle$. Then $\bar{v}_{a_2} = 2\bar{z}^3 = 2\bar{z}(\bar{z}^2)$, and choose $h = 2z(z^2)$ and set $c_2 := -7x + 2z$ and $f := 2x^3$.

Finally we set $r := f = 2x^3$ and get a final answer of $f = (-7x+2z)z^2+2x^3$.

□

The existence of Gröbner bases solves the problem of having a unique remainder when working in a polynomial ring. Therefore we want to define a similar notion in the context of a general algebra with a valuation map. The following definition gives such a general notion of a Gröbner basis which we will be calling a Khovanskii-Gröbner basis.

Definition 3.0.5. (Khovanskii-Gröbner Basis) Let $J \subset A$ be an ideal of A. We say that $G = \{g_1, \ldots, g_m\} \subset J$ is a *Khovanskii-Gröbner basis* for J if

$$\langle \bar{G} \rangle = \langle \bar{g}_1, \ldots, \bar{g}_m \rangle = \langle \bar{J} \rangle = \langle \bar{j} \mid j \in J \rangle$$

where $\bar{j}$ is the image of j in $\mathrm{gr}_{\mathfrak{v}}(A)$.

Proposition 3.0.6. *If G is a Khovanskii-Gröbner basis and the valuation has one dimensional leaves, then the remainder term r in the division algorithm (Algorithm 3.0.2) is unique. In particular, $r = 0$ if and only if $f \in \langle G \rangle$.*

Proof: Let $G = \{g_1, \ldots, g_m\}$ be a Khovanskii-Gröbner basis for A. Consider $f \in A$, and assume that the division algorithm gives us $f = \sum b_i g_i + r_1 = \sum c_i g_i + r_2$. Then consider $r = r_1 - r_2 = \sum c_i g_i - \sum b_i g_i \in \langle G \rangle$, thus, $\bar{r} \in \langle \bar{G} \rangle = \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$ since G is a Khovanskii-Gröbner basis. Let $r = \underset{a}{\Sigma} r_a$ where $r_a \in V_a$, and $\bar{r} = \bar{r}_b$. Let $r_i = \underset{a_i}{\Sigma} r_{a_i}$ for $i = 1, 2$, and note that $r_b = r_{\alpha_1} - r_{\beta_2}$ where $\mathfrak{v}(r_b) = \mathfrak{v}(r_{\alpha_1}) = \mathfrak{v}(r_{\beta_2})$. Since the valuation has one dimensional leaves then we have that $r_{\alpha_1} \in \langle r_b \rangle$, therefore $r_{\alpha_1} = c r_b$ for some $c \in \mathbf{k}$. By construction of the remainder we have that $\bar{r}_{\alpha_1} = c\bar{r}_b = c\bar{r} \notin \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$, a contradiction unless $\bar{r} = 0$. Using the fact that $\mathfrak{v}(\bar{r}) = \mathfrak{v}(0) = \infty$, we have that $r = r_1 - r_2 = 0$.

$\square$

Since we have a unique remainder (Proposition 3.0.6), then the division algorithm and a Khovanskii-Gröbner basis gives a solution to the ideal membership problem for an ideal J in a general algebra A with a finite Khovanskii basis. The following code in Macaulay2 determines if a function $f \in A \cong \mathbf{k}[x_1, \ldots, x_n]/\langle h_1, \ldots, h_r \rangle$ is a member of an ideal J with Khovanskii-Gröbner basis $G = \{g_1, \ldots, g_m\}$ and valuation on A determined by a weight matrix M.

```
w_1 = -1* (row 1 of matrix M)

.

.

.

w_s = -1*(last row of matrix M)


IMP = (R,C,G,f) -> while true do (
    I:=ideal C;
```

```
A:=R/I;
Gbar:=ideal(G/leadTerm);
r:=f%Gbar;
if r == 0
then (
    print true;
    break)
else (
    print false;
    break
    )
)
```

The following will give us a true result if $f \in J$ and a false result if $f \notin J$.

```
IMP(QQ[x_1,...,x_n,MonomialOrder=>
     {Weights=>w_1,...,Weights=> w_s}],
           {h_1,...,h_r},{g_1,...,g_m},f)
```

Remark 3.0.7. $w_1, \ldots, w_s$ in the Macaulay2 code are defined by multiplying the rows by -1 because Macaulay2 uses the maximum convention instead of the minimum convention used in this paper.

Similar to the usual Gröbner bases in a polynomial ring, the Khovanskii-Gröbner bases computed with the analogous Buchberger algorithm (Algorithm 3.0.23) are often larger than necessary. Thus it is useful to have notations of a minimal and reduced Khovanskii-Gröbner basis.

Definition 3.0.8. (Minimal Khovanskii-Gröbner Basis) Let $J \subset A$ be an ideal and $G \subset J$ a Khovanskii-Gröbner basis for J. We say that G is a *minimal Khovanskii-Gröbner basis* if for any element $g \in G$, $\bar{g} \notin \langle \bar{G} \setminus \{\bar{g}\} \rangle$.

Definition 3.0.9. (Reduced Khovanskii-Gröbner Basis) Let $J \subset A = \bigoplus V_a$ be an ideal, and $G \subset J$ a Khovanskii-Gröbner basis for J. We say that a minimal Khovanskii-Gröbner basis G is a *reduced Khovanskii-Gröbner basis* if for any element $g \in G$, with $g = \sum_a v_a$ and $v_a \in V_a$, $\bar{v}_a \notin \langle \bar{G} \setminus \{\bar{g}\} \rangle$ for all a.

Proposition 3.0.10. *For every ideal $J \subset A$ there exists a reduced Khovanskii-Gröbner basis G for J if the value semigroup $S(A, \mathfrak{v})$ is maximum well-ordered.*

Proof: Recall that by assumption $\mathrm{gr}_{\mathfrak{v}}(A)$ is finitely generated and hence Noetherian. Suppose $\bar{J} \subset \mathrm{gr}_{\mathfrak{v}}(A)$ is generated by $\{\bar{k}_1, \ldots, \bar{k}_m\}$. For each $1 \leq i \leq m$, let $g_i \in J$ such that the image of g_i in $\mathrm{gr}_{\mathfrak{v}}(A)$ is $\bar{k}_i$. Then the set $\{g_1, \ldots, g_m\}$ is a Khovanskii-Gröbner basis for J. Using reduced Khovanskii-Gröbner basis algorithm (Algorithm 3.0.11) the Khovanskii-Gröbner basis $\{g_1, \ldots, g_m\}$ can be turned into a reduced Khovanskii-Gröbner basis.

□

Algorithm 3.0.11. (Reduced Khovanskii-Gröbner basis Algorithm)
Input: A Khovanskii-Gröbner basis G for an ideal $J \subset A$, with $A = \bigoplus V_a$.
Output: A reduced Khovanskii-Gröbner basis G for J.

(1) Write $G = \{g_1, \dots, g_m\}$.

Set $i = 1$.

(2) If $i \leq m$,

(a) If $\bar{g}_i \in \langle \bar{G} \setminus \{\bar{g}_i\} \rangle$,

Set $G := G \setminus \{g_i\}$.

Go to step (1).

(b) Else set $i := i + 1$.

Go to step (2).

(3) If $i > n$ set $j := 1$,

(4) If $j \leq n$ compute $g_j \xrightarrow{G \setminus \{g_j\}} g'_j$,

Set $G := (G \setminus \{g_j\}) \cup \{g'_j\}$.

Set $j := j + 1$.

Go to step (4).

(5) If $j > m$, Print G.

Corollary 3.0.12. *The reduced Khovanskii-Gröbner basis algorithm (Algorithm 3.0.11) produces a reduced Khovanskii-Gröbner basis and terminates in a finite number of steps if the value semigroup $S(A, \mathfrak{v})$ is maximum well-ordered.*

Proof: Since G is a finite set, and we never add any elements to G, while $i \leq m$ we can only take away a finite number of elements in step (2). This will give us a minimal Khovanskii-Gröbner basis by construction.

We claim that once an element g_j is reduced in a minimal Khovanskii-Gröbner basis $G = \{g_1, \dots, g_m\}$ it will remain reduced for any other minimal Khovanskii-Gröbner basis with the same $\{\bar{g}_1, \dots, \bar{g}_m\}$. The proof is as follows:

Let $G = \{g_1, \ldots, g_m\}$ be a minimal Khovanskii-Gröbner basis. Let $g_j \xrightarrow{G \setminus \{g_j\}} g'_j$. Note that $\bar{g}_j = \bar{g}'_j$, since if they did not equal, G would not be minimal. Thus $\langle \bar{g}_1, \ldots, \bar{g}_j, \ldots, \bar{g}_m \rangle = \langle \bar{g}_1, \ldots, \bar{g}'_j, \ldots, \bar{g}_m \rangle$, and $G' = (G \setminus \{g_j\}) \cup \{g'_j\}$ is a Khovanskii-Gröbner basis. The element g'_j is reduced in G' by the construction of the division algorithm (Algorithm 3.0.2), since the division algorithm only uses $\bar{G}$.

□

Remark 3.0.13. If the value semigroup is not maximum well-ordered then the division algorithm (Algorithm 3.0.2) may not terminate. Note in the above algorithm (Algorithm 3.0.11) when applying the division algorithm to reduce the elements of the Khovanskii-Gröbner basis, one element is removed from the Khovanskii-Gröbner basis, thus we are not guaranteed a unique remainder; hence, a reduced Khovanskii-Gröbner basis is not necessarily unique.

Example 3.0.14. We use the same A, valuation map, Khovanskii basis and adapted basis as in Example 2.7.2. In Example 3.0.24 below we will show that

$$G = \{2x^3, -x^2z, -y^2 + x^2, xy + yz, z^2\}$$

is a Khovanskii-Gröbner basis. Note that for each element $g \in G$, $g \xrightarrow{G \setminus \{g\}} g$. Thus, G is already a reduced Khovanskii-Gröbner basis.

□

Note that the reduced Khovanskii-Gröbner basis algorithm (Algorithm 3.0.11) first turns our Khovanskii-Gröbner basis into a minimal Khovanskii-Gröbner basis. Thus to compute a minimal Khovanskii-Gröbner basis we only need to apply the first half of the reduced Khovanskii-Gröbner basis algorithm. The following propositions tell us when these are minimal and reduced Khovanskii-Gröbner bases are unique and what restrictions are needed.

Proposition 3.0.15. *Suppose an algebra A, valuation map $\mathfrak{v}$ with one dimensional leaves, and an adapted basis $\mathbb{B}$ are given. Let $J \subset A$ be an ideal, then $\bar{G}$ for a minimal Khovanskii-Gröbner basis G of J is unique up to multiplication by a unit.*

Proof: Let $G \neq G'$ be minimal Khovanskii-Gröbner bases for J. Let $g_1 \in G \setminus G'$, since G' is a Gröbner basis then $\bar{g}_1 \in \langle \bar{G}' \rangle$. Thus we have that $\bar{g}_1 = \sum \bar{h}_i \bar{g}'_i$ were $\bar{g}'_i$ is the image of $g'_i \in G'$ in $\mathrm{gr}_{\mathfrak{v}}(A)$. Since our valuation has one dimensional leaves, then $\exists i$ such that $\bar{g}_1 = \bar{h}_i \bar{g}'_i$. Since G is minimal we know that $g'_i \notin G$. Similarly we know that $\exists g_j \in G$ such that $\bar{g}'_i = \bar{k}_j \bar{g}_j$. Then $\bar{g}_1 = \bar{h}_i \bar{k}_j \bar{g}_j$, since G is minimal then we know that $\bar{g}_1 = \bar{g}_j$. Thus, $\bar{h}_i$ and $\bar{k}_j$ are units and $\bar{G} = \bar{G}'$ up to multiplication by a unit.

$\square$

We introduce the following alternative definition for a reduced Khovanskii-Gröbner basis when $S(A, \mathfrak{v}) \subset \mathbb{Z}^s_{\leq 0}$. Notice that the two definitions (Definition 3.0.9 and Definition 3.0.16) are identical when $S(A, \mathfrak{v}) \subset \mathbb{Z}^s_{\leq 0}$.

Definition 3.0.16. (Reduced Khovanskii-Gröbner basis - Negative Orthant Case)

Let $J \subset A = \bigoplus V_a$ be an ideal, $G \subset J$ a Khovanskii-Gröbner basis for J, and $\mathfrak{v}(A \setminus \{0\}) \subset \mathbb{Z}^s_{\leq 0}$, then a minimal Khovanskii-Gröbner basis G is a *reduced*

Khovanskii-Gröbner basis if for any element $g \in G$, with $g = \sum v_a$, $v_a \in V_a$, we have $\bar{v}_a \notin \langle \bar{G} \rangle \; \forall a \neq \mathfrak{v}(g)$.

Remark 3.0.17. To see why we require $S(A, \mathfrak{v}) \subset \mathbb{Z}^s_{\leq 0}$ consider the example where $g = x + x^2 \in \mathbf{k}[x] = A$ with the valuation map $\mathfrak{v}(x^\alpha) = \alpha$. It is clear that $S(A, \mathfrak{v}) \subset \mathbb{Z}^s_{\geq 0}$ where $s = 1$. Then $\mathfrak{v}(g) = 1$ and $\bar{g} = \bar{x}$, thus we have that $\bar{x}^2 \in \langle \bar{g} \rangle$. Therefore, if we did not require $\mathfrak{v}(A \setminus \{0\}) \subset \mathbb{Z}^s_{\leq 0}$ the reduced Khovanskii-Gröbner basis for $\langle g \rangle$ is $\langle x \rangle$, but we clearly have that $\langle g \rangle \subsetneq \langle x \rangle$, a contradiction.

Proposition 3.0.18. *Suppose an algebra A, valuation map $\mathfrak{v}$ with one dimensional leaves such that $\mathfrak{v}(A \setminus \{0\}) \subset \mathbb{Z}^s_{\leq 0}$, and adapted basis $\mathbb{B}$ are given. Let $J \subset A$ be an ideal, then the reduced Khovanskii-Gröbner basis G of J is unique up to scalar multiplication.*

Proof: Let $G \neq G'$ be reduced Khovanskii-Gröbner basis for $J \subset A$. Since we know that both G and G' are minimal, then we have that $\bar{G}$ is equal to $\bar{G}'$ up to units. Let $\bar{g} = \alpha \bar{g}'$ where α is a unit, and consider $g - \alpha g' \in J$. Assume $g - \alpha g' \neq 0$, since the valuation has one-dimensional leaves, $\overline{g - \alpha g'}$ is equal to an element $\bar{g}_b$ which is the image of an element of g or $\alpha g'$ in $\mathrm{gr}_{\mathfrak{v}}(A)$. Without loss of generality, let $\overline{g - \alpha g'} = \bar{g}_b$ be the image of an element of g. Then $g = \sum_a g_a + g_b$ with $g_a \in V_a$ and $g_b \in V_b$. Since G is reduce then $\bar{g}_b \notin \langle \bar{G} \rangle$ and clearly $g_b \in \langle G \rangle$, thus $\bar{g}_b = \overline{g - \alpha g'} = 0$. Using the fact that $\mathfrak{v}(0) = \infty$, we have that $g - \alpha g' = 0$.

□

We now look at how to compute a Khovanskii-Gröbner basis using the analog of the Buchberger criterion (Theorem 3.0.20) and the analog to the Buchberger algorithm (Algorithm 3.0.23). Following [Mil96] we introduce the following notation. Let A be an algebra and let $G = \{g_1, \ldots, g_m\} \subset A$ and $f = a_1 g_1 + \cdots + a_m g_m \in \langle g_1, \ldots, g_m \rangle$, then define

$$\mathrm{ht}(f)_G = \min_i \{\mathfrak{v}(a_i g_i)\}.$$

This depends on the choice of representation of f as $\sum\limits_{i=1}^{m} a_i g_i$, even though it is not reflected in the notation. We will drop the G when it is obvious.

Definition 3.0.19. (Syzygy) A *syzygy* on a finite subset $\bar{G}$ of a graded ring R is an m-tuple $\bar{\mathbf{h}} = (\bar{h}_1, \ldots, \bar{h}_m) \in R^m$ such that $\sum\limits_{i=1}^{m} \bar{h}_i \bar{g}_i = 0$. Define $\mathrm{Syz}(\bar{G}) \subseteq R^m$ to be the set of all syzygies on $\bar{G}$.

For a generating set H for $\mathrm{Syz}(\bar{G})$ we can make it so that every element $(\bar{h}_1, \ldots, \bar{h}_m) \in H$ is such that $\mathfrak{v}(\bar{h}_i \bar{g}_i) = \mathfrak{v}(\bar{h}_j \bar{g}_j)$, $\forall i, j$ with $\bar{h}_i, \bar{h}_j \neq 0$. Syzygies will be used in the same way as the S-polynomials are used in the usual Gröbner theory in polynomial rings.

Theorem 3.0.20. *(Buchberger Criterion) Let $G = \{g_1, \ldots, g_m\} \subset A$ and $H \subset A^m = \bigoplus\limits_{i=1}^{m} A$ such that $\{(\bar{h}_1, \ldots, \bar{h}_m) \mid (h_1, \ldots, h_m) \in H\}$ generates the $\mathrm{gr}_{\mathfrak{v}}(A)$-module $\mathrm{Syz}(\bar{G})$. Then G is a Khovanskii-Gröbner basis for $\langle G \rangle$ with respect to the valuation $\mathfrak{v}$ if and only if $\forall \mathbf{h} \in H$, $h_1 g_1 + \cdots + h_m g_m \xrightarrow{G} 0$.*

Proof: Let G be a Khovanskii-Gröbner basis for $\langle G \rangle$. Then for $\mathbf{h} \in H$ consider $h_1 g_1 + \cdots + h_m g_m = f \in \langle G \rangle$. Since G is a Khovanskii-Gröbner basis we know that $\bar{f} \in \langle \bar{G} \rangle$, thus we have $\bar{f} = \sum \bar{a}_i \bar{g}_i$. Let $f_1 = f - \sum a_i g_i \in \langle G \rangle$. Clearly, $\mathfrak{v}(f) < \mathfrak{v}(f_1)$. If $f_1 \neq 0$ then $\bar{f}_1 \in \langle \bar{G} \rangle$ since G is a Khovanskii-Gröbner basis. Thus $\bar{f}_1 = \sum \bar{b}_i \bar{g}_i$ and we can consider $f_2 = f_1 - \sum b_i g_i$. If we continue this process, since the value semigroup is maximum well-ordered and $\mathfrak{v}(f_j) < \mathfrak{v}(f_{j+1})$, we must hit 0 eventually, therefore $f \xrightarrow{G} 0$.

Conversely, suppose the conditions in the theorem hold, we would like to show G is a Khovanskii-Göbner basis. Let $f \in \langle G \rangle$, we want to show that $\bar{f} \in \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$. We can write $f = a_1 g_1 + \cdots + a_m g_m$, with $a_i \in A$, such that $\mathrm{ht}(f) = t_0$ is maximal with respect to all representations of f as a combination of elements of G. Note that by the definition of a valuation map (Definition 2.3.1), $\mathfrak{v}(f) \geq \mathrm{ht}(f) = t_0$. Assume $\mathfrak{v}(f) > t_0$. Reorder G such that $\mathfrak{v}(a_1 g_1) = \cdots = \mathfrak{v}(a_\gamma g_\gamma) = t_0$ and define $\mathbf{a} = (a_1, a_2, \ldots, a_\gamma, 0, \ldots, 0)$. Note that $\bar{a}_1 \bar{g}_1 + \cdots + \bar{a}_\gamma \bar{g}_\gamma = 0$, thus $\bar{\mathbf{a}} \in \mathrm{Syz}(\bar{G})$. If this was not the case then $\bar{f} = \bar{a}_1 \bar{g}_1 + \cdots + \bar{a}_\gamma \bar{g}_\gamma$ and $\mathfrak{v}(f) = t_0$. Therefore there exists $c_1, \ldots, c_\beta \in A$ and $\mathbf{h_1}, \ldots, \mathbf{h}_\beta \in H$ such that $\bar{\mathbf{a}} = \sum\limits_{j=1}^{\beta} \bar{c}_j \bar{\mathbf{h}}_{\mathbf{j}}$ with $\bar{\mathbf{h}}_{\mathbf{j}} = (\bar{h}_{j1}, \ldots, \bar{h}_{jm})$. Since $\bar{a}_i = \sum\limits_j \bar{c}_j \bar{h}_{ji} \in \mathrm{gr}_{\mathfrak{v}}(A)$ is a homogeneous element we have $\mathfrak{v}(\sum\limits_j \bar{c}_j \bar{h}_{ji}) = \mathfrak{v}(\bar{a}_i)\ \forall i$. We know by the definition of a generating set of $\mathrm{Syz}(\bar{G})$ we have $\mathfrak{v}(\bar{c}_j \bar{h}_{ji} \bar{g}_i) = \mathfrak{v}(\bar{a}_i \bar{g}_i) = t_0\ \forall i, j$.

Rewrite

$$
\begin{aligned}
f &= \sum_{i=1}^{m} a_i g_i - \sum_{i=1}^{m}\sum_{j=1}^{\beta} c_j h_{ji} g_i + \sum_{j=1}^{\beta}\sum_{i=1}^{m} c_j h_{ji} g_i \\
&= \sum_{i=1}^{m} a_i g_i - \sum_{i=1}^{m}(\sum_{j=1}^{\beta} c_j h_{ji}) g_i + \sum_{j=1}^{\beta} c_j (\sum_{i=1}^{m} h_{ji} g_i) \\
&= \sum_{i=1}^{m}(a_i - \sum_{j=1}^{\beta} c_j h_{ji}) g_i + \sum_{j=1}^{\beta} c_j (\sum_{i=1}^{m} k_{ji} g_i)
\end{aligned}
$$

where $\sum\limits_{i=1}^{m} k_{ji} g_i$ is a G-representation for $\sum\limits_{i=1}^{m} h_{ji} g_i$, such that $\text{ht}(\sum\limits_{i=1}^{m} k_{ij} g_i) = \mathfrak{v}(\sum\limits_{i=1}^{m} k_{ij} g_i)$. This is possible because by assumption $\sum\limits_{i} h_{ji} g_i \xrightarrow{G} 0$. Let $\mathfrak{v}(\sum\limits_{i=1}^{m} k_{ji} g_i) = t_j$. Since $(\bar{h}_{1j}, \ldots, \bar{h}_{mj}) \in \text{Syz}(\bar{G})$, we know $t_j = \mathfrak{v}(\sum\limits_{i=1}^{m} k_{ji} g_i) = \mathfrak{v}(\sum\limits_{i=1}^{m} h_{ji} g_i) > \text{ht}(\sum\limits_{i=1}^{m} h_{ji} g_i)\ \forall j$.

Consider $\sum\limits_{j=1}^{\beta} c_j (\sum\limits_{i=1}^{m} k_{ji} g_i) = \sum\limits_{i=1}^{m}(\sum\limits_{j=1}^{\beta} c_j k_{ji}) g_i$,

$$
\begin{aligned}
\text{ht}(\sum_{j=1}^{\beta} c_j (\sum_{i=1}^{m} k_{ji} g_i)) &= \min_i\{\mathfrak{v}((\sum_{j=1}^{\beta} c_j k_{ji}) g_i)\} &&> \min_{i,j}\{\mathfrak{v}(c_j k_{ji} g_i)\} \\
= \min_{i,j}\{\mathfrak{v}(c_j) + \mathfrak{v}(k_{ji} g_i)\} &= \min_j\{\mathfrak{v}(c_j) + t_j\} &&> \min_{i,j}\{\mathfrak{v}(c_j) + \mathfrak{v}(h_{ji} g_i)\} \\
&= \min_{i,j}\{\mathfrak{v}(c_j h_{ji} g_i)\} &&= \min_i\{\mathfrak{v}(a_i g_i)\} = t_0.
\end{aligned}
$$

Consider $\sum\limits_{i=1}^{m}(a_i - \sum\limits_{j=1}^{\beta} c_j h_{ji}) g_i$. For $i \le n$, $\bar{a}_i = \sum\limits_{j=1}^{\beta} \bar{c}_j \bar{h}_{ji}$ thus, $\mathfrak{v}(a_i - \sum\limits_{j=1}^{\beta} c_j h_{ji}) > \mathfrak{v}(a_i)$, which implies that $\mathfrak{v}((a_i - \sum\limits_{j=1}^{\beta} c_j h_{ji}) g_i) > \mathfrak{v}(a_i g_i) = t_0$. For $i > n$ we have $\mathfrak{v}(a_i g_i) > t_0$ and $\sum\limits_{j=1}^{\beta} c_j h_{ji} = 0$. Therefore, $\text{ht}(\sum\limits_{i=1}^{m}(a_i - \sum\limits_{j=1}^{\beta} c_j h_{ji}) g_i) > t_0$. Thus, $f = \sum\limits_{i=1}^{m}(a_i - \sum\limits_{j=1}^{\beta} c_j h_{ji}) g_i + \sum\limits_{j=1}^{\beta} c_j (\sum\limits_{i=1}^{m} k_{ji} g_i)$ is a representation for f, with $\text{ht}(f) > t_0$, a contradiction to t_0 being maximal.

Therefore, $\mathfrak{v}(f) = t_0$. Let $N = \{i \mid \mathfrak{v}(a_i g_i) = t_0\} \subset \{1, \ldots, m\}$. Then $\bar{f} = \sum\limits_{i \in N} \bar{a}_i \bar{g}_i \in \langle \bar{g}_1, \ldots, \bar{g}_m \rangle$.

□

In preparation for the analogue of the Buchberger algorithm (Algorithm 3.0.23), we first need to compute a set of generators for the syzygies of the associated graded algebra (Algorithm 3.0.21). This algorithm is written in such a way that we can compute the syzygies for any graded ring R. First consider the natural homomorphism $\Phi : \mathbf{k}[\mathbf{x}] \to R$ with $\Phi(x_i) = \bar{b}_i$, where $\bar{\mathcal{B}} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a set of algebra generators for R and let $I_B = \ker(\Phi)$. Using the subduction algorithm (Algorithm 2.6.6) we can write any element $f \in R$ as a polynomial in the algebra generators.

Algorithm 3.0.21. Computing the syzygies for a graded ring R.
Input: $\bar{S} = (\bar{s}_1, \ldots, \bar{s}_m) \in R^m$, with $\bar{s}_i$ a homogeneous elements of R with respect to the grading, and a set of algebra generators $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ for R.
Output: A generating set L for $\mathrm{Syz}(\bar{S})$.

(1) Set $L := \emptyset$.
(2) For each $\bar{s}_i$, write $\bar{s}_i = p_i(\bar{b}_1, \ldots, \bar{b}_n)$.
(3) Let $f_i = p_i(x_1, \ldots, x_n)$, $\forall i$ in the polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$, compute a generating set N for $\mathrm{Syz}(f_1, \ldots, f_m)$.[1]
(4) For each $\mathbf{n} = (n_1, \ldots, n_m) \in N$, set $L := L \cup \{(\Phi(n_1), \ldots, \Phi(n_m))\}$.
(5) Compute a generating set G for $\langle f_1, \ldots, f_r \rangle \cap I_B$.
(6) For each $g \in G$ do:

[1]For computing the generating set for syzygies in a polynomial ring, if the f_i's are monomials then it is generated by the *Koszul syzygies*: $f_i f_j - f_j f_i$.

(a) Write $g = \sum\limits_{i=1}^{m} h_i f_i$.
(b) Set $L := L \cup \{(\Phi(h_1), \ldots, \Phi(h_m))\}$.

Proof: We will show that any element of $\mathrm{Syz}(\bar{s}_1, \ldots, \bar{s}_m)$ can be generated by S. Let $\mathbf{Q} = (q_1, \ldots, q_m) \in \mathrm{Syz}(\bar{s}_1, \ldots, \bar{s}_m) \subset R^m$, thus $q_1\bar{s}_1 + \cdots + q_m\bar{s}_m = 0$. For each $\bar{s}_i = p_i(\bar{b}_1, \ldots, \bar{b}_n)$, let $f_i = p_i(x_1, \ldots, x_n) \in \Phi^{-1}(\bar{s}_i)$ and pick $k_i \in \Phi^{-1}(q_i)$. Consider $F = \sum\limits_{i=1}^{m} k_i f_i$, note that since Φ is a homomorphism, $\Phi(F) = q_1\bar{s}_1 + \cdots + q_m\bar{s}_m = 0$.

If $F = 0$ then $(k_1, \ldots, k_m) \in \mathrm{Syz}(f_1, \ldots, f_m)$, thus we can find $\mathbf{n_1}, \ldots, \mathbf{n}_\alpha \in N \subset (\mathbf{k}[\mathbf{x}])^m$ such that $p_1\mathbf{n_1} + \cdots + p_\alpha\mathbf{n}_\alpha = (k_1, \ldots, k_m)$. Therefore,

$$\mathbf{Q} = \Phi(p_1)(\Phi(n_{11}), \ldots, \Phi(n_{1m})) + \cdots + \Phi(p_\alpha)(\Phi(n_{\alpha 1}), \ldots, \Phi(n_{\alpha m}))$$

and $(\Phi(n_{j1}), \ldots, \Phi(n_{jm})) \in L\ \forall j \in \{1, \ldots, \alpha\}$.

If $F \neq 0$, note that since $\Phi(F) = 0$ then $F \in \langle f_1, \ldots, f_m\rangle \cap I_B$. Then since we have a generating set $G = \{g_1, \ldots, g_\beta\}$ for $\langle f_1, \ldots, f_m\rangle \cap I_B$, then $F \in \langle g_1, \ldots, g_\beta\rangle$. Thus we can say that

$$F = p_1 g_1 + \cdots + p_\beta g_\beta = \sum_{i=1}^{m} p_1 h_{1i} f_i + \cdots + \sum_{i=1}^{m} p_\beta h_{\beta i} f_i =$$

$$\sum_{j=1}^{\beta}\sum_{i=1}^{m} p_j h_{ji} f_i = \sum_{i=1}^{m}(\sum_{j=1}^{\beta} p_j h_{ji}) f_i.$$

Note that $\sum\limits_{i=1}^{m}(k_i - \sum\limits_{j=1}^{\beta} p_j h_{ji}) f_i = 0$ which implies that $(k_1 - \sum\limits_{j=1}^{\beta} p_j h_{j1}, \ldots, k_m - \sum\limits_{j=1}^{\beta} p_j h_{jm}) \in \mathrm{Syz}(f_1, \ldots, f_m)$. Let $k_i - \sum\limits_{j=1}^{\beta} p_j h_{ji} = n_i$. Then

$$\mathbf{Q} = (\Phi(k_1), \ldots, \Phi(k_m)) = (\Phi(n_1 + \sum_{j=1}^{\beta} p_j h_{j1}), \ldots, \Phi(n_m + \sum_{j=1}^{\beta} p_j h_{jm})) =$$

$$(\Phi(n_1), \ldots, \Phi(n_m)) + \sum_{j=1}^{\beta} \Phi(p_j)(\Phi(h_{j1}), \ldots, \Phi(h_{jm})),$$

and we know $(\Phi(n_1), \ldots, \Phi(n_m)) \in L$ by above and $(\Phi(h_{i1}), \ldots, \Phi(h_{im})) \in L$, $\forall i \in \{1, \ldots, \beta\}$.

$\square$

Example 3.0.22. Once again we use the same A, valuation map, Khovanskii basis, and adapted basis as in Example 2.7.2. Let us consider $\bar{S} = \{-\bar{y}^2, \bar{x}\bar{y}, \bar{z}^2\}$ and $R = \mathrm{gr}_{\mathfrak{v}}(A)$. Each element of $\bar{S}$ is already of the right form constructed in step (2). Then we compute the generating set for $\mathrm{Syz}(-y^2, xy, z^2) = \langle (x, y, 0), (z^2, 0, y^2), (0, -z^2, xy) \rangle$. Thus we can set $L = \{(\bar{x}, \bar{y}, \bar{0}), (\bar{z}^2, \bar{0}, \bar{y}^2), (\bar{0}, -\bar{z}^2, \bar{x}\bar{y})\}$. We next consider $\langle -y^2, xy, z^2 \rangle \cap \langle y^2 z - x^3 \rangle = \langle x^3 y - y^3 z, x^3 z^2 - y^2 z^3 \rangle$. With $x^3 y - y^3 z = yz(-y^2) + x^2(xy)$ and $x^3 z^2 - y^2 z^3 = z^3(-y^2) + x^3(z^2)$. Giving us $\mathrm{Syz}(\bar{S}) = \langle (\bar{x}, \bar{y}, \bar{0}), (\bar{z}^2, \bar{0}, \bar{y}^2), (\bar{0}, -\bar{z}^2, \bar{x}\bar{y}), (\bar{y}\bar{z}, \bar{x}^2, \bar{0}), (\bar{z}^3, \bar{0}, \bar{x}^3) \rangle$.

$\square$

Algorithm 3.0.23. (Buchberger Algorithm for Computing a Khovanskii-Gröbner basis.)
Input: A generating set S for an ideal $J \subset A$, and a valuation map.
Output: A Khovanskii-Gröbner basis for J.

(1) Let $S = \{s_1, \ldots, s_m\}$, and compute the generating set L for $\mathrm{Syz}(\bar{s}_1, \ldots, \bar{s}_m) \subset (\mathrm{gr}_v(A))^m$.
(2) Set $N := \{0\}$.
(3) For each $\bar{\mathbf{f}} = (\bar{f}_1, \ldots, \bar{f}_m) \in L$ do:
    (a) Find $\mathbf{f} = (f_1, \ldots, f_m) \in A^m$ such that the image of f_i in $\mathrm{gr}_{\mathfrak{v}}(A)$ is $\bar{f}_i$.

(b) Set $h = f_1 s_1 + \cdots + f_m s_m$ and compute h' where $h \xrightarrow{S} h'$.

(c) Set $N := N \cup \{h'\}$.

(4) If $N \neq \{0\}$, set $S := S \cup \{N \setminus \{0\}\}$, return to step (1).

(5) If $N = \{0\}$, Print S.

Proof: We can see that the set S changes after step (4). Let us label these sets S_i where S_i is the set S we get after doing i, note that $S = S_0$. We have $\langle \bar{S}_i \rangle \subset \mathrm{gr}_{\mathfrak{v}}(A)$. Since $\mathrm{gr}_{\mathfrak{v}}(A)$ is a finitely generated algebra it is Noetherian, then the increasing sequence of ideals $\langle \bar{S}_0 \rangle \subseteq \langle \bar{S}_1 \rangle \subseteq \ldots$ stabilizes. Note that if $\langle \bar{S}_N \rangle = \langle \bar{S}_{N+1} \rangle$ then by the construction of the division algorithm 3.0.2 we can see that $S_N = S_{N+1}$, because if $S_N \subset S_{N+1}$ then the element added in step (3.c.) would be a new element for $\bar{S}_N$, a contradiction. Hence the algorithm will terminate after some finite number of steps.

Consider the set S that we obtain after the algorithm terminates. It is clear that S generates J since the starting generating set is a subset of S. Let $\{(\bar{f}_1, \ldots, \bar{f}_m)\} = L$ the generating set for $\mathrm{Syz}(\bar{S})$. Consider $f_1 s_1 + \cdots + f_m s_m \xrightarrow{S} h$ where the image of $(f_1, \ldots, f_m)$ in $(\mathrm{gr}_{\mathfrak{v}}(A))^m$ is $(\bar{f}_1, \ldots, \bar{f}_m)$. If $h \neq 0$ then our set $N \neq \{0\}$ and our algorithm has not terminated yet, a contradiction. Therefore $h = 0$ and the resulting S is a Khovanskii-Gröbner basis for J by the analogue of the Buchberger criterion (Theorem 3.0.20).

$\square$

Example 3.0.24. We use the same A, valuation map, Khovanskii basis, and adapted basis as in Example 2.7.2. Consider $J = \langle -y^2 + x^2, xy + yz, z^2 \rangle \subset A$. Note that in Example 3.0.22 we computed $\mathrm{Syz}(-\bar{y}^2, \bar{x}\bar{y}, \bar{z}^2) = \langle (\bar{x}, \bar{y}, \bar{0}), (\bar{z}^2, \bar{0}, \bar{y}^2), (\bar{0}, -\bar{z}^2, \bar{x}\bar{y}), (\bar{y}\bar{z}, \bar{x}^2, \bar{0}), (\bar{z}^3, \bar{0}, \bar{x}^3) \rangle$.

During step (3) of the analogous Buchberger algorithm (Algorithm 3.0.23) for $\bar{m} = (\bar{x}, \bar{y}, \bar{0})$ we set $m = (x, y, 0)$ and get $h = x(-y^2 + x^2) + y(xy + yz) = x^3 + y^2z \cong 2x^3 - 7xz^2 + 2z^3$. In Example 3.0.4 we see that $h \xrightarrow{S} 2x^3$.

We notice that for the other elements of M, $h \xrightarrow{S} 0$, thus we set $S = \{2x^3, -y^2 + x^2, xy + yz, z^2\}$.

If we repeat this process two more times we get that the Khovanskii-Gröbner basis for J is $G = \{2x^3, -x^2z, -y^2 + x^2, xy + yz, z^2\}$. This is a reduced Khovanskii-Gröbner basis.

□

Remark 3.0.25. Recall that there is more than one correct adapted basis for each valuation map that comes from a prime cone. The adapted basis that is chosen is extremely important when computing a Khovanskii-Gröbner basis for an ideal. If we consider the previous example (Example 2.7.2 and Example 3.0.24) and only change the adapted basis that we work with such that $\mathbb{B} = \{x^\alpha y^\beta z^\gamma \mid \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}, \alpha \leq 2\}$, being the standard monomials from the cone corresponding to $in_\omega(E) = -x^3$. We can see that the reduced Khovanskii-Gröbner basis for the ideal $J = \langle -y^2 + x^2, xy + yz, z^2 \rangle \subset A$ is $G = \{-y^2 + x^2, xy + yz, z^2\}$.

The following is the Macaulay2 code for computing a Khovanskii-Gröbner basis for an ideal $J = \langle f_1, \ldots, f_t \rangle \subset A \cong \mathbf{k}[x_1, \ldots, x_n]/\langle h_1, \ldots, h_r \rangle$ with a valuation map that comes from a matrix M corresponding to a prime cone.

```
w_1=-1*{row 1 of M};
.
.
```

```
.
w_s=-1*{last row of M};

KGB = (R,C,G,i) -> while true do (
    I:=ideal C;
    A:=R/I;
    Gbar:=ideal(G/leadTerm);
    S:=syz gens Gbar;
    L := sub(image S,A);
    Int:= intersect(Gbar, ideal leadTerm(i,I));
    Madded:=sub(image(gens Int//gens Gbar),A);
    L = L +Madded;
    LR := lift(gens L,R);
    J := ideal G;
    H := (gens J)*LR;
    HA := sub(image H,A);
    H' := lift(gens HA,R);
    N := H'%Gbar;
    if N == 0
    then (
        print G;
        break)
    else (
        N' := flatten entries N;
        G = G|N';
        )
    );
```

Then the following will give the Khovanskii-Gröbner basis [2] for $J = \{f_1, \dots, f_t\} \subset A \cong \mathbf{k}[x_1, \dots, x_n]/\langle h_1, \dots, h_r\rangle$.

```
KGB(QQ[x_1,..,x_n,MonomialOrder=>{Weights=>w_1,..,Weights=>w_s}],
    {h_1,...,h_r},{f_1,...,f_t},s)
```

The following two examples illustrate algebras where we are unable to apply SAGBI theory but can apply the theory of Khovanskii bases and Khovanskii-Gröbner bases.

Example 3.0.26. Consider the algebra $A = \mathbf{k}[x+y, xy, xy^2]$. For any term order on A this algebra does not contain a finite SAGBI basis [RS90]. If we consider the rank 2 valuation whose first component is the negative of the degree and the second component is the order of division by $x+y$. Then we can see that $\mathfrak{v}(x+y) = (-1,1)$, $\mathfrak{v}(xy) = (-2,0)$ and $\mathfrak{v}(xy^2) = (-3,0)$.

We will show that the generators $\{x+y, xy, xy^2\}$ for A are a Khovanskii basis for $\mathfrak{v}$ by showing that their valuations generate the value semigroup. Note that the second entry of the valuation of any monomial is 0 and every graded component of A will contain a monomial except degree 1. Thus our value semigroup will have a dot on each coordinate $(-n,0)$, where $n \neq 1$. Also note that $(x+y)^n = x^n + y^n +$ monomials of smaller degree. Hence $x^n + y^n \in A$, which implies that $(-n,n)$ is in the value semigroup. The remaining points of the semigroup can be made by adding $(-m,0)+(-n,n)$. Therefore $S(A,\mathfrak{v}) = \{(-m-n,n) \mid n,m \in \mathbb{Z}_{\geq 0}, n \neq 1\}$. It is not hard to check that $\mathfrak{v}(x+y) = (-1,1)$, $\mathfrak{v}(xy) = (-2,0)$ and $\mathfrak{v}(xy^2) = (-3,0)$ generates the value semigroup $S(A,\mathfrak{v})$, hence by the definition of a Khovanskii basis

[2] Note that this code does not produce a reduced Khovanskii-Gröbner basis.

(Definition 2.6.5), $\mathcal{B} = \{x+y, xy, xy^2\}$.

Consider the change of coordinates $a = x+y$, $b = xy$, and $c = xy^2$, and note that $abc - b^3 - c^2 = 0$. One can show $A \cong \mathbf{k}[a,b,c]/\langle abc - b^3 - c^2\rangle$ and the valuation is constructed using the matrix

$$M = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \end{bmatrix}.$$

Under this valuation we see that $\mathfrak{v}(b^3) = \mathfrak{v}(c^2)$ and thus the prime cone corresponds to the ideal generated by $\langle b^3 - c^2\rangle$. The we have that $\mathrm{gr}_{\mathfrak{v}}(A) \cong \mathbf{k}[a,b,c]/\langle b^3 - c^2\rangle$.

We will use the above Macaulay2 code to compute the Khovanskii-Gröbner basis for the ideal $J = \langle a - bc, a^3 - c, ab + c\rangle$;

```
KGB(QQ[a,b,c,MonomialOrder=>
    {Weights=>{1,2,3},Weights=>{-1,0,0}}],
    a*b*c-b^3-c^2,{a-b*c,a^3-c,a*b+c},2)
```

$= \{-bc + a, -c + a^3, c + ab, ab + a^3, a^3b - a, a^3, -a, a^2, a^5 + a\}$.

If we apply the reduced Khovanskii-Gröbner basis algorithm (Algorithm 3.0.11), we get a reduced Khovanskii-Gröbner basis $G = \{c, a\}$.

□

Example 3.0.27. The following example was first introduced by Göbel [Gob95]. Consider the invariants of the alternating group $A = \mathbf{k}[x_1, x_2, x_3]^{A_3}$. It is easily see that there is no term order that gives the algebra A a finite SAGBI basis. Kaveh and Manon [KM19] found that with a change of coordinates there does exist a finite Khovanskii basis. Consider the elementary symmetric polynomials $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, and $e_3 = x_1x_2x_3$ and the

Vandermonde determinant form $y = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, then we get that the principle ideal that vanishes is generated by

$$f = e_1^2 e_2^2 - 4e_2^3 - 4e_3 e_1^3 + 18e_1 e_2 e_3 - 27e_3^2 - y^2$$

Therefore $A \cong \mathbf{k}[e_1, e_2, e_3, y]/\langle f \rangle$, with a Khovanskii basis $\mathcal{B} = \{e_1, e_2, e_3, y\}$. Following the construction of a valuation on A from a prime cone of $\operatorname{trop}(f)$. Maclagan and Sturmfels showed that $\operatorname{trop}(f)$ has the structure of a fan [MS15], this fan is generated by

$$\{(-19, -4, 11, -23), (-3, -6, 8, -9), (-1, 32, -3, -3),$$

$$(19, 4, 6, 6), (-3, -6, -9, 8)\}.$$

Since A is positively graded then then first row of the matrix M is $(-1, -2, -3, -3)$, the degrees of $\{e_1, e_2, e_3, y\}$ respectfully. The other two rows come from the generators of a prime cone. The initial ideals corresponding to prime cones are $\langle 4e_2^3 - y^2 \rangle$, $\langle 4e_2^3 - 27e_3^2 \rangle$, and $\langle 4e_3 e_1^3 - y^2 \rangle$. Consider the prime cone that corresponds to $\langle 4e_2^3 - 27e_3^2 \rangle$. Therefore the matrix for the construction of the valuation is

$$M = \begin{bmatrix} -1 & -2 & -3 & -3 \\ 19 & 4 & 6 & 6 \\ -3 & -6 & -9 & 8 \end{bmatrix}.$$

Under this valuation map we have $\operatorname{gr}_{\mathfrak{v}}(A) \cong \mathbf{k}[e_1, e_2, e_3, y]/\langle 4e_2^3 - 27e_3^2 \rangle$.

Consider the homogenous ideal $J = \langle y - e_1^3, e_1 e_2 + y, e_3 - e_1 e_2 + y \rangle$. Using Macauly2 code we compute the Khovanskii-Gröbner basis,

```
KGB (QQ[e_1,e_2,e_3,y,MonomialOrder=>{Weights=>{1,2,3,3},
   Weights=>{-19,-4,-6,-6},Weights=>{3,6,9,-8}}],
   e_1^2*e_2^2-4*e_2^3-4*e_3*e_1^3+18*e_1*e_2*e_3-27*e_3^2-y^2,
   {y-e_1^3,e_1*e_2+y,e_3-e_1*e_2+y},3)
```

$= \{y - e_1^3, y + e_1e_2, e_3 + y - e_1e_2, e_1e_2 + e_1^3, -e_1^5e_2^2, -\frac{1}{4}e_1^5e_2^2, e_1^3\}$.

Applying the reduced Khovanskii-Gröbner basis algorithm (Algorithm 3.0.11), we get a reduced Khovanskii-Gröbner basis $G = \{y, e_3, e_1e_2, e_1^3\}$.

$\square$

We conclude this section with a new property about Khovanskii bases that comes in handy sometimes. Suppose A is an algebra with a valuation map $\mathfrak{v}$ with one dimensional leaves, and a Khovanskii basis $\mathcal{B} = \{b_1, ..., b_n\}$ such that $\mathfrak{v}(b_1) < \mathfrak{v}(b_2) < \cdots < \mathfrak{v}(b_n)$.

Proposition 3.0.28. *The set $\mathcal{B}_i = \{b_{i+1}, \ldots, b_n\}$ is a Khovanskii basis for $A_i = A \cap \mathbf{k}[b_{i+1}, \ldots, b_n]$ if the set $\{\mathfrak{v}(b_j), \mathfrak{v}(b_{i+1}), \mathfrak{v}(b_{i+2}), \ldots, \mathfrak{v}(b_n)\} \subset \mathbb{Q}^n$ is linearly independent $\forall j \in \{1, \ldots, i\}$.*

Proof: Let $f \in A_i \subset A$ such that $\mathfrak{v}(f) \in \langle \mathfrak{v}(b_j), \mathfrak{v}(b_{i+1}), \ldots, \mathfrak{v}(b_n) \rangle$ for some $j \leq i$. Since $f \in A_i$ then $f \in \mathbf{k}[b_{i+1}, \ldots, b_n]$, which implies that $\mathfrak{v}(f) \in \langle \mathfrak{v}(b_{i+1}), \ldots, \mathfrak{v}(b_n) \rangle$. Therefore we have

$$\mathfrak{v}(f) = a_j\mathfrak{v}(b_j) + a_{i+1}\mathfrak{v}(b_{i+1}) + \cdots + a_n\mathfrak{v}(b_n) = c_{i+1}\mathfrak{v}(b_{i+1}) + \cdots + c_n\mathfrak{v}(b_n)$$

$$\Rightarrow a_j\mathfrak{v}(b_j) + (a_{i+1} - c_{i+1})\mathfrak{v}(b_{i+1}) + \cdots + (a_n - c_n)\mathfrak{v}(b_n) = 0$$

Since $\{\mathfrak{v}(b_j), \mathfrak{v}(b_{i+1}), \mathfrak{v}(b_{i+2}), \ldots, \mathfrak{v}(b_n)\}$ is linearly independent $\forall j \in [1, i]$, then $a_j = 0$. Therefore $\{\mathfrak{v}(b_{i+1}), \ldots, \mathfrak{v}(b_n)\}$ generates the semigroup $\{\mathfrak{v}(f) \mid f \in A_i\}$. Thus, $\mathcal{B}_i$ is a Khovanskii basis for A_i.

$\square$

4.0 ELIMINATION THEORY

One of the applications of Gröbner theory is in developing algorithmic approaches to to elimination theory. Namely, to eliminate some variables between polynomials in several variables, in order to solve systems of polynomial equations. There is a well-known algorithm to solve a system of polynomial equations in a polynomial ring using Gröbner bases (Theorem 2.1.13). In this section we use Khovanskii-Gröbner bases to extend this and give an algorithm for solving systems of equations on a projective variety whose coordinate ring admits a finite Khovanskii basis with respect to a valuation with one dimensional leaves. For this section we will assume that the base field $\mathbf{k}$ is algebraically closed.

Let X be a projective variety with homogeneous coordinate ring $A = \mathbf{k}[X] = \bigoplus_{i\geq 0}\mathbf{k}[X]_i$. Fix a smooth point $p \in X$, and consider its local ring $\mathcal{O}_p$ with $\mathfrak{m}_p\mathcal{O}_p$ its maximal ideal. Since p is a smooth point, $\mathcal{O}_p$ is a regular local ring and we can write $\mathfrak{m}_p\mathcal{O}_p = \langle u_1, \ldots, u_d\rangle$ where $d = \dim X$. One calls $\{u_1, \ldots, u_d\}$ a system of parameters at the point p. By the Cohen structure theorem $\mathcal{O}_p \cong \mathbf{k}[[u_1, \ldots, u_d]]$, the ring of formal power series in the u_i, and any $f \in \mathcal{O}_p$ can be written as a formal power series $f = \sum_{\alpha} c_\alpha u^\alpha$. Fix $0 \neq h \in A_1$ such that $h(p) \neq 0$. We have an embedding $A \hookrightarrow \mathbf{k}[[u_1, \ldots, u_d]][t] = \bigoplus_{m\geq 0}\mathbf{k}[[u_1, \ldots, u_d]]t^m$ by $f \in A_m \mapsto \frac{f}{h^m}t^m$. Let $\mathfrak{v}'$ be the minimum term valuation (Example

2.3.2) corresponding to the choice of $\{u_1, \ldots, u_d\}$ with $u_1 \prec \cdots \prec u_d$ (i.e. $(1, 0, \ldots, 0) \prec (0, 1, \ldots, 0) \prec \cdots \prec (0, \ldots, 0, 1)$), and for $f \in \mathbf{k}[X]_m$ define $\mathfrak{v}(f) = (-m, \mathfrak{v}'(\frac{f}{h^m}))$. We can see that with the valuation the value semigroup $S(A, \mathfrak{v})$ is maximum well ordered. We make the standing assumption that:

(1) We have a finite Khovanskii basis with respect to $\mathfrak{v}$.

(2) A is a finite module over $\mathbf{k}[u_1, \ldots, u_d, t]$.

We will now introduce the notion of an elimination ideal in this setting.

Definition 4.0.1. (Elimination Ideal) With notation as above, let $J \subset A$ be an ideal. Define the i^{th} *elimination ideal* $J_i = J \cap \mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$. It is an ideal in $\mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$, where $\mathbf{k}[[u_{i+1}, \ldots, u_d]]$ is the ring of formal power series in the variables $\{u_{i+1}, \ldots, u_d\}$.

Proposition 4.0.2. *(Elimination Theory) Consider the homogeneous ideal $J = \langle f_1, \ldots, f_d \rangle \subset A$ with a Khovanskii-Gröbner basis G and minimum term valuation $\mathfrak{v}$ defined by a system of parameters $\{u_1, \ldots, u_d\}$. Then $G_i = G \cap \mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$ is a Khovanskii-Gröbner basis for J_i.*

Proof: Fix i between 0 and $d - 1$. By definition $G_i \subset J_i$ and thus we have that $\langle \bar{G}_i \rangle \subset \langle \bar{J}_i \rangle$. Let $f \in J_i \subset J$, then since G is a Khovanskii-Gröbner basis there exists $\{g_1, \ldots, g_s\} \subset G$ with g_j homogeneous in t such that $\bar{f} = \sum_{j=1}^{s} \bar{c}_j \bar{g}_j$ where $c_j \neq 0$ for all j and moreover $\mathfrak{v}(f) = \mathfrak{v}(c_j g_j)$, $\forall j$. Since $f \in J_i$, then $\bar{f} \in \mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$. Since we use a minimum term valuation with $u_1 \prec u_2 \prec \cdots \prec u_d$ we see that if $g_j \in \mathbf{k}[[u_\alpha, \ldots, u_d]][t]$ and $g_j \notin \mathbf{k}[[u_{\alpha+1}, \ldots, u_d]][t]$ then $\bar{g}_j \in \mathbf{k}[[u_\alpha, \ldots, u_d]][t]$ and $\bar{g}_j \notin \mathbf{k}[[u_{\alpha+1}, \ldots, u_d]][t]$.

Thus we can conclude that $g_j \in \mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$, $\forall j$. Note that we can also assume $c_i \in \mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$ by replacing c_i with $c_i(0, \ldots, 0, u_{i+1}, \ldots, u_d)(t)$. Therefore $\{g_1, \ldots, g_s\} \subset G_i$ and $\bar{f} \in \langle \bar{G}_i \rangle$, which implies $\langle \bar{G}_i \rangle = \langle \bar{J}_i \rangle$.

□

We recall the well-known Noether Normalization theorem.

Theorem 4.0.3. *(Noether Normalization) Let A be an affine ring of dimension n over a field k. If $I_1 \subset \cdots \subset I_\alpha$ is a chain of ideals of A with $dim(I_j) = d_j$ and $n \geq d_1 > d_2 > \cdots > d_\alpha > 0$, then A contains a polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$ in such a way that A is a finitely generated S-module and*

$$I_j \cap S = \langle x_{d_{j+1}}, \ldots, x_n \rangle.$$

If the ideals I_i are homogeneous, then the x_i may be chosen to be homogeneous. In fact, if $\mathbf{k}$ is infinite, and A is generated over $\mathbf{k}$ by $y_1, \ldots, y_n$, then for $j \leq d_\alpha$ the element x_j may be chosen to be a $\mathbf{k}$-linear combination of the y_i.

Proof: The proof can be found in [Eis95] page 283.

□

From the Noether normalization theorem (Theorem 4.0.3) our standing assumption that A is a finite module over $\mathbf{k}[u_1, \ldots, u_n, t]$ holds for almost all choices of $\{u_1, \ldots, u_d\}$.

Let $f \in G_i = G \cap \mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$. We know there exists a polynomial $q(w) \in \mathbf{k}[u_1, \ldots, u_d, t][w]$ such that $q(f) = 0$. We can write $q(f)$ as a series with coefficients in $\mathbf{k}[u_1, \ldots, u_i]$ i.e., $q(f) = \sum c_\alpha(u_1, \ldots, u_i)\mathbf{u}_*^\alpha = 0$, where $\mathbf{u}_* = (u_{i+1}, \ldots, u_d, t)$. Since $f \in \mathbf{k}[[u_{i+1}, \ldots, u_d]][t]$ we see that f being a solution of $q(w)$ is independent of the value of $\{u_1, \dots, u_i\}$, hence we can set $u_1 = \cdots = u_i = 0$ and see that $q(f) = \sum c_\alpha(0, \dots, 0)\mathbf{u}_*^\alpha = 0$.

Thus there exists an $h(w) \in \mathbf{k}[u_{i+1}, \dots, u_d, t][w]$ such that $h(f) = \sum_{i=0}^{\beta} a_i(u_{i+1}, \dots, u_d) f^i = 0$ for $a_i(u_{i+1}, \dots, u_d) \in \mathbf{k}[u_{i+1}, \dots, u_d]$. We notice that if there exists a solution $(u_{i+1}, \dots, u_d) = (x_{i+1}, \dots, x_d)$ such that $f(x_{i+1}, \dots, x_d, t) = 0$ then we must have that $a_0(x_{i+1}, \dots, x_d) = 0$, thus if we solve for all $(u_{i+1}, \dots, u_d) = (x_{i+1}, \dots, x_d)$ such that $a_0(x_{i+1}, \dots, x_d) = 0$ then these are the only possible solutions for f. Therefore we need to find a function $h(w) \in \mathbf{k}[u_{i+1}, \dots, u_d, t][w]$ such that $h(f) = 0$, which will be done in the algorithm below (Algorithm 4.0.4).

Algorithm 4.0.4. Computing algebraic polynomials in local system of parameters that elements of A satisfy.
Input: A a finite module over $\mathbf{k}[u_1, \dots, u_d, t]$ with $u_i \in A$, and $f \in A$.
Output: A polynomial $h \in \mathbf{k}[u_1, \dots, u_d, t][w]$ such that $h(f) = 0$.

(1) If $f \in \mathbf{k}[u_1, \dots, u_d, t]$, output $h = (w - f)$.
(2) Else, find a vector space basis, G, for $\mathrm{Frac}(A)$ over $\mathbf{k}(u_1, \dots, u_d, t)$.
   (a) Define the map $m_f : \mathrm{Frac}(A) \to \mathrm{Frac}(A)$ to be multiplication by f. Compute the matrix M of m_f with respect to the basis G.
   (b) Output $h(w) = \det(wI - M)$.

Remark 4.0.5. We can see that the multiplication map m_f clearly maps $A \to A$, thus we have that for any $f \in A$ the equation $q(w) = \det(wI - M)$ has the property that $q(f) = 0$ by the Cayley-Hamilton theorem.

Recall that we assume that the algebra A has a finite Khovanskii basis with respect to the minimum term valuation associated to the system of parameters $\{u_1, \ldots, u_d\}$ and we can approximate the Khovanskii basis elements $\{b_1, \ldots, b_n\}$ with polynomials $\{h_1, \ldots, h_n\}$ of sufficiently high degree in terms of the local parameter u_i. With these assumptions we present the following proposal for solving a system of equations.

Proposal 4.0.6. Solving a system of equations in an algebra A in terms of the algebra generators of A with $\dim(A) = d$.

Input: A set $\{f_1, \ldots, f_d\} \in A$, and a Khovanskii basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ for A with respect to the valuation $\mathfrak{v}$ as above.[1]

Output: A solution for the system $f_1 = \cdots = f_d = 0$ for a point in $\operatorname{Spec}(A) \subset \mathbb{A}^n$.

(1) Let $J = \langle f_1, \ldots, f_d \rangle$.

Compute a Khovanskii-Gröbner basis G for J with respect to the valuation $\mathfrak{v}$.

(2) For each i, use Algorithm 4.0.4 to find a polynomial $q_i(w) = \sum_{j=0}^{m_i} a_{ij}(\mathbf{u}) w^j$ for each $g_i \in G$ such that $q_i(g_i) = 0$, where $\mathbf{u} = (u_1, \ldots, u_d, t)$.

(a) Set $\alpha = 1$ and $S = \emptyset$.

(b) While $\alpha < d + 1$,

For all $g_i \in G_{d-\alpha} = G \cap \mathbf{k}[[u_{d-\alpha+1}, \ldots, u_d]][t]$ and let $a_{i0}(u_{d-\alpha+1}, \ldots, u_d)$ be the constant term of $q_i(g_i)$ as a polynomial in $u_1, \ldots, u_{d-\alpha}$.

Solve $a_{i0}(u_{d-\alpha+1}, \ldots, u_d) = 0$ for the variable $u_{d-\alpha+1}$[2] and set $S :=$

[1] The choice of $\mathcal{B}$ gives an embedding of $\operatorname{Spec}(A) \subset \mathbb{A}^n$.

[2] When $\alpha > 1$ we can use the solutions of S to solve for the new variable $u_{d-\alpha+1}$. If $S = \emptyset$ when $\alpha > 1$ then there is no solution.

$S \cup \{\mathbf{u} \mid \mathbf{u}$ is a solution$\}$.

(c) For each $\mathbf{u} \in S$ compute $g_j(\mathbf{u})$ for all $g_j \in G_{d-\alpha}$,

If there exists a $g_j \in G_{d-\alpha}$ such that $g_j(\mathbf{u}) \neq 0$[3] set $S := S \setminus \{\mathbf{u}\}$.

Set $\alpha := \alpha + 1$ and go to step (b).

(4) If $S \neq \emptyset$, for all $\mathbf{u} \in S$ Print $b_i = h_i(\mathbf{u})$, $\forall i \in [1, n]$.

Else, Print no solution exists.

[3]Since g_i might be a power series we can approximate g_i with a polynomial, p, of sufficiently high degree. Given a solution $\mathbf{x} = (x_1, \ldots, x_n)$, if $|p(\mathbf{x})| > \epsilon$ for some chosen $\epsilon > 0$ then we will say $g_i(\mathbf{x}) \neq 0$.

5.0 GENERALIZED TROPICAL VARIETIES

We start by recalling the Proj construction of a graded domain $R = \bigoplus_{k\geq 0} R_k$. Let h be a homogenous element of R. Then we define

$$(R_h)_0 = \{\frac{f}{h^{\frac{\deg(f)}{\deg(h)}}} \mid f \in R\} \subset \mathrm{Frac}(R),$$

where $\mathrm{Frac}(R)$ is the field of fractions of R. Then $\mathrm{Proj}(R)$ is defined as:

$$\mathrm{Proj}(R) = \bigcup_h \mathrm{Spec}((R_h)_0),$$

where the union is over nonzero homogeneous h and the $\mathrm{Spec}((R_h)_0)$ are glued to each other along appropriate gluing maps. Note that it is sufficient to take the union over a finite generating set, furthermore if R is generated by R_1 then it would suffice to take only the degree 1 h.

We start with an algebra $A \cong \mathbf{k}[\mathbf{x}]/I$, a prime cone $C \subset \mathrm{trop}(I) \cap \mathbb{Z}^n_{\leq 0}$, a valuation constructed from that prime cone and an adapted basis $\mathbb{B}$ for $(A, \mathfrak{v})$. We consider an ideal $J \subset A$ and let $Y = V(J) \subset \mathrm{Spec}(A) = X$. For $\omega \in C$ we define $\mathfrak{v}_\omega$ to be the weight valuation defined by ω. For an arbitrary vector this will give us a quasivaluation; however, since $\omega \in C$, which is a prime cone, we know that $\mathfrak{v}_\omega$ is a valuation. Define $\mathbb{B}_{\omega,\geq -k} = \{b \in \mathbb{B} \mid \mathfrak{v}_\omega(b) \geq -k\}$ and the subspace $F_\omega(k) \subset A$ by:

$$F_\omega(k) = \mathrm{span}(\mathbb{B}_{\omega,\geq -k}).$$

Define the associated graded $\mathrm{gr}_\omega(A)$, similar to before as:

$$\mathrm{gr}_\omega(A) = \bigoplus_{k\geq 0} F_\omega(k)/F_\omega(k-1),$$

and initial ideal $in_\omega(J)$ by:

$$in_\omega(J) = \bigoplus_{k\geq 0}(F_\omega(k)\cap J)/(F_\omega(k-1)\cap J).$$

Let $\mathfrak{R} = \bigoplus_{k\geq 0} F_\omega(k)t^k$ be the Reese algebra of the increasing filtration $\{F_\omega(k)\}_{k\geq 0}$ and let

$$\tilde{X}_\omega = \mathrm{Proj}(\mathfrak{R}).$$

The variety $\tilde{X}_\omega$ is a partial compactification of X obtained by adding the divisor $D_\omega = \mathrm{Proj}(\mathrm{gr}_\omega(A))$, noting that $I(D_\omega) = \langle t\rangle$. The compactification of Y in $\tilde{X}_\omega$ is $\tilde{Y}_\omega = V(\hat{J}) = \mathrm{Proj}(\bigoplus_{k\geq 0} F_\omega(k)t^k/\bigoplus_{k\geq 0}(F_\omega(k)\cap J)t^k)$; with $\hat{J} = \bigoplus_{k\geq 0}(F_\omega(k)\cap J)t^k$. We can see that $D_\omega \cap \tilde{Y}_\omega = V(\langle t\rangle + \hat{J}) \subset \tilde{X}_\omega$.

We now propose the following definition of a tropical variety for an ideal $J \subset A$ with respect to a prime cone C. Note that for $f \in A$, $in_\omega(f)$ is an invertible element of $\mathrm{gr}_\omega(A)$ if there exists an element $g \in A$ such that $in_\omega(f)in_\omega(g) \in \mathbf{k}$. We can see that $\mathfrak{v}_\omega(f) = -\mathfrak{v}_\omega(g)$ and we will denote the inverse of $in_\omega(f)$ by $(in_\omega(f))^{-1}$.

Definition 5.0.1. (Relative Tropical Variety) Let X be an affine variety, $J \subset A \simeq \mathbf{k}[X]$ and $C \subset \mathrm{trop}(X) \cap \mathbb{Z}^n_{\leq 0}$ a prime cone. The *tropical variety of J relative to C* is

$$\mathrm{trop}_C(J) = \{\omega \in C \mid in_\omega(J) \text{ does not contain an invertible element}\}.$$

Remark 5.0.2. This definition generalizes the notion of spherical tropical varieties of a subvariety in an affine homogenous space which was introduced by Kaveh, Manon, and Vogiannou [Vog15, KM19].

The next theorem is the main result of this section. It shows how $\mathrm{trop}_C(J)$ encodes asymptotic directions of the subvariety Y defined by J. It states that given a subvariety Y of X, the relative tropical variety $\mathrm{trop}_C(J)$ gives us information about the closure of Y in any partial compactification of X corresponding to any $\omega \in C$.

Theorem 5.0.3. *Take $\omega \in C$ and let $\tilde{X}_\omega$ be the corresponding partial compactification of $X = Spec(A)$. Let $J \subset A$ be an ideal with $V(J) = Y \subset X$, and $\tilde{Y}_\omega$ its closure in $\tilde{X}_\omega$. Then $\tilde{Y}_\omega \setminus Y \neq \emptyset$ if and only if $\omega \in trop_C(J)$.*

Proof: We will prove this by proving two claims and using the fact that if $\tilde{Y}_\omega \setminus Y \neq \emptyset$ then $\tilde{Y}_\omega \cap D_\omega \neq \emptyset$.

Claim 1: $\tilde{Y}_\omega \cap D_\omega \cong \mathrm{Proj}(\mathrm{gr}_\omega(A)/in_\omega(J))$. Proof: Consider the natural map $\pi : \mathfrak{R} \to \mathrm{gr}_\omega(A)$ which sends $F_\omega(k) \to F_\omega(k)/F_\omega(k-1)$. We immediately see that the kernel of this map is $\bigoplus_{k\geq 1} F_\omega(k-1)t^k$ from the definition of π. Factor t from the kernel and see that $\bigoplus_{k\geq 1} F_\omega(k-1)t^k = t\mathfrak{R} = \langle t\rangle$, hence $\mathfrak{R}/\langle t\rangle \cong \mathrm{gr}_\omega(A)$.

Let us consider this claim on the affine cone and let $X^{\mathbb{A}}_\omega = \mathrm{Spec}(\mathfrak{R})$ and $D^{\mathbb{A}}_\omega = \mathrm{Spec}(\mathrm{gr}_\omega(A))$. We can see that

$$\mathrm{Spec}(\mathrm{gr}_\omega(A)/in_\omega(J)) = V(in_\omega(J)) \subset D^{\mathbb{A}}_\omega \subset \tilde{X}^{\mathbb{A}}_\omega,$$

and the ideal for $V((\langle t\rangle+\hat{J})/\langle t\rangle) = D_\omega^{\mathbb{A}} \cap \tilde{Y}_\omega^{\mathbb{A}} \subset \mathfrak{R}/\langle t\rangle$. Note that the restriction of the map π on $\hat{J}$ is clearly contained in $in_\omega(J) \subset \mathrm{gr}_\omega(A)$. We would like to know $\pi^{-1}(in_\omega(J))$. Note that since $\mathfrak{R}$ and $\mathrm{gr}_\omega(A)$ are graded algebras then π is a graded homomorphism, thus is suffices to find π^{-1} of the graded pieces of $in_\omega(J)$. Fix $k \geq 0$, then $\pi^{-1}((F_\omega(k) \cap J)/(F_\omega(k-1) \cap J)) = (F_\omega(k) \cap J + F_\omega(k-1))t^k \subset \hat{J} + \langle t\rangle$. Therefore we have $(\langle t\rangle + \hat{J})/\langle t\rangle \cong in_\omega(J)$ from the above isomorphism. Since the ideals are isomorphic then the varieties they define are isomorphic, hence $\tilde{Y}_\omega^{\mathbb{A}} \cap D_\omega^{\mathbb{A}} \cong \mathrm{Spec}(\mathrm{gr}_\omega(A)/in_\omega(J))$. Since this is true for the affine cone, then this is true for the Proj as well.

Claim 2: $\mathrm{Proj}(\mathrm{gr}_\omega(A)/in_\omega(J)) \neq \emptyset$ if and only if $\omega \in \mathrm{trop}_C(J)$. Proof: Recall $\omega \in \mathrm{trop}_C(J)$ if and only if $in_\omega(J)$ does not contain any invertible elements. This is equivalent to $\mathrm{gr}_\omega(A)/in_\omega(J) \neq \{0\}$. But Proj of a $\mathbf{k}$-algebra is empty if and only if the algebra is trivial (i.e., equal to $\{0\}$).

$\square$

Remark 5.0.4. We expect that one can prove a generalization of the tropical compactification theorem (Theorem 2.8.1) in this context. More precisely, with notation as above, fix a prime cone C and let Σ be a fan whose support lies in C. In a fashion similar to the construction of toric varieties, one should be able to construct a variety $\tilde{X}_\Sigma$ that contains $X = \mathrm{Spec}(A)$ as an open subset. We then expect that if Σ is a fan whose support is $\mathrm{trop}_C(J)$ then one can then try to prove an analogue of the tropical compactification theorem [Tev07] for the compactification of $Y \subset X$ in $\tilde{X}_\Sigma$. We plan to address this in future work.

Example 5.0.5. As before, let $A \cong \mathbf{k}[x,y,z]/\langle y^2z - x^3 + 7xz^2 - 2z^3\rangle$ with prime cone $C = \mathbb{Q}(1,1,1) + \mathbb{Q}_{\geq 0}(-2,-3,0)$, which gives us a valuation from the matrix

$$M = \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & 0 \end{bmatrix}$$

and an adapted basis $\mathbb{B} = \{x^\alpha y^\beta z^\gamma \mid \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}, \alpha \leq 2\}$. Note for an ideal J and an element $f \in J \subset A$ if there does not exists $g \in A$ such that $\mathfrak{v}_\omega(f) + \mathfrak{v}_\omega(g) = 0$ then we know that f is not invertible. We can see, after rescaling, that any interior $\omega \in C$ is of the form $\omega = (-2+a, -3+a, a)$ where $a \in \mathbb{Z}$ and when $a < 0$ any non-constant element $g \in A$ has $\mathfrak{v}_\omega(g) < 0$. Therefore $C \cap \mathbb{Z}^3_{<0} \subset \mathrm{trop}_C(J)$ for every non-trival ideal $J \subset A$. Since we are only considering the part of $C \subset \mathbb{Z}^3_{\leq 0}$, then only $\omega = (-2,-3,0)$ is interesting for this algebra. Consider $J = \langle -y^2 + x^2, xy + yz, z^2 \rangle$, it was shown above that the Khovanskii-Gröbner basis for this ideal is $G = \{-y^2 + x^2, xy + yz, z^2\}$. For $\omega = (-2,-3,0)$ the valuation of any element of J will be generated by the valuations of G, namely the set $\{-6,-5,0\}$, as a semigroup ideal. We can see that for any adapted basis element b in this algebra to be invertible we must have $\bar{b} \in \mathbf{k}$ or $\mathfrak{v}_\omega(b) \geq 1$. Since this cannot happen with these Khovanskii-Gröbner basis elements we have that $\omega = (-2,-3,0) \in \mathrm{trop}_C(J)$.

BIBLIOGRAPHY

[Buc06] Bruno Buchberger. Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal, *Journal of Symbolic Computation* **41**, 475-511, 2006.

[CLO15] David A. Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms.* An introduction to computational algebraic geometry and commutative algebra, Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015.

[CLS10] David A. Cox, John Little, and Hal Schenck. *Toric Varieties*, volume 124 of *Graduate studies in Mathematics.* American Mathematical Society, Providence, RI, 2010.

[Eis95] David Eisenbud. *Commutative algebra. With a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[FJ08] Michael D. Fried, Moshe Jarden. Field Arithmetic. *A Series of Modern Surveys in Mathematics.* Springer, Third Edition, 2008.

[Gob95] Manfred Göbel. Computing bases for rings of permutation-invariant polynomials, *Journal of Symbolic Computation*, **19**, 285-291, 1995.

[Har97] Robin Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics. Springer, New York, 1997.

[KM19] Kiumars Kaveh and Christopher Manon. Gröbner theory and tropical geometry on spherical varieties. *Transformation Groups* **24**, 1095-1145, 2019

[KM19] Kiumars Kaveh and Christopher Manon. Khovanskii Bases, Higher Rank Valuations and Tropical Geometry. *SIAM Journal on Applied Algebra and Geometry.* **3**, 292-336, 2019.

[Mil94] J. Lyn Miller. *Analogs of Gröbner Bases in Polynomial Rings over a Ring.* ArXiv:9411209v1 [math.AC]. 1994.

[Mil96] J. Lyn Miller. Analogs of Gröbner Bases in Polynomial Rings over a Ring, *Journal of Symbolic Computation* **21**, 139-153, 1996.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry,* volume 161 of *Graduate studies in Mathematics.* American Mathematical Society, Providence, RI, 2015.

[Oll91] F. Ollivier. Canonical bases: relations with standard bases, finiteness conditions, and applications to tame automorphisms, in *Effective Methods in Algebraic Geometry* (T. Mora, C. Traverso, eds). Birkäuser, Boston, pp. 379-400, 1991.

[RS90] Lorenzo Robbiano and Moss Sweedler. Subalgebra bases. *Commutative algebra (Salvador, 1988)*, volume 1430 of *Lecture Notes in Math.*, pages 61-87. Springer, Belin, 1990.

[Stu93] Bernd Sturmfels. *Algoritms in Invariant Theory.* Springer Verlag, Vienna, 1993.

[Stu96] Bernd Sturmfels. *Grobner bases and convex polytopes*, volume 8 of *University Lecture Series.* American Mathematical Society, Providence, RI, 1996.

[Tev07] Jenia Tevelev. Compactifications of subvarieties of tori. *Amer. J. Math.*, 129(4):1087-1104, 2007.

[Vak13] Ravi Vakil. *Foundations of Algebraic Geometry.* math216.wordpress.com, 2013.

[VGSEH97] Wolmer V. Vasconcelos, Daniel R. Grayson, Michael Stillman, David Eisenbud, and Jüurgen Herzog. *Computational Methods in Commutative Algebra and Algebraic Geometry.* Springer, 1997.

[Vog15] Tassos Vogiannou. *Spherical Tropicalization.* ArXiv:1511.02203v1 [math.AG]. 2015.