# Cesàro averaging and extension of functionals on infinite dimensional spaces

by

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**Cesàro averaging and extension of functionals on infinite dimensional spaces**

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On the sequence space $\ell^\infty$, we construct Banach limits that are invariant under the Cesàro averaging operator. On the function space $L^\infty(0,\infty)$, we start by defining a new operator $J^\alpha$, for each $\alpha > 0$. This new operator extends the definition of $J^n$, with $n \in \mathbb{N}$, which is the operator obtained by composing the Cesàro averaging operator with itself $n$ times. We show that the family of operators $(J^\alpha)_{\alpha>0}$ has the semigroup property. We also construct Banach limits on $L^\infty(0,\infty)$ that are invariant under the members of this family of operators. Finally, on the operator space $\mathcal{B}(\ell^2(\mathbb{N}_0))$, we define a Cesàro averaging operator from this space to itself. We also discuss known results about vector-valued Banach limits on $\ell^\infty(\ell^2(\mathbb{Z}))$ that preserve Cesàro convergence, and use them to construct a continuous linear functional on $\mathcal{B}(\ell^2(\mathbb{N}_0))$ with Cesàro-invariance-like properties.

**Keywords:** Functional Analysis, Cesàro averaging operators, invariant Banach limits, fractional powers.

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## Preface

This thesis is dedicated to my husband, Ivan Ramirez, with whom I embarked on this PhD adventure. Now we continue our journey, together always.

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## 1.0 Introduction

In the present work we study the Cesàro averaging operator and generalizations of it in different Banach spaces, as well as Banach limits invariant under these operators.

We start, in Chapter 2, by studying the Cesàro averaging operator defined on the sequence space $\ell^\infty$: Let $C$ be the Cesàro averaging operator that takes a bounded sequence $(x_n)_n = (x_1, x_2, x_3, \ldots)$ as input, and gives the output $C\,(x_n)_n = \left(\frac{x_1+x_2+\ldots+x_n}{n}\right)_n = (\frac{x_1}{1}, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \ldots)$. Whenever $C\,(x_n)_n$ is a convergent sequence, we say $(x_n)_n$ is Cesàro convergent. It is a well known result due to Cauchy [4], that if $(x_n)_n$ converges in the classical sense to the value $L$, then it is also Cesàro convergent and to the same value $L$.

Now, since there are sequences that are not convergent in the classical sense, but are Cesàro convergent, we can think of Cesàro convergence as a generalization of the concept of classical convergence. Also, since there exist bounded sequences that are not Cesàro convergent, we could ask the following question: "Does there exist a concept that generalizes Cesàro convergence such that all bounded sequences have a limit?". Using the Hahn-Banach Extension Theorem, the existence of such generalizations can be shown. Furthermore, by carefully employing a generalization of this theorem, J. Sivek, [23], showed the existence of particular generalizations, that are both Banach limits and Cesàro invariant, and that also enjoy other desirable invariance properties. In our work, we were able to construct a generalization of this type: Cesàro invariant that is also a Banach limit.

Next, in Chapter 3, we consider the space of (essentially) bounded functions $f(x)$ with domain $(0, \infty)$. An analogous concept to Cesàro averaging would be $Jf(x) = \frac{1}{x}\int_0^x f(t)dt$, and an analogous concept of Cesàro convergent would be whenever $\lim_{x\to\infty} Jf(x)$ converges to a real number. Our main results are in this chapter, where we were able to define an operator that generalizes the concept of iterates of Cesàro averaging, to an operator of the form $J^r f(x)$, where now $r$ is any positive real number. Our definition of fractional powers of Cesàro averaging is such that $(J^r)_{r>0}$ has the semigroup property, that is, $J^r(J^s f) = J^{r+s}(f)$, for all $r, s > 0$, and for all $f \in L^\infty(0, \infty)$. We also construct Banach limits on $L^\infty(0, \infty)$, such that they are invariant under our continuous generalization of Cesàro iterates $J^r$, $r > 0$.

In [7], Banach limits invariant under Cesàro averaging were first studied, for the sequence space $\ell^\infty$. In [21], the authors E. Semenov and F. Sukochev, gave sufficient conditions for a linear operator on $\ell^\infty$, to guarantee the existence of Banach limits that would be invariant under the given operator. Also, they gave necessary and sufficient conditions for a sequence to have the same output under any Cesàro invariant Banach limit. In [22], the authors E. Semenov, F. Sukochev, A. Usachev, and D. Zanin, studied Banach limits on $\ell^\infty$ that are invariant under the Cesàro operator and the Dilation operator. Sukochev, Usachev, and Zanin [25], have also studied generalized limits on $L^\infty$ invariant under the Cesàro operator and related operators. Moreover, their work has applications to non-commutative geometry. Also, their work does not discuss fractional powers of the Cesàro operator. A more recent paper related to [25] is [14], where convolution invariant linear functionals on $L^\infty$ are studied, Cesàro invariant functionals are also considered.

We conclude Chapter 3 by proving the following theorem: For any $f \in L^\infty(0,\infty)$, $J^r f(x)$ has a limit at infinity for some $r > 0$, if and only if $J^s f(x)$ has a limit at infinity for any $s > 0$. In this case, the limit values are all the same. In fact, our result says more, since we were able to obtain a quantitative version of the statement above. Our theorem is analogous to the quantitative result obtained in [23] for the sequence space $\ell^\infty$. The qualitative version of the result in [23] follows from a theorem due to Frobenius [9], and a classical theorem of Hardy and Littlewood (see Theorem 7.3 of [13]).

Finally, in Chapter 4, we define a Cesàro averaging operator on the operator space $\mathcal{B}(\mathcal{H})$. Here, $\mathcal{B}(\mathcal{H})$ is the space of bounded linear operators on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}_0)$. When we first considered this problem, we turned our attention to Toeplitz matrices (see e.g. [2]), which are matrices that are constant down each diagonal parallel to the main diagonal. We were interested in Toeplitz matrices since constant sequences on $\ell^\infty$ are fixed points of the Cesàro operator. However, it turns out that Toeplitz matrices not always correspond to bounded operators on $\mathcal{H}$. So, we employed an operator defined in [6] to guarantee our definition of Cesàro operator on $\mathcal{B}(\mathcal{H})$ maps back to $\mathcal{B}(\mathcal{H})$. We also discuss known results presented in [1], about vector-valued Banach limits on $\ell^\infty(\ell^2(\mathbb{Z}))$ that preserve Cesàro convergence, and use them to construct a continuous linear functional on $\mathcal{B}(\mathcal{H})$ with Cesàro-invariance-like properties.

## 2.0 Cesàro averaging and new invariant Banach limits on $\ell^\infty$

### 2.1 Notation and preliminaries

Let $\mathbb{N}$ be the set of all natural numbers, and let $\ell^\infty = \ell^\infty(\mathbb{N})$ be the Banach space of all bounded real-valued sequences $x = (x_1, x_2, x_3, \ldots)$, with the supremum norm

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

The dual space of $\ell^\infty$, $(\ell^\infty)^*$, consisting of all continuous linear functionals on $\ell^\infty$, is isometrically isomorphic to $\ell^1 \bigoplus_1 (c_0)^\perp$ (see [27]). Here

$$c_0 := \{x = (x_n)_n \in \ell^\infty : \lim_{n\to\infty} x_n = 0\},\ \ell^1 := \{x = (x_n)_n \in \ell^\infty : \sum_{n=1}^{\infty} |x_n| < \infty\},$$
$$\text{and } (c_0)^\perp := \{\psi \in (\ell^\infty)^* : \psi(x) = 0,\ \textit{for all } x = (x_n)_n \in c_0\}.$$

The following is The Hahn-Banach Theorem as it is stated in Rudin's *Real and Complex Analysis* [20]:

**Theorem 2.1.1.** *If $M$ is a subspace of a normed linear space $X$ and if $f$ is a bounded linear functional on $M$, then $f$ can be extended to a bounded linear functional $F$ on $X$ so that $\|F\| = \|f\|$.*

We next present two examples on how this theorem can be applied to obtain extensions of functionals on $\ell^\infty$:

On $(\ell^\infty, \|\cdot\|_\infty)$ we consider the Banach subspace over $\mathbb{R}$

$$c := \{x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}\ \forall n \textit{ and } \lambda(x) := \lim_{n\to\infty} x_n \textit{ exists in } \mathbb{R}\}.$$

By Hahn-Banach Extension Theorem, the linear functional $\lambda : c \to \mathbb{R}$ can be extended to all $\ell^\infty$. That is, there exists $\Phi \in (\ell^\infty)^*$ with

$$\|\Phi\|_{(\ell^\infty)^*} = \|\lambda\|_{c^*} = 1, \text{ and such that } \Phi\,|_c = \lambda.$$

This functional is called *a generalized limit* on $\ell^\infty$.

We can also consider the subspace

$$Ces := \{w = (w_n)_{n\in\mathbb{N}} \in \ell^\infty : w_n \in \mathbb{R}\ \forall n \textit{ and } \psi(w) := \lim_{n\to\infty} (Cw)_n \in \mathbb{R}\},$$

where $(Cw)_n := \frac{1}{n}\sum_{k=1}^{n} w_k = \frac{w_1 + \ldots + w_n}{n}$ for all $n$. The sequence $Cw$ is called *the Cesàro average of* $w$.

By Hahn-Banach Extension Theorem, the linear functional $\psi : Ces \to \mathbb{R}$ can be extended to all $\ell^\infty$. That is, there exists $\Psi \in (\ell^\infty)^*$ with

$$\|\Psi\|_{(\ell^\infty)^*} = \|\psi\|_{Ces^*} = 1, \text{ and such that } \Psi\,|_{Ces} = \psi.$$

*Remark.* $c \subsetneq Ces \subsetneq \ell^\infty$. To see the first strict inclusion, notice $(1, 0, 1, 0, \ldots) \in Ces$ but it is not convergent. On the other hand, consider the sequence

$$a = (1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, \ldots),$$

which has monotonically increasingly long sections of 1s and $-1$s in the pattern indicated, that is, the length of the n-th section is $2^{n-1}$. This is a bounded sequence with divergent Cesàro average.

In this work, we are interested in obtaining extensions of functionals that are also Banach limits. We define Banach limits next:

**Definition 2.1.1.** A function $\Lambda : \ell^\infty \to \mathbb{R}$ is called a *Banach limit* if

1. $\Lambda$ is continuous and linear, and $\|\Lambda\|_{(\ell^\infty)^*} = 1$.
2. $\Lambda(x) = \lim_{n\to\infty} x_n$ , for all $x = (x_n)_{n\in\mathbb{N}} \in c$.
3. $\Lambda(Sx) = \Lambda x$, for all $x \in \ell^\infty$. Here $S(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots)$ for all $(x_n)_{n\in\mathbb{N}} \in \ell^\infty$, and $S$ is known as *the left shift operator.*

It can be verified that the previous definition is equivalent to the one originally given by Banach [3], which we present next:

**Definition 2.1.2.** A *Banach limit* $\Lambda : \ell^\infty \to \mathbb{R}$ can also be defined as a linear functional such that

1. $\Lambda(x) \geq 0$ for all $x = (x_n)_n \in \ell^\infty$ such that $x_n \geq 0$ for all $n \in \mathbb{N}$.
2. $\Lambda(\mathbb{1}) = 1$.
3. $\Lambda(Sx) = \Lambda x$, for all $x \in \ell^\infty$.

Definition 2.1.1 admits complex-valued sequences, and even vector-valued sequences, which we will consider in Chapter 4.

We will use definition 2.1.1 hereafter, when working with Banach limits.

**Lemma 2.1.2.** *$\Psi : \ell^\infty \to \mathbb{R}$ above is a Banach limit.*

*Proof.* We already know that $\|\Psi\|_{(\ell^\infty)^*} = 1$. Also, since for any $x \in c$, we have that $x \in Ces$ and $\psi(x) = \lim_{n\to\infty} x_n$, we see the second condition of definition 2.1.1 also holds.

Next, consider $y = x - Sx = (x_1 - x_2, x_2 - x_3, \ldots)$. For all $n \in \mathbb{N}$, we have that

$$\begin{aligned}(Cy)_n &= \frac{y_1 + y_2 + \ldots + y_n}{n} \\ &= \frac{x_1 - x_2 + x_2 - x_3 + \ldots + x_n - x_{n+1}}{n} \\ &= \frac{x_1 - x_{n+1}}{n}.\end{aligned}$$

So,

$$|(Cy)_n| \leq \frac{2\|x\|_\infty}{n} \to 0 \text{ as } n \to \infty.$$

Therefore $y \in Ces_0 := \{z \in Ces : \lim_{n\to\infty} (Cz)_n = 0\}$. Thus, $\Psi(x - Sx) = \psi(x - Sx) = 0$, implying $\Psi(x) = \Psi(Sx)$. □

*Remark.* Recall that by definition all Banach limits are left-shift invariant, but not every Banach limit is Cesàro averaging invariant, or even preserves Cesàro convergence. In Jeromy Sivek's PhD Thesis [23], the following results were discussed:

*Definition* 2.1.3. A bounded sequence $(x_n)_n$ is called *almost convergent* if there is some number $s$ such that $\Lambda((x_n)_n) = s$ for every Banach limit $\Lambda$.

The following criterion for almost convergent is due to Lorentz [15]:

*Theorem* 2.1.3. *The bounded sequence $(x_n)_n$ is almost convergent if and only if*

$$\lim_{p\to\infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = s$$

*holds uniformly in $n$.*

We use this criterion to present an example of a Cesàro convergent sequence for which there exists a Banach limit which does not preserve Cesàro convergence: we will study a sequence $a$ which is not almost convergent, that is, for any number $s$ there exists at least one Banach limit $\Lambda$ for which $\Lambda(a) = s$ does not hold. The sequence $a$ is given by

$$a = (1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, 1, \ldots),$$

which has monotonically increasingly long sections of 1s and $-1$s in the pattern indicated. We check that $a$ fails to be almost convergent: let $p$ be arbitrarily large and choose $n$ to be the first index of a section of 1s of length $p$ or greater. As $p \to \infty$, $\dfrac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p}$ goes to 1. Similarly, letting $n$ be the first index of a similarly long block of $-1$s, we get that $\dfrac{a_n + a_{n+1} + \cdots + a_{n+p-1}}{p}$ goes to $-1$ as $p \to \infty$. This shows $a$ fails the second condition of Theorem 2.1.3, and is therefore not almost convergent. So, for $s = 0$ there is at least one Banach limit $\Lambda$ for which $\Lambda(a) \neq 0$. But it turns out that the Cesàro average of $a$, $Ca$, converges to 0, as it is shown in [23].

In [23], the following version of the Hahn-Banach extension theorem from Royden's *Real Analysis* [18] was used to prove the existence of a Banach limit that preserves Cesàro convergence:

**Theorem 2.1.4.** *Let $X$ be a vector space and let $V$ be a subspace of $X$. Let $p : X \longrightarrow \mathbb{R}$ be subadditive and positive homogeneous. Let $f : V \longrightarrow \mathbb{R}$ be linear with the property that $f(v) \leq p(v)\ \forall v \in V$.*
*Let $G$ be an Abelian semigroup of linear operators on $X$ such that for every $A \in G$ we have $p(Ax) \leq p(x)\ \forall x \in X$, while for every $v \in V$ we have $Av \in V$ and $f(Av) = f(v)$. Then there exists an extension $F$ of $f$ to a linear functional on $X$ such that $F(x) \leq p(x)$ and $F(Ax) = F(x)$ for all $x \in X$.*

In fact, this Theorem was used to prove the existence of a Banach limit on the space ${}^{\ell^\infty}/_{c_0}$, which induces a Banach limit on $\ell^\infty$, such that it is invariant under any composition of any number of the Cesàro operator $C$, and the Pascal operator $P$. This was obtained by applying the theorem to $X = {}^{\ell^\infty}/_{c_0}$, $V := \{[x] \in X : x = (a, a, a, \ldots)\}$ (which is the equivalence classes of elements of $c$ inside $X$), the semigroup $G$ acting on $X$, where $G$ is the associated semigroup of $G_0$ -the semigroup of linear operators on $\ell^\infty$ generated by $\{I, S, C, P\}$, $p = \|\cdot\|_X$, and $f : V \longrightarrow \mathbb{R}$ defined by $f[x] = a$, where $x = (a, a, a, \ldots)$. The definition of the Pascal operator is as follows:

Given any sequence $x = (x_1, x_2, \ldots)$, define $P(x)$ to be the sequence whose $n$th term is

$$P_n x = \sum_{k=1}^{n} \frac{1}{2^{n-1}} \binom{n-1}{k-1} x_k.$$

We say $x$ is Pascal convergent if $\lim_{n\to\infty} P_n x$ exists. It can be shown that if $x \in c$, then $\lim_{n\to\infty} P_n x = \lim_{n\to\infty} x_n$.

*Remark.* The construction of an element $x \in \ell^\infty$ and a corresponding Banach limit $\Psi$, such that $\Psi$ preserves Cesàro convergence, but still $\Psi(x) \neq \Psi(Cx)$ (and therefore $\Psi$ is not Cesàro invariant), can be found in [5].

### 2.2 A stronger Banach limit on $\ell^\infty$

Using similar ideas to the ones discussed so far, we next introduce a "stronger" Banach limit on $\ell^\infty$:

**Claim 2.2.1.** *For all $x \in \ell^\infty$ we define*

$$H(x) := \Psi(\Psi(x), \Psi(Cx), \Psi(C^2x), \Psi(C^3x), \ldots). \tag{†}$$

*$H$ is a Banach limit.*

*Remark.* Before presenting the proof of this claim, we make the remark that $H$ is a Cesàro invariant Banach limit. This is easy to see, since for any $x \in \ell^\infty$ we have that

$$\begin{aligned} H(Cx) &= \Psi(\Psi(Cx), \Psi(C^2x), \Psi(C^3x), \ldots) \\ &= \Psi(\Psi(x), \Psi(Cx), \Psi(C^2x), \Psi(C^3x), \ldots) \\ &= H(x), \end{aligned}$$

where the second equality holds since $\Psi$ is left-shift invariant. Therefore, $H$ preserves Cesàro convergence: if $x \in Ces$, that is $x \in \ell^\infty$ and $\lim_{n\to\infty}(Cx)_n$ exists, then $H(x) = H(Cx) = \lim_{n\to\infty}(Cx)_n$. This last statement is true since $H$ is a Banach limit and therefore it preserves classical convergence.

*Proof.* Fix $x \in \ell^\infty$,

$$\begin{aligned}
|H(x)| &\leq \|\Psi\|_{(\ell^\infty)^*} \| \big(\Psi(x), \Psi(Cx), \Psi(C^2x), \dots\big) \|_\infty \\
&= 1 \cdot \| \big(\Psi(x), \Psi(Cx), \Psi(C^2x), \dots\big) \|_\infty \\
&= \sup\{|\Psi(x)|, |\Psi(Cx)|, |\Psi(C^2x)|, \dots\} \\
&\leq \sup\{\|\Psi\|_{(\ell^\infty)^*}\|x\|_\infty, \|\Psi\|_{(\ell^\infty)^*}\|Cx\|_\infty, \|\Psi\|_{(\ell^\infty)^*}\|C^2x\|_\infty, \dots\} \\
&= \sup\{\|x\|_\infty, \|Cx\|_\infty, \|C^2x\|_\infty, \dots\}.
\end{aligned}$$

Notice $\|Cx\|_\infty \leq \|x\|_\infty$, therefore $\|C^n x\|_\infty \leq \|x\|_\infty$ for all $n \in \mathbb{N}$, so $\|H\|_{(\ell^\infty)^*} \leq 1$. On the other hand, let $\mathbb{1} := (1, 1, 1, \dots) \in \ell^\infty$, then

$$H(\mathbb{1}) = \Psi\left(\Psi(\mathbb{1}), \Psi(C\mathbb{1}), \Psi(C^2\mathbb{1}), \dots\right).$$

Since $C^n\mathbb{1} = \mathbb{1}$ for all $n$, and $\Psi(\mathbb{1}) = \psi(\mathbb{1}) = 1$, we get $H(\mathbb{1}) = \Psi(\mathbb{1}) = 1$. Therefore $\|H\|_{(\ell^\infty)^*} = 1$. So, condition 1 of the Banach limit definition holds.

To verify condition 2 holds, first notice that for an arbitrary $x \in c$ we have that $\lim_{n\to\infty}(Cx)_n = \lim_{n\to\infty} x_n$. Therefore, for every $x \in c$

$$\lim_{n\to\infty}(C^k x)_n = \lim_{n\to\infty} x_n, \text{ for all } k \in \mathbb{N}.$$

Thus $\Psi(C^k x) = \lim_{n\to\infty} x_n$, for all $k \in \mathbb{N}$. Denote $L := \lim_{n\to\infty} x_n$. Then,

$$H(x) = \Psi(\Psi(x), \Psi(Cx), \Psi(C^2x), \Psi(C^3x), \dots) = \Psi(L, L, L, \dots) = L.$$

Finally, to verify condition 3 holds, we use the following fact, which we later present as a claim: $\Psi(C^n Sx) = \Psi(C^n x)$, *for all* $n \in \mathbb{N}$ *and for all* $x \in \ell^\infty$.

It is clear then that condition 3 holds, since

$$\begin{aligned}
H(Sx) &:= \Psi(\Psi(Sx), \Psi(CSx), \Psi(C^2Sx), \Psi(C^3Sx), \dots) \\
&= \Psi(\Psi(x), \Psi(Cx), \Psi(C^2x), \Psi(C^3x), \dots) \\
&= H(x),
\end{aligned}$$

where the second identity holds by the claim we are assuming, and by the fact that $\Psi(Sx) = \Psi(x)$, since $\Psi$ is a Banach limit, and therefore left-shift invariant. So, $H$ is a Banach limit. □

It remains to prove the claim used to prove condition 3, which we present next:

**Claim 2.2.2.** *$\Psi(C^nSx) = \Psi(C^nx)$, for all $n \in \mathbb{N}$ and for all $x \in \ell^\infty$.*

*Proof.* We proceed by induction over $n \in \mathbb{N}$. We start by showing that for arbitrary $x \in \ell^\infty$, $w_x := CSx - SCx \in c_0$. This will imply that $CSx = SCx + w_x$, and so we get $\Psi(CSx) = \Psi(SCx + w_x) = \Psi(SCx) + \Psi(w_x) = \Psi(SCx)$. Where the last equality holds since $w_x \in c_0$ implies that $\Psi(w_x) = 0$. Since $\Psi$ is left-shift invariant, we would have that $\Psi(CSx) = \Psi(SCx) = \Psi(Cx)$.

Fix $x \in \ell^\infty$, to prove $w_x := CSx - SCx \in c_0$, notice that

$$CSx - SCx = \left(x_2, \frac{x_2 + x_3}{2}, \frac{x_2 + x_3 + x_4}{3}, \ldots\right) - \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots\right).$$

Thus,

$$\begin{aligned}
|(CSx - SCx)_n| &= \left|\frac{x_2 + \ldots + x_{n+1}}{n} - \frac{x_1 + \ldots + x_{n+1}}{n+1}\right| \\
&= \left|\frac{n(x_2 + \ldots + x_{n+1}) + (x_2 + \ldots + x_{n+1}) - (nx_1 + n(x_2 + \ldots + x_{n+1}))}{n(n+1)}\right| \\
&= \left|\frac{-nx_1 + x_2 + \ldots + x_{n+1}}{n(n+1)}\right| \\
&\leq \frac{n\|x\|_\infty + n\|x\|_\infty}{n(n+1)} \\
&= \frac{2\|x\|_\infty}{n+1} \to 0 \textit{ as } n \to \infty.
\end{aligned}$$

Next, for the inductive step we present two proofs:

First proof: Assume that for a fixed $n \in \mathbb{N}$ we have that $\Psi(C^nSz) = \Psi(C^nz)$, for each $z \in \ell^\infty$. Then, for arbitrary $x \in \ell^\infty$ we have that

$$\begin{aligned}
\Psi(C^{n+1}Sx) &= \Psi\left(C^n(CSx)\right) \\
&= \Psi\left(C^n(SCx + w_x)\right) \\
&= \Psi\left(C^n(SCx)\right) + \Psi\left(C^n(w_x)\right).
\end{aligned}$$

Here $w_x := CSx - SCx$. We showed in the base case that $w_x \in c_0$, therefore $C^n(w_x) \in c_0$. So, we have that

$$\begin{aligned}
\Psi(C^{n+1}Sx) &= \Psi\left(C^n(SCx)\right) \\
&= \Psi\left(C^n(Cx)\right) \\
&= \Psi\left(C^{n+1}x\right),
\end{aligned}$$

where the second to last equality holds by the inductive hypothesis.

Second proof: We want to show that, if for a fixed $n \in \mathbb{N}$ we have $(C^n S - SC^n)(\ell^\infty) \subseteq c_0$, then this implies that $(C^{n+1}S - SC^{n+1})(\ell^\infty) \subseteq c_0$. Fix $x \in \ell^\infty$, then

$$\begin{aligned} C^{n+1}Sx - SC^{n+1}x &= C(C^n Sx) - CSC^n x + CSC^n x - SC(C^n x) \\ &= C(C^n Sx - SC^n x) + (CS - SC)(C^n x). \end{aligned}$$

We can see this last expression is in $c_0$ since by base case the second term of the addition is in $c_0$, and by inductive hypothesis and the fact that the operator $C$ preserves limits, we have that the first term in the sum is also in $c_0$.

By Principle of Mathematical Induction, we have that for any $n \in \mathbb{N}$ and any $x \in \ell^\infty$, $C^n Sx - SC^n x \in c_0$ holds. This implies $\Psi(C^n Sx) = \Psi(SC^n x)$, and since $\Psi$ is shift invariant, we get $\Psi(C^n Sx) = \Psi(C^n x)$. The Claim has been proven. □

**Open Question 2.2.3.** *In the next chapter, we define fractional powers of the Cesàro averaging operator $J$ corresponding to the function space $L^\infty(0, \infty)$. We define these fractional powers, $J^r$ for $r > 0$, in such a way that they extend the concept of taking iterates of the Cesàro operator. Also, this new family of operators have the semigroup property, that is $J^r(J^s(\cdot)) = J^{r+s}(\cdot)$, for all $r, s > 0$. What is a possible definition of fractional powers of the Cesàro averaging operator $C$ for the sequence space $\ell^\infty$? We require for these operators, to have the semigroup property.*

## 3.0 Cesàro averaging and new invariant Banach limits on $L^\infty(0,\infty)$

### 3.1 Notation and preliminaries

Let $L^\infty = L^\infty(0,\infty)$ be the Banach space of (classes of) essentially bounded, real-valued Lebesgue measurable functions $f$ on $(0,\infty)$, equipped with the uniform norm

$$\|f\|_\infty := \operatorname{ess\,sup} |f(t)|,$$

where the supremum is taking over all $t > 0$.

The integrals we consider in this document, are with respect to the Lebesgue measure on $(0,\infty)$.

The dual space of $L^\infty$, $(L^\infty)^*$, consisting of all continuous linear functionals on $L^\infty$, is isometrically isomorphic to $fa(m)$, the space of finitely additive measures on the Borel subsets of $(0,\infty)$ that vanish on $m$-null sets (here $m$ is Lebesgue measure). Moreover, $fa(m)$ is isometrically isomorphic to $L^1(0,\infty)\bigoplus_1 pfa(m)$, where $pfa(m)$ is the space of purely finitely additive measures on the Borel subsets of $(0,\infty)$ that vanish on $m$-null sets (see [27] or [8]).

We next present two examples on how the Hahn-Banach theorem can be applied to obtain extensions of functionals on $L^\infty$:

On $(L^\infty, \|\cdot\|_\infty)$, we consider the closed vector subspace

$$BC := \{f \in L^\infty : f \textit{ is continuous } \},$$

and the closed vector subspace

$$BC_L := \Big\{ f \in L^\infty : f \textit{ is continuous and } \lim_{x\to\infty} f(x) \textit{ exists in } \mathbb{R} \Big\}.$$

For an element $f \in L^\infty$, when we write $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$ , we mean the following: For every $\epsilon > 0$, there exists $N > 0$ such that $\operatorname{ess\,sup}_{x\geq N} |f(x) - L| \leq \epsilon$.

On $BC_L$, we can define the linear map $\varphi : BC_L \to \mathbb{R}$ given by $\varphi(f) = \lim_{x\to\infty} f(x)$. By Hahn-Banach Extension Theorem, there exists $\Phi \in (L^\infty)^*$ such that

$$\|\Phi\|_{(L^\infty)^*} = \|\varphi\|_{BC_L^*} = 1 \text{ and } \Phi\,|_{BC_L} = \varphi.$$

Now consider the closed vector subspace of $L^\infty$

$$Ces_L := \left\{ g \in L^\infty : \psi(g) := \lim_{x\to\infty} \frac{1}{x}\int_0^x g(t)dt \text{ exists in } \mathbb{R} \right\}.$$

We define the linear operator $J : L^\infty \to BC$ by

$$(Jf)(x) := \frac{1}{x}\int_0^x f(t)dt, \text{ for all } x \in (0,\infty), \text{ for all } f \in L^\infty.$$

$Jf$ is called *the Cesàro average of* $f$. It is easy to see $Jf \in L^\infty$ since

$$\left| \frac{1}{x}\int_0^x f(t)dt \right| \leq \frac{1}{x}x\|f\|_\infty = \|f\|_\infty \text{ for almost all } x \in (0,\infty).$$

$(Jf)(x)$ is continuous since both $\frac{1}{x}$ and $\int_0^x f(t)dt$ are continuous on $(0,\infty)$ (the last expression is Lipschitz continuous with Lipschitz constant $\|f\|_\infty$).

By Hahn-Banach Extension Theorem, there exists $\Psi \in (L^\infty)^*$ such that

$$\|\Psi\|_{(L^\infty)^*} = \|\psi\|_{Ces_L^*} = 1 \text{ and } \Psi\mid_{Ces_L} = \psi.$$

**Claim 3.1.1.** *If* $\lim_{x\to\infty} f(x)$ *exists and equals* $L \in \mathbb{R}$*, then* $\lim_{x\to\infty}(Jf)(x) = L$.

*Proof.* For an arbitrary $N > 0$,

$$\begin{aligned}\left| \frac{1}{x}\int_{t=0}^x f(t)dt - L \right| &= \left| \frac{1}{x}\int_{t=0}^x (f(t) - L)dt \right| \\ &\leq \frac{1}{x}\int_{t=0}^N \mid f(t) - L \mid dt + \frac{1}{x}\int_{t=N}^x \mid f(t) - L \mid dt.\end{aligned}$$

Now, for any $\epsilon > 0$, there exists $N_\epsilon := N > 0$ such that $\mid f(t) - L \mid < \epsilon$ for almost all $t \geq N$, and therefore

$$\left| \frac{1}{x}\int_{t=0}^x f(t)dt - L \right| \leq \frac{N}{x}(2\|f\|_\infty) + \frac{x-N}{x}\epsilon.$$

Therefore, by taking limit superior as $x$ approaches to $\infty$ and then letting $\epsilon$ go to 0, we get that $\psi(f) = \lim_{x\to\infty}(Jf)(x) = \lim_{x\to\infty} f(x)$. ∎

**Definition 3.1.1.** For every $r \in [0,\infty)$, and every $f \in L^\infty$, we define

$$S_r f := f_r := (f(x+r))_{x\in(0,\infty)},$$
$$f_r(x) := f(x+r) \text{ for all } x \in (0,\infty).$$

So $S_r$ is *the left shift by* $r$ *operator* on $L^\infty$.

**Claim 3.1.2.** *For every* $r \in [0,\infty)$ *and every* $f \in L^\infty$*,* $\Psi(f) = \Psi(S_r f)$.

*Proof.* We will show that for every $r \in [0, \infty)$, and every $f \in L^\infty$,

$$K_f := f - f_r = f - S_r f \in Ces_{L,0}.$$

Here $Ces_{L,0} := \{g \in Ces_L : \psi(g) = 0\}$. So, we wish to show that

$$\frac{1}{x}\left[\int_{t=0}^{x} (f(t) - f(t+r))\, dt\right] \to 0 \text{ as } x \to \infty.$$

Notice

$$\begin{aligned}
\frac{1}{x}\left[\int_{t=0}^{x} f(t)dt - \int_{t=0}^{x} f(t+r)dt\right] &= \frac{1}{x}\left[\int_{t=0}^{r} f(t)dt + \int_{t=r}^{x} f(t)dt - \int_{u=r}^{x+r} f(u)du\right] \\
&= \frac{1}{x}\left[\int_{t=0}^{r} f(t)dt - \int_{t=x}^{x+r} f(t)dt\right] \\
&\leq \frac{1}{x}\left[r\|f\|_\infty + r\|f\|_\infty\right] \to 0 \;\; as \; x \to \infty.
\end{aligned}$$

So, $\psi(f - S_r f) = 0$, which implies $\Psi(f - S_r f) = 0$, and therefore $\Psi(f) = \Psi(S_r f)$. □

As in the previous chapter, we are interested in obtaining extensions of functionals that are also Banach limits. We define Banach limits on $L^\infty$ next:

**Definition 3.1.2.** A continuous linear functional $\Lambda : L^\infty \to \mathbb{R}$ is a *Banach limit on* $L^\infty$ if

1. $\|\Lambda\|_{(L^\infty)^*} = 1$.
2. $\Lambda(f) = \lim_{x\to\infty} f(x)$ for all $f \in BC_L$.
3. $\Lambda(S_r f) = \Lambda(f)$, for all $f \in L^\infty$, and for all $r \in (0, \infty)$; where $S_r f(x) = f(x + r)$.

**Lemma 3.1.3.** *The linear functional* $\Psi$ *defined above is a Banach limit on* $L^\infty$*.*

*Proof.* We already know condition 1 from the definition of Banach limit on $L^\infty$ holds. By the previous claim, we also have condition 3. Next recall that by Claim 3.1.1, if $f \in BC_L$, then $f \in Ces_L$ with $\psi(f) = \lim_{x\to\infty} f(x)$. Therefore $\Psi(f) = \psi(f) = \lim_{x\to\infty} f(x)$. □

### 3.2 A stronger Banach limit on $L^\infty$

Consider an arbitrary Banach limit $\sigma$ on $\ell^\infty$. We will use $\sigma$ to define a Banach limit on $L^\infty$, in analogy with equation (†) defined in the previous chapter:

**Theorem 3.2.1.** *Fix $\sigma$ a Banach limit on $\ell^\infty$. Define $\Delta : L^\infty \to \mathbb{R}$ by*

$$\Delta(f) := \sigma(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \dots) \text{ for all } f \in L^\infty.$$

*$\Delta$ is a Banach limit on $L^\infty$.*

*Proof.* Take an arbitrary $f \in L^\infty$.

$$\begin{aligned}
|\Delta(f)| &\leq \|\sigma\|_{(\ell^\infty)^*} \|(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \dots)\|_\infty \\
&= 1 \|(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \dots)\|_\infty \\
&= \sup\{|\Psi(f)|, |\Psi(Jf)|, |\Psi(J^2 f)|, |\Psi(J^3 f)|, \dots\} \\
&\leq \sup\{\|\Psi\|_{(L^\infty)^*} \|f\|_\infty, \|\Psi\|_{(L^\infty)^*} \|Jf\|_\infty, \|\Psi\|_{(L^\infty)^*} \|J^2 f\|_\infty, \dots\} \\
&= \sup\{\|f\|_\infty, \|Jf\|_\infty, \|J^2 f\|_\infty, \dots\}.
\end{aligned}$$

Since $\|Jf\|_\infty \leq \|f\|_\infty$ for every $f \in L^\infty$, we have that $\|J^n f\|_\infty \leq \|f\|_\infty$, for all $n \in \mathbb{N}$ and all $f \in L^\infty$. Therefore

$$|\Delta(f)| \leq \|f\|_\infty \text{ for all } f \in L^\infty,$$

which implies $\|\Delta\|_{(L^\infty)^*} \leq 1$. On the other hand, let $\mathbb{1}(x) = 1$, for all $x \in (0, \infty)$. Then,

$$\Delta(\mathbb{1}) = \sigma(\Psi(\mathbb{1}), \Psi(\mathbb{1}), \Psi(\mathbb{1}), \Psi(\mathbb{1}), \dots) = \sigma(1, 1, 1, \dots) = 1.$$

Therefore $\|\Delta\|_{(L^\infty)^*} \geq 1$. Thus $\|\Delta\|_{(L^\infty)^*} = 1$.

Next, let $f \in BC_L$ and let $L := \lim_{x\to\infty} f(x)$, then $\lim_{x\to\infty} Jf(x) = L$, and therefore $\lim_{x\to\infty} J^n f(x) = L$, for each $n \in \mathbb{N}$. This implies that $\Psi(J^n f) = L$, for each $n \in \mathbb{N}$. So, we get that

$$\Delta(f) = \sigma(L, L, L, \dots) = L.$$

Finally, to show that condition 3 of the definition of Banach limit on $L^\infty$ holds, we use the following fact, which we will prove later as a claim: *For every $r > 0$ and $f \in L^\infty$ we have that $\Psi(J^n S_r f) = \Psi(J^n f)$, for all $n \in \mathbb{N}$.* Also notice $\Psi(S_r f) = \Psi(f)$ since $\Psi$ is a Banach limit, and therefore left-shift invariant.

Condition 3 follows from this, since we have that

$$\begin{aligned}\Delta(S_r f) &:= \sigma(\Psi(S_r f), \Psi(JS_r f), \Psi(J^2 S_r f), \Psi(J^3 S_r f), \dots) \\ &= \sigma(\Psi(f), \Psi(Jf), \Psi(J^2 f), \Psi(J^3 f), \dots) \\ &= \Delta(f).\end{aligned}$$

□

It only remains to prove the claim used for condition 3. We present its proof next:

**Claim 3.2.2.** *For every $r > 0$ and $f \in L^\infty$ we have that $\Psi(J^n S_r f) = \Psi(J^n f)$, for all $n \in \mathbb{N}$.*

*Proof.* Fix $r > 0$ and $f \in L^\infty$.

We proceed by induction over $n$: For $n = 1$, we first want to show that $h := S_r Jf - JS_r f \in BC_{L,0}$, the set of continuous bounded functions on $(0, \infty)$ whose limit is 0 as $x$ tends to infinity.

$$\begin{aligned}h(x) &= S_r\left(\frac{1}{x}\int_{t=0}^{x} f(t)dt\right) - \frac{1}{x}\int_{t=0}^{x}(S_r f)(t)dt \\ &= \frac{1}{x+r}\int_{t=0}^{x+r} f(t)dt - \frac{1}{x}\int_{t=0}^{x} f(t+r)dt \\ &= \frac{1}{x+r}\int_{t=0}^{x+r} f(t)dt - \frac{1}{x}\int_{s=r}^{x+r} f(s)ds \\ &= \frac{1}{x+r}\int_{t=0}^{x+r} f(t)dt - \frac{1}{x}\left(-\int_{t=0}^{r} f(t)dt + \int_{t=0}^{x+r} f(t)dt\right) \\ &= \left(\frac{1}{x+r} - \frac{1}{x}\right)\int_{t=0}^{x+r} f(t)dt + \frac{1}{x}\int_{t=0}^{r} f(t)dt \\ &= \frac{-r}{x(x+r)}\int_{t=0}^{x+r} f(t)dt + \frac{1}{x}\int_{t=0}^{r} f(t)dt.\end{aligned}$$

Therefore

$$|h(x)| \leq \frac{r}{x(x+r)}(x+r)\|f\|_\infty + \frac{r}{x}\|f\|_\infty = \frac{2r}{x}\|f\|_\infty \to 0 \text{ as } x \to \infty.$$

So, $\lim_{x\to\infty} h(x) = 0$. Now, since $\Psi$ preserves limit as $x$ tends to infinity, we get that $\Psi(h) = 0$, which implies $\Psi(JS_rf) = \Psi(S_rJf)$. Finally since $\Psi$ is left-shift invariant we get $\Psi(JS_rf) = \Psi(Jf)$, which is the desired conclusion for the base case of our induction.

For the inductive step, we present two proofs, similar to the ones presented for $\ell^\infty$:

First proof: Assume that for a fixed $n \in \mathbb{N}$ we have that $\Psi(J^nS_rf) = \Psi(J^nf)$ holds for every $f \in L^\infty$, and notice that

$$\begin{aligned}\Psi(J^{n+1}S_rf) &= \Psi(J^n(JS_rf)) \\ &= \Psi(J^n(S_rJf - h)) \\ &= \Psi(J^n(S_rJf)) - \Psi(J^nh) \\ &= \Psi(J^{n+1}f) - \Psi(J^nh),\end{aligned}$$

where $h$ is defined as in the base case, $h(x) := (S_rJf - JS_rf)(x)$. We already checked that $\lim_{x\to\infty} h(x) = 0$, and this implies that $\lim_{x\to\infty} (J^nh)(x) = 0$ for each $n \in \mathbb{N}$. Therefore $\Psi(J^nh) = 0$, so we get what we wanted.

Second proof: We want to show that, if for a fixed $n \in \mathbb{N}$ we have that $J^nS_r - S_rJ^n$ is an operator on $L^\infty$ to $BC_{L,0}$, then $J^{n+1}S_r - S_rJ^{n+1}$ is also an operator on $L^\infty$ to $BC_{L,0}$: Let $f \in L^\infty$ arbitrary.

$$\begin{aligned}J^{n+1}S_rf - S_rJ^{n+1}f &= J(J^nS_rf) - JS_rJ^nf + JS_rJ^nf - S_rJ(J^nf) \\ &= J(J^nS_rf - S_rJ^nf) + (JS_r - S_rJ)(J^nf),\end{aligned}$$

and this last expression is clearly in $BC_{L,0}$ by base case and inductive hypothesis.

By Principle of Mathematical Induction we have that for any $r > 0$ and any $f \in L^\infty$, $J^nS_rf - S_rJ^nf \in BC_{L,0}$ holds for all $n \in \mathbb{N}$, which implies $\Psi(J^nS_rf) = \Psi(S_rJ^nf)$, and since $\Psi$ is shift invariant we get $\Psi(J^nS_rf) = \Psi(J^nf)$ for all $f \in L^\infty$. □

*Remark.* The Banach limit $\Delta$ defined in Theorem 3.2.1, is Cesàro averaging invariant. That is, $\Delta(f) = \Delta(Jf)$. This is true since $\sigma$ is a Banach limit and therefore it is left-shift invariant. Therefore, we obtain that $\Delta$ is a Banach limit that preserves Cesàro convergence.

## 3.3 Defining the fractional Cesàro averaging operator on $L^\infty$

**Goal:** We want to define the $r$-power of the operator $J$ on $L^\infty$, for each $r > 0$, to generalize the concept of iterates of the $J$ operator, and in such a way that the family of these operators indexed by $r > 0$, forms a commutative semigroup of operators on $L^\infty$. For this, we present two different approaches:

### 3.3.1 Approach 1: Generalizing a formula for iterated Cesàro integration

In the literature, the following well known operator [10] can be found:

**Definition 3.3.1.** For $\alpha > 0$, the *Hadamard fractional integral* is an operator that is defined by

$$\left(\mathcal{F}_+^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-\alpha}} f(t) \frac{dt}{t}, \text{ for } x > a.$$

This definition comes from generalizing the following formula:

$$\int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} \cdots \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n = \frac{1}{(n-1)!} \int_a^x \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-n}} f(t) \frac{dt}{t}.$$

It is known that the operators $\mathcal{F}_+^\alpha$ admit the semigroup property, that is $\mathcal{F}_+^\alpha \mathcal{F}_+^\beta f = \mathcal{F}_+^{\alpha+\beta} f$, under appropriate assumptions of the function $f$ and the exponents $\alpha$ and $\beta$.

In [12], the author introduced a new operator that generalizes both the Hadamard and the Riemann-Liouville fractional integral. The latter is another well known fractional integral obtained by generalizing the Cauchy's formula for repeated integration. An overview of fractional calculus can be found in the book [17].

Inspired by these fractional integrals, we show the following Claim:

**Claim 3.3.1.** *For $n \in \mathbb{N}$, the nth iteration of applying the Cesàro averaging operator to a function $f \in L^\infty$ is given by the following formula:*

$$\int_0^x \frac{dt_1}{x} \int_0^{t_1} \frac{dt_2}{t_1} \cdots \int_0^{t_{n-1}} f(t_n) \frac{dt_n}{t_{n-1}} = \frac{1}{(n-1)!} \frac{1}{x} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{n-1} f(t) dt.$$

*Proof.* We proceed by induction on n:

If $n = 1$, then

$$\int_0^x f(t_1)\frac{dt_1}{x} = \frac{1}{0!}\frac{1}{x}\int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^0 f(t)dt.$$

Next, fix $n \in \mathbb{N}$, $n > 1$, and let

$$J^{(n-1)}f(x) := \int_0^x \frac{dt_1}{x}\int_0^{t_1}\frac{dt_2}{t_1}\dots\int_0^{t_{n-2}} f(t_{n-1})\frac{dt_{n-1}}{t_{n-2}}.$$

Assume

$$J^{(n-1)}f(x) = \frac{1}{(n-2)!}\frac{1}{x}\int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{n-2} f(t)dt.$$

Then

$$\begin{aligned} J^n f(x) &:= J(J^{(n-1)}f)(x) \\ &= \frac{1}{x}\int_0^x \frac{1}{(n-2)!}\frac{1}{u}\int_0^u \left[\ln\left(\frac{u}{t}\right)\right]^{n-2} f(t)dtdu \\ &= \frac{1}{x}\frac{1}{(n-2)!}\int_0^x\int_t^x \frac{1}{u}\left[\ln\left(\frac{u}{t}\right)\right]^{n-2} f(t)dudt. \end{aligned}$$

In this last step, we were able to apply Fubini-Tonelli to change the order of integration since the integrand is a measurable function, and

$$\begin{aligned} \frac{1}{x}\int_0^x\int_0^u \frac{1}{u}\left|\left[\ln\left(\frac{u}{t}\right)\right]^{n-2} f(t)\right| dtdu &\le \|f\|_\infty \frac{1}{x}\int_0^x\int_0^u \frac{1}{u}\left[\ln\left(\frac{u}{t}\right)\right]^{n-2} dtdu \\ &= \|f\|_\infty \frac{1}{x}\int_0^x\int_0^\infty \frac{1}{u}y^{n-2}ue^{-y}dydu \\ &= \|f\|_\infty \frac{1}{x}\int_0^x \Gamma(n-1)du \\ &= \|f\|_\infty\Gamma(n-1) < \infty. \end{aligned}$$

Here, we made the substitution $y = \ln\left(\frac{u}{t}\right)$. We will continue to make use of this substitution throughout this document. We also note that the expression $\Gamma(n-1)$ is the Gamma function evaluated at $n-1$, which equals $(n-2)!$, and therefore is finite. We will discuss the Gamma function in more detail later on in this subsection.

So, after applying Fubini-Tonelli we obtain

$$\begin{aligned} J^n f(x) &:= J(J^{(n-1)}f)(x) \\ &= \frac{1}{x}\frac{1}{(n-2)!}\int_0^x\int_t^x \frac{1}{u}\left[\ln\left(\frac{u}{t}\right)\right]^{n-2} f(t)dudt \\ &= \frac{1}{x}\frac{1}{(n-2)!}\int_0^x\int_0^{\ln\left(\frac{x}{t}\right)} [y]^{n-2} f(t)dydt \\ &= \frac{1}{x}\frac{1}{(n-1)!}\int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{n-1} f(t)dt. \end{aligned}$$

Where again we made the substitution $y = \ln\left(\frac{u}{t}\right)$. $\square$

Based on the previous result, we will define the $r$ power of the operator $J$, where $r > 0$. But first, following [19], we will give an overview of the Gamma function -which is the extension of the factorial function to positive real numbers, and the Beta function.

**The Gamma and the Beta function.** For $0 < x < \infty$, *the Gamma function* is defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt.$$

This integral converges for these values of $x$.

The equation

$$\Gamma(x+1) = x\Gamma(x)$$

holds for $0 < x < \infty$.

We also have that

$$\Gamma(n+1) = n! \text{ for } n = 1, 2, 3, \ldots.$$

For $x > 0$ and $y > 0$, *the Beta function* is defined by

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

The following equation holds for $x > 0$ and $y > 0$:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

As mentioned before, the Gamma function is the extension of the factorial function to positive real numbers, and so we use it to define the $r$ power of the operator $J$, where $r > 0$. In analogy with the identity in Claim 3.3.1, we give the following definition for fractional powers of the operator $J$:

**Definition 3.3.2.** For $r > 0$, define the operator $\mathcal{I}^r$ by

$$(\mathcal{I}^r f)(x) := \frac{1}{\Gamma(r)} \frac{1}{x} \int_{t \in (0,x)} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-r}} dt,$$

for $f \in L^\infty$, and $x \in (0, \infty)$.

Next, we show that the family of operators $(\mathcal{I}^r)_{r>0}$ has the semigroup property:

**Theorem 3.3.2.** *$\mathcal{I}^r\mathcal{I}^p f = \mathcal{I}^{r+p} f$ for all $f \in L^\infty$, for all $p, r > 0$.*

*Proof.* Fix $f \in L^\infty$, and fix $p, r > 0$.

By making use of the identity $\Gamma(p)\Gamma(r) = B(p,r)\Gamma(p+r)$, we can express $\left(\mathcal{I}^{p+r} f\right)(x)$ in the following way

$$\begin{aligned}
&\left(\mathcal{I}^{p+r} f\right)(x) \\
&= \frac{1}{\Gamma(p+r)} \frac{1}{x} \int_{s=0}^{x} f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds \\
&= \frac{1}{\Gamma(p)\Gamma(r)} \frac{1}{x} \int_{s=0}^{x} B(p,r) f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds \\
&= \frac{1}{\Gamma(p)} \frac{1}{\Gamma(r)} \frac{1}{x} \int_{s=0}^{x} \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left(\mathcal{I}^{p}(\mathcal{I}^{r} f)\right)(x) \\
&= \frac{1}{\Gamma(p)} \frac{1}{x} \int_{t=0}^{x} (\mathcal{I}^r f)(t) \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \\
&= \frac{1}{\Gamma(p)} \frac{1}{x} \int_{t=0}^{x} \frac{1}{\Gamma(r)} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \\
&= \frac{1}{\Gamma(p)} \frac{1}{\Gamma(r)} \frac{1}{x} \int_{t=0}^{x} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt.
\end{aligned}$$

So, we wish to show

$$\int_{s=0}^{x} \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds = \int_{t=0}^{x} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt.$$

Denote

$$I_1 := \int_{s=0}^{x} \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds,$$

and

$$I_2 := \int_{t=0}^{x} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt.$$

For $I_2$, we make the substitution $u = \frac{t}{x}$, to get

$$\begin{aligned}
&\int_{t=0}^{x} \frac{1}{t} \int_{s=0}^{t} f(s) \left[\ln\left(\frac{t}{s}\right)\right]^{r-1} ds \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \\
&= \int_{u=0}^{1} \frac{1}{u} \int_{s=0}^{xu} f(s) \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \left[\ln\left(\frac{1}{u}\right)\right]^{p-1} dsdu \\
&= \int_{s=0}^{x} f(s) \int_{u=\frac{s}{x}}^{1} \frac{1}{u} \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \left[\ln\left(\frac{1}{u}\right)\right]^{p-1} duds.
\end{aligned}$$

We were able to change the order of integration since the function

$$g(s,u) := \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \frac{(-\ln(u))^{p-1}}{u} \chi_{[0,ux]}(s)$$

is continuous a.e. on $[0,\infty) \times [0,1]$, $f$ is measurable, therefore $g(s,u)f(s)$ is measurable. Also, we have that

$$\begin{aligned}
&\int_{u=0}^{1} \int_{s=0}^{ux} |f(s)| \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} (-\ln(u))^{p-1} ds \frac{1}{u} du \\
&\leq \|f\|_\infty \int_{u=0}^{1} \int_{s=0}^{ux} \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} ds (-\ln(u))^{p-1} \frac{1}{u} du \\
&= \|f\|_\infty \int_{u=0}^{1} xu\Gamma(r)(-\ln(u))^{p-1} \frac{1}{u} du \\
&= \|f\|_\infty x\Gamma(r) \int_{u=0}^{1} (-\ln(u))^{p-1} du \\
&= \|f\|_\infty x\Gamma(r)\Gamma(p) < \infty;
\end{aligned}$$

where the Gamma function evaluated at $r$, $\Gamma(r)$, is obtained by making the substitution $y = \ln\left(\frac{xu}{s}\right)$ and the Gamma function evaluated at $p$, $\Gamma(p)$, is obtained by making the substitution $y = \ln\left(\frac{1}{u}\right)$.

Next, we rewrite $I_1$ as

$$\begin{aligned}
&\int_{s=0}^{x} \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} du f(s) \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} ds \\
&= \int_{s=0}^{x} f(s) \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} duds.
\end{aligned}$$

Let

$$Q(s,x) := \int_{u=\frac{s}{x}}^{1} \left[\ln\left(\frac{xu}{s}\right)\right]^{r-1} \frac{(-\ln(u))^{p-1}}{u} du,$$

which is the inner integral on $I_2$. Therefore, it is enough to show that

$$Q(s,x) = \int_{u=0}^{1} u^{p-1}(1-u)^{r-1} \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} du.$$

Notice

$$\begin{aligned} Q(s,x) &= \int_{u=\frac{s}{x}}^{1} \left[\ln\left(\frac{x}{s}\right) - (-\ln(u))\right]^{r-1} \frac{(-\ln(u))^{p-1}}{u} du \\ &= \int_{u=\frac{s}{x}}^{1} \left[\ln\left(\frac{x}{s}\right)\right]^{r-1+p} \left[1 - \frac{(-\ln(u))}{\ln\left(\frac{x}{s}\right)}\right]^{r-1} \left(\frac{-\ln(u)}{\ln\left(\frac{x}{s}\right)}\right)^{p} \frac{1}{-\ln(u)} \frac{du}{u} \\ &= \left[\ln\left(\frac{x}{s}\right)\right]^{p+r-1} \int_{1}^{0} (1-q)^{r-1} q^{p} \frac{-1}{q} dq. \end{aligned}$$

Where $q = \dfrac{-\ln(u)}{\ln\left(\frac{x}{s}\right)}$, therefore $dq = \dfrac{-1}{u} \dfrac{1}{\ln\left(\frac{x}{s}\right)} du$ and so $\dfrac{1}{u(-\ln(u))} du = \dfrac{-1}{q} dq$. Thus we get the desired result.

□

### 3.3.2 Approach 2: Emulating an identity for real numbers

Consider the following identity for real numbers:

**Lemma 3.3.3.** *For each fixed $p \in (0,1)$ there exists $K_p \in (0,\infty)$ such that, for every $t \in (0,\infty)$,*

$$t^p = \frac{1}{K_p} \int_{\lambda \in (0,\infty)} \frac{t\lambda}{1+t\lambda} \frac{1}{\lambda^{p+1}} d\lambda. \qquad (\star)$$

*In fact $K_p = \dfrac{\pi}{\sin(\pi p)}$.*

*Proof.* Let

$$Q(t) := \int_{\lambda \in (0,\infty)} \frac{t\lambda}{1+t\lambda} \frac{1}{\lambda^{p+1}} d\lambda \in (0,\infty).$$

Using the substitution $\mu = t\lambda$ we get

$$Q(t) = t^p \int_{\mu \in (0,\infty)} \frac{\mu}{1+\mu} \frac{1}{\mu^{p+1}} d\mu.$$

Define $K_p$ as

$$K_p = \int_{\mu \in (0,\infty)} \frac{\mu}{1+\mu} \frac{1}{\mu^{p+1}} d\mu.$$

We will check later on, that in fact $K_p = \dfrac{\pi}{\sin(p\pi)}$. □

We will use the identity from the previous Lemma as inspiration to give a second definition for the $p$-power of the operator $J$, first for $p \in (0,1)$. We proceed as follows:

Fix $f \in L^\infty$. Let $\alpha = \dfrac{1}{\lambda} \in (0,\infty)$. Let $q(t)$ denote the integrand in the identity of the previous lemma; we regard $\dfrac{1}{\lambda^{p+1}} d\lambda$ as our measure of integration. Notice $q(t) = \dfrac{t\lambda}{1+t\lambda} = \dfrac{t}{\alpha+t}$ is such that $(\alpha+t)q(t) = t$. We pose the following integral equation:

$$\text{Find } q \in L^\infty \text{ such that } (\alpha I + J)\, q = Jf.$$

**Claim 3.3.4.** *For fixed $f \in L^\infty$, the integral equation*

$$(\alpha I + J)\, q = Jf$$

*has a solution $q \in L^\infty$.*

Before proving this claim, we make the following observations:

We are looking for a solution of this equation given an arbitrary $f \in L^\infty$. If such a solution $q$ exists, then for all $x \in (0,\infty)$ we would have

$$\alpha q(x) + (Jq)(x) = (Jf)(x).$$

This is equivalent to

$$\alpha q(x) + \frac{1}{x}\int_0^x q(t)dt = \frac{1}{x}\int_0^x f(t)dt.$$

By applying classical methods of solving differential equations, we make the following ansatz:

$$q(x) = q_{\alpha,f}(x) = x^{-\frac{\alpha+1}{\alpha}} \int_0^x \frac{1}{\alpha} t^{\frac{1}{\alpha}} f(t)dt = \frac{1}{\alpha}\frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{\frac{1}{\alpha}} f(t)dt.$$

Notice

$$|q(x)| \leq \frac{1}{\alpha}\frac{1}{x}\|f\|_\infty \int_0^x 1^{\frac{1}{\alpha}} dt = \frac{1}{\alpha}\|f\|_\infty < \infty.$$

Therefore $q \in L^\infty$.

Now, in order to prove the claim, we will check that this definition of $q$ solves the integral equation we posed:

*Proof.* We want to show that for $f \in L^\infty$,

$$\alpha q(x) + (Jq)(x) = (Jf)(x) \text{ for all } x \in (0, \infty).$$

Notice

$$\alpha q(x) + (Jq)(x) = \frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{\frac{1}{\alpha}} f(t)dt + \frac{1}{x}\int_0^x q(s)ds.$$

Also

$$\begin{aligned}
\int_0^x q(s)ds &= \frac{1}{\alpha}\int_0^x \frac{1}{s}\int_0^s \left(\frac{t}{s}\right)^{\frac{1}{\alpha}} f(t)dtds \\
&= \frac{1}{\alpha}\int_0^x \int_t^x \frac{1}{s}\frac{t^{1/\alpha}}{s^{1/\alpha}} f(t)dsdt \\
&= \frac{1}{\alpha}\int_0^x t^{1/\alpha} f(t)\alpha\left(\frac{1}{t^{1/\alpha}} - \frac{1}{x^{1/\alpha}}\right)dt \\
&= \int_0^x f(t)dt - \int_0^x \left(\frac{t}{x}\right)^{1/\alpha} f(t)dt.
\end{aligned}$$

Where we apply Fubini-Tonelli to change the order of integration. We can do this since $g(t,s) := \frac{1}{s}\left(\frac{t}{s}\right)^{\frac{1}{\alpha}} f(t)\chi_{(0,s)}(t)$ is measurable given that $\frac{1}{s}\left(\frac{t}{s}\right)^{\frac{1}{\alpha}}$ is a continuous function on $(0,\infty) \times (0, x)$ and $f(t)\chi_{(0,s)}(t)$ is measurable. Also, given that

$$\begin{aligned}
\int_0^\infty \int_0^x |g(t,s)|dsdt &= \int_0^x \int_t^x \frac{1}{s}\frac{t^{1/\alpha}}{s^{1/\alpha}}|f(t)|dsdt \\
&= \alpha\int_0^x t^{1/\alpha}|f(t)|\left(\frac{1}{t^{1/\alpha}} - \frac{1}{x^{1/\alpha}}\right)dt \\
&\le \|f\|_\infty \alpha \int_0^x \left(1 - \left(\frac{t}{x}\right)^{1/\alpha}\right)dt < \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{\frac{1}{\alpha}} f(t)dt + \frac{1}{x}\int_0^x q(s)ds \\
&= \frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{1/\alpha} f(t)dt + \frac{1}{x}\int_0^x f(t)dt - \frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{1/\alpha} f(t)dt \\
&= \frac{1}{x}\int_0^x f(t)dt.
\end{aligned}$$

We conclude that $q$ does solve the integral equation posed. □

*Remark.* We see then that $\alpha I + J$ is invertible on the range of $J$. So $(\alpha I + J)^{-1}$ exists on $J(L^\infty)$. Then

$$q(x) = q_{\alpha,f}(x) = (T_\alpha f)(x) = \frac{1}{\alpha}\frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{\frac{1}{\alpha}} f(t)dt,$$

where $T_\alpha$ is the operator $(\alpha I + J)^{-1}J$.

Next, we present two results concerning $q(x)$, the solution of the integral equation.

**Claim 3.3.5.** *For $\alpha \in (0, \infty)$ and $f \in L^1$, we have that $q(x) = q_{\alpha,f}(x) \in L^1$.*

*Proof.* We write

$$\int_0^\infty |q(s)|ds = \lim_{N\to\infty} \int_0^N |q(s)|ds,$$

and

$$\begin{aligned}
\int_0^N |q(s)|ds &= \frac{1}{\alpha}\int_0^N \left| \frac{1}{s}\int_0^s \left(\frac{t}{s}\right)^{1/\alpha} f(t)dt \right| ds \\
&\le \frac{1}{\alpha}\int_0^N \frac{1}{s}\int_0^s \left(\frac{t}{s}\right)^{1/\alpha} |f(t)|\, dtds \\
&= \frac{1}{\alpha}\int_0^N \int_t^N \frac{t^{1/\alpha}}{s^{1+1/\alpha}} |f(t)|dsdt \\
&= \int_0^N t^{1/\alpha}|f(t)| \left(\frac{1}{t^{1/\alpha}} - \frac{1}{N^{1/\alpha}}\right) dt.
\end{aligned}$$

Where again we are applying Fubini-Tonelli to the measurable function

$$g(t, s) := \frac{1}{s}\left(\frac{t}{s}\right)^{\frac{1}{\alpha}} f(t)\chi_{(0,s)}(t),$$

defined on $(0, \infty) \times (0, N)$. Therefore

$$\int_0^N |q(s)|ds \le \int_0^N |f(t)|dt.$$

Therefore, letting $N$ go to infinity we get

$$\int_0^\infty q(s)ds \le \int_0^\infty |f(t)|dt < \infty.$$

□

**Claim 3.3.6.** *For every $x > 0$ such that $f$ is continuous at $x$, we have that $q(x) \to f(x)$, as $\alpha \to 0^+$, for all $f \in L^\infty(0, \infty)$.*

*Proof.* We have that

$$q(x) = \frac{1}{x}\int_0^x \frac{1}{\alpha}\left(\frac{t}{x}\right)^{\frac{1}{\alpha}} f(t)dt.$$

By making the substitution $s = \frac{t}{x}$, followed by the substitution $u = s^{\frac{1}{\alpha}}$, we obtain

$$\begin{aligned} q(x) &= \int_0^1 \frac{1}{\alpha} s^{\frac{1}{\alpha}} f(xs)ds \\ &= \int_0^1 u^\alpha f(xu^\alpha)du. \end{aligned}$$

We can do these change of variables since $g(s) := xs$ and $h(u) := u^\alpha$ are continuously differentiable, one-to-one mappings on $(0,1)$, which also implies that the integrands in the last two expressions are measurable functions (see e.g. Theorem 3, section 9.3 of [16]).

Then,

$$\begin{aligned} \left|\frac{1}{\alpha x}\int_0^x \left(\frac{t}{x}\right)^{\frac{1}{\alpha}} f(t)dt - f(x)\right| &= \left|\int_0^1 \left[u^\alpha f(xu^\alpha) - f(x)\right] du\right| \\ &\leq \int_0^1 |u^\alpha f(xu^\alpha) - f(x)|du. \end{aligned}$$

Notice that for $u \in (0,1)$ we have that

$$|u^\alpha f(xu^\alpha) - f(x)| \leq 2\|f\|_\infty,$$

and the constant function $2\|f\|_\infty$ is integrable on $(0,1)$. So, we can apply Dominated Convergence Theorem when taking the limit as $\alpha \to 0^+$, to obtain

$$0 \leq \lim_{\alpha\to 0^+}\left|\frac{1}{\alpha x}\int_0^x \left(\frac{t}{x}\right)^{\frac{1}{\alpha}} f(t)dt - f(x)\right| \leq \int_0^1 \lim_{\alpha\to 0^+} |u^\alpha f(xu^\alpha) - f(x)|du.$$

Since $f$ is continuous at $x$, we have that

$$|u^\alpha f(xu^\alpha) - f(x)| \longrightarrow |f(x) - f(x)| = 0, \text{ as } \alpha \to 0^+,$$

and so we get the desired result.

□

Now, we will use the operator $T_\alpha$ to give another definition of the $p$-power of the operator $J$. We start by considering powers $p \in (0,1)$, and give a definition in analogy with equation ($\star$), and then we extend the definition to arbitrary powers $p > 0$. We will denote this second definition by $\tilde{J}^p$, since a priori, it is different to the one given in the previous subsection:

**Definition 3.3.3.** For $p \in (0,1)$ we define the operator $\tilde{J}^p$ by

$$\tilde{J}^p(\cdot) := \frac{1}{K_p} \int_{\lambda \in (0,\infty)} T_{\frac{1}{\lambda}}(\cdot) \frac{1}{\lambda^{p+1}} d\lambda.$$

Next, we extend the previous definition of $\tilde{J}^p$ for $p \in (0,1)$, to arbitrary $r > 0$ by

$$\tilde{J}^r f := J^{\lfloor r \rfloor} \left( \tilde{J}^{\langle r \rangle} f \right), \text{ for all } f \in L^\infty.$$

Where $\langle r \rangle = r - \lfloor r \rfloor$.

### 3.3.3 The fractional Cesàro averaging operator

In this subsection we will see that the two different approaches we took to define fractional powers of $J$, gave us in fact the same definition. We start by proving the following result:

**Claim 3.3.7.** *For $p \in (0,1)$ we have that $\Gamma(p)\Gamma(1-p) = K_p$.*
*This will imply that $K_p = \dfrac{\pi}{\sin(\pi p)}$, since $p \in (0,1)$.*

*Proof.* Recall,

$$\Gamma(p)\Gamma(1-p) = B(p, 1-p) = \frac{\pi}{\sin(\pi p)} \text{ and } B(p,1-p) = \int_{t \in (0,1)} \frac{t^{p-1}}{(1-t)^p} dt.$$

Also, recall that in the proof of Lemma 3.3.3, we showed that

$$K_p = \int_{\mu=0}^{\infty} \frac{\mu}{1+\mu} \frac{1}{\mu^{p+1}} d\mu.$$

By making the substitution $t = \dfrac{\mu}{1+\mu}$, we obtain

$$K_p = \int_0^1 t \left( \frac{1-t}{t} \right)^{p+1} \frac{1}{(1-t)^2} dt = \int_0^1 \frac{(1-t)^{p-1}}{t^p} dt.$$

Next, let $x = 1 - t$, then

$$K_p = \int_0^1 \frac{x^{p-1}}{(1-x)^p} dx.$$

So, we see that $\Gamma(p)\Gamma(1-p) = K_p$. □

**Theorem 3.3.8.** *The operator defined in Definition 3.3.2, and the operator defined in Definition 3.3.3 are the same. That is, $\mathcal{I}^r = \tilde{J}^r$ for all $r > 0$.*

*Proof.* Fix $f \in L^\infty$, and fix $x \in (0,\infty)$. Then, for $p \in (0,1)$ we have that

$$\begin{aligned}
\left(\tilde{J}^p f\right)(x) &:= \frac{1}{K_p}\int_{\lambda\in(0,\infty)} \left(T_{\frac{1}{\lambda}} f\right)(x)\frac{1}{\lambda^{p+1}}d\lambda \\
&= \frac{1}{K_p}\int_{\lambda\in(0,\infty)} \left(\frac{\lambda}{x}\int_{t=0}^{x}\left(\frac{t}{x}\right)^{\lambda} f(t)dt\right)\frac{1}{\lambda^{p+1}}d\lambda \\
&= \frac{1}{K_p}\frac{1}{x}\int_{\lambda=0}^{\infty}\int_{t=0}^{x}\frac{t^\lambda f(t)}{x^\lambda \lambda^p}dtd\lambda.
\end{aligned}$$

Next, we would like to change the order of integration, so we verify the hypotheses from the Fubini-Tonelli Theorem hold: Notice that we have a measurable integrand. Also, for $t \in (0,x)$ we have

$$\begin{aligned}
0 < I(x,t,p) &:= \int_{\lambda=0}^{\infty}\left(\frac{t}{x}\right)^{\lambda}\frac{1}{\lambda^p}d\lambda \\
&= \int_{\lambda=0}^{\infty} e^{\lambda\ln\left(\frac{t}{x}\right)}\lambda^{-p}d\lambda \\
&= \int_{\lambda=0}^{\infty} e^{-\lambda\ln\left(\frac{x}{t}\right)}\lambda^{-p}d\lambda.
\end{aligned}$$

Next, we make the change of variables $y = \lambda\ln\left(\frac{x}{t}\right)$, to get

$$\begin{aligned}
I(x,t,p) &= \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}}\int_{y=0}^{\infty} e^{-y}y^{-p}dy \\
&= \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}}\Gamma(1-p).
\end{aligned}$$

Therefore, by making the usual substitution $u = \ln\left(\frac{x}{t}\right)$, we obtain

$$\begin{aligned}
\frac{1}{x}\int_{t=0}^{x}|f(t)I(x,t,p)|\,dt &= \frac{1}{x}\Gamma(1-p)\int_{t=0}^{x}\frac{|f(t)|}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}}dt \\
&= \Gamma(1-p)\int_{u=0}^{\infty}\left|f\left(\frac{x}{e^u}\right)\right|u^{p-1}\frac{1}{e^u}du \\
&\le \Gamma(1-p)\Gamma(p)\|f\|_\infty < \infty.
\end{aligned}$$

We can do this change of variables since $g(u) := xe^{-u}$ is a continuously differentiable, one-to-one mapping on $(0, \infty)$, which also implies the last integrand is a measurable function (see e.g. Theorem 3, section 9.3 of [16]).

So, we can apply Fubini-Tonelli to the expression for $(\tilde{J}^p f)(x)$, and get

$$\begin{aligned}
(\tilde{J}^p f)(x) &= \frac{1}{K_p}\frac{1}{x}\int_{t=0}^{x} f(t)\int_{\lambda=0}^{\infty}\left(\frac{t}{x}\right)^{\lambda}\frac{1}{\lambda^p}d\lambda dt \\
&= \frac{\Gamma(1-p)}{K_p}\frac{1}{x}\int_{t\in(0,x)} f(t)\frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}}dt \\
&= \frac{1}{\Gamma(p)}\frac{1}{x}\int_{t\in(0,x)} f(t)\frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}}dt \\
&= \mathcal{I}^p f(x),
\end{aligned}$$

where we are using the previous Claim to replace $\dfrac{\Gamma(1-p)}{K_p}$ with $\dfrac{1}{\Gamma(p)}$.

Therefore, we have

$$(\tilde{J}^p f)(x) = \mathcal{I}^p f(x), \text{ for } p \in (0, 1).$$

Now, for arbitrary $r > 0$, recall that $\tilde{J}^r f := J^{\lfloor r\rfloor}\left(\tilde{J}^{\langle r\rangle} f\right)$. Therefore $\tilde{J}^r f = J^{\lfloor r\rfloor}\left(\mathcal{I}^{\langle r\rangle} f\right)$. By Claim 3.3.1, we have that $J^{\lfloor r\rfloor} = \mathcal{I}^{\lfloor r\rfloor}$, so $\tilde{J}^r f = \mathcal{I}^{\lfloor r\rfloor}\left(\mathcal{I}^{\langle r\rangle} f\right)$. Since we already proved the semigroup property for $(\mathcal{I}^r)_{r>0}$, we get that $\mathcal{I}^{\lfloor r\rfloor}\left(\mathcal{I}^{\langle r\rangle} f\right) = \mathcal{I}^r f$. Thus, $\tilde{J}^r f = \mathcal{I}^r f$. □

### 3.4 Properties of the fractional Cesàro averaging operator $J^r$

Given the results from the previous section, we can now present the definition and notation that we will use hereafter for the $p$-power of the $J$ operator:

**Definition 3.4.1.** For arbitrary $p > 0$, we denote by $J^p$ the following operator:

$$(J^p f)(x) := \frac{1}{\Gamma(p)}\frac{1}{x}\int_{t\in(0,x)} f(t)\frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}}dt, \text{ for } f \in L^\infty, \text{ and } x \in (0, \infty).$$

We refer to this operator as *the fractional Cesàro averaging operator* on $L^\infty$.

**Claim 3.4.1.** *For any $p > 0$, we have that $\|J^p\|_{op} = 1$.*

*Proof.* Fix $p > 0$. First notice that

$$(J^p \mathbb{1})(x) = \frac{1}{\Gamma(p)} \frac{1}{x} \int_0^x \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}} dt = \frac{1}{\Gamma(p)} \frac{1}{x} x \int_0^\infty u^{p-1} e^{-u} du = 1.$$

Here the last integral is obtained by making the usual substitution $u = \ln\left(\frac{x}{t}\right)$. So we see that $\|J^p\|_{op} \geq 1$. On the other hand, for arbitrary $f \in L^\infty$ and $x > 0$, we have that

$$|(J^p f)(x)| \leq \frac{\Gamma(p)}{\Gamma(p)} \|f\|_\infty = \|f\|_\infty.$$

Therefore $\|J^p f\|_\infty \leq \|f\|_\infty$, which implies $\|J^p\|_{op} \leq 1$. Thus $\|J^p\|_{op} = 1$. $\square$

The following Hardy-like inequality tells us that the operator $J^r$ maps $L^p(0,\infty)$ into $L^p(0,\infty)$:

**Claim 3.4.2.** *If $f \in L^p(0,\infty)$ for some $p > 1$, then for every $r > 0$ we have that $J^r f \in L^p(0,\infty)$. In fact,*

$$\|J^r f\|_p \leq \left(\frac{p}{p-1}\right)^r \|f\|_p.$$

*Proof.* Fix $p > 1$. Fix $r > 0$ and $f \in L^p(0,\infty)$.

$$\begin{aligned}
\|J^r f\|_p &= \left(\int_0^\infty \left|\frac{1}{\Gamma(r)} \frac{1}{x} \int_0^x f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-r}} dt\right|^p dx\right)^{1/p} \\
&\leq \frac{1}{\Gamma(r)} \left(\int_0^\infty \left(\frac{1}{x} \int_0^x |f(t)| \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-r}} dt\right)^p dx\right)^{1/p} \\
&= \frac{1}{\Gamma(r)} \left(\int_0^\infty \left(\int_0^1 |f(ux)| \frac{1}{\left[\ln\left(\frac{1}{u}\right)\right]^{1-r}} du\right)^p dx\right)^{1/p}.
\end{aligned}$$

Where the last equality was obtained by making the substitution $u = t/x$ for the inner integral. We can do this change of variables since $g(u) := xu$ is a continuously differentiable, one-to-one mapping on $(0,1)$, which also implies the inner integrand in the last expression is a measurable function (see e.g. Theorem 3, section 9.3 of [16]).

Next, we apply Minkowski's integral inequality (see e.g. [11]), followed by the substitution $s = ux$ for the inner integral (which we are allow to do since $h(s) := s/u$ is continuously differentiable and one-to-one), to obtain:

$$\frac{1}{\Gamma(r)}\left(\int_0^\infty\left(\int_0^1 |f(ux)|\frac{1}{\left[\ln\left(\frac{1}{u}\right)\right]^{1-r}}du\right)^p dx\right)^{1/p}$$
$$\leq \frac{1}{\Gamma(r)}\int_0^1\left(\int_0^\infty |f(ux)|^p\frac{1}{\left[\ln\left(\frac{1}{u}\right)\right]^{p(1-r)}}dx\right)^{1/p} du$$
$$= \frac{1}{\Gamma(r)}\int_0^1\left(\frac{1}{u}\int_0^\infty |f(s)|^p\frac{1}{|\ln(u)|^{p(1-r)}}ds\right)^{1/p} du$$
$$= \frac{1}{\Gamma(r)}\int_0^1 \frac{1}{u^{1/p}}\frac{1}{|\ln(u)|^{(1-r)}}\left(\int_0^\infty |f(s)|^p ds\right)^{1/p} du$$
$$= \frac{1}{\Gamma(r)}\|f\|_p\int_0^1 \frac{1}{u^{1/p}}\frac{1}{|\ln(u)|^{(1-r)}}du.$$

Next, we make the substitution $y = \ln u$, followed by the substitution $y = -s$:

$$\begin{aligned}\frac{1}{\Gamma(r)}\|f\|_p\int_0^1 \frac{1}{u^{1/p}}\frac{1}{|\ln(u)|^{(1-r)}}du &= \frac{1}{\Gamma(r)}\|f\|_p\int_{-\infty}^0 \frac{1}{e^{y/p}}\frac{1}{|y|^{(1-r)}}e^y dy\\ &= \frac{1}{\Gamma(r)}\|f\|_p\int_{-\infty}^0 |y|^{(r-1)}\, e^{y(1-1/p)}dy\\ &= \frac{1}{\Gamma(r)}\|f\|_p\int_0^\infty s^{(r-1)}e^{s(1/p-1)}ds.\end{aligned}$$

Finally, by denoting $q = 1 - 1/p = \frac{p-1}{p}$, and making the substitution $z = qs$, we obtain:

$$\begin{aligned}\frac{1}{\Gamma(r)}\|f\|_p\int_0^\infty s^{(r-1)}e^{s(1/p-1)}ds &= \frac{1}{\Gamma(r)}\|f\|_p\int_0^\infty s^{(r-1)}e^{-qs}ds\\ &= \frac{1}{\Gamma(r)}\|f\|_p\int_0^\infty \left(\frac{z}{q}\right)^{(r-1)} e^{-z}\frac{1}{q}dz\\ &= \frac{1}{\Gamma(r)}\|f\|_p\frac{1}{q^r}\int_0^\infty z^{(r-1)}e^{-z}dz\\ &= \|f\|_p\frac{1}{q^r}\\ &= \|f\|_p\left(\frac{p}{p-1}\right)^r.\end{aligned}$$

Therefore

$$\|J^r f\|_p \leq \left(\frac{p}{p-1}\right)^r \|f\|_p.$$

□

*Remark.* Consider the function $f := \chi_{[0,1]}$. Clearly $f \in L^1$. Nevertheless

$$Jf(x) = \begin{cases} 1, & 0 < x \leq 1 \\ \frac{1}{x}, & 1 \leq x \end{cases}$$

is not in $L^1(0,\infty)$. So, we see the previous Claim does not necessarily hold for $p = 1$.

The following Theorem will imply one more result concerning $L^p$ spaces, for a particular type of function:

**Theorem 3.4.3.** *For every $s > 0$ there exists a constant $K_s := K > 0$ such that, for every $f \in L^\infty$, we have that*

$$|(J^{s/2} f)(x)| \leq \left(\frac{\|f\|_\infty^2}{K}\right)^{\frac{1}{3}} \max\left\{((J^s f_1)(x))^{\frac{1}{3}}, ((J^s f_2)(x))^{\frac{1}{3}}\right\},$$

*where*

$$f_1 := max\{f, 0\} \text{ and } f_2 := max\{-f, 0\}.$$

*Proof.* Fix $s > 0$. First assume that $f \geq 0$. Recall

$$(J^s f)(x) = \frac{1}{\Gamma(s)} \frac{1}{x} \int_{t\in(0,x)} \left[\ln\left(\frac{x}{t}\right)\right]^{s-1} f(t)dt.$$

We make the usual substitution $u = \ln\left(\frac{x}{t}\right)$, to get

$$(J^s f)(x) = \frac{1}{\Gamma(s)} \int_{u\in(0,\infty)} [u]^{s-1} e^{-u} f(xe^{-u})du.$$

Clearly, the desired inequality holds when $f$ is the constant function 0, for any value we choose for $K > 0$. Therefore we may assume $f \in L^\infty \setminus \{0\}$. Let $g := \frac{f}{\|f\|_\infty}$, then $\|g\|_\infty = 1$ and $g \geq 0$. Fix $x > 0$, and fix $\tau \in (0, s)$. Then

$$\begin{aligned}(J^s g)(x) &= \frac{1}{\Gamma(s)} \int_{u\in(0,\infty)} u^{s-1} e^{-u} g(xe^{-u})du \\ &= \frac{1}{\Gamma(s)} \int_{u\in(0,\infty)} u^\tau u^{s-\tau-1} e^{-u} g(xe^{-u})du.\end{aligned}$$

Observe that $\int_{u\in(0,\infty)} u^{s-\tau-1}e^{-u}du = \Gamma(s-\tau)$. So, we can re-write $J^s g$ as

$$(J^s g)(x) = \frac{\Gamma(s-\tau)}{\Gamma(s)} \int_{u\in(0,\infty)} u^{\tau} g(xe^{-u}) d\nu_\tau(u),$$

where $d\nu_\tau(u) := \dfrac{u^{s-\tau-1}e^{-u}du}{\Gamma(s-\tau)}$, with $\nu_\tau$ being a probability measure on $\Delta_{[0,\infty)}$.

Fix $\sigma > 1$. Notice that that the integrand $u^\tau g(xe^{-u})$ is non-negative, and so we apply Jensen's inequality:

$$\begin{aligned}(J^s g)(x) &= \frac{\Gamma(s-\tau)}{\Gamma(s)} \int_{u\in(0,\infty)} \left(u^{\frac{\tau}{\sigma}} (g(xe^{-u}))^{\frac{1}{\sigma}}\right)^\sigma d\nu_\tau(u) \\ &\geq \frac{\Gamma(s-\tau)}{\Gamma(s)} \left(\int_{u\in(0,\infty)} u^{\frac{\tau}{\sigma}} (g(xe^{-u}))^{\frac{1}{\sigma}} d\nu_\tau(u)\right)^\sigma \\ &= \frac{\Gamma(s-\tau)}{\Gamma(s)} \left(\int_{u\in(0,\infty)} u^{\frac{\tau}{\sigma}} (g(xe^{-u}))^{\frac{1}{\sigma}} \frac{u^{s-\tau-1}e^{-u}du}{\Gamma(s-\tau)}\right)^\sigma .\end{aligned}$$

Since $0 \leq g(xe^{-u}) \leq 1$ and $0 < \frac{1}{\sigma} < 1$, we have that $(g(xe^{-u}))^{\frac{1}{\sigma}} \geq g(xe^{-u})$. Thus

$$\begin{aligned}(J^s g)(x) &\geq \frac{\Gamma(s-\tau)}{\Gamma(s)(\Gamma(s-\tau))^\sigma} \left(\int_{u\in(0,\infty)} u^{s-\tau-1+\frac{\tau}{\sigma}} g(xe^{-u}) e^{-u} du\right)^\sigma \\ &= \frac{\Gamma(s-\tau)\Gamma(s-\tau+\frac{\tau}{\sigma})^\sigma}{\Gamma(s)(\Gamma(s-\tau))^\sigma} \left(\left(J^{s-\tau+\frac{\tau}{\sigma}} g\right)(x)\right)^\sigma .\end{aligned}$$

So, we choose $\tau \in (0, s)$ and $\sigma > 1$ in such a way that we get the desired conclusion: take $\tau = \frac{3}{4}s$ and $\sigma = 3$. We get

$$(J^s g)(x) \geq \frac{\Gamma(s/4)\Gamma(s/2)^3}{\Gamma(s)\Gamma(s/4)^3} \left(\left(J^{s/2} g\right)(x)\right)^3 .$$

This, after substituting $g = \dfrac{f}{\|f\|_\infty}$, gives us

$$\left(J^s \frac{f}{\|f\|_\infty}\right)(x) \geq \frac{\Gamma(s/2)^3}{\Gamma(s)\Gamma(s/4)^2} \left(\left(J^{s/2} \frac{f}{\|f\|_\infty}\right)(x)\right)^3 .$$

Therefore, by letting $K := \dfrac{\Gamma(s/2)^3}{\Gamma(s)\Gamma(s/4)^2}$, we obtain

$$\left(J^s f\right)(x) \geq K \frac{\left(\left(J^{s/2} f\right)(x)\right)^3}{\|f\|_\infty^2}.$$

Next, consider an arbitrary $f \in L^\infty \setminus \{0\}$. Then $f_1, f_2$ as defined above, are elements of $L^\infty$ and $f_1, f_2 \geq 0$. Consequently, by linearity of $J^{s/2}$, we have that

$$0 \leq \left(J^{s/2} f_1\right)(x) \leq \left(\frac{\|f_1\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_1\right)(x)\right)^{\frac{1}{3}},$$

and

$$-\left(\frac{\|f_2\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_2\right)(x)\right)^{\frac{1}{3}} \leq -\left(J^{s/2} f_2\right)(x) \leq 0.$$

Notice that $\|f_1\|_\infty, \|f_2\|_\infty \leq \|f\|_\infty$, and $f = f_1 - f_2$. So using again the linearity of $J^{s/2}$, we obtain

$$-\left(\frac{\|f\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_2\right)(x)\right)^{\frac{1}{3}} \leq (J^{s/2} f)(x) \leq \left(\frac{\|f\|_\infty^2}{K}\right)^{\frac{1}{3}} \left(\left(J^s f_1\right)(x)\right)^{\frac{1}{3}},$$

which implies the desired conclusion. □

**Lemma 3.4.4.** *Let $f \in L^\infty$ such that $f \geq 0$ and $J^s f \in L^p(0,\infty)$, for some $p > 0$ and some $s > 0$. Then we have that $J^{s/2} f \in L^{3p}(0,\infty)$.*

*Proof.* We can assume $f$ is not the constant function 0, since in this case the conclusion holds trivially.

Since we are assuming $f \geq 0$, then $f_1 = f$, and $f_2 = 0$, and so, by Theorem 3.4.3, we have that

$$\left(\frac{\|f\|_\infty^2}{K}\right)^{1/3} \left(\left(J^s f\right)(x)\right)^{1/3} \geq \left(J^{s/2} f\right)(x) \geq 0,$$

which implies

$$\left(J^s f\right)(x) \geq \frac{K}{\|f\|_\infty^2} \left(\left(J^{s/2} f\right)(x)\right)^3.$$

Therefore,

$$\infty > \int_0^\infty |\left(J^s f\right)(x)|^p dx \geq \left(\frac{K}{\|f\|_\infty^2}\right)^p \int_0^\infty |\left(J^{s/2} f\right)(x)|^{3p} dx.$$

Thus, $J^{s/2} \in L^{3p}(0,\infty)$. □

*Remark.* We already mentioned that for the function $f := \chi_{[0,1]}$, we have that $Jf \notin L^1(0,\infty)$. Still

$$J^{1/2} f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ \frac{1}{\Gamma(1/2)} \int_{\ln(x)}^\infty u^{-1/2} e^{-u} du, & 1 \leq x \end{cases}$$

is an element of $L^3(0,\infty)$, given that

$$\int_1^\infty \left| \int_{\ln(x)}^\infty u^{-1/2} e^{-u} du \right|^3 dx = -\sqrt{\pi}\left(6(-2+\sqrt{2})+\pi\right).$$

Hence, we see the converse of the previous Lemma does not necessarily hold.

In the remainder of this section, we will discuss continuity results for the operator $J^p$.

**Claim 3.4.5.** *For all $f \in L^\infty$ and $x \in (0,\infty)$, we have that*

$$\lim_{p\to 1-} (J^p f)(x) = (Jf)(x).$$

*Proof.* Fix $f \in L^\infty$, and fix $x \in (0,\infty)$. Recall $(J^p f)(x) = \dfrac{1}{\Gamma(p)} \dfrac{1}{x} \displaystyle\int_{t\in(0,x)} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}} dt.$

Notice $\displaystyle\lim_{p\to 1-} \frac{1}{\Gamma(p)} = \frac{1}{\Gamma(1)} = 1$, since the $\Gamma$ is continuous for positive input. So it is enough to show

$$\lim_{p\to 1-} \frac{1}{x} \int_{t\in(0,x)} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}} dt = \frac{1}{x} \int_{t\in(0,x)} f(t) dt$$

We will use Dominated Convergence Theorem. Fix $(p_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of positive values, converging to 1. Let

$$g_n(t) = f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p_n}}.$$

We want to find a function $h_x := h$ in $L^1(0,x)$ such that

$$|g_n(t)| \le h(t), \text{ for all } t \in (0,x) \text{ and for all } n \in \mathbb{N}.$$

Let $q_n := 1 - p_n$ for each $n \in \mathbb{N}$. Notice $q_n \searrow 0$ and $q_n \in (0,1)$ for each $n$. Fix $t \in (0,x)$, then $\dfrac{x}{t} > 1$. Let $\alpha := \ln(x/t) > 0$.

Notice

$$\frac{1}{\alpha^{q_n}} \le \frac{1}{\alpha^{q_{n+1}}} \Leftrightarrow \alpha^{q_{n+1}} \le \alpha^{q_n} \Leftrightarrow 1 \le \alpha \Leftrightarrow e \le \frac{x}{t} \Leftrightarrow t \le \frac{x}{e}.$$

Therefore, we consider two cases:

**Case 1.** If $t \in \left(0, \frac{x}{e}\right]$, then $\left(\left(\frac{1}{\alpha}\right)^{q_n}\right)_n$ is increasing to 1. We see then,

$$\frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{q_n}} = \left(\frac{1}{\alpha}\right)^{q_n} \le 1 \text{ , for all } t \in \left(0, \tfrac{x}{e}\right] \text{ and for all } n \in \mathbb{N}.$$

**Case 2.** If $t \in \left(\frac{x}{e}, x\right)$, then $\left(\left(\frac{1}{\alpha}\right)^{q_n}\right)_n$ is decreasing. So

$$\frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{q_n}} \leq \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{q_1}} \text{ , for all } t \in \left(\frac{x}{e}, x\right) \text{ and for all } n \in \mathbb{N}.$$

Define $h$ as follows:

$$h(t) := |f(t)|g(t) \text{ for all } t \in [0, x].$$

Where

$$g(t) = \begin{cases} 1, & 0 < t \leq \frac{x}{e} \\ \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{q_1}}, & \frac{x}{e} < t < x \end{cases}$$

Since by hypothesis $f \in L^\infty$, we only need to verify now that $g$ is integrable on $(0, x)$ :

$$\int_0^x g(t)dt = \frac{x}{e} + \int_{\frac{x}{e}}^x \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{q_1}} dt.$$

By making our usual substitution $u = \ln\left(\frac{x}{t}\right)$, we get

$$\begin{aligned} \int_{\frac{x}{e}}^x \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{q_1}} dt &= x \int_{u=0}^1 u^{-q_1} e^{-u} du \\ &\leq x\Gamma(1 - q_1). \end{aligned}$$

We know $0 < \Gamma(1 - q_1) < \infty$ since $q_1 \in (0, 1)$. Therefore, by applying Dominated convergence Theorem, we get

$$\lim_{p \to 1-} \frac{1}{x} \int_{t \in (0,x)} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}} dt = \frac{1}{x} \int_{t \in (0,x)} f(t) dt.$$

□

**Claim 3.4.6.** *For all $f \in L^\infty$, we have that $\lim_{p \to 0+} (J^p f)(x) = f(x)$ for every $x > 0$ such that $f$ is continuous at $x$.*

*Proof.* We present two proofs for this Claim:

First proof: Let $x > 0$ such that $f$ is continuous at $x$.

After making our usual substitution $u = \ln(\frac{x}{t})$, we get

$$\begin{aligned}
|(J^p f)(x) - f(x)| &= \left| \frac{1}{\Gamma(p)} \frac{1}{x} \int_{t=0}^{x} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}} dt - f(x) \right| \\
&= \left| \frac{1}{\Gamma(p)} \int_{u=0}^{\infty} f\left(\frac{x}{e^u}\right) \frac{1}{u^{1-p}} \frac{1}{e^u} du - f(x) \right| \\
&= \left| \frac{1}{\Gamma(p)} \left\{ \int_{u=0}^{\infty} f\left(\frac{x}{e^u}\right) \frac{1}{u^{1-p}} \frac{1}{e^u} du - f(x)\Gamma(p) \right\} \right| \\
&= \left| \frac{1}{\Gamma(p)} \left\{ \int_{u=0}^{\infty} f\left(\frac{x}{e^u}\right) \frac{1}{u^{1-p}} \frac{1}{e^u} du - f(x) \int_{u=0}^{\infty} \frac{1}{u^{1-p}} \frac{1}{e^u} du \right\} \right| \\
&\leq \frac{1}{\Gamma(p)} \int_{u=0}^{\infty} \mid f\left(\frac{x}{e^u}\right) - f(x) \mid \frac{1}{u^{1-p}} \frac{1}{e^u} du.
\end{aligned}$$

Again, we can make the previous change of variables since $g(u) := xe^{-u}$ is continuously differentiable and one-to-one on $(0, \infty)$, which also implies that the integrand is a measurable function.

Next, fix $\epsilon > 0$ arbitrary. By continuity of $f$ at $x$, we know there exists $\sigma > 0$ such that $|f(x) - f(z)| < \epsilon$ whenever $|x - z| < \sigma$. Letting $\delta = \ln\left(\frac{\sigma}{x} + 1\right)$, we get that if $u \in [0, \delta)$ then $|f\left(\frac{x}{e^u}\right) - f(x)| < \epsilon$.

Assume $p < 1$, then we get:

$$\begin{aligned}
\frac{1}{\Gamma(p)} \int_{u=0}^{\infty} \mid f\left(\frac{x}{e^u}\right) - f(x) \mid \frac{1}{u^{1-p}} \frac{1}{e^u} du &\leq \frac{\epsilon}{\Gamma(p)} \int_{u=0}^{\delta} \frac{1}{u^{1-p}} \frac{1}{e^u} du + \frac{2\|f\|_\infty}{\Gamma(p)} \int_{\delta}^{\infty} \frac{1}{u^{1-p}} \frac{1}{e^u} du \\
&\leq \frac{\epsilon}{\Gamma(p)} \int_{u=0}^{\infty} \frac{1}{u^{1-p}} \frac{1}{e^u} du + \frac{2\|f\|_\infty}{\Gamma(p)} \int_{\delta}^{\infty} \frac{1}{\delta^{1-p}} \frac{1}{e^u} du \\
&= \frac{\epsilon}{\Gamma(p)} \Gamma(p) + \frac{2\|f\|_\infty}{\Gamma(p)} \delta^{p-1} e^{-\delta}.
\end{aligned}$$

Next, we take the limit superior as $p \longrightarrow 0^+$, and since $\Gamma(p) \to \infty$ as $p \longrightarrow 0^+$, we get that $\frac{\epsilon}{\Gamma(p)}\Gamma(p) + \frac{2\|f\|_\infty}{\Gamma(p)} \delta^{p-1} e^{-\delta} \longrightarrow \epsilon \; as \; p \longrightarrow 0^+$. Then we let $\epsilon$ go to 0 to get the desired result.

Second proof: Fix $x > 0$ such that $f$ is continuous at $x$. We want to show

$$\lim_{p \to 0+} \frac{1}{\Gamma(p)} \frac{1}{x} \int_{t \in (0,x)} f(t) \frac{1}{\left[\ln\left(\frac{x}{t}\right)\right]^{1-p}} dt = f(x).$$

Define

$$\varphi_\epsilon(t) := \tfrac{1}{x}\tfrac{1}{\Gamma(\epsilon)}\frac{1}{\left[\ln\left(\dfrac{x}{x-t}\right)\right]^{1-\epsilon}}\chi_{[0,x]}(x-t).$$

Then

$$\varphi_\epsilon(x-t) = \tfrac{1}{x}\tfrac{1}{\Gamma(\epsilon)}\frac{1}{\left[\ln\left(\dfrac{x}{t}\right)\right]^{1-\epsilon}}\chi_{[0,x]}(t).$$

Let $g(t) := f(t)\chi_{[0,x]}(t)$, then $g \in L^1(-\infty,\infty)$ since

$$\int_{-\infty}^{\infty} |g(t)|dt \le x\|f\|_\infty < \infty.$$

So, we can rewrite our desired conclusion as

$$\int_{-\infty}^{\infty} g(t)\varphi_\epsilon(x-t)dt \to g(x) \text{ as } \epsilon \to 0^+.$$

Or equivalently

$$g * \varphi_\epsilon(x) \to g(x) \text{ as } \epsilon \to 0^+.$$

To prove this, we will show that $\varphi_\epsilon$ is a good kernel, according to the definition in [24]. That is:

1. $\displaystyle\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = 1.$
2. $\displaystyle\int_{-\infty}^{\infty} |\varphi_\epsilon(t)|dt \le A$, for some constant $A$ independent of $\epsilon$.
3. For any $\nu > 0$, $\displaystyle\int_{|x|\ge\nu} \varphi_\epsilon(t)dt$ tends to 0 as $\epsilon \to 0^+$.

This will imply that

$$g * \varphi_\epsilon(x) \to g(x) = f(x) \text{ as } \epsilon \to 0^+, \text{ at every point } x \text{ of continuity of } f.$$

We first check $\displaystyle\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = 1$:

$$\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = \frac{1}{x\Gamma(\epsilon)}\int_0^x \frac{1}{\left[\ln\left(\dfrac{x}{x-t}\right)\right]^{1-\epsilon}}dt.$$

Let $s = \ln\left(\frac{x}{x-t}\right)$, then

$$\int_{-\infty}^{\infty} \varphi_\epsilon(t)dt = \frac{1}{\Gamma(\epsilon)}\int_0^{\infty} s^{\epsilon-1}e^{-s}ds = \frac{\Gamma(\epsilon)}{\Gamma(\epsilon)} = 1.$$

Thus the first condition holds.

Next notice that $\varphi_\epsilon(t) \geq 0$, and so $\displaystyle\int_{-\infty}^{\infty} |\varphi_\epsilon(t)|dt = 1$.

Finally, notice that for any $\nu \in (0, x)$, by making again the substitution $s = \ln\left(\frac{x}{x-t}\right)$ we obtain

$$\frac{1}{x\Gamma(\epsilon)} \int_\nu^x \frac{1}{\left[\ln\left(\dfrac{x}{x-t}\right)\right]^{1-\epsilon}} dt = \frac{1}{\Gamma(\epsilon)} \int_{\ln\left(\frac{x}{x-\nu}\right)}^{\infty} s^{\epsilon-1} e^{-s} ds.$$

Let $R := \ln\left(\frac{x}{x-\nu}\right)$, notice $R > 0$. Since we are going to let $\epsilon$ tend to 0, we may assume $\epsilon < 1$. Then for $s \in (R, \infty)$ we have that $s^{\epsilon-1} \leq R^{\epsilon-1}$. Therefore

$$\int_{\ln\left(\frac{x}{x-\nu}\right)}^{\infty} s^{\epsilon-1} e^{-s} ds \leq R^{\epsilon-1} \int_R^\infty e^{-s} ds = R^{\epsilon-1} e^{-R}.$$

Thus,

$$\int_{(-\infty,-\nu)\cup(\nu,\infty)} \varphi_\epsilon(t) dm(t) \text{ tends to 0 as } \epsilon \to 0^+, \text{ since } \tfrac{1}{\Gamma(\epsilon)} \to 0.$$

Therefore the desired conclusion holds. □

**Claim 3.4.7.** *Let $f \in L^\infty$. Then*

$$\liminf_{x\to\infty} f(x) \leq \liminf_{x\to\infty} (J^p f)(x) \leq \limsup_{x\to\infty} (J^p f)(x) \leq \limsup_{x\to\infty} f(x),$$

*for all $p > 0$.*

*Proof.* Let $f \in L^\infty$. Let $\ell := \liminf_{x\to\infty} f(x)$. This means the following: For every $\epsilon > 0$, there exists $N > 0$ such that $\ell - \epsilon \leq f(x)$ for almost all $x > N$.

Fix $p > 0$. For each $x > 0$ we have that

$$\begin{aligned}
\ell - (J^p f)(x) &= \ell - \frac{1}{x\Gamma(p)} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} f(t)dt \\
&= \frac{1}{x\Gamma(p)} \left(\ell x\Gamma(p) - \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} f(t)dt\right) \\
&= \frac{1}{x\Gamma(p)} \left(\ell x \int_0^\infty u^{p-1} e^{-u} du - \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} f(t)dt\right) \\
&= \frac{1}{\Gamma(p)} \left(\ell \int_0^\infty u^{p-1} e^{-u} du - \int_0^\infty [u]^{p-1} f\left(\frac{x}{e^u}\right) e^{-u} du\right) \\
&= \frac{1}{\Gamma(p)} \int_0^\infty \left(\ell - f\left(\frac{x}{e^u}\right)\right) u^{p-1} e^{-u} du.
\end{aligned}$$

Where the previous to last expression comes from the usual substitution $u = \ln\left(\frac{x}{t}\right)$.

Now, fix $\epsilon > 0$. Since $\Gamma(p) < \infty$, there is $S > 0$ such that $\int_S^\infty u^{p-1}e^{-u}du < \frac{\Gamma(p)}{2\|f\|_\infty}\epsilon$. In fact, for any $R > S$, we have that

$$\int_R^\infty u^{p-1}e^{-u}du < \int_S^\infty u^{p-1}e^{-u}du < \frac{\Gamma(p)}{2\|f\|_\infty}\epsilon.$$

Let $x > Ne^S$, then $S < \ln\left(\frac{x}{N}\right)$. Next, choose $R$ such that $R \in (S, \ln(\frac{x}{N}))$. Notice that, if $u < R$, then

$$\frac{x}{e^u} > \frac{x}{e^R} > \frac{x}{e^{\ln(x/N)}} = N, \text{ and so } \ell - \epsilon \leq f\left(\frac{x}{e^u}\right).$$

Consequently,

$$\begin{aligned}
\ell - (J^p f)(x) &= \frac{1}{\Gamma(p)}\left(\int_0^R \left(\ell - f\left(\frac{x}{e^u}\right)\right) u^{p-1}e^{-u}du + \int_R^\infty \left(\ell - f\left(\frac{x}{e^u}\right)\right) u^{p-1}e^{-u}du\right) \\
&\leq \frac{1}{\Gamma(p)}\epsilon \int_0^R u^{p-1}e^{-u}du + \frac{2\|f\|_\infty}{\Gamma(p)} \int_R^\infty u^{p-1}e^{-u}du \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

As $\epsilon \to 0^+$, we obtain

$$\ell \leq \liminf_{x\to\infty}(J^p f)(x).$$

Similarly we can verify that

$$\limsup_{x\to\infty}(J^p f)(x) \leq \limsup_{x\to\infty} f(x).$$

$\square$

**Corollary 3.4.8.** *If* $\lim_{x\to\infty} f(x) = L$ *then* $\lim_{x\to\infty}(J^p f)(x) = L$ *for all* $p > 0$.

*Here,* $\lim_{x\to\infty} f(x) = L$ *for* $f \in L^\infty$ *means the following: For every* $\epsilon > 0$*, there exists* $N > 0$ *such that* $\operatorname{ess\,sup}_{x\geq N} |f(x) - L| \leq \epsilon$.

**Theorem 3.4.9.** *For each* $f \in L^\infty$*, the map* $p \mapsto J^p f : (0,\infty) \to L^\infty$ *is norm-to-norm continuous.*

*Proof.* First consider $p \in (0, 1)$. Consider a sequence of positive real numbers $(p_n)_n$ such that $p_n \to p$. We can assume that $p_n \in (0, 1)$, for each $n \in \mathbb{N}$. So, for a fixed $f \in L^\infty$ we have that

$$\|J^{p_n}f - J^p f\|_\infty = \operatorname*{ess\,sup}_{x\in(0,\infty)} \left| \frac{1}{x\Gamma(p_n)} \int_{t=0}^{x} f(t) \left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1} dt - \frac{1}{x\Gamma(p)} \int_{t=0}^{x} f(t) \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} dt \right|.$$

We add and subtract the term $\dfrac{1}{x\Gamma(p)} \displaystyle\int_{t=0}^{x} f(t) \left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1} dt$ and obtain

$$\|J^{p_n}f - J^p f\|_\infty \le \operatorname*{ess\,sup}_{x\in(0,\infty)} \left\{ \left| \frac{1}{\Gamma(p_n)} - \frac{1}{\Gamma(p)} \right| \|f\|_\infty \frac{1}{x} \int_{t=0}^{x} \left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1} dt + \frac{1}{x\Gamma(p)} \int_{t=0}^{x} \left| f(t) \left( \left[\ln\left(\frac{x}{t}\right)\right]^{p_n-1} - \left[\ln\left(\frac{x}{t}\right)\right]^{p-1} \right) dt \right| \right\}.$$

Now, let $u = \ln\left(\frac{x}{t}\right)$, our usual substitution. Then

$$\|J^{p_n}f - J^p f\|_\infty \le \left| \frac{1}{\Gamma(p_n)} - \frac{1}{\Gamma(p)} \right| \|f\|_\infty \Gamma(p_n) + \|f\|_\infty \frac{1}{\Gamma(p)} \int_{u=0}^{\infty} \left| [u]^{p_n-1} - [u]^{p-1} \right| e^{-u} du.$$

Since $\frac{1}{\Gamma(\cdot)}$ is continuous for positive values, it is enough to show that

$$\lim_{n\to\infty} \int_{u=0}^{\infty} \left| [u]^{p_n-1} - [u]^{p-1} \right| e^{-u} du = 0.$$

We apply Dominated Convergence Theorem:

Since $p_n - 1 \to p - 1$ and $p - 1 < 0$, there exists $N \in \mathbb{N}$ such that $p - 1 - \frac{p}{2} \le p_n - 1 < 0$ for all $n \ge N$.

If $u \ge 1$ then we have that $\left|u^{p_n-1} - u^{p-1}\right| e^{-u} \le (u^{p_n-1} + u^{p-1})e^{-u} \le 2e^{-u}$ for all $n \ge N$.

If $u \in (0,1)$, then $u^y$ is decreasing as function of $y$. Then $u^{p_n-1} \le u^{\frac{p}{2}-1}$ for all $n \ge N$, and so $\left|u^{p_n-1} - u^{p-1}\right| e^{-u} \le \left|u^{p_n-1} - u^{p-1}\right| \le u^{p_n-1} + u^{p-1} \le u^{\frac{p}{2}-1} + u^{p-1}$.

Define

$$g(u) = \begin{cases} u^{\frac{p}{2}-1} + u^{p-1}, & 0 < u < 1 \\ 2e^{-u}, & 1 \le u \end{cases}$$

Notice

$$\int_0^\infty g(u)du = \int_0^1 \left(u^{\frac{p}{2}-1} + u^{p-1}\right) du + 2\int_1^\infty e^{-u} du.$$

We know this integral is finite since $p-1$, $\frac{p}{2}-1 \in (-1,0)$. Therefore we reach the desired conclusion.

Now, for $p \geq 1$ such that $\langle p \rangle > 0$, we know there exists $N \in \mathbb{N}$ such that $\lfloor p_n \rfloor = \lfloor p \rfloor$ for all $n \geq N$. Assume, without loss of generality, that $N = 1$. Since $\langle p_n \rangle \to \langle p \rangle$, $\langle p_n \rangle$,$\langle p \rangle$ $\in (0,1)$ for all $n$, we have that $(J^{\langle p_n \rangle} f)(x) \to (J^{\langle p \rangle} f)(x)$ uniformly in $x$ as $n \to \infty$. Therefore $J^{\lfloor p_n \rfloor}(J^{\langle p_n \rangle} f)(x) \to J^{\lfloor p \rfloor}(J^{\langle p \rangle} f)(x)$ uniformly in $x$ as $n \to \infty$.

Next, consider when $p \in \mathbb{N}$. If $s \to p$, then we can assume that $s > p-1$. So, we notice that

$$\|J^s f - J^p f\|_\infty \leq \|J^{p-1}\|_{op} \|J^{s-p+1} f - Jf\|_\infty,$$

where, in case $p = 1$, we define $J^0 f := f$ for every $f \in L^\infty$. Then

$$\begin{aligned}\lim_{s\to p} \|J^s f - J^p f\| &\leq \|J^{p-1}\|_{op} \lim_{s \to p} \|(J^{s-p+1/2})(J^{1/2} f) - (J^{1/2})(J^{1/2} f)\|_\infty \\ &= \|J^{p-1}\|_{op} \lim_{r \to 1/2} \|(J^r)(J^{1/2} f) - (J^{1/2})(J^{1/2} f)\|_\infty.\end{aligned}$$

Now, for $r \in (0,1)$ and fixed $g \in L^\infty$, we already checked that the mapping $r \mapsto J^r g : (0,1) \to L^\infty$ is norm-to-norm continuous. Since $J^{1/2} f \in L^\infty$, we can apply this result to the last expression, and get the desired result. □

**Theorem 3.4.10.** *Fix $r > 0$. Fix $f \in L^\infty$. Then $(J^r f)(x)$ is continuous on $x \in (0, \infty)$.*

*Proof.* First, we fix $r \in (0,1)$. Let $x, y \in (0,\infty)$. Without loss of generality, assume $y > x$. Then

$$|(J^r f)(x) - (J^r f)(y)| = \left| \frac{1}{\Gamma(r)} \left\{ \frac{1}{x} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{r-1} f(t)dt - \frac{1}{y} \int_0^y \left[\ln\left(\frac{y}{t}\right)\right]^{r-1} f(t)dt \right\} \right|.$$

Therefore

$$\Gamma(r)|(J^r f)(x) - (J^r f)(y)| =$$

$$\left| \frac{1}{x} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{r-1} f(t)dt - \left(\frac{1}{x} + \frac{x-y}{xy}\right) \left( \int_0^x \left[\ln\left(\frac{y}{t}\right)\right]^{r-1} f(t)dt + \int_x^y \left[\ln\left(\frac{y}{t}\right)\right]^{r-1} f(t)dt \right) \right| =$$

$$|I_1 - I_2 - I_3| \leq |I_1| + |I_2| + |I_3|.$$

Where

$$\begin{aligned}
I_1 &:= \int_0^x \left\{ \frac{1}{x} \left[ \ln \left( \frac{x}{t} \right) \right]^{r-1} - \frac{1}{x} \left[ \ln \left( \frac{y}{t} \right) \right]^{r-1} \right\} f(t)dt, \\
I_2 &:= \frac{x-y}{xy} \int_0^x \left[ \ln \left( \frac{y}{t} \right) \right]^{r-1} f(t)dt, \, and \\
I_3 &:= \frac{1}{y} \int_x^y \left[ \ln \left( \frac{y}{t} \right) \right]^{r-1} f(t)dt.
\end{aligned}$$

For the second integral, we make the usual substitution $u = \ln\left(\frac{y}{t}\right)$ to get

$$\left| \frac{x-y}{xy} \int_0^x \left[ \ln \left( \frac{y}{t} \right) \right]^{r-1} f(t)dt \right| \leq \|f\|_\infty \left| \frac{x-y}{x} \right| \Gamma(r),$$

and this last expression goes to 0 as $y - x \to 0$.

For the last integral, again we make the substitution $u = \ln\left(\frac{y}{t}\right)$ to get

$$\begin{aligned}
\frac{1}{y} \left| \int_x^y \left[ \ln \left( \frac{y}{t} \right) \right]^{r-1} f(t)dt \right| &\leq \|f\|_\infty \int_0^{\ln(y/x)} u^{r-1} e^{-u} du \\
&\leq \|f\|_\infty \int_0^{\ln(y/x)} u^{r-1} du \\
&= \|f\|_\infty \frac{1}{r} \left[ \ln (y/x) \right]^r,
\end{aligned}$$

and this last expression goes to 0 as $y - x \to 0$.

For the first integral, since $r - 1 < 0$ and $y > x$, then

$$\left| \frac{1}{x} \left[ \ln \left( \frac{y}{t} \right) \right]^{r-1} - \frac{1}{x} \left[ \ln \left( \frac{x}{t} \right) \right]^{r-1} \right| \leq 2 \frac{1}{x} \left[ \ln \left( \frac{x}{t} \right) \right]^{r-1},$$

and recall that

$$2 \int_0^x \frac{1}{x} \left[ \ln \left( \frac{x}{t} \right) \right]^{r-1} dt = 2\Gamma(r) < \infty.$$

Therefore, we can apply Dominated Convergence Theorem to get

$$\int_0^x \left\{ \frac{1}{x} \left[ \ln \left( \frac{x}{t} \right) \right]^{r-1} f(t) - \frac{1}{x} \left[ \ln \left( \frac{y}{t} \right) \right]^{r-1} f(t) \right\} dt \to 0 \text{ as } y - x \to 0.$$

Finally, for $r \geq 1$, we have that

$$|J^r f(y) - J^r f(x)| = |J^{\lfloor r \rfloor} J^{\langle r \rangle} f(y) - J^{\lfloor r \rfloor} J^{\langle r \rangle} f(x)|.$$

Let $h(x) := J^{\langle r \rangle} f(x)$, let $\lfloor r \rfloor = n$, and let $g := J^{n-1}h$, where $J^0 h := h$. Then

$$\begin{aligned}
|J^r f(y) - J^r f(x)| &= |J^n h(y) - J^n h(x)| \\
&= |Jg(y) - Jg(x)| \\
&= \left| \frac{1}{y} \int_0^y g(t)dt - \frac{1}{x} \int_0^x g(t)dt \right| \\
&= \left| \left( \frac{1}{x} + \frac{x-y}{xy} \right) \left( \int_0^x g(t)dt + \int_x^y g(t)dt \right) - \frac{1}{x} \int_0^x g(t)dt \right| \\
&= \left| \frac{x-y}{xy} \int_0^x g(t)dt + \frac{1}{y} \int_x^y g(t)dt \right| \\
&\leq \left| \frac{x-y}{y} \right| \|g\|_\infty + \frac{y-x}{y} \|g\|_\infty,
\end{aligned}$$

and this last expression goes to 0 as $y - x \to 0$. □

## 3.5 The vector space $Ces^r$

**Definition 3.5.1.** For each $r > 0$, we define the vector subspace of $L^\infty$:

$$Ces^r := \{f \in L^\infty : \zeta_r(f) := \lim_{x\to\infty} (J^r f)(x) \in \mathbb{R}\}.$$

*Remark.* We check that $\|\zeta_r\|_{(Ces^r)^*} = 1$: Recall that we already know $\|J^r\|_{op} = 1$, therefore for any $f \in L^\infty$ we have that

$$|\zeta_r(f)| = |\lim_{x\to\infty} (J^r f)(x)| = \lim_{x\to\infty} |(J^r f)(x)| \leq \|f\|_\infty.$$

So $\|\zeta_r\|_{(Ces^r)^*} \leq 1$. Also, $\zeta_r(\mathbb{1}) = \lim_{x\to\infty} (J^r \mathbb{1})(x) = \lim_{x\to\infty} \mathbb{1}(x) = 1$. Thus $\|\zeta_r\|_{(Ces^r)^*} = 1$.

Jeromy Sivek, in his PhD Thesis [23], obtained a quantitative version of the following qualitative theorem:

*For $x \in \ell^\infty$, $Cx$ is convergent if and only if $C^2 x$ is convergent.*

This qualitative result follows from a theorem due to Frobenius [9], and a classical theorem of Hardy and Littlewood (see Theorem 7.3 of [13]).

We followed closely Sivek's arguments in the proof of Theorem II.9 from [23], and obtained the exact analogue result for the space $L^\infty$, including the same quantitative outcome.

The qualitative version of our result tells us that $Ces^n = Ces^m$ for every $n, m \in \mathbb{N}$. We present our quantitative result next:

**Theorem 3.5.1.** *Let $f \in L^\infty$ with $Jf \notin BC_L$. Define*

$$q := \limsup_{x\to\infty} f(x) \text{ and } p := \liminf_{x\to\infty} f(x);$$

*and also*

$$b := \limsup_{x\to\infty} Jf(x) \text{ and } a := \liminf_{x\to\infty} Jf(x);$$

*Let $d := b - a$ and $m := (a-p) \vee (q-b)$. Then*

$$\limsup_{x\to\infty} J^2 f(x) - \liminf_{x\to\infty} J^2 f(x) \geq \frac{d^2}{10d + 8m + \sqrt{(10d+8m)^2 - 4d^2}}.$$

*In particular, $h(x) := J^2 f(x) \notin BC_L$.*

*Proof.* Let $f \in L^\infty$. Assume $Jf(x) \notin BC_L$.

Since $q := \limsup_{x\to\infty} f(x)$, then for each $\epsilon > 0$, there exists $N > 0$ such that $f(x) < q + \epsilon$ for almost all $x \geq N$, therefore

$$\frac{1}{x}\int_{t=0}^{x} f(t)dt \leq \frac{1}{x}\int_{t=0}^{N} f(t)dt + \frac{1}{x}\int_{t=N}^{x} (q+\epsilon)dt \leq \frac{1}{x}\int_{t=0}^{N} \|f\|_\infty dt + \frac{x-N}{x}(q+\epsilon).$$

Next, by taking limit superior as $x$ approaches $\infty$ and then letting $\epsilon$ go to 0, we get that

$$\limsup_{x\to\infty} Jf(x) \leq q = \limsup_{x\to\infty} f(x).$$

That is, $b \leq q$. Similarly we see that $p \leq a$.

So, we have

$$-\infty < -\|f\|_\infty \leq p \leq a < b \leq q \leq \|f\|_\infty < \infty, \text{ and}$$
$$d := b - a > 0.$$

Fix real numbers $\theta$, $\tau$, $\theta'$ and $\tau'$ with $0 < \theta < \tau < 1/2$ and $0 < \theta' < \tau' < 1/2$. We will later choose these values in an optimal way to achieve the inequality stated in the theorem.

Fix $\epsilon \in (0, (1/2 - \tau) \wedge (1/2 - \tau'))$. Then, there exists $K > 0$ such that

for all $x \geq K$ we have that $a - \epsilon d < Jf(x) < b + \epsilon d$, and

for almost all $x \geq K$ we have that $p - \epsilon d < f(x) < q + \epsilon d$.

Fix $x_0 \geq K$ arbitrary. We consider two cases: $J^2 f(x_0) := \dfrac{1}{x_0}\displaystyle\int_0^{x_0} Jf(t)dt \leq \dfrac{a+b}{2}$; and $J^2 f(x_0) \geq \dfrac{a+b}{2}$.

**Case 1.** $J^2 f(x_0) := \dfrac{1}{x_0}\displaystyle\int_0^{x_0} Jf(t)dt \leq \dfrac{a+b}{2}$.

Since $b := \limsup_{x\to\infty} Jf(x)$, then there exists $x_1 > x_0$ such that

$$Jf(x_1) > b - \epsilon d = \frac{a+b}{2} + \left(\tfrac{1}{2} - \epsilon\right) d > \frac{a+b}{2} + \tau d.$$

Since $a := \liminf_{x\to\infty} Jf(x)$, then there exists $\tilde{x} > x_1$ such that

$$Jf(\tilde{x}) < a + \epsilon d = \frac{a+b}{2} + \left(\epsilon - \tfrac{1}{2}\right) d < \frac{a+b}{2} - \tau d < \frac{a+b}{2} + \tau d.$$

We see then, that the set $A := \left\{ x \in \mathbb{R} : x > x_1 \text{ } and \text{ } Jf(x) = \dfrac{1}{x}\displaystyle\int_0^x f(t)dt < \dfrac{a+b}{2} + \tau d \right\}$ is not empty, and is bounded below by $x_1$. So we let $x_2 := \inf A$.

Notice $x_1 \neq x_2$, since

$$Jf(x_1) > \frac{a+b}{2} + \tau d \text{ and } Jf(x_2) \leq \frac{a+b}{2} + \tau d, \text{ by continuity of } Jf(x).$$

Therefore $x_1 < x_2$, and we have that

$$Jf(x) = \frac{1}{x}\int_0^x f(t)dt \geq \frac{a+b}{2} + \tau d, \text{ for all } x \in [x_1, x_2].$$

Let $\Delta := x_2 - x_1 > 0$. Then

$$\begin{aligned}
\frac{a+b}{2} + \tau d &\geq Jf(x_2) \\
&= \frac{1}{x_2}\int_0^{x_2} f(t)dt \\
&= \frac{1}{x_1+\Delta}\int_0^{x_1} f(t)dt + \frac{1}{x_1+\Delta}\int_{x_1}^{x_1+\Delta} f(t)dt \\
&> \frac{x_1}{x_1+\Delta} Jf(x_1) + \frac{\Delta}{x_1+\Delta}(p - \epsilon d) \\
&> \frac{x_1}{x_1+\Delta}(b - \epsilon d) + \frac{\Delta}{x_1+\Delta}(p - \epsilon d).
\end{aligned}$$

Therefore

$$(x_1 + \Delta)\left(\frac{a+b}{2} + \tau d\right) > x_1(b - \epsilon d) + \Delta(p - \epsilon d);$$

and so,

$$\Delta\left(\frac{a+b}{2}+\tau d-(p-\epsilon d)\right)>x_1\left[(b-\epsilon d)-\left(\frac{a+b}{2}+\tau d\right)\right].$$

Thus,

$$\Delta\left[\left(\tfrac{1}{2}+\tau+\epsilon\right)d+a-p\right]>x_1\left(\tfrac{1}{2}-\tau-\epsilon\right)d;$$

and since $p<a$ and $\epsilon<\frac{1}{2}-\tau$ we get that

$$\Delta>\frac{x_1(\frac{1}{2}-\epsilon-\tau)}{\left[\frac{1}{2}+\epsilon+\tau+\frac{a-p}{d}\right]}>0.$$

Now, recall that $\theta\in(0,\tau)$.

**Sub-case 1.a.** $J^2f(x_1)\geq\dfrac{a+b}{2}+\theta d.$

Since $J^2f(x_0)\leq\dfrac{a+b}{2}$, then

$$J^2f(x_1)-J^2f(x_0)\geq\theta d.$$

**Sub-case 1.b.** $J^2f(x_1)<\dfrac{a+b}{2}+\theta d.$

Then

$$\begin{aligned}
J^2f(x_1+\Delta)-J^2f(x_1)&=\frac{1}{x_1+\Delta}\int_0^{x_1+\Delta}Jf(t)dt-J^2f(x_1)\\
&=\frac{1}{x_1+\Delta}\int_0^{x_1}Jf(t)dt+\frac{1}{x_1+\Delta}\int_{x_1}^{x_1+\Delta}Jf(t)dt-J^2f(x_1)\\
&\geq\frac{x_1}{x_1+\Delta}J^2f(x_1)+\frac{\Delta}{x_1+\Delta}\left(\frac{a+b}{2}+\tau d\right)-J^2f(x_1)\\
&=\frac{\Delta}{x_1+\Delta}\left(\frac{a+b}{2}+\tau d\right)-\frac{\Delta}{x_1+\Delta}J^2f(x_1)\\
&\geq\frac{\Delta}{x_1+\Delta}\left(\frac{a+b}{2}+\tau d\right)-\frac{\Delta}{x_1+\Delta}\left(\frac{a+b}{2}+\theta d\right)\\
&=\left(1-\frac{x_1}{x_1+\Delta}\right)(\tau-\theta)d.
\end{aligned}$$

So, we have that

$$\begin{aligned}
J^2f(x_1+\Delta)-J^2f(x_1)&>\left(1-\frac{x_1}{x_1+\frac{x_1(\frac{1}{2}-\epsilon-\tau)}{\left[\frac{1}{2}+\epsilon+\tau+\frac{a-p}{d}\right]}}\right)(\tau-\theta)d\\
&=\left(1-\frac{1}{1+\frac{(\frac{1}{2}-\epsilon-\tau)}{\left[\frac{1}{2}+\epsilon+\tau+\frac{a-p}{d}\right]}}\right)(\tau-\theta)d.
\end{aligned}$$

It follows from Sub-cases 1.a and 1.b that

$$\sup_{x \geq x_0} J^2 f(x) - \inf_{y \geq x_0} J^2 f(y) \geq U_\epsilon,$$

where

$$U_\epsilon := min \left\{ \theta d, \left( 1 - \frac{1}{1 + \frac{(\frac{1}{2} - \epsilon - \tau)}{\left[\frac{1}{2} + \epsilon + \tau + \frac{a-p}{d}\right]}} \right) (\tau - \theta) d \right\}.$$

**Case 2.** $J^2 f(x_0) \geq \dfrac{a+b}{2}$.

In this case $J^2 g(x_0) \leq \dfrac{(-b)+(-a)}{2}$, where $g(x) := -f(x)$.

Notice that $\limsup_{x \to \infty} g(x) = -p$ and $\liminf_{x \to \infty} g(x) = -q$. From Case 1, replacing $\theta$ with $\theta'$, $\tau$ with $\tau'$, $a$ with $-b$, $b$ with $-a$, and $p$ with $-q$, we get that

$$\sup_{x \geq x_0} J^2 f(x) - \inf_{y \geq x_0} J^2 f(y) = \sup_{y \geq x_0} J^2 g(y) - \inf_{x \geq x_0} J^2 g(x) \geq U_\epsilon',$$

where

$$U_\epsilon' := min \left\{ \theta' d, \left( 1 - \frac{1}{1 + \frac{(\frac{1}{2} - \epsilon - \tau')}{\left[\frac{1}{2} + \epsilon + \tau' + \frac{-b-(-q)}{d}\right]}} \right) (\tau' - \theta') d \right\}.$$

It follows from Case 1 and Case 2 that for all real numbers $x_0 \geq K$,

$$\sup_{x \geq x_0} J^2 f(x) - \inf_{y \geq x_0} J^2 f(y) \geq U_\epsilon \wedge U_\epsilon'.$$

Letting $x_0 \to \infty$ we get that

$$\limsup_{x \to \infty} J^2 f(x) - \liminf_{x \to \infty} J^2 f(x) \geq U_\epsilon \wedge U_\epsilon'.$$

But $\epsilon \in (0, (1/2 - \tau) \wedge (1/2 - \tau'))$ is completely arbitrary, so we can let $\epsilon \to 0^+$ to get

$$\limsup_{x \to \infty} J^2 f(x) - \liminf_{x \to \infty} J^2 f(x) \geq U \wedge U',$$

where

$$U := min \left\{ \theta d, \left( 1 - \frac{1}{1 + \frac{(\frac{1}{2} - \tau)}{\left[\frac{1}{2} + \tau + \frac{a-p}{d}\right]}} \right) (\tau - \theta) d \right\} = min \left\{ \theta, \frac{\left(\frac{1}{2} - \tau\right)}{1 + \frac{a-p}{d}} (\tau - \theta) \right\} d,$$

and

$$U^{'} := min\left\{\theta^{'}, \frac{\left(\frac{1}{2}-\tau^{'}\right)}{1+\frac{q-b}{d}}(\tau^{'}-\theta^{'})\right\}d.$$

Recall that the real numbers $\theta$, $\tau$, $\theta^{'}$ and $\tau^{'}$ are still to be chosen. Fix $\theta \in (0, 1/2)$. Fix $\tau \in (\theta, 1/2)$. Consider positive real numbers $u := 1/2 - \tau$ and $v := \tau - \theta$, so that $u + v = 1/2 - \theta$. The product $uv$ is maximal when $u = v = (1/2)(1/2 - \theta)$.

Thus, when $\tau \in (\theta, 1/2)$, we have that $(1/2-\tau)(\tau-\theta) \leq (1/4)(1/2-\theta)^2$, with equality when $\tau = (1/2)(1/2+\theta)$. We may argue similarly using $\theta^{'}$ and $\tau^{'}$. So, for all $\theta, \theta^{'} \in (0, 1/2)$,

$$\limsup_{x\to\infty} J^2 f(x) - \liminf_{x\to\infty} J^2 f(x) \geq W_\theta \wedge W_\theta^{'},$$

where

$$W_\theta := min\left\{\theta, \frac{\frac{1}{4}\left(\frac{1}{2}-\theta\right)^2}{1+\frac{a-p}{d}}\right\}d, \text{ and } W_\theta^{'} := min\left\{\theta^{'}, \frac{\frac{1}{4}\left(\frac{1}{2}-\theta^{'}\right)^2}{1+\frac{q-b}{d}}\right\}d.$$

Since $\theta \mapsto \theta$ is strictly increasing on $(0, 1/2)$, and $\theta \mapsto \dfrac{\frac{1}{4}\left(\frac{1}{2}-\theta\right)^2}{1+\frac{a-p}{d}}$ is strictly decreasing on $(0, 1/2)$, we have that the function $\theta \mapsto min\left\{\theta, \dfrac{\frac{1}{4}\left(\frac{1}{2}-\theta\right)^2}{1+\frac{a-p}{d}}\right\}$ is maximal on $(0, 1/2)$ precisely when

$$\theta = \frac{\frac{1}{4}\left(\frac{1}{2}-\theta\right)^2}{1+\frac{a-p}{d}}.$$

Solving this quadratic equation for $\theta \in (0, 1/2)$ yields:

$$\theta = \frac{1}{2}\left[5+4\left(\frac{a-p}{d}\right)\right] - \frac{1}{2}\sqrt{\left[5+4\left(\frac{a-p}{d}\right)\right]^2 - 1}.$$

Making a similar argument for $\theta^{'}$, we conclude that

$$\limsup_{x\to\infty} J^2 f(x) - \liminf_{x\to\infty} J^2 f(x) \geq (\Theta \wedge \Theta^{'})d;$$

where

$$\Theta := \frac{1}{2}\left[5+4\left(\frac{a-p}{d}\right)\right] - \frac{1}{2}\sqrt{\left[5+4\left(\frac{a-p}{d}\right)\right]^2 - 1},$$

and

$$\Theta^{'} := \frac{1}{2}\left[5+4\left(\frac{q-b}{d}\right)\right] - \frac{1}{2}\sqrt{\left[5+4\left(\frac{q-b}{d}\right)\right]^2 - 1},$$

Let $m := (a-p) \vee (q-b) = \max\{a-p, q-b\}$. It follows that

$$(\Theta \wedge \Theta^{'})d = \frac{d/4}{\frac{1}{2}\left[5+\frac{4m}{d}\right]+\frac{1}{2}\sqrt{\left[5+\frac{4m}{d}\right]^2-1}}$$
$$= \frac{d^2}{10d+8m+\sqrt{(10d+8m)^2-4d^2}}.$$

□

The following Corollary not only follows from Theorem 3.5.1, but also generalizes it:

**Corollary 3.5.2.** *For any $1 \le r, s$ we have that $Ces^r = Ces^s$.*

*Proof.* Assume, without loss of generality, that $r < s$. First notice that we already know $Ces^r \subset Ces^s$. Indeed, from the semigroup property of $\{J^p\}_{p>0}$ we have that $J^s f(x) = J^{s-r}(J^r f(x))$. By the fact that the operator $J^p$ preserves limits at infinity for every $p > 0$, and in particular for $p = s - r$, the result follows. Next, consider $f \in Ces^s$. This will imply that $f \in Ces^{\lfloor s \rfloor + 1}$ since $s < \lfloor s \rfloor + 1$. Then by Theorem 3.5.1 this implies that $f \in Ces^{\lfloor r \rfloor}$, which in turn implies $f \in Ces^r$. □

To obtain the most general form of the previous results, we first present the following theorem, whose proof is a variation of the proof for Theorem 3.5.1:

**Theorem 3.5.3.** *Let $r \in (0,1]$. Let $f \in L^\infty$ with $J^r f \notin BC_L$. Define*

$$q := \limsup_{x\to\infty} f(x) \text{ and } p := \liminf_{x\to\infty} f(x);$$

*and also*

$$b := \limsup_{x\to\infty} J^r f(x) \text{ and } a := \liminf_{x\to\infty} J^r f(x);$$

*Let $d := b - a$. Then for every $\tau \in (0, 1/2)$ there exists $\Theta := \Theta(r, d, \|f - \frac{a+b}{2}\|_\infty, \tau) > 0$ such that*

$$\delta_{1+r} f := \limsup_{x\to\infty} J^{1+r} f(x) - \liminf_{x\to\infty} J^{1+r} f(x) \ge \Theta.$$

*In particular, $J^{1+r} f(x) \notin BC_L$.*

*Moreover, we may choose $\Theta$ to be*

$$\Theta := \left(\frac{r\gamma}{2V_r^{-1}(\gamma)}\right)^{1/r} \frac{(\tau/2)}{G_r^{-1}(\gamma)\|f - \frac{a+b}{2}\|_\infty^{1/r}} d^{1+1/r}.$$

*Here,*

$$\gamma := \gamma(\tau, r) := \frac{1/2 - \tau}{2\tau + \frac{1}{\Gamma(r)}}.$$

*Also,*

$$G_r(w) := w \int_0^{\ln(w)} u^{r-1} e^{-u} du, \text{ and } \; V_r(w) := w \frac{(\ln(w))^r}{r}, \text{ for all } w \in [1, \infty).$$

*Proof.* Without loss of generality, we may assume that $\dfrac{a+b}{2} = 0$. Since, if this is not the case, we can consider $f_1 := f - \dfrac{a+b}{2}$. Then we would have that

$$b_1 := \limsup_{x\to\infty} J^r f_1(x) = \limsup_{x\to\infty} J^r f(x) - \frac{a+b}{2} = b - \frac{a+b}{2} = \frac{b-a}{2} = \frac{d}{2} > 0,$$

and

$$a_1 := \liminf_{x\to\infty} J^r f_1(x) = \liminf_{x\to\infty} J^r f(x) - \frac{a+b}{2} = a - \frac{a+b}{2} = \frac{a-b}{2} = -\frac{d}{2} < 0.$$

Therefore

$$\frac{a_1 + b_1}{2} = 0 \text{ and } d_1 := b_1 - a_1 = d > 0.$$

Also

$$q_1 := \limsup_{x\to\infty} f_1(x) = \limsup_{x\to\infty} f(x) - \frac{a+b}{2} = q - \frac{a+b}{2},$$

and

$$p_1 := \liminf_{x\to\infty} f_1(x) = \liminf_{x\to\infty} f(x) - \frac{a+b}{2} = p - \frac{a+b}{2},$$

By Claim 3.4.7,

$$\frac{d}{2} \le q_1 \le \|f - \frac{a+b}{2}\|_\infty \text{ and } -\|f - \frac{a+b}{2}\|_\infty \le p_1 \le -\frac{d}{2}.$$

Moreover,

$$\begin{aligned}
\delta_{1+r} f_1 &= \limsup_{x\to\infty} J^{1+r} f_1(x) - \liminf_{x\to\infty} J^{1+r} f_1(x) \\
&= \limsup_{x\to\infty} J^{1+r} \left( f(x) - \frac{a+b}{2} \right) - \liminf_{x\to\infty} J^{1+r} \left( f(x) - \frac{a+b}{2} \right) \\
&= \limsup_{x\to\infty} \left( J^{1+r} f(x) - \frac{a+b}{2} J^{1+r} \mathbb{1}(x) \right) - \liminf_{x\to\infty} \left( J^{1+r} f(x) - \frac{a+b}{2} J^{1+r} \mathbb{1}(x) \right) \\
&= \limsup_{x\to\infty} J^{1+r} f(x) - \frac{a+b}{2} - \liminf_{x\to\infty} J^{1+r} f(x) + \frac{a+b}{2} \\
&= \limsup_{x\to\infty} J^{1+r} f(x) - \liminf_{x\to\infty} J^{1+r} f(x) = \delta_{1+r} f.
\end{aligned}$$

Hence, we can relabel $f_1$ as $f$, $a_1$ as $a$, $b_1$ as $b$, $p_1$ as $p$, $q_1$ as $q$, and $d_1$ as $d$ (and $d_1 = d$). So, without loss of generality, we have that $a = -b$, and $d = 2b > 0$.

Recall, by Claim 3.4.7, we know that

$$-\infty < -\|f\|_\infty \le p \le a = \tfrac{-d}{2} < 0 \text{ and } 0 < b = \tfrac{d}{2} \le q \le \|f\|_\infty < \infty.$$

Fix real numbers $\theta$, $\tau$, $\theta'$ and $\tau'$ with $0 < \theta < \tau < 1/2$ and $0 < \theta' < \tau' < 1/2$, arbitrary. The relationships between $\tau, \theta, \tau'$ and $\theta'$ will be chosen later.

Fix $\epsilon \in (0, (1/2 - \tau) \wedge (1/2 - \tau'))$. Then, there exists $K := K_\epsilon > 0$ such that for all $x \ge K$,

$$a - \epsilon d < J^r f(x) < b + \epsilon d.$$

Fix $x_0 > K$ arbitrary. We consider two cases: $J^{1+r} f(x_0) = \dfrac{1}{x_0} \displaystyle\int_0^{x_0} J^r f(t) dt \le 0$; and $J^{1+r} f(x_0) \ge 0$.

**Case 1.** $J^{1+r} f(x_0) \le 0$.

Since $b := \limsup_{x\to\infty} J^r f(x)$, then there exists $x_1 > x_0$ such that

$$J^r f(x_1) > b - \epsilon d = \frac{a+b}{2} + \left(\tfrac{1}{2} - \epsilon\right) d = \left(\tfrac{1}{2} - \epsilon\right) d > \tau d.$$

Since $a := \liminf_{x\to\infty} J^r f(x)$, then there exists $\tilde{x} > x_1$ such that

$$J^r f(\tilde{x}) < a + \epsilon d = \frac{a+b}{2} + \left(\epsilon - \tfrac{1}{2}\right) d = \left(\epsilon - \tfrac{1}{2}\right) d < -\tau d < \tau d.$$

We see then, that the set $A := \{x \in (0, \infty) : x > x_1 \textit{ and } J^r f(x) < \tau d\}$ is not empty, and is bounded below by $x_1$. So we let $x_2 := \inf A$.

Notice $x_1 \ne x_2$ since

$$J^r f(x_1) > \tau d \text{ and } J^r f(x_2) \leq \tau d \text{ by continuity of } J^r f.$$

Therefore $x_1 < x_2$, and we have that

$$J^r f(x) \geq \tau d \textit{ for all } x \in [x_1, x_2], \textit{ and in fact } J^r f(x_2) = \tau d. \qquad (\diamond \diamond)$$

Let $\Delta := x_2 - x_1 > 0$. Then,

$$\begin{aligned}
&\tau d - \frac{x_1}{x_2} J^r f(x_1) \\
&= J^r f(x_2) - \frac{x_1}{x_2} J^r f(x_1) \\
&= \frac{1}{x_2 \Gamma(r)} \left( \int_0^{x_2} f(t) \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt - \int_0^{x_1} f(t) \left[ \ln\left(\frac{x_1}{t}\right) \right]^{r-1} dt \right) \\
&= \frac{1}{x_2 \Gamma(r)} \left( \int_0^{x_1} f(t) \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt + \int_{x_1}^{x_2} f(t) \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt - \int_0^{x_1} f(t) \left[ \ln\left(\frac{x_1}{t}\right) \right]^{r-1} dt \right) \\
&= \frac{1}{x_2 \Gamma(r)} \left( \int_{x_1}^{x_2} f(t) \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt - \int_0^{x_1} f(t) \left[ \left[ \ln\left(\frac{x_1}{t}\right) \right]^{r-1} - \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} \right] dt \right) \\
&\geq \frac{1}{x_2 \Gamma(r)} \left( -\|f\|_\infty \int_{x_1}^{x_2} \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt - \|f\|_\infty \int_0^{x_1} \left[ \left[ \ln\left(\frac{x_1}{t}\right) \right]^{r-1} - \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} \right] dt \right).
\end{aligned}$$

Therefore

$$\tau d - \frac{x_1}{x_2} J^r f(x_1) \geq \frac{-\|f\|_\infty}{x_2 \Gamma(r)} \left( \int_{x_1}^{x_2} \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt + \int_0^{x_1} \left[ \left[ \ln\left(\frac{x_1}{t}\right) \right]^{r-1} - \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} \right] dt \right).$$

Denote by

$$\Phi_1(r, x_1, x_2) := \int_{x_1}^{x_2} \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt > 0 \text{ and}$$
$$\Phi_2(r, x_1, x_2) := \int_0^{x_1} \left[ \left[ \ln\left(\frac{x_1}{t}\right) \right]^{r-1} - \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} \right] dt \geq 0.$$

Also denote

$$\Phi(r, x_1, x_2) := \Phi_1(r, x_1, x_2) + \Phi_2(r, x_1, x_2).$$

Notice that, by making the usual substitution $u = \ln\left(\frac{x_2}{t}\right)$ for $\Phi_1(r, x_1, x_2)$, we obtain

$$\Phi_1(r, x_1, x_2) = \int_{x_1}^{x_2} \left[ \ln\left(\frac{x_2}{t}\right) \right]^{r-1} dt = x_2 \int_0^{\ln\left(\frac{x_2}{x_1}\right)} u^{r-1} e^{-u} du.$$

Also, by making the substitutions $u = \ln\left(\frac{x_1}{t}\right)$ and $v = \ln\left(\frac{x_2}{t}\right)$ for $\Phi_2(r, x_1, x_2)$,

$$\begin{aligned}
\Phi_2(r, x_1, x_2) &= \int_0^{x_1} \left[\left[\ln\left(\frac{x_1}{t}\right)\right]^{r-1} dt - \int_0^{x_1} \left[\ln\left(\frac{x_2}{t}\right)\right]^{r-1}\right] dt \\
&= x_1 \int_0^{\infty} u^{r-1} e^{-u} du - x_2 \int_{\ln\left(\frac{x_2}{x_1}\right)}^{\infty} v^{r-1} e^{-v} dv \\
&= (x_1 - x_2 + x_2)\Gamma(r) - x_2 \int_{\ln\left(\frac{x_2}{x_1}\right)}^{\infty} v^{r-1} e^{-v} dv \\
&= -\Delta\Gamma(r) + x_2 \int_0^{\ln\left(\frac{x_2}{x_1}\right)} v^{r-1} e^{-v} dv.
\end{aligned}$$

Therefore

$$\Phi(r, x_1, x_2) = -\Delta\Gamma(r) + 2x_2 \int_0^{\ln\left(\frac{x_2}{x_1}\right)} u^{r-1} e^{-u} du = -\Delta\Gamma(r) + 2\Phi_1(r, x_1, x_2).$$

Here, we make the observation that for $r = 1$, we have that $\Phi(1, x_1, x_2) = x_2 - x_1 = \Delta$.

Now, for $r \in (0, 1]$, we have the following inequality

$$\begin{aligned}
\tau d - \frac{x_1}{x_2} J^r f(x_1) &\geq \frac{-\|f\|_\infty}{x_2 \Gamma(r)} \Phi(r, x_1, x_2) \\
&= \frac{-\|f\|_\infty}{x_2 \Gamma(r)} \left(-\Delta\Gamma(r) + 2\Phi_1(r, x_1, x_2)\right),
\end{aligned}$$

or equivalently

$$\tau d x_2 \geq x_1 J^r f(x_1) + \frac{-\|f\|_\infty}{\Gamma(r)} \left(-\Delta\Gamma(r) + 2\Phi_1(r, x_1, x_2)\right).$$

Since $r \in (0, 1]$, then $\Gamma(r) \geq 1$, and so $\tau d x_2 = \tau d(x_1 + \Delta) \leq \tau d(x_1 + \Gamma(r)\Delta)$, given that $\tau d > 0$. Also recall that $J^r f(x_1) > (\frac{1}{2} - \epsilon)d$. Then,

$$\tau d(x_1 + \Gamma(r)\Delta) \geq x_1(\tfrac{1}{2} - \epsilon)d + \frac{-\|f\|_\infty}{\Gamma(r)} \Phi(r, x_1, x_2).$$

Next, since $\Phi(r, x_1, x_2) \geq 0$, and also $\Gamma(r)\Delta > 0$, we have that

$$\tau d(x_1 + \Gamma(r)\Delta + \Phi(r, x_1, x_2)) \geq x_1(\tfrac{1}{2} - \epsilon)d - \frac{\|f\|_\infty}{\Gamma(r)} \left(\Gamma(r)\Delta + \Phi(r, x_1, x_2)\right).$$

Recall that $\Gamma(r)\Delta + \Phi(r, x_1, x_2) = 2\Phi_1(r, x_1, x_2)$. Thus

$$\tau d(x_1 + 2\Phi_1(r, x_1, x_2)) \geq x_1(\tfrac{1}{2} - \epsilon)d - \frac{\|f\|_\infty}{\Gamma(r)} 2\Phi_1(r, x_1, x_2).$$

This implies

$$2\Phi_1(r, x_1, x_2) \left(\tau d + \frac{\|f\|_\infty}{\Gamma(r)}\right) \geq x_1(\tfrac{1}{2} - \epsilon - \tau)d.$$

Therefore

$$\Delta_r := \Phi_1(r, x_1, x_2) \geq \frac{x_1(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

Equivalently

$$\frac{\Delta_r}{x_1} = \frac{1}{x_1}\Phi_1(r, x_1, x_2) = \frac{x_2}{x_1}\int_0^{\ln\left(\frac{x_2}{x_1}\right)} u^{r-1}e^{-u}du \geq \frac{(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

Denote

$$P_\epsilon := P_{r,d,\|f\|_\infty,\tau,\epsilon} := \frac{(\frac{1}{2} - \epsilon - \tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)},$$

and consider the function

$$G_r(w) := w\int_0^{\ln(w)} u^{r-1}e^{-u}du, \text{ for all } w \in [1, \infty).$$

Notice that $G_r$ is a strictly increasing continuous function. Also $G_r(1) = 0$, and $\lim_{w\to\infty} G_r(w) = \infty$, thus $G_r : [1, \infty) \to [0, \infty)$ has an inverse. So, $(G_r)^{-1} : [0, \infty) \to [1, \infty)$ is well defined and it is also a strictly increasing continuous function. In particular $(G_r)^{-1}(s) > 1$ for all $s \in (0, \infty)$.

Also notice $G_r(\frac{x_2}{x_1}) = \dfrac{\Delta_r}{x_1} \geq P_\epsilon > 0$. Therefore

$$\tfrac{x_2}{x_1} \geq (G_r)^{-1}(P_\epsilon) > 1,$$

which implies

$$\Delta = x_2 - x_1 \geq x_1\left((G_r)^{-1}(P_\epsilon) - 1\right).$$

Here we make the following observation: $G_1(w) = w(1 - e^{-\ln(w)}) = w(1 - \frac{1}{w}) = w - 1$, for all $w \in [1, \infty)$. So $G_1^{-1}(s) = s + 1$, for all $s \in [0, \infty)$.

Now, recall that $0 < \theta < \tau < 1/2$.

**Sub-case 1.a.** $J^{1+r}f(x_1) \geq \theta d$.

Since $J^{1+r}f(x_0) \leq 0$, then

$$J^{1+r}f(x_1) - J^{1+r}f(x_0) \geq \theta d.$$

**Sub-case 1.b.** $J^{1+r}f(x_1) < \theta d$.

Then by fact $(\diamond\,\diamond)$ above,

$$\begin{aligned}
J^{1+r}f(x_1+\Delta) - J^{1+r}f(x_1) &= \frac{1}{x_1+\Delta}\int_0^{x_1+\Delta} J^r f(t)dt - J^{1+r}f(x_1) \\
&= \frac{1}{x_1+\Delta}\int_0^{x_1} J^r f(t)dt + \frac{1}{x_1+\Delta}\int_{x_1}^{x_1+\Delta} J^r f(t)dt - J^{1+r}f(x_1) \\
&\geq \frac{x_1}{x_1+\Delta}J^{1+r}f(x_1) + \frac{\Delta}{x_1+\Delta}\tau d - J^{1+r}f(x_1) \\
&= \frac{\Delta}{x_1+\Delta}\tau d - \frac{\Delta}{x_1+\Delta}J^{1+r}f(x_1) \\
&\geq \frac{\Delta}{x_1+\Delta}(\tau-\theta)d \\
&= \left(1 - \frac{x_1}{x_1+\Delta}\right)(\tau-\theta)d \\
&\geq \left(1 - \frac{x_1}{x_1 + x_1[(G_r)^{-1}(P_\epsilon) - 1]}\right)(\tau-\theta)d \\
&= \left(1 - \frac{1}{(G_r)^{-1}(P_\epsilon)}\right)(\tau-\theta)d \\
&= \left(1 - \frac{1}{(G_r)^{-1}\left(\dfrac{(\frac{1}{2}-\epsilon-\tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}\right)}\right)(\tau-\theta)d.
\end{aligned}$$

It follows from Sub-cases 1.a and 1.b that

$$\sup_{x\geq x_0} J^{1+r}f(x) - \inf_{y\geq x_0} J^{1+r}f(y) \geq U_\epsilon,$$

where

$$U_\epsilon := \min\left\{\theta d, \left(1 - \frac{1}{(G_r)^{-1}(P_\epsilon)}\right)(\tau-\theta)d\right\}, \text{ and } P_\epsilon := \frac{(\frac{1}{2}-\epsilon-\tau)d}{2\left(\tau d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

**Case 2.** $J^{1+r}f(x_0) \geq 0$.

Let $g := -f$. Then $g \in L^\infty$, and $\|g\|_\infty = \|f\|_\infty$. Let $x_0 \geq K$ be fixed and arbitrary.

Notice

$$\liminf_{x\to\infty} J^r g(x) = \liminf_{x\to\infty} -(J^r f(x)) = \limsup_{x\to\infty} J^r f(x) = -b \text{ and } \limsup_{x\to\infty} J^r g(x) = -a.$$

Thus $-a-(-b) = b-a = d$. Also $\frac{-b+-a}{2} = 0$. Further $J^{1+r}g(x_0) = -(J^{1+r}f(x_0)) \leq 0$.

So, we are back in Case 1, with $f$ replaced by $g := -f$. Also replace $\theta$ by $\theta'$, and $\tau$ by $\tau'$. Then,

$$\sup_{x\geq x_0} J^{1+r}f(x) - \inf_{y\geq x_0} J^{1+r}f(y) = \sup_{y\geq x_0} J^{1+r}g(y) - \inf_{x\geq x_0} J^{1+r}g(x) \geq U'_\epsilon,$$

where

$$U'_\epsilon := \min\left\{\theta' d, \left(1 - \frac{1}{(G_r)^{-1}(P'_\epsilon)}\right)(\tau' - \theta')d\right\}, \text{ and } P'_\epsilon := \frac{(\frac{1}{2} - \epsilon - \tau')d}{2\left(\tau' d + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

It follows from Case 1 and Case 2 that for all real numbers $x_0 \geq K$,

$$\sup_{x\geq x_0} J^{1+r}f(x) - \inf_{y\geq x_0} J^{1+r}f(y) \geq U_\epsilon \wedge U'_\epsilon.$$

Letting $x_0 \to \infty$ we get that

$$\delta_{1+r}f := \limsup_{x\to\infty} J^{1+r}f(x) - \liminf_{x\to\infty} J^{1+r}f(x) \geq U_\epsilon \wedge U'_\epsilon.$$

But $\epsilon \in (0, (1/2-\tau) \wedge (1/2-\tau'))$ is completely arbitrary. So we can let $\epsilon \to 0^+$, and by continuity of $G_r^{-1}$, we get

$$\delta_{1+r}f = \limsup_{x\to\infty} J^{1+r}f(x) - \liminf_{x\to\infty} J^{1+r}f(x) \geq U \wedge U',$$

where

$$U := \min\left\{\theta, \left(1 - \frac{1}{(G_r)^{-1}(P)}\right)(\tau - \theta)\right\} d,\ P := P_{\tau,d,r,f} := \frac{(\frac{1}{2} - \tau)d}{2\left(d\tau + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)},$$

and

$$U' := \min\left\{\theta', \left(1 - \frac{1}{(G_r)^{-1}(P')}\right)(\tau' - \theta')\right\} d, \text{ and } P' := P'_{\tau',d,r,f} := \frac{(\frac{1}{2} - \tau')d}{2\left(d\tau' + \dfrac{\|f\|_\infty}{\Gamma(r)}\right)}.$$

Recall that the $0 < \theta < \tau < 1/2$ and $0 < \theta' < \tau' < 1/2$ are fixed and arbitrary. Let $\theta' = \theta$ and $\tau' = \tau$. Then $U' = U$. Then $\delta_{1+r} \geq U$, and $U > 0$.

Fix $\tau \in (0, 1/2)$ arbitrary. Let $\theta = \frac{\tau}{2}$. Then $\tau - \theta = \frac{\tau}{2}$ also. Thus

$$U = \left(1 - \frac{1}{G_r^{-1}(P)}\right)\frac{\tau}{2}d.$$

For example, for $r = 1$, we have $G_1^{-1}(s) = s + 1$, for all $s \in [0, \infty)$. So

$$U = \left(1 - \frac{1}{P+1}\right)\frac{\tau}{2}d.$$

Let $\tau = 1/4$. Then in this case $P = \dfrac{d/4}{d/2 + 2\|f\|_\infty}$, and $U = \left(\dfrac{d/4}{3d/4 + 2\|f\|_\infty}\right)\dfrac{d}{8}$.

Therefore,

$$\delta_2 f \geq U = \frac{d^2/32}{3d/4 + 2\|f\|_\infty} = \frac{d^2}{24d + 64\|f\|_\infty} > 0.$$

This is similar to the optimized lower bound for $\delta_2 f$ given in Theorem 3.5.1.

Note also that $d = 2b \leq 2\|f\|_\infty$. So

$$\delta_2 f \geq \frac{d^2}{112\|f\|_\infty}.$$

Now, going back to the general situation $r \in (0, 1]$, we have that

$$\delta_{1+r} f \geq U := \left(1 - \frac{1}{G_r^{-1}(P)}\right)\frac{\tau}{2}d, \text{ where } P := \frac{(1/2 - \tau)d}{2\tau d + \frac{2\|f\|_\infty}{\Gamma(r)}}.$$

Since $d \leq 2\|f\|_\infty$, we have that

$$P \geq \frac{(1/2 - \tau)d}{4\tau\|f\|_\infty + \frac{2\|f\|_\infty}{\Gamma(r)}}.$$

Let $Q := Q_{r,d,\|f\|_\infty,\tau} := \dfrac{(1/2 - \tau)d}{4\tau\|f\|_\infty + \frac{2\|f\|_\infty}{\Gamma(r)}}$. Then

$$\delta_{1+r} f \geq \left(1 - \frac{1}{G_r^{-1}(Q)}\right)\frac{\tau}{2}d = \frac{G_r^{-1}(Q) - 1}{G_r^{-1}(Q)}\frac{\tau}{2}d.$$

Also notice that

$$Q \leq \frac{(1/2 - \tau)2\|f\|_\infty}{4\tau\|f\|_\infty + \frac{2\|f\|_\infty}{\Gamma(r)}} = \frac{2(1/2 - \tau)}{4\tau + \frac{2}{\Gamma(r)}} =: \gamma(r, \tau),$$

thus

$$\delta_{1+r} f \geq \frac{G_r^{-1}(Q) - 1}{G_r^{-1}(\gamma(r, \tau))}\frac{\tau}{2}d.$$

Also,

$$Q := Q_{r,d,\|f\|_\infty,\tau} = \frac{\gamma(r, \tau)}{2}\frac{d}{\|f\|_\infty} \leq \gamma(r, \tau).$$

Fix $w_0 > 1$. For all $w \in (1, w_0]$,

$$G_r(w) := w\int_0^{\ln(w)} u^{r-1}e^{-u}du \leq w_0\int_0^{\ln(w)} u^{r-1}du = w_0\frac{(\ln(w))^r}{r}.$$

Consider the mapping $s = H_r(w) := w_0 \frac{(\ln(w))^r}{r}$, for $w \geq 1$. $H_r$ is strictly increasing and continuous, and such that $H_r(1) = 0$. Therefore $H_r : [1, w_0] \to [0, H_r(w_0)]$ has an inverse, and $w = H_r^{-1}(s) = \exp\left(\left(\frac{rs}{w_0}\right)^{1/r}\right)$. So, we have that for $w \in (1, w_0]$,

$$w \leq G_r^{-1}(H_r(w)).$$

Now, for all $s \in (0, H_r(w_0)]$,

$$G_r^{-1}(s) - 1 \geq w - 1 = \exp\left(\left(\frac{rs}{w_0}\right)^{1/r}\right) - 1 \geq \left(\frac{rs}{w_0}\right)^{1/r}.$$

Recall that $0 < Q \leq \gamma := \gamma(r, \tau)$, and also $H_r^{-1} : [0, H_r(w_0)] \to [1, w_0]$. We wish to find $w_0 > 1$ such that $H_r(w_0) = \gamma$. If we define $V_r(w) := w\frac{(\ln(w))^r}{r}$, for all $w \in [1, \infty)$, then we know its inverse $V_r^{-1}$ is well defined on $[0, \infty)$, since $V_r$ strictly increasing to infinity. Then $H_r(w_0) = \gamma$ is equivalent to $w_0 = V_r^{-1}(\gamma)$. In this case, since we have that $Q \in (0, \gamma]$, then

$$G_r^{-1}(Q) - 1 \geq \left(\frac{rQ}{V_r^{-1}(\gamma)}\right)^{1/r} = \left(\frac{r\gamma d}{2V_r^{-1}(\gamma)\|f\|_\infty}\right)^{1/r}.$$

Thus

$$\delta_{1+r} f \geq \left(\frac{r\gamma}{2V_r^{-1}(\gamma)\|f\|_\infty}\right)^{1/r} \frac{1}{G_r^{-1}(\gamma)} \frac{\tau}{2} d^{1+1/r} > 0.$$

□

Now we have all the tools to prove the main theorem of this section:

**Theorem 3.5.4.** *For any $0 < r, s$ we have that $Ces^r = Ces^s$.*

*Proof.* Without loss of generality assume $0 < r < s$. If $1 \leq r$, then by Corollary 3.5.2 we already have the conclusion. So, assume $0 < r \leq 1$ and $r < s$. We wish to show $Ces^s \subset Ces^r$, since the other inclusion is already known. Let $f \in Ces^s$, then $f \in Ces^{\lfloor s \rfloor + 1 + r}$, since $s < \lfloor s \rfloor + 1 + r$. This is equivalent to $J^r f \in Ces^{\lfloor s \rfloor + 1}$. By Theorem 3.5.1 this implies $J^r f \in Ces^1$, that is $f \in Ces^{1+r}$. Thus, by Theorem 3.5.3 we have that $f \in Ces^r$. □

*Remark.* Notice that the previous result not only says that $Ces^r = Ces^s$ for any $s, r > 0$ but also $\lim_{x\to\infty} J^s f(x) = \lim_{x\to\infty} J^r f(x)$ for any $0 < r < s$ whenever one of these two limits exist. This is clear when we assume $f \in Ces^r$, since the operator $J^{s-r}$ preserves limit at infinity.

But also, notice that if we assume $\lim_{x\to\infty} J^s f(x) = L \in \mathbb{R}$, then $\lim_{x\to\infty} J^r f(x)$ exists, which implies $\lim_{x\to\infty} J^r f(x) = \lim_{x\to\infty} J^s f(x) = L$.

*Remark.* Notice that the function sine is an element of $Ces^1$, but sine $\notin BC_L$. So, $BC_L \subsetneq Ces^1$.

## 3.6 Banach limits on $L^\infty$ invariant under $J^r$

In this section, we construct Banach limits on $L^\infty$ that are invariant under our definition of fractional powers of the Cesàro averaging operator. That is, Banach limits $\Lambda$ invariant under $J^r$, for each $r > 0$; and therefore these Banach limits preserve $J^r$-convergence. Given the results from the previous section, for $f \in L^\infty$ such that $f \in Ces^r$, for some $r > 0$, there exists $L \in \mathbb{R}$ such that $\lim_{x\to\infty} J^s f(x) = L$ for all $s > 0$. In this case, for any Banach limit $\Lambda$ that is fractional-power-Cesàro-invariant, we have that $\Lambda(f) = L$.

We follow the same approach used in Chapter 2, to construct the desired $J^r$-invariant Banach limits. We will make use of $\Psi$, the extension of the Cesàro limit functional, obtained through Hahn-Banach Theorem:

Recall the closed vector subspace of $L^\infty$

$$Ces_L := \left\{ g \in L^\infty : \psi(g) := \lim_{x\to\infty} Jg(x) \text{ exists in } \mathbb{R} \right\}.$$

By Hahn-Banach Extension Theorem, there exists $\Psi \in (L^\infty)^*$ such that

$$\|\Psi\|_{(L^\infty)^*} = \|\psi\|_{Ces_L^*} = 1 \text{ and } \Psi\mid_{Ces_L} = \psi.$$

**Definition 3.6.1.** For each $f \in L^\infty$, we define the map $\Gamma_f : (0,\infty) \to \mathbb{R}$ by

$$\Gamma_f(r) := \Psi(J^r f), \text{ for all } r \in (0,\infty)\ .$$

**Claim 3.6.1.** *For each $f \in L^\infty$, $\Gamma_f \in BC(0,\infty) \subseteq L^\infty(0,\infty)$.*

*Proof.* Fix $f \in L^\infty$. We already establish on Theorem 3.4.9 that the map $r \mapsto J^r f$ is continuous, therefore the composition $\Psi(J^r f)$ is also a continuous function of $r$, since $\Psi$ is a continuous functional.

We see then $\Gamma_f \in BC(0,\infty)$ since for all $r \in (0,\infty)$, we have that

$$|\Gamma_f(r)| \leq \|\Psi\|_{(L^\infty)^*}\|J^r\|_{op}\|f\|_\infty = \|f\|_\infty < \infty.$$

$\square$

**Definition 3.6.2.** In analogy with equation (†) we define the linear mapping $\Lambda : L^\infty \to \mathbb{R}$ by

$$\Lambda(f) := \Psi\Gamma_f, \text{ for every } f \in L^\infty.$$

**Theorem 3.6.2.** *$\Lambda$ is a Banach limit in $L^\infty$ that is invariant under fractional powers of the Cesàro operator, i.e. $\Lambda(J^r f) = \Lambda(f)$, for all $f \in L^\infty$, for all $r > 0$.*

*Proof.* Fix $f \in L^\infty$ and $r_0 > 0$. We first check that $\Lambda$ in fact is invariant under fractional powers of the Cesàro averaging operator:

Recall that for every $r \in [0, \infty)$, we define the left shift operator by $r$ on $L^\infty$, by $S_r f(\cdot) := f(\cdot + r) \in L^\infty$, for every $f \in L^\infty$. Also recall $\Psi$ is left-shift invariant, since it is a Banach limit. So,

$$\begin{aligned}
\Lambda(J^{r_0} f) &= \Psi(r \mapsto \Psi(J^r J^{r_0} f)) \\
&= \Psi(r \mapsto \Psi(J^{r_0 + r} f)) \\
&= \Psi(r \mapsto \Gamma_f(r_0 + r)) \\
&= \Psi(r \mapsto (S_{r_0}\Gamma_f)(r)) \\
&= \Psi(S_{r_0}\Gamma_f) \\
&= \Psi(\Gamma_f) \\
&= \Lambda(f).
\end{aligned}$$

Next we check that $\|\Lambda\|_{(L^\infty)^*} = 1$:

Notice that for any $f \in L^\infty$ we have that $|\Lambda(f)| \leq \|\Psi\|_{(L^\infty)^*}\|\Gamma_f\|_\infty \leq \|f\|_\infty$. Therefore, $\|\Lambda\|_{(L^\infty)^*} \leq 1$. On the other hand,

$$\Lambda(\mathbb{1}) = \Psi(r \mapsto \Psi(J^r \mathbb{1})) = \Psi(r \mapsto \Psi(\mathbb{1})) = \Psi(r \mapsto 1) = \Psi(\mathbb{1}) = 1.$$

Thus $\|\Lambda\|_{(L^\infty)^*} = 1$.

Next, we check that if $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$ then $\Lambda(f) = L$:

Recall that by Corollary 3.4.8, we know that if $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to\infty} (J^r f)(x) = L$ for all $r > 0$.

Therefore

$$\begin{aligned}
\Lambda(f) &= \Psi(r \mapsto \Psi(J^r f)) \\
&= \Psi(r \mapsto L) \\
&= \Psi(L\mathbb{1}) \\
&= L.
\end{aligned}$$

Finally, we verify that $\Lambda(f) = \Lambda(S_r f)$ for all $r > 0$:

Notice that $\Lambda(S_r f) = \Psi(s \mapsto \Psi(J^s S_r f))$. We claim that

$$\Psi(J^s S_r f) = \Psi(J^s f), \text{ for all } r > 0 \text{ and all } s > 0.$$

We will prove the claim stated above later. Assume this claim holds, then we have

$$\begin{aligned}
\Lambda(S_r f) &= \Psi(s \mapsto \Psi(J^s S_r f)) \\
&= \Psi(s \mapsto \Psi(J^s f)) \\
&= \Psi(s \mapsto \Gamma_f(s)) \\
&= \Psi(\Gamma_f) \\
&= \Lambda(f).
\end{aligned}$$

□

The only thing remaining to justify is the claim used in the previous proof, which we do next:

**Claim 3.6.3.** *For any $f \in L^\infty$, we have that $\Psi(J^s S_r f) = \Psi(J^s f)$ for all $r > 0$ and all $s > 0$.*

*Proof.* We first start by considering $s \in (0,1)$.

Let $h := J^s S_r f - S_r J^s f$. We want to show $h \in BC_{L,0}$, the set of bounded continuous functions, whose limit at infinity is 0. Notice

$$\begin{aligned} h(x) &= \frac{1}{x\Gamma(s)} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{s-1} f(t+r)dt - S_r \left[\frac{1}{x\Gamma(s)} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{s-1} f(t)dt\right] \\ &= \frac{1}{\Gamma(s)} \left(\frac{1}{x} \int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{s-1} f(t+r)dt - \frac{1}{x+r} \int_0^{x+r} \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1} f(t)dt\right). \end{aligned}$$

By making the substitution $u = t + r$, we obtain

$$\int_0^x \left[\ln\left(\frac{x}{t}\right)\right]^{s-1} f(t+r)dt = \int_r^{x+r} \left[\ln\left(\frac{x}{u-r}\right)\right]^{s-1} f(u)du.$$

Next, we write

$$\int_0^{x+r} \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1} f(t)dt = \int_0^{r} \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1} f(t)dt + \int_r^{x+r} \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1} f(t)dt.$$

By writing $\frac{1}{x} = \frac{r}{x(x+r)} + \frac{1}{x+r}$, we can express $h(x)$ as

$$\begin{aligned} &\frac{1}{\Gamma(s)} \frac{r}{x(x+r)} \int_r^{x+r} \left[\ln\left(\frac{x}{t-r}\right)\right]^{s-1} f(t)dt \\ &+ \frac{1}{\Gamma(s)} \frac{1}{x+r} \int_r^{x+r} \left(\left[\ln\left(\frac{x}{t-r}\right)\right]^{s-1} - \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1}\right) f(t)dt \\ &- \frac{1}{\Gamma(s)} \frac{1}{x+r} \int_0^{r} \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1} f(t)dt. \end{aligned}$$

The last integral is easy to show it goes to 0 as $x$ tends to infinity: Let $u = \ln\left(\frac{x+r}{t}\right)$, then $t = \frac{x+r}{e^u}$. We get,

$$\begin{aligned} \frac{1}{x+r} \left| \int_0^r \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1} f(t)dt \right| &\le \frac{\|f\|_\infty}{x+r}(x+r) \int_{\ln\left(\frac{x+r}{r}\right)}^\infty u^{s-1}e^{-u}du \\ &= \|f\|_\infty \int_{\ln\left(\frac{x+r}{r}\right)}^\infty u^{s-1}e^{-u}du. \end{aligned}$$

This last expression is the tail of the convergent integral $\Gamma(s)$, therefore it goes to 0 as $x$ tends to infinity.

For the first integral we first make the substitution $u = t - r$, and then we make the substitution $w = \ln\left(\frac{x}{u}\right)$:

$$\left|\frac{1}{\Gamma(s)}\frac{r}{x(x+r)}\int_r^{x+r}\left[\ln\left(\frac{x}{t-r}\right)\right]^{s-1}f(t)dt\right| = \left|\frac{1}{\Gamma(s)}\frac{r}{x(x+r)}\int_0^{x}\left[\ln\left(\frac{x}{u}\right)\right]^{s-1}f(r+u)du\right|$$
$$\leq \frac{1}{\Gamma(s)}\frac{r\|f\|_\infty}{x+r}\int_0^\infty w^{s-1}e^{-w}dw$$
$$= \frac{r\|f\|_\infty}{x+r},$$

and this last expression goes to 0 as $x$ tends to infinity.

Next, we consider the second integral

$$I = \frac{1}{x+r}\int_r^{x+r}\left[\left[\ln\left(\frac{x}{t-r}\right)\right]^{s-1} - \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1}\right]f(t)dt.$$

Notice $\frac{x+r}{t} \leq \frac{x}{t-r}$ since

$$\begin{aligned}\frac{x+r}{t} \leq \frac{x}{t-r} &\iff (t-r)(x+r) \leq xt \\ &\iff xt + tr - rx - r^2 \leq xt \\ &\iff r(t-x-r) \leq 0 \\ &\iff t-x-r \leq 0 \\ &\iff t \leq x+r \textit{ which is true!}\end{aligned}$$

Therefore $\ln\left(\frac{x+r}{t}\right) \leq \ln\left(\frac{x}{t-r}\right)$, which implies $\left[\ln\left(\frac{x}{t-r}\right)\right]^{s-1} \leq \left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1}$. So,

$$|I| \leq \frac{\|f\|_\infty}{x+r}\int_r^{x+r}\left[\left[\ln\left(\frac{x+r}{t}\right)\right]^{s-1} - \left[\ln\left(\frac{x}{t-r}\right)\right]^{s-1}\right]dt.$$

Let $u = x+r-t$, that is $t = x+r-u$. Then we get

$$|I| \leq \frac{\|f\|_\infty}{x+r}\int_0^{x}\left[\left[\ln\left(\frac{x+r}{x+r-u}\right)\right]^{s-1} - \left[\ln\left(\frac{x}{x-u}\right)\right]^{s-1}\right]du.$$

Notice that, since $\alpha := \frac{x-u}{x} - 1 = \frac{-u}{x} \in (-1,0)$ and by the Taylor series expansion of the logarithmic function, we have

$$-q := \ln\left(\frac{x-u}{x}\right) = \ln\left(1+\left(\frac{x-u}{x}-1\right)\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\alpha^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^{2k+1}u^k}{kx^k} = -\sum_{k=1}^{\infty}\frac{u^k}{kx^k}.$$

Let $\delta := \sum_{k=2}^{\infty} \frac{u^k}{kx^k}$. Therefore $q = \ln\left(\frac{x}{x-u}\right) = \frac{u}{x} + \delta$.

Similarly, since $\beta := \frac{x+r-u}{x+r} - 1 = \frac{-u}{x+r} \in (-1, 0)$, and by the Taylor series expansion of the logarithmic function, we have

$$-p := \ln\left(\frac{x+r-u}{x+r}\right) = \ln\left(1 + \left(\frac{x+r-u}{x+r} - 1\right)\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\beta^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}u^k}{k(x+r)^k} = -\sum_{k=1}^{\infty} \frac{u^k}{k(x+r)^k}.$$

Let $\epsilon := \sum_{k=2}^{\infty} \frac{u^k}{k(x+r)^k}$. Therefore $p = \ln\left(\frac{x+r}{x+r-u}\right) = \frac{u}{x+r} + \epsilon$.

**Claim 3.6.4.** *For $0 < p \leq q$ and $s \in (0, 1)$ we have that*

$$p^{s-1} - q^{s-1} \leq 2\frac{p^{-1} - q^{-1}}{p^{-s} + q^{-s}}.$$

*Proof.* Notice

$$\begin{aligned}
\left(p^{s-1} - q^{s-1}\right)\left(p^{-s} + q^{-s}\right) &= p^{-1} + \frac{1}{q^s}\frac{1}{p^{1-s}} - \frac{1}{p^s}\frac{1}{q^{1-s}} - q^{-1} \\
&\leq p^{-1} + \frac{1}{p^s}\frac{1}{p^{1-s}} - \frac{1}{q^s}\frac{1}{q^{1-s}} - q^{-1} \\
&= 2\left(p^{-1} - q^{-1}\right).
\end{aligned}$$

□

Now, by the previous claim we have that $|I|$ is bounded above by

$$
\begin{aligned}
&\frac{\|f\|_\infty}{x+r}\int_0^x \left[p^{s-1}-q^{s-1}\right]du\\
&\le \frac{2\|f\|_\infty}{x+r}\int_0^x \frac{\frac{1}{p}-\frac{1}{q}}{p^{-s}+q^{-s}}du\\
&= \frac{2\|f\|_\infty}{x+r}\int_0^x \frac{q-p}{p^{1-s}q+pq^{1-s}}du\\
&= \frac{2\|f\|_\infty}{x+r}\int_0^x \frac{\frac{u}{x}+\delta-\frac{u}{x+r}-\epsilon}{\left(\frac{u}{x+r}+\epsilon\right)^{1-s}\left(\frac{u}{x}+\delta\right)+\left(\frac{u}{x+r}+\epsilon\right)\left(\frac{u}{x}+\delta\right)^{1-s}}du\\
&\le \frac{2\|f\|_\infty}{x+r}\int_0^x u\frac{\frac{1}{x}-\frac{1}{x+r}+\frac{\delta-\epsilon}{u}}{\left(\frac{u}{x+r}\right)^{1-s}\left(\frac{u}{x}\right)+\left(\frac{u}{x+r}\right)\left(\frac{u}{x}\right)^{1-s}}du\\
&= \frac{2\|f\|_\infty}{x+r}\int_0^x \frac{u}{u^{2-s}}\frac{\left(\frac{1}{x}-\frac{1}{x+r}+\frac{\delta-\epsilon}{u}\right)}{\left[\left(\frac{1}{x+r}\right)^{1-s}\left(\frac{1}{x}\right)+\left(\frac{1}{x+r}\right)\left(\frac{1}{x}\right)^{1-s}\right]}du\\
&= \frac{2\|f\|_\infty}{x+r}(x+r)^{1-s}x^{1-s}\int_0^x \frac{u}{u^{2-s}}\frac{\left(\frac{1}{x}-\frac{1}{x+r}+\frac{\delta-\epsilon}{u}\right)}{\left[\left(\frac{1}{x}\right)^{s}+\left(\frac{1}{x+r}\right)^{s}\right]}du\\
&= \frac{2\|f\|_\infty}{x+r}(x+r)x\int_0^x u^{s-1}\frac{\frac{r}{x(x+r)}+\frac{\delta-\epsilon}{u}}{(x+r)^s+(x)^s}du\\
&= \frac{2\|f\|_\infty}{x+r}\int_0^x u^{s-1}\frac{r+(x+r)x\frac{\delta-\epsilon}{u}}{(x+r)^s+(x)^s}du\\
&\le \frac{2\|f\|_\infty}{(x+r)2x^s}\int_0^x u^{s-1}\left(r+(x+r)x\frac{\delta-\epsilon}{u}\right)du\\
&= \frac{\|f\|_\infty}{(x+r)x^s}r\frac{x^s}{s}+\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x u^{s-2}(x+r)x(\delta-\epsilon)du\\
&= \frac{\|f\|_\infty}{(x+r)}\frac{r}{s}+\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x u^{s-2}(x+r)x\delta du-\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x u^{s-2}(x+r)x\epsilon du.
\end{aligned}
$$

Now, we substitute $\delta$ and $\epsilon$ with the respective series, to get

$$
\begin{aligned}
&\frac{\|f\|_\infty}{x+r}\int_0^x \left[p^{s-1}-q^{s-1}\right]du\\
&\le \frac{\|f\|_\infty}{(x+r)}\frac{r}{s}+\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x u^{s-2}(x+r)x\sum_{k=2}^\infty\frac{u^k}{kx^k}du-\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x u^{s-2}(x+r)x\sum_{k=2}^\infty\frac{u^k}{k(x+r)^k}du\\
&= \frac{\|f\|_\infty}{(x+r)}\frac{r}{s}+\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x \sum_{k=2}^\infty(x+r)x\frac{u^{k+s-2}}{kx^k}du-\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x \sum_{k=2}^\infty(x+r)x\frac{u^{k+s-2}}{k(x+r)^k}du.
\end{aligned}
$$

Notice the functions

$$f(k,u) := (x+r)x\frac{u^{k+s-2}}{kx^k} \text{ and } g(k,u) := (x+r)x\frac{u^{k+s-2}}{k(x+r)^k}$$

are non-negative measurable on their domain $(\mathbb{N}\setminus\{1\})\times(0,x)$, therefore we can apply Tonelli to get

$$\frac{\|f\|_\infty}{(x+r)}\frac{r}{s}+\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x\sum_{k=2}^\infty (x+r)x\frac{u^{k+s-2}}{kx^k}du-\frac{\|f\|_\infty}{(x+r)x^s}\int_0^x\sum_{k=2}^\infty (x+r)x\frac{u^{k+s-2}}{k(x+r)^k}du=$$
$$\frac{\|f\|_\infty}{(x+r)}\frac{r}{s}+\frac{\|f\|_\infty}{(x+r)x^s}\sum_{k=2}^\infty \frac{(x+r)x}{kx^k}\int_0^x u^{k+s-2}du-\frac{\|f\|_\infty}{(x+r)x^s}\sum_{k=2}^\infty \frac{(x+r)x}{k(x+r)^k}\int_0^x u^{k+s-2}du=$$
$$\frac{\|f\|_\infty}{(x+r)}\frac{r}{s}+\frac{\|f\|_\infty}{(x+r)x^s}\sum_{k=2}^\infty \frac{(x+r)x}{kx^k}\frac{x^{k+s-1}}{k+s-1}-\frac{\|f\|_\infty}{(x+r)x^s}\sum_{k=2}^\infty \frac{(x+r)x}{k(x+r)^k}\frac{x^{k+s-1}}{k+s-1}=$$
$$\frac{\|f\|_\infty}{(x+r)}\frac{r}{s}+\|f\|_\infty\left(\sum_{k=2}^\infty\frac{1}{k(k+s-1)}-\sum_{k=2}^\infty\left(\frac{x}{x+r}\right)^k\frac{1}{k(k+s-1)}\right).$$

As $x$ tends to infinity, this expression tends to 0. We can move the limit inside the last integral by Lebesgue Dominated Convergence Theorem for the counting measure, since for $x>0$ and for each $k\in\mathbb{N}\setminus\{1\}$ we have that

$$\left(\frac{x}{x+r}\right)^k\frac{1}{k(k+s-1)}\leq\frac{1}{k(k+s-1)}$$

and

$$\sum_{k=2}^\infty\frac{1}{k(k+s-1)}<\infty.$$

Therefore, we obtained that $h=J^sS_rf-S_rJ^sf\to 0$ as $x$ tends to infinity. So, $\Psi(J^sS_rf)=\Psi(S_rJ^sf)=\Psi(J^sf)$ holds for $0<s<1$.

Finally, for arbitrary $s>0$, notice that

$$\begin{aligned}\Psi(J^sS_rf-J^sf)&=\Psi(J^sS_rf-J^{\lfloor s\rfloor}S_rJ^{\langle s\rangle}f+J^{\lfloor s\rfloor}S_rJ^{\langle s\rangle}f-J^sf)\\&=\Psi(J^{\lfloor s\rfloor}J^{\langle s\rangle}S_rf-J^{\lfloor s\rfloor}S_rJ^{\langle s\rangle}f+J^{\lfloor s\rfloor}S_rJ^{\langle s\rangle}f-J^{\lfloor s\rfloor}J^{\langle s\rangle}f)\\&=\Psi(J^{\lfloor s\rfloor}\left[J^{\langle s\rangle}S_rf-S_rJ^{\langle s\rangle}f\right])+\Psi((J^{\lfloor s\rfloor}S_r-J^{\lfloor s\rfloor})\left[J^{\langle s\rangle}f\right]).\end{aligned}$$

In the last expression, the first term in the addition equals 0 since we already proved that $J^{\langle s\rangle}S_rf-S_rJ^{\langle s\rangle}f\in BC_{L,0}$ and we also showed that $J^{\lfloor s\rfloor}$ preserves limits at infinity, and so does $\Psi$ by definition of Banach limit. The second expression in the addition also equals 0, we showed this in Claim 3.2.2. □

*Remark.* We can modify Theorem 3.6.2 to get a more general one, using the definition of $\Lambda$ below:

Theorem 3.6.2* Fix $\Delta$ and $T$ any two Banach limits in $L^\infty$. Then $\Lambda : L^\infty \to \mathbb{R}$ defined by

$$\Lambda(f) := T(r \mapsto \Delta(J^r f)).$$

is a Banach limit that is invariant under fractional powers of the Cesàro averaging operator.

Proof of Theorem 3.6.2*: As in the proof of Theorem 3.6.2, we can verify that $\Lambda(J^{r_0} f) = \Lambda(f)$, for all $f \in L^\infty$ and for all $r_0 > 0$. Also, we check that $\|\Lambda\|_{(L^\infty)^*} = 1$ and that $\Lambda(f) = \lim_{x\to\infty} f(x)$ for all $f \in BC_L$.

To show that $\Lambda$ is invariant under the left shift operator, it is enough to show that

$$\Delta(J^s S_r f) = \Delta(J^s f) \text{ for all } r > 0, \text{ for all } s > 0.$$

We already showed in the proof of Claim 3.6.3 that $J^s S_r f - S_r J^s f \in BC_{L,0}$, for $s \in (0,1)$. Since $\Delta$ preserves classical convergence, we have that for $s \in (0,1)$ the following holds: $\Delta(J^s S_r f) = \Delta(S_r J^s f)$. Next, recall $\Delta$ is invariant under the left shift operator, therefore we get $\Delta(J^s S_r f) = \Delta(J^s f)$.

Now, for $s \geq 1$, notice that

$$\begin{aligned}\Delta(J^s S_r f - J^s f) &= \Delta(J^{\lfloor s \rfloor} J^{\langle s \rangle} S_r f - J^{\lfloor s \rfloor} S_r J^{\langle s \rangle} f + J^{\lfloor s \rfloor} S_r J^{\langle s \rangle} f - J^{\lfloor s \rfloor} J^{\langle s \rangle} f) \\ &= \Delta(J^{\lfloor s \rfloor}(J^{\langle s \rangle} S_r f - S_r J^{\langle s \rangle} f)) + \Delta((J^{\lfloor s \rfloor} S_r - J^{\lfloor s \rfloor}) \left[J^{\langle s \rangle} f\right]).\end{aligned}$$

Recall $\lim_{x\to\infty} (J^n f)(x) = \lim_{x\to\infty} f(x)$, for all $f \in BC_L$, for all $n \in \mathbb{N}$, and since $\Delta$ preserves classical convergence, we get that the first term in the previous addition is 0. Also, Claim 3.2.2 can be generalized to show $\Delta(J^n S_r g) = \Delta(J^n g)$ for all $r > 0$, $n \in \mathbb{N}$ and $g \in L^\infty$, therefore the second term of the addition is also 0. So we get the desired result.

**Open Question 3.6.5.** *It was proven in [23], that the Pascal Operator defined on $\ell^\infty$ commutes with the Cesàro averaging operator. In fact, along with the left shift operator and the identity operator, they generate an abelian semigroup of linear operators on $\ell^\infty/_{c_0}$.*

*Then, using a generalization of the Hahn Banach Extension Theorem [18], it was shown the existence of a Banach limit invariant under any number of compositions of these operators.*

*What could be an analogue definition of the Pascal Operator $P$ for the space $L^\infty(0,\infty)$? Can continuous iterates of this operator be defined? If this new operator, and its iterations, commute with our definition of iterates of Cesàro averaging, one could use the generalization of the Hahn Banach Extension Theorem to get the existence of Banach limits invariant under compositions of $J^r$ and $P^s$, for any $r, s > 0$.*

**Open Question 3.6.6.** *The quantitative result from Theorem 3.5.3 could be improved by choosing $\tau$ in an appropriate way. What value of $\tau$ would give us an optimal lower bound for the inequality of this theorem?*

## 4.0 Cesàro averaging on $\mathcal{B}(\mathcal{H})$ and related Banach limits and continuous linear functionals

### 4.1 Notation and preliminaries

Let $\mathbb{N}_0 = \{0, 1, 2, 3 \ldots\}$ denote the non-negative integers and $\mathbb{Z}$ denote the set of all integers. Let $\mathcal{H} := \ell^2(\mathbb{N}_0, \mathbb{C})$ be the usual Hilbert space of all sequences $x = (x_0, x_1, x_2, \ldots)$ of complex numbers satisfying

$$x_i \in \mathbb{C} \text{ for each } i \in \mathbb{N}_0 \text{ and } \sum_{i=0}^{\infty} |x_i|^2 < \infty,$$

and equipped with the norm

$$\|x\|_2 = \left( \sum_{i=0}^{\infty} |x_i|^2 \right)^{1/2}, \text{ for } x \in \mathcal{H}.$$

We denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. The operator norm is defined by

$$\|E\|_{op} = \sup_{\|x\|_2 \leq 1} \|Ex\|_2, \text{ for all } E \in \mathcal{B}(\mathcal{H}).$$

We know that $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{op})$ is a Banach space.

The dual space of $\mathcal{B}(\mathcal{H})$, $(\mathcal{B}(\mathcal{H}))^*$, consisting of all continuous linear functionals on $\mathcal{B}(\mathcal{H})$, is isometrically isomorphic to $C_1(\mathcal{H}) \bigoplus_1 (\mathcal{K}(\mathcal{H}))^\perp$, where $\mathcal{K}(\mathcal{H})^\perp$ is the annihilator of $\mathcal{K}(\mathcal{H})$ inside $\mathcal{B}(\mathcal{H})^*$. Also, $C_1(\mathcal{H})$ is the trace class, and $\mathcal{K}(\mathcal{H})$ is the space of all compact operators on the underlying Hilbert space $\mathcal{H}$.

We let $\mathcal{P}(\mathcal{H})$ be the subset of $\mathcal{B}(\mathcal{H})$ consisting of all orthogonal projections.

We also consider the Hilbert space $\mathcal{G} := \ell^2(\mathbb{Z}, \mathbb{C})$.

Consider the matrix with infinitely many rows and infinitely many columns $E = (a_{j,k})_{j,k}$, indexed by $j, k \in \mathbb{N}_0$:

$$E := \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ \vdots & & & \ddots \end{pmatrix}.$$

Also, each $a_{j,k} \in \mathbb{C}$. For each $n \in \mathbb{N}_0$, we define the two sided sequence $\vec{\delta^n}$ as follows:

$$\vec{\delta^n} := \left(\delta^n_j\right)_{j=-\infty}^{\infty} := (\ldots, a_{n+2,n},\ a_{n+1,n},\ a_{n,n},\ a_{n,n+1},\ a_{n,n+2}, \ldots), \text{ where } \delta^n_0 = a_{n,n}.$$

That is,

$$\begin{array}{c} \textit{0th-entry} \\ \downarrow \end{array}$$

$$\vec{\delta^0} := (\ldots, a_{2,0},\ a_{1,0},\ a_{0,0},\ a_{0,1},\ a_{0,2}, \ldots)$$
$$\vec{\delta^1} := (\ldots, a_{3,1},\ a_{2,1},\ a_{1,1},\ a_{1,2},\ a_{1,3}, \ldots)$$
$$\vec{\delta^2} := (\ldots, a_{4,2},\ a_{3,2},\ a_{2,2},\ a_{2,3},\ a_{2,4}, \ldots)$$
$$\vdots$$

Notice that $\vec{\delta^0}$ is the two sided sequence formed by concatenating two sequences: the first column of the matrix $E$, and its first row. We make the concatenation at the entry $a_{0,0}$ (which is the entry they have in common) in such a way that entries of the column go along the entries of $\vec{\delta^0}$ corresponding to non-positive indices, and the entries of the row go along the entries of $\vec{\delta^0}$ corresponding to non-negative indices. In general, $\vec{\delta^n}$ is the two sided sequence formed by concatenating two sequences: the $(n+1)$-column of the matrix $E$ starting at the entry $a_{n,n}$, and its $(n+1)$-row starting at the entry $a_{n,n}$ as well, making the concatenation at this common entry.

**Definition 4.1.1.** For a matrix $E$, we will refer to the two sided sequence $\vec{\delta^n}$, for each $n \in \mathbb{N}_0$, described above, as the *n-th Gamma shell of E*. We name it this way because it resembles the letter $\Gamma$.

We illustrate below the Gamma shell corresponding to $\vec{\delta^n}$, circled in blue:

$$\begin{pmatrix}
a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} & a_{0,n+1} & a_{0,n+2} & \cdots \\
a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & a_{1,n+1} & a_{1,n+2} & \cdots \\
a_{2,0} & a_{2,1} & & & & & & & \\
a_{3,0} & a_{3,1} & & & & & & & \\
\vdots & \vdots & & & \ddots & & & & \\
a_{n,0} & a_{n,1} & \cdots & & & a_{n,n} & a_{n,n+1} & a_{n,n+2} & \cdots \\
a_{n+1,0} & a_{n+1,1} & \cdots & & & a_{n+1,n} & a_{n+1,n+1} & a_{n+1,n+2} & \cdots \\
a_{n+2,0} & a_{n+2,1} & \cdots & & & a_{n+2,n} & a_{n+2,n+1} & \ddots & \\
a_{n+3,0} & a_{n+3,1} & \cdots & & & a_{n+3,n} & a_{n+3,n+1} & & \\
\vdots & \vdots & & & & \vdots & \vdots & &
\end{pmatrix}$$

Notice that any matrix $E = (a_{j,k})_{j,k}$ can be represented as the sequence of its Gamma shells, $E = (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots)$. We next verify that whenever $E \in \mathcal{B}(\mathcal{H})$ (where $E$ acting on $\mathcal{H}$ is identified with its matrix with respect to the usual basis on $\mathcal{H}$: $e_0, e_1, e_2, \ldots$), then $E \in \ell^\infty(\mathcal{G})$, and therefore $\mathcal{B}(\mathcal{H}) \subset \ell^\infty(\mathcal{G})$. We will discuss later that, in fact, this inclusion is known to be strict.

**Claim 4.1.1.** $\mathcal{B}(\mathcal{H}) \subset \ell^\infty(\mathcal{G})$. *In fact, the inclusion mapping $i : E \hookrightarrow E$ is continuous, with*

$$\|E\|_{\ell^\infty(\mathcal{G})} := \sup_{n \in \mathbb{N}_0} \|\vec{\delta^n}\|_{\mathcal{G}} \leq \sqrt{2}\|E\|_{op} \ ,$$

*for all $E \in \mathcal{B}(\mathcal{H})$.*

*Proof.* Fix $E = (a_{j,k})_{j,k} \in \mathcal{B}(\mathcal{H})$.

Fix $k \in \mathbb{N}_0$, and define $x(k) := x = (0, \ldots, 0 \ , \ \overline{a_{k,k}} \ , \ \overline{a_{k,k+1}} \ , \ \overline{a_{k,k+2}} \ , \ldots)^T$, where the first $k$ positions of $x$ are zeros, and $T$ indicates "transpose".

Notice that

$$\begin{aligned}
&\|E\|_{op}\|x\|_2\\
&\geq \|E(0,\dots,0\ ,\ \overline{a_{k,k}}\ ,\ \overline{a_{k,k+1}}\ ,\ \overline{a_{k,k+2}}\ ,\dots)^T\|_2\\
&= \left[\left|\sum_{n=0}^{\infty} a_{0,k+n}\ \overline{a_{k,k+n}}\ \right|^2 + \dots + \left|\sum_{n=0}^{\infty} a_{k,k+n}\ \overline{a_{k,k+n}}\ \right|^2 + \left|\sum_{n=0}^{\infty} a_{k+1,k+n}\ \overline{a_{k,k+n}}\ \right|^2 + \dots\right]^{1/2}\\
&\geq \sum_{n=0}^{\infty} |a_{k,k+n}|^2\\
&= \|x\|_2^2.
\end{aligned}$$

Thus $x \in \mathcal{H}$ and $\|x\|_2 \leq \|E\|_{op}$.

We also define $y(k) := y = (0,\dots,0,a_{k+1,k},a_{k+2,k},a_{k+3,k},\dots)^T$, where the first $k+1$ positions of $y$ are zeros. Then

$$\begin{aligned}
&\|E^*\|_{op}\|y\|_2\\
&\geq \|E^*(0,\dots,0,a_{k+1,k},a_{k+2,k},a_{k+3,k},\dots)^T\|_2\\
&= \left[\left|\sum_{n=1}^{\infty} \overline{a_{k+n,0}}\ a_{k+n,k}\ \right|^2 + \dots + \left|\sum_{n=1}^{\infty} \overline{a_{k+n,k}}\ a_{k+n,k}\ \right|^2 + \left|\sum_{n=1}^{\infty} \overline{a_{k+n,k+1}}\ a_{k+n,k}\ \right|^2 + \dots\right]^{1/2}\\
&\geq \sum_{n=1}^{\infty} |a_{k+n,k}|^2\\
&= \|y\|_2^2.
\end{aligned}$$

Hence $y \in \mathcal{H}$ and $\|y\|_2 \leq \|E^*\|_{op} = \|E\|_{op}$.

Therefore

$$\|E\|_{op}^2 \geq \sum_{n=0}^{\infty} |a_{k,k+n}|^2 \text{ and } \|E\|_{op}^2 = \|E^*\|_{op}^2 \geq \sum_{n=1}^{\infty} |a_{k+n,k}|^2.$$

Thus

$$2\|E\|_{op}^2 \geq \sum_{n=0}^{\infty} |a_{k,k+n}|^2 + \sum_{n=1}^{\infty} |a_{k+n,k}|^2 = \|\vec{\delta^k}\|_{\mathcal{G}}^2.$$

So,

$$\sqrt{2}\|E\|_{op} \geq \|\vec{\delta^k}\|_{\mathcal{G}}, \text{ for every } k \in \mathbb{N}_0,$$

which implies the desired conclusion. □

To see that $\mathcal{B}(\mathcal{H}) \subsetneq \ell^\infty(\mathcal{G})$, we first discuss the following results that were presented in [2]:

**Definition 4.1.2.** A Toeplitz matrix is of the form

$$E := \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & \\ a_{-2} & a_{-1} & a_0 & a_1 & \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Where $a_n \in \mathbb{C}$, for all $n \in \mathbb{Z}$.

So, the Gamma shells of a Toeplitz matrix form a constant sequence. That is, a Toeplitz matrix is of the sequence form $(\vec{\delta}, \vec{\delta}, \vec{\delta}, \ldots)$. Or put differently, the matrix $E$ is constant down each diagonal parallel to the main diagonal.

In [2], a characterization of Toeplitz matrices corresponding to bounded linear operators is presented:

**Theorem 4.1.2.** *A Toeplitz matrix corresponding to a two sided sequence $(a_n)_{n=-\infty}^{\infty}$ of complex numbers, is the matrix of a bounded operator on $\mathcal{H}$ if and only if $(a_n)_{n=-\infty}^{\infty}$ corresponds to the Fourier coefficients of a function $f$ in $L^\infty(\partial D, \sigma)$.*

*Here $D$ is the unit disk in the complex plane, and $\sigma$ denotes the usual arc length measure of $\partial D$, normalized.*

So, we can show that $\mathcal{B}(\mathcal{H}) \subsetneq \ell^\infty(\mathcal{G})$ by considering $(a_n)_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z}, \mathbb{C})$ such that $(a_n)_{n=-\infty}^{\infty}$ are the Fourier coefficients of a unbounded function $f \in L^2([0, 2\pi], \frac{m}{2\pi}; \mathbb{C})$.

## 4.2 The explosion operator

We will use the representation every bounded linear operator on $\mathcal{H}$ has in the space $\ell^\infty(\mathcal{G})$, to define a Cesàro averaging operator on $\mathcal{B}(\mathcal{H})$. In order to guarantee this operator maps back into $\mathcal{B}(\mathcal{H})$, we first introduce the following operator from [6]:

**Definition 4.2.1.** From an arbitrary element $E = (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots) \in \ell^\infty(\mathcal{G})$, with matrix representation $E = (a_{j,k})_{j,k \in \mathbb{N}_0}$, we construct the following matrix, denoted by $\tilde{E}$, and called the *exploded version of* $E$:

$$\tilde{E} := \begin{pmatrix} a_{0,0} & 0 & a_{0,1} & 0 & a_{0,2} & 0 & a_{0,3} & 0 & a_{0,4} & 0 & a_{0,5} & 0 & a_{0,6} & \cdots \\ 0 & a_{1,1} & 0 & 0 & 0 & a_{1,2} & 0 & 0 & 0 & a_{1,3} & 0 & 0 & 0 & \\ a_{1,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{2,3} & 0 & \\ a_{2,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & a_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ a_{3,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{3,3} & 0 & 0 & 0 & 0 & 0 & \\ a_{4,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & a_{3,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ a_{5,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ a_{6,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \vdots & & & & & & & & & & & & & \ddots \end{pmatrix}$$

When writing $\tilde{E}$ as the sequence of its Gamma shells, we get

$$\tilde{E} := (\vec{\gamma^0}, \vec{\gamma^1}, \vec{\gamma^2}, \vec{\gamma^3}, \vec{\gamma^4}, \vec{\gamma^5}, \vec{\gamma^6}, \vec{\gamma^7}, \ldots) = (\vec{\Delta^0}, \vec{\Delta^1}, \vec{0}, \vec{\Delta^2}, \vec{0}, \vec{0}, \vec{0}, \vec{\Delta^3}, \ldots),$$

where, for $k \notin \{0, 1, 3, 7, \ldots\} = \{2^n - 1 : n \in \mathbb{N}_0\}$ we have that $\vec{\gamma^k} = \vec{0}$; and for $k$ such that $k = 2^n - 1$ for some $n \in \mathbb{N}_0$, $\vec{\gamma^k} = \gamma^{\vec{2^n}-1} = \vec{\Delta^n}$, where each $\vec{\Delta^n}$ is obtained by inserting $2^{n+1} - 1$ zeros in between the entries of $\vec{\delta^n}$.

This operator is introduced in [6], and it was used to define a new predual of the trace class, in such a way that this new predual is an analogue of the sequence space $c$, the space of convergent sequences. Following ideas discussed in [6], we present the following result:

**Claim 4.2.1.** *The linear map defined by $k : E \mapsto \tilde{E}$ is such that $k : \ell^\infty(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$, and*

$$\tfrac{1}{\sqrt{2}}\|E\|_{\ell^\infty(\mathcal{G})} \leq \|\tilde{E}\|_{op} \leq \|E\|_{\ell^\infty(\mathcal{G})}.$$

*Remark.* This claim, together with Claim 4.1.1, tells us that not only does $\mathcal{B}(\mathcal{H})$ continuously and linearly embed into $\ell^\infty(\mathcal{G})$, but also there is a closed vector subspace of $\mathcal{B}(\mathcal{H})$ that is an isomorphic copy of $\ell^\infty(\mathcal{G})$.

Before presenting the proof of this claim, we will need first the following two lemmas from [6], which are included here for completeness.

**Lemma 4.2.2.** *Let $P, Q \in \mathcal{P}(\mathcal{H})$ with $QP = 0$, then for all $A \in \mathcal{B}(\mathcal{H})$ we have that*

$$(\max\{\|PA\|_{op}\ , \|QA\|_{op}\})^2 \leq \|PA + QA\|_{op}^2 \leq \|PA\|_{op}^2 + \|QA\|_{op}^2.$$

*Proof.* We first notice that $PQ = P^*Q^* = (QP)^* = QP$, and so $PQ = 0$ as well.

Also note that for all $E \in \mathcal{B}(\mathcal{H})$, $\|E\|_{op}^2 = \|E^*E\|_{op}$. Hence,

$$\begin{aligned}
\|PA + QA\|_{op}^2 &= \|(PA + QA)^*(PA + QA)\|_{op} \\
&= \|(A^*P + A^*Q)(PA + QA)\|_{op} \\
&= \|A^*PPA + A^*PQA + A^*QPA + A^*QQA\|_{op} \\
&= \|(PA)^*PA + 0 + 0 + (QA)^*QA\|_{op} \\
&\leq \|(PA)^*PA\|_{op} + \|(QA)^*QA\|_{op} \\
&= \|PA\|_{op}^2 + \|QA\|_{op}^2.
\end{aligned}$$

Next, we define $R := P + Q$ in $\mathcal{B}(\mathcal{H})$. In fact $R \in \mathcal{P}(\mathcal{H})$ since $R^* = P^* + Q^* = P + Q = R$, and $R^2 = (P + Q)^2 = P^2 + PQ + QP + Q^2 = P + Q = R$.

Also, $0 \leq P \leq R$. Here $\leq$ is the usual quadratic form ordering on the self-adjoint members of $\mathcal{B}(\mathcal{H})$. Thus $0 \leq A^*PA \leq A^*RA$ as well.

For non-negative definite operators, the norm $\|\cdot\|_{op}$ is monotone; that is, $0 \leq U \leq V$ in $\mathcal{B}(\mathcal{H})$ implies $\|U\|_{op} \leq \|V\|_{op}$.

Consequently,

$$\begin{aligned}
\|PA\|_{op}^2 &= \|(PA)^*(PA)\|_{op} \\
&= \|A^*PPA\|_{op} \\
&= \|A^*PA\|_{op} \\
&\leq \|A^*RA\|_{op} \\
&= \|A^*RRA\|_{op} \\
&= \|(RA)^*RA\|_{op} \\
&= \|RA\|_{op}^2 \\
&= \|(P+Q)A\|_{op}^2 \\
&= \|PA+QA\|_{op}^2.
\end{aligned}$$

Mutatis mutandis, we obtain $\|QA\|_{op}^2 \leq \|PA+QA\|_{op}^2$. □

**Lemma 4.2.3.** *Consider a two sided sequence* $\vec{\delta} := (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ *in* $\ell^2(\mathbb{Z}, \mathbb{C})$, *and define the linear operator* $B$ *acting on* $\mathcal{H}$ *as the following sequence of Gamma shells* $B := (\vec{\delta}, \vec{0}, \vec{0}, \ldots)$. *Then* $B \in \mathcal{B}(\mathcal{H})$ *and in fact* $\|B\|_{op} \leq \|\vec{\delta}\|_{\ell^2(\mathbb{Z},\mathbb{C})} \leq \sqrt{2}\|B\|_{op}$.

*Proof.* Let $P := \langle \cdot, e_0 \rangle e_0 \in \mathcal{P}(\mathcal{H})$. Let $Q = I - P = \sum_{j=1}^{\infty} \langle \cdot, e_j \rangle e_j \in \mathcal{P}(\mathcal{H})$. Notice that $B = PB + QB$. Therefore, by Lemma 4.2.2 we have that

$$(\max\{\|PB\|_{op}\,, \|QB\|_{op}\})^2 \leq \|B\|_{op}^2 \leq \|PB\|_{op}^2 + \|QB\|_{op}^2.$$

This implies

$$\begin{aligned}
\|B\|_{op}^2 &\leq \left(|a_0|^2 + |a_1|^2 + |a_2|^2 + \ldots\right) + \left(|a_{-1}|^2 + |a_-2|^2 + |a_{-3}|^2 + \ldots\right) \\
&= \|\vec{\delta}\|_{\mathcal{G}}^2 \\
&\leq 2\max\{\left(|a_0|^2 + |a_1|^2 + |a_2|^2 + \ldots\right), \left(|a_{-1}|^2 + |a_-2|^2 + |a_{-3}|^2 + \ldots\right)\} \\
&\leq 2\|B\|_{op}^2.
\end{aligned}$$

Therefore, we get the desired conclusion. □

Now we can present the proof of Claim 4.2.1:

*Proof.* Fix $E = (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \dots) \in \ell^\infty(\mathcal{G})$. We also consider $E$ as an operator acting on $\mathcal{H}$ with respect to the usual basis, and with matrix representation $E = (a_{j,k})_{j,k\in\mathbb{N}_0}$. So, for each $n \in \mathbb{N}_0$, $\vec{\delta^n}$ is the $n$-th Gamma shell of the matrix $E$.

Consider an arbitrary $x = (x_0, x_1, x_2, x_3, \dots) \in \mathcal{H}$. Then

$$\begin{aligned}\|\tilde{E}x\|_2^2 = &|a_{0,0}\ x_0 + a_{0,1}\ x_2 + a_{0,2}x_4 + a_{0,3}\ x_6 + \dots|^2 + |x_0|^2(|a_{1,0}|^2 + |a_{2,0}|^2 + |a_{3,0}|^2 + \dots) \\ &+ |a_{1,1}\ x_1 + a_{1,2}\ x_5 + a_{1,3}x_9 + a_{1,4}\ x_{13} + \dots|^2 + |x_1|^2(|a_{2,1}|^2 + |a_{3,1}|^2 + |a_{4,1}|^2 + \dots) \\ &+ |a_{2,2}\ x_3 + a_{2,3}\ x_{11} + a_{2,4}x_{19} + a_{2,5}\ x_{27} + \dots|^2 + |x_3|^2(|a_{3,2}|^2 + |a_{4,2}|^2 + |a_{5,2}|^2 + \dots) \\ &+ |a_{3,3}\ x_7 + a_{3,4}\ x_{23} + a_{3,5}x_{39} + \dots|^2 + |x_7|^2(|a_{4,3}|^2 + |a_{5,3}|^2 + |a_{6,3}|^2 + \dots) + \dots\end{aligned}$$

Next, we define

$$B^0 := \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \dots \\ a_{1,0} & 0 & 0 & 0 & \\ a_{2,0} & 0 & 0 & 0 & \\ a_{3,0} & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

For arbitrary $z = (z_0, z_1, z_2, z_3, \dots) \in \mathcal{H}$ we have that

$$\|B^0 z\|_2^2 = |a_{0,0}\ z_0 + a_{0,1}\ z_1 + a_{0,2}z_2 + a_{0,3}\ z_3 + \dots|^2 + |z_0|^2(|a_{1,0}|^2 + |a_{2,0}|^2 + |a_{3,0}|^2 + \dots)$$

Let

$$B^1 := \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \dots \\ a_{2,1} & 0 & 0 & 0 & \\ a_{3,1} & 0 & 0 & 0 & \\ a_{4,1} & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Then for arbitrary $z = (z_0, z_1, z_2, z_3, \dots) \in \mathcal{H}$,

$$\|B^1 z\|_2^2 = |a_{1,1}\ z_0 + a_{1,2}\ z_1 + a_{1,3}z_2 + a_{0,3}\ z_3 + \dots|^2 + |z_0|^2(|a_{2,1}|^2 + |a_{3,1}|^2 + |a_{4,1}|^2 + \dots).$$

We continue this process, and for each $n \in \mathbb{N}_0$ we define

$$B^n := \begin{pmatrix} a_{n,n} & a_{n,n+1} & a_{n,n+2} & a_{n,n+3} & \cdots \\ a_{n+1,n} & 0 & 0 & 0 & \\ a_{n+2,n} & 0 & 0 & 0 & \\ a_{n+3,n} & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Then for arbitrary $z = (z_0, z_1, z_2, z_3, \ldots) \in \mathcal{H}$, we have that

$$\|B^n z\|_2^2 = |a_{n,n}\ z_0 + a_{n,n+1}\ z_1 + a_{n,n+2}z_2 + \ldots|^2 + |z_0|^2(|a_{n+1,n}|^2 + |a_{n+2,n}|^2 + |a_{n+3,n}|^2 + \ldots).$$

Therefore,

$$\begin{aligned} &\|\tilde{E}x\|_2^2 = \\ &\|B^0(x_0, x_2, x_4, \ldots)^T\|_2^2 + \|B^1(x_1, x_5, x_9, \ldots)^T\|_2^2 + \|B^2(x_3, x_{11}, x_{19}, \ldots)^T\|_2^2 + \ldots + \\ &\qquad \|B^n(x_{2^n-1}, x_{2^n-1+2^{n+1}}, x_{2^n-1+2\times 2^{n+1}}, \ldots)^T\|_2^2 + \ldots. \end{aligned}$$

Recall that, by Claim 4.2.3 we know that $\|B^n\|_{op} \leq \|\vec{\delta^n}\|_{\mathcal{G}}$, for every $n \in \mathbb{N}_0$.

Since $\sup_{n\in\mathbb{N}_0} \|\vec{\delta^n}\|_{\ell^2(\mathbb{Z},\mathbb{C})} = \|(\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots)\|_{\ell^\infty(\mathcal{G})} = \|E\|_{\ell^\infty(\mathcal{G})}$, we obtain that for arbitrary $x = (x_1, x_2, x_3, \ldots) \in \mathcal{H}$

$$\|\tilde{E}x\|_2^2 \leq \sup_{n\in\mathbb{N}_0} \|B^n\|_{op}\|x\|_2^2 \leq \|E\|^2_{\ell^\infty(\mathcal{G})}\|x\|_2^2.$$

So,

$$\|\tilde{E}\|_{op}^2 \leq \|E\|^2_{\ell^\infty(\mathcal{G})}.$$

Now, on the other hand, given an arbitrary $\epsilon > 0$, there exists $z^0 := (z_n)_{n\in\mathbb{N}_0} \in \mathcal{H}$ with $\|z^0\|_{\mathcal{H}} \leq 1$, such that

$$\|B^0 z^0\|_2^2 > \|B^0\|_{op}^2(1-\epsilon)^2.$$

Then

$$\begin{aligned} \|\tilde{E}\|_{op}^2 &\geq \|\tilde{E}(z_0, 0, z_1, 0, z_2, 0, \ldots)^T\|_2^2 \\ &= |a_{0,0}\ z_0 + a_{0,1}\ z_1 + a_{0,2}z_2 + a_{0,3}\ z_3 + \ldots|^2 + |z_0|^2(|a_{1,0}|^2 + |a_{2,0}|^2 + |a_{3,0}|^2 + \ldots) \\ &= \|B^0 z^0\|_2^2 > \|B^0\|_{op}^2(1-\epsilon)^2. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain $\|\tilde{E}\|_{op}^2 \geq \|B^0\|_{op}^2$.

Again, for an arbitrary $\epsilon > 0$, there exists $z^1 := (z_n)_{n\in\mathbb{N}_0} \in \mathcal{H}$ with $\|z^1\|_{\mathcal{H}} \leq 1$, such that

$$\|B^1 z^1\|_2^2 > \|B^1\|_{op}^2 (1-\epsilon)^2.$$

Then

$$\begin{aligned}\|\tilde{E}\|_{op}^2 &\geq \|\tilde{E}(0, z_0, 0, 0, 0, z_1, 0, 0, 0, z_2, 0, 0, 0, z_3, \dots)^T\|_2^2 \\ &= |a_{1,1}\ z_0 + a_{1,2}\ z_1 + a_{1,3} z_2 + a_{1,4}\ z_3 + \dots|^2 + |z_0|^2(|a_{2,1}|^2 + |a_{3,1}|^2 + |a_{4,1}|^2 + \dots) \\ &= \|B^1 z^1\|_2^2 > \|B^1\|_{op}^2 (1-\epsilon)^2.\end{aligned}$$

Therefore, we also get $\|\tilde{E}\|_{op}^2 \geq \|B^1\|_{op}^2$. Similarly, for each $n \in \mathbb{N}_0$ we can verify that $\|\tilde{E}\|_{op}^2 \geq \|B^n\|_{op}^2$. So, we obtain

$$\|\tilde{E}\|_{op}^2 = \sup_{n \in \mathbb{N}_0} \|B^n\|_{op}^2.$$

By Lemma 4.2.3 we know that for each $n \in \mathbb{N}_0$ we have that $\|B^n\|_{op}^2 \geq \frac{1}{2}\|\vec{\delta^n}\|_{\ell^2(\mathbb{Z},\mathbb{C})}^2$. Thus

$$\|\tilde{E}\|_{op}^2 \geq \tfrac{1}{2} \sup_{n \in \mathbb{N}_0} \|\vec{\delta^n}\|_{\ell^2(\mathbb{Z},\mathbb{C})}^2 = \frac{1}{2}\|E\|_{\ell^\infty(\mathcal{G})}^2.$$

□

## 4.3 Cesàro averaging on $\mathcal{B}(\mathcal{H})$

We start by defining the following map on $\ell^\infty(\mathcal{G})$:

**Definition 4.3.1.** Consider $h : \ell^\infty(\mathcal{G}) \to \ell^\infty(\mathcal{G})$ defined by

$$E := (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \dots) \mapsto \left(\vec{\delta^0}, \frac{\vec{\delta^0}+\vec{\delta^1}}{2}, \frac{\vec{\delta^0}+\vec{\delta^1}+\vec{\delta^2}}{3}, \dots\right).$$

The matrix representation of $h(E)$ is given by

$$h(E) := \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} & \cdots \\ a_{1,0} & \frac{a_{0,0}+a_{1,1}}{2} & \frac{a_{0,1}+a_{1,2}}{2} & \frac{a_{0,2}+a_{1,3}}{2} & \frac{a_{0,3}+a_{1,4}}{2} & \\ a_{2,0} & \frac{a_{1,0}+a_{2,1}}{2} & \frac{a_{0,0}+a_{1,1}+a_{2,2}}{3} & \frac{a_{0,1}+a_{1,2}+a_{2,3}}{3} & \frac{a_{0,2}+a_{1,3}+a_{2,4}}{3} & \\ a_{3,0} & \frac{a_{2,0}+a_{3,1}}{2} & \frac{a_{1,0}+a_{2,1}+a_{3,2}}{3} & \frac{a_{0,0}+a_{1,1}+a_{2,2}+a_{3,3}}{4} & \frac{a_{0,1}+a_{1,2}+a_{2,3}+a_{3,4}}{4} & \\ a_{4,0} & \frac{a_{3,0}+a_{4,1}}{2} & \frac{a_{2,0}+a_{3,1}+a_{4,2}}{3} & \frac{a_{1,0}+a_{2,1}+a_{3,2}+a_{4,3}}{4} & \frac{a_{0,0}+a_{1,1}+a_{2,2}+a_{3,3}+a_{4,4}}{5} & \\ \vdots & & & & & \ddots \end{pmatrix}$$

Next, we define the Cesàro averaging operator on $\mathcal{B}(\mathcal{H})$ as the following composition of maps:

**Definition 4.3.2.** We define the mapping $C : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$C(E) := k(h(E)),$$

where $k$ is the operator from definition 4.2.1, and $h$ is the operator from definition 4.3.1.

So, the matrix representation of $C(E)$ is given by

$$
\begin{pmatrix}
a_{0,0} & 0 & a_{0,1} & 0 & a_{0,2} & 0 & a_{0,3} & 0 & a_{0,4} & 0 & a_{0,5} & 0 & \cdots \\
0 & \frac{a_{0,0}+a_{1,1}}{2} & 0 & 0 & 0 & \frac{a_{0,1}+a_{1,2}}{2} & 0 & 0 & 0 & \frac{a_{0,2}+a_{1,3}}{2} & 0 & 0 & \\
a_{1,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & \frac{a_{0,0}+a_{1,1}+a_{2,2}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_{0,1}+a_{1,2}+a_{2,3}}{3} & \\
a_{2,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \frac{a_{1,0}+a_{2,1}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
a_{3,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}\sum_{k=0}^{3} a_{k,k} & 0 & 0 & 0 & 0 & \\
a_{4,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \frac{a_{2,0}+a_{3,1}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
a_{5,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & \frac{a_{1,0}+a_{2,1}+a_{3,2}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & & & & & & & \ddots
\end{pmatrix}
$$

To define “quasi-iterates” of the Cesàro averaging operator on $\mathcal{B}(\mathcal{H})$, we proceed as follows:

**Definition 4.3.3.** Let $n \in \mathbb{N}$, then we define $C^{(n)} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$C^{(n)}(E) := k(h^n(E)),$$

where the operator $h^n$ denotes applying the operator $h$ $n$ times to the vector $E \in \mathcal{B}(\mathcal{H})$.

## 4.4 Vector valued Banach limits on $\ell^\infty(\mathcal{G})$

Consider a Banach space $(X, \|\cdot\|_X)$. We define $B_X := \{x \in X : \|x\|_X \leq 1\}$, the closed unit ball of $X$.

We denote by $\ell^\infty(\mathbb{N}_0, X) := \ell^\infty(X)$ the space of sequences $(x_n)_{n\in\mathbb{N}_0}$ such that $x_n \in X$ for each $n \in \mathbb{N}_0$, and such that $\sup_{n\in\mathbb{N}_0} \|x_n\|_X < \infty$. Notice that $\left(\ell^\infty(X), \|\cdot\|_{\ell^\infty(X)}\right)$ is a Banach space, with norm $\|(x_n)_{n\in\mathbb{N}_0}\|_{\ell^\infty(X)} := \sup_{n\in\mathbb{N}_0} \|x_n\|_X$, for each $(x_n)_{n\in\mathbb{N}_0} \in \ell^\infty(X)$.

We denote by $c(\mathbb{N}_0, X) := c(X)$ the subspace of $\ell^\infty(X)$ consisting of sequences $(x_n)_n$ for which there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$ in norm.

The following definition of vector-valued Banach limit was given in [1]:

**Definition 4.4.1.** Let $(X, \|\cdot\|_X)$ be a Banach space. A bounded linear operator $\Lambda : \ell^\infty(X) \to X$ is called a *vector-valued Banach limit on* $X$ if

1. $\|\Lambda\|_{\ell^\infty(X)\to X} = 1$.
2. $\Lambda\left((x_n)_n\right) = \lim_{n\to\infty} x_n$, for all $(x_n)_n \in c(X)$.
3. $\Lambda\left((x_n)_{n=0}^\infty\right) = \Lambda\left((x_n)_{n=1}^\infty\right)$, for all $(x_n)_{n=0}^\infty \in \ell^\infty(X)$.

Next, we show the existence of a vector-valued Banach limit with domain $\ell^\infty(\mathcal{G})$, that preserves weak Cesàro convergence. This result was presented in [1]:

**Theorem 4.4.1.** *There exists a vector-valued Banach limit $\Psi$ with domain $\ell^\infty(\mathcal{G})$, such that if $E = (\vec{\delta}^0, \vec{\delta}^1, \vec{\delta}^2, \dots) \in \ell^\infty(\mathcal{G})$ is such that $h(E) = \left(\vec{\delta}^0, \dfrac{\vec{\delta}^0 + \vec{\delta}^1}{2}, \dfrac{\vec{\delta}^0 + \vec{\delta}^1 + \vec{\delta}^2}{3}, \dots\right)$ is weakly convergent to $x$, for some $x \in \mathcal{G}$, then $\Psi(E) = x$.*

*Proof.* Let $E := (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots) \in \ell^\infty(\mathcal{G})$. Then $(\vec{\delta^n})_{n\in\mathbb{N}_0} \subset \|E\|_{\ell^\infty(\mathcal{G})} B_{\mathcal{G}}$. By the Banach-Alaoglu theorem, we know that the closed unit ball $B_{\mathcal{G}}$ is weak*-compact, and since $\mathcal{G}$ is reflexive, this is equivalent to weakly compact.

Fix $\mathcal{U}$ a free ultrafilter over $\mathbb{N}_0$. We use it to define the following linear map:

$$\Psi : \ell^\infty(\mathcal{G}) \to \mathcal{G} \cong \mathcal{H}$$
$$E = (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots) \mapsto \mathcal{U} - \lim_{n\in\mathbb{N}_0} \vec{r^n} = \vec{L} \in \|E\|_{\ell^\infty(\mathcal{G})} B_{\mathcal{G}}, \text{ in the weak sense.}$$

This means that for all weak neighborhoods $V$ of $\vec{L}$ in $\mathcal{G}$, $\{n \in \mathbb{N}_0 : \vec{r^n} \in V\} \in \mathcal{U}$.

Here for each $n \in \mathbb{N}_0$ we define $\vec{r^n}$ by:

$$\vec{r^n} := \frac{\vec{\delta^0} + \vec{\delta^1} + \ldots + \vec{\delta^n}}{n+1}.$$

It is easy to verify that $\|(\vec{r^0}, \vec{r^1}, \vec{r^2}, \ldots)\|_{\ell^\infty(\mathcal{G})} \leq \|(\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots)\|_{\ell^\infty(\mathcal{G})} = \|E\|_{\ell^\infty(\mathcal{G})}$. Therefore $(\vec{r^n})_{n\in\mathbb{N}_0} \subset \|E\|_{\ell^\infty(\mathcal{G})} B_{\mathcal{G}}$, which is weakly-compact, then $\mathcal{U} - \lim_{n\in\mathbb{N}} \vec{r^n}$ exists in $\|E\|_{\ell^\infty(\mathcal{G})} B_{\mathcal{G}}$, and it is unique. So, the mapping is well defined.

Since $\Psi(E) \in \|E\|_{\ell^\infty(\mathcal{G})} B_{\mathcal{G}}$, then $\|\Psi(E)\|_{\mathcal{G}} \leq \|E\|_{\ell^\infty(\mathcal{G})}$. Therefore

$$\|\Psi\|_{\ell^\infty(\mathcal{G})\to\mathcal{G}} \leq 1.$$

Also $\Psi(I) = \lim_{\mathcal{U}} e_0 = e_0$, where $e_0 = (\ldots, 0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$, with the entry value 1 in the zero position. Therefore $\|\Psi(I)\|_{\mathcal{G}} = 1$. Thus $\|\Psi\|_{\ell^\infty(\mathcal{G})\to\mathcal{G}} = 1$.

Next, we check that $\Psi$ preserves norm convergence:

Let $(\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots) \in \ell^\infty(\mathcal{G})$, such that there exists $x \in \mathcal{G}$ such that $\lim_{n\to\infty} \vec{\delta^n} = x$ in norm. This implies that

$$\lim_{n\to\infty} \vec{r^n} = \lim_{n\to\infty} \frac{\vec{\delta^0} + \vec{\delta^1} + \ldots + \vec{\delta^n}}{n+1} = x, \text{ in norm.}$$

Therefore $\lim_{n\to\infty} \vec{r^n} = x$, weakly. So,

$$\Psi((\vec{\delta^n})_n) = x.$$

Next, we check that $\Psi$ is left-shift invariant: Let

$$(\vec{s^n})_{n\in\mathbb{N}_0} := (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots) - (\vec{\delta^1}, \vec{\delta^2}, \vec{\delta^3}, \ldots).$$

Then $\dfrac{\vec{s^0} + \vec{s^1} + \ldots + \vec{s^n}}{n+1} = \dfrac{\vec{\delta^0} - \vec{\delta^{n+1}}}{n+1}$.

Therefore

$$\left\| \frac{\vec{s^0} + \vec{s^1} + \ldots + \vec{s^n}}{n+1} \right\|_{\mathcal{G}} \leq \frac{2\|(\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots)\|_{\ell^\infty(\mathcal{G})}}{n+1} \to 0 \text{ as } n \to \infty.$$

So, $\dfrac{\vec{s^0} + \vec{s^1} + \ldots + \vec{s^n}}{n+1} \to 0$ in norm, as $n \to \infty$. This implies $\dfrac{\vec{s^0} + \vec{s^1} + \ldots + \vec{s^n}}{n+1} \to 0$ weakly, as $n \to \infty$.

Therefore $\Psi((\vec{s^n})_{n\in\mathbb{N}_0}) = 0$, implying $\Psi(\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots) = \Psi(\vec{\delta^1}, \vec{\delta^2}, \vec{\delta^3}, \ldots)$.

Finally, notice that if $(\vec{\delta^n})_{n\in\mathbb{N}_0}$ is weakly-Cesàro convergent to some $x \in \mathcal{G}$, that is,

$$\frac{\vec{\delta^0} + \vec{\delta^1} + \ldots + \vec{\delta^n}}{n+1} \to x \text{ weakly,}$$

then $\Psi((\vec{\delta^n})_{n\in\mathbb{N}_0}) = x$. □

## 4.5 A stronger vector valued Banach limit on $\ell^\infty(\mathcal{G})$

**Definition 4.5.1.** Following the same approach as in Section 2.2, we define

$$H(E) := \Psi(\Psi(E), \Psi(hE), \Psi(h^2E), \Psi(h^3E), \ldots), \text{ for all } E \in \ell^\infty(\mathcal{G}),$$

where $\Psi$ is as defined in the proof of Theorem 4.4.1, and , $h$ is the operator from definition 4.3.1.

**Claim 4.5.1.** *$H$, defined above, is a vector-valued Banach limit with domain $\ell^\infty(\mathcal{G})$ such that it is invariant under $h$.*

*Proof.* Since $\Psi$ is left-shift invariant, we see that $H$ is $h$-Cesàro averaging invariant, that is $H(h(E)) = H(E)$.

Let $I = (\vec{\delta}, \vec{\delta}, \vec{\delta}, \ldots) \in \ell^\infty(\mathcal{G})$, with $\vec{\delta} := (\ldots, 0, 0, 0, 1, 0, 0, 0, \ldots)$, where the non-zero entry occurs in the 0-position. Notice $\|\vec{\delta}\|_{\mathcal{G}} = 1$ and therefore $\|I\|_{\ell^\infty(\mathcal{G})} = 1$. Also notice $h^n(I) = I$ for every $n \in \mathbb{N}_0$. So, $\Psi(h^n(I)) = \vec{\delta}$, for every $n \in \mathbb{N}_0$, since $(\vec{\delta}, \vec{\delta}, \vec{\delta}, \ldots)$ converges to $\vec{\delta}$. Therefore

$$H(I) := \Psi(\vec{\delta}, \vec{\delta}, \vec{\delta}, \ldots) = \vec{\delta}.$$

Thus $\|H(I)\|_{\mathcal{G}} = 1$, and this implies $1 \leq \|H\|_{op}$.

Next, notice that for arbitrary $E \in \ell^\infty(\mathcal{G})$, we have that

$$
\begin{aligned}
\|H(E)\|_{\mathcal{G}} &\leq \|\Psi\|_{op}\| \left(\Psi(E), \Psi(hE), \Psi(h^2E), \ldots\right) \|_{\ell^\infty(\mathcal{G})} \\
&= 1 \cdot \| \left(\Psi(E), \Psi(hE), \Psi(h^2E), \ldots\right) \|_{\ell^\infty(\mathcal{G})} \\
&= \sup\{\|\Psi(E)\|_{\mathcal{G}}, \|\Psi(hE)\|_{\mathcal{G}}, \|\Psi(h^2E)\|_{\mathcal{G}}, \ldots\} \\
&\leq \sup\{\|\Psi\|_{op}\|E\|_{\ell^\infty(\mathcal{G})}, \|\Psi\|_{op}\|hE\|_{\ell^\infty(\mathcal{G})}, \|\Psi\|_{op}\|h^2E\|_{\ell^\infty(\mathcal{G})}, \ldots\} \\
&= \sup\{\|E\|_{\ell^\infty(\mathcal{G})}, \|hE\|_{\ell^\infty(\mathcal{G})}, \|h^2E\|_{\ell^\infty(\mathcal{G})}, \ldots\} \\
&\leq \|E\|_{\ell^\infty(\mathcal{G})}.
\end{aligned}
$$

Hence $\|H\|_{op} = 1$.

Next, notice that, if $E = (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots)$ is such that this sequence converges in norm to $x \in \mathcal{G}$, then $h(E) = \left(\vec{\delta^0}, \frac{\vec{\delta^0} + \vec{\delta^1}}{2}, \frac{\vec{\delta^0} + \vec{\delta^1} + \vec{\delta^2}}{3}, \ldots\right)$ converges to $x$, and so does $h^n(E)$ for every $n \in \mathbb{N}_0$. So $\Psi(h^n(E)) = x$, for every $n \in \mathbb{N}_0$. This implies that $H(E) = \Psi(x, x, x, \ldots) = x$. Thus $H$ preserves convergence.

Next, given $E = (\vec{\delta^0}, \vec{\delta^1}, \vec{\delta^2}, \ldots) \in \ell^\infty(\mathcal{G})$, denote by $S(E) := (\vec{\delta^1}, \vec{\delta^2}, \vec{\delta^3}, \ldots)$. We want to show that $H(E) = H(S(E))$. For this, we will use the following claim: $\Psi(h^n S(E)) = \Psi(h^n E)$, for all $n \in \mathbb{N}_0$ and for all $E \in \ell^\infty(\mathcal{G})$. Then,

$$
\begin{aligned}
H(S(E)) &= \Psi(\Psi(S(E)), \Psi(hS(E)), \Psi(h^2S(E)), \Psi(h^3S(E)), \ldots) \\
&= \Psi(\Psi(E), \Psi(hE), \Psi(h^2E), \Psi(h^3E), \ldots) \\
&= H(E).
\end{aligned}
$$

□

**Claim 4.5.2.** *$\Psi(h^n S(E)) = \Psi(h^n E)$, for all $n \in \mathbb{N}$ and for all $E \in \ell^\infty(\mathcal{G})$.*

*Remark.* This claim also holds for $n = 0$ (where $h^n := id$ is the identity operator on $\ell^\infty(\mathcal{G})$), since $\Psi$ is a vector-valued Banach limit, and therefore, left-shift invariant.

*Proof.* Let $E := (\delta^0, \delta^1, \delta^2, \ldots)$ We proceed by induction over $n \in \mathbb{N}$.

Base case:

Consider $W := hSE - ShE$. We will show that $W$ as an element of $\ell^\infty(\mathcal{G})$ converges to $\vec{0}$ in norm. This would imply that $\Psi(hSE) = \Psi(ShE) = \Psi(hE)$, where the last equality is true since $\Psi$ is left-shift invariant.

Notice that

$$hSE - ShE = \left(\delta^1, \frac{\delta^1 + \delta^2}{2}, \frac{\delta^1 + \delta^2 + \delta^3}{3}, \ldots\right) - \left(\frac{\delta^0 + \delta^1}{2}, \frac{\delta^0 + \delta^1 + \delta^2}{3}, \ldots\right).$$

Thus, for a fixed $n \in \mathbb{N}_0$

$$\begin{aligned}(hSE - ShE)_n &= \frac{\delta^1 + \ldots + \delta^{n+1}}{n+1} - \frac{\delta^0 + \ldots + \delta^{n+1}}{n+2} \\ &= \frac{-(n+1)\delta^0 + \delta^1 + \ldots + \delta^{n+1}}{(n+1)(n+2)}.\end{aligned}$$

So

$$\begin{aligned}\|(hSE - ShE)_n\|_{\mathcal{G}} &= \frac{\| - (n+1)\delta^0 + \delta^1 + \ldots + \delta^{n+1}\|_{\mathcal{G}}}{(n+1)(n+2)} \\ &\leq \frac{(n+1)\|\delta^0\|_{\mathcal{G}} + \|\delta^1\|_{\mathcal{G}} + \ldots + \|\delta^{n+1}\|_{\mathcal{G}}}{(n+1)(n+2)} \\ &\leq \frac{(n+1)\|E\|_{\ell^\infty(\mathcal{G})} + (n+1)\|E\|_{\ell^\infty(\mathcal{G})}}{(n+1)(n+2)} = \frac{2\|E\|_{\ell^\infty(\mathcal{G})}}{n+2} \to 0 \ \textit{as}\ n \to \infty.\end{aligned}$$

Next, for the inductive step: We want to show that if for a fixed $n \in \mathbb{N}$, we have that the elements of $(h^n S - Sh^n)(\ell^\infty(\mathcal{G}))$ are sequences in $\ell^\infty(\mathcal{G})$ that converge in norm to $\vec{0}$, then this implies that the set $(h^{n+1}S - Sh^{n+1})(\ell^\infty(\mathcal{G}))$ has the same property:

Fix $E \in \ell^\infty(\mathcal{G})$, then

$$\begin{aligned}(h^{n+1}S - Sh^{n+1})E &= h(h^n SE) - hSh^n E + hSh^n E - Sh(h^n E) \\ &= h(h^n SE - Sh^n E) + (hS - Sh)(h^n E).\end{aligned}$$

This last expression converges in norm to $\vec{0}$, by base case and inductive hypothesis, and the fact that the operator $h$ preserves limits. □

### 4.6 A continuous linear functional on $\mathcal{B}(\mathcal{H})$ with Cesàro-invariance-like properties

**Definition 4.6.1.** For every $\lambda \in \mathcal{G}$, $\lambda \neq 0$, we define

$$\gamma_\lambda(E) := \gamma(E) := \langle H(E), \lambda \rangle_{\mathcal{G}}, \text{ for every } E \in \ell^\infty(\mathcal{G}),$$

where $H$ is the vector-valued Banach limit from definition 4.5.1.

This defines a continuous linear functional, that is $\gamma_\lambda$ is an element of $(\ell^\infty(\mathcal{G}))^*$, since

$$|\gamma_\lambda(E)| \leq \|H\|_{op} \|E\|_{\ell^\infty(\mathcal{G})} \|\lambda\|_{\mathcal{G}} = \|E\|_{\ell^\infty(\mathcal{G})} \|\lambda\|_{\mathcal{G}},$$

and so $\|\gamma_\lambda\|_{(\ell^\infty(\mathcal{G}))^*} \leq \|\lambda\|_{\mathcal{G}}$.

Observe $\gamma_\lambda \neq 0$ since, without loss of generality, we assume that $\lambda_0 \neq 0$. Then $\gamma_\lambda(I) = \lambda_0$.

We notice that $\gamma_\lambda(hE) = \gamma_\lambda(E)$, for all $E \in \ell^\infty(\mathcal{G})$, where $h$ is the Cesàro averaging operator defined on $\ell^\infty(\mathcal{G})$ (Definition 4.3.1).

Also notice $\gamma_\lambda \in (\mathcal{B}(\mathcal{H}))^*$ since $i : \mathcal{B}(\mathcal{H}) \hookrightarrow \ell^\infty(\mathcal{G})$ is continuous. We use this observation, to define the following operator:

**Definition 4.6.2.** Define the continuous linear functional $\Gamma_\lambda : Y \subset \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by

$$\Gamma_\lambda(D) = \gamma_\lambda\left(k^{-1}D\right) = \langle H(k^{-1}D), \lambda \rangle_{\mathcal{G}},$$

where $Y := k(\ell^\infty(\mathcal{G}))$, which is a closed vector subspace of $\mathcal{B}(\mathcal{H})$. Recall $k$ is the operator from Definition 4.2.1. Also, recall $Y \approx \ell^\infty(\mathcal{G})$.

Notice that

$$|\Gamma_\lambda(D)| \leq \|k^{-1}D\|_{\ell^\infty(\mathcal{G})} \|\lambda\|_{\mathcal{G}} \leq \sqrt{2}\|\lambda\|_{\mathcal{G}} \|D\|_{\mathcal{B}(\mathcal{H})}, \text{ for all } D \in Y.$$

So, $\|\Gamma_\lambda\|_{Y^*} \leq \sqrt{2}\|\lambda\|_{\mathcal{G}}$.

Since $\Gamma_\lambda \in Y^*$, by Hahn-Banach Extension Theorem, we know there exists $\Lambda_\lambda \in (\mathcal{B}(\mathcal{H}))^*$ such that $\Lambda_\lambda|_Y = \Gamma_\lambda$ and $\|\Lambda_\lambda\|_* = \|\Gamma_\lambda\|_*$.

Recall, $C = k \circ h$ is the Cesàro operator defined on $\mathcal{B}(\mathcal{H})$ (Definition 4.3.2). We have that

$$\Lambda_\lambda(C(E)) = \Lambda_\lambda(k(h(E))) = \Gamma_\lambda(k(h(E))) = \gamma_\lambda\left(h(E)\right) = \gamma_\lambda(E) = \Gamma_\lambda(k(E)) = \Lambda_\lambda(k(E)).$$

*Remark.* $k(E)$ is always an element of $\mathcal{B}(\mathcal{H})$.

**Open Question 4.6.1.** *The explosion operator was introduced in this chapter since it is not clear that the naive way of taking Cesàro average maps bounded operators on* $\mathcal{H}$ *back into* $\mathcal{B}(\mathcal{H})$*. What is an example, if there exists one, of an element in* $\mathcal{B}(\mathcal{H})$ *whose naive Cesàro average is not an element of* $\mathcal{B}(\mathcal{H})$*? What other definitions of Cesàro averaging on* $\mathcal{B}(\mathcal{H})$ *could be studied, to guarantee outputs in* $\mathcal{B}(\mathcal{H})$*?*

# Appendix A A Banach limit in $L^\infty$ that preserves Cesàro convergence, but is not Cesàro invariant

We start by stating the following version of The Hahn-Banach extension Theorem for seminorms from [26]:

**Theorem A.0.1** (Bohnenblust-Sobczyk)**.** *Let $Y$ be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $W$ be a vector subspace of $Y$. Let $p : Y \to [0, \infty)$ be a seminorm, and $f : W \to \mathbb{K}$ be a linear functional such that $|f(w)| \leq p(w)$, for all $w \in W$. Then there exists a linear functional $F : Y \to \mathbb{K}$ such that $F|_W = f$, and $|F(y)| \leq p(y)$, for all $y \in Y$.*

Now, we can prove the following theorem:

**Theorem A.0.2.** *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $V$ be a vector subspace of $X$, and $a, z \in X$. Consider the affine subspace $M$ generated by $a$ and $z$, that is, $M := \{(1-t)z + ta : t \in \mathbb{K}\}$. Assume there is a positive distance from $M$ to $V$, that is, $\eta := d(M, V) := \inf\{\|(1-t)z + ta - v\| : t \in \mathbb{K}, v \in V\} > 0$. Then, there exists $\Psi \in X^*$ such that $\|\Psi\|_{X^*} = 1$, $\Psi|_V = 0$, and $\Psi(z) = \Psi(a) = \eta$.*

*Proof.* Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $V$ be a vector subspace of $X$, and $a, z \in X$. Consider the affine subspace $M$ generated by $a$ and $z$, that is, $M := \{(1-t)z + ta : t \in \mathbb{K}\}$. Assume there is a positive distance from $M$ to $V$, that is,

$$\eta := d(M, V) := \inf\{\|(1-t)z + ta - v\| : t \in \mathbb{K}, v \in V\} > 0.$$

Let $p(x) := \|x\|$. Since $p$ is a norm, it is therefore a seminorm. We consider the following cases:

**Case 1.** Suppose $z = a$.

Let $W := \{v + \beta z : \beta \in \mathbb{K} \text{ and } v \in V\}$. $W$ is a vector subspace of $X$ and $V \subset W$. We notice that elements in $W$ have unique representation: Let $w \in W$ and suppose $w = v + \beta z = v_1 + \beta_1 z$, where $v, v_1 \in V$ and $\beta, \beta_1 \in \mathbb{K}$. Then $(\beta - \beta_1)z = v_1 - v \in V$. But $z \notin V$, which implies $\beta - \beta_1 = 0$, and therefore $\beta = \beta_1$ and $v = v_1$.

Therefore we can define the function $f : W \to \mathbb{K}$ by

$$f(v+\beta z) := \beta\eta, \text{ for all } w = v+\beta z \in W.$$

$f$ is well-defined and a linear functional on $W$.

Fix an arbitrary $w = v+\beta z \in W$,

$$|f(w)| = |\beta|\eta = |\beta| \inf\{\|z-u\| : u \in V\}.$$

If $\beta = 0$, then $|f(w)| = 0 \leq p(w)$. If $\beta \neq 0$, then

$$|f(w)| = \inf\{\|\beta z - \beta u\| : u \in V\} \leq \|\beta z + v\|, \text{ by taking } u = \tfrac{-v}{\beta} \in V.$$

So,

$$|f(w)| \leq \|w\|, \text{ for all } w \in W.$$

By Theorem A.0.1, there exists a linear functional $\Psi : X \to \mathbb{K}$ such that

$$\Psi|_W = f, \text{ and } |\Psi(x)| \leq \|x\|, \text{ for all } x \in X.$$

Thus $\Psi \in X^*$, in fact $\|\Psi\|_{X^*} \leq 1$. Also $\Psi|_V = f|_V = 0$, and $\Psi(z) = f(z) = \eta$. Since $a = z$, then $\Psi(a) = \eta$ as well.

Further, $\|\Psi\|_{X^*} = 1$. To see this, we will show that $\|f\|_{W^*} = 1$:

Notice, we already know $\|f\|_{W^*} \leq 1$.

Fix an arbitrary $\epsilon > 0$. Then there exists $v_0 \in V$ such that

$$0 < \eta \leq \|z - v_0\| < \eta(1+\epsilon).$$

Let $w_0 = z - v_0$, and let $\tau_0 = \frac{w_0}{\|w_0\|}$. Then

$$|f(\tau_0)| = \frac{|f(z-v_0)|}{\|z-v_0\|} = \frac{|\eta|}{\|z-v_0\|}.$$

Consequently,

$$\frac{\eta}{\eta(1+\epsilon)} < |f(\tau_0)| \leq \frac{\eta}{\eta}.$$

Therefore

$$\frac{1}{1+\epsilon} < |f(\tau_0)| \leq 1.$$

This implies $\|f\|_{W^*} = \sup_{\tau \in B_W} |f(\tau)| = 1$.

**Case 2.** Suppose $z \neq a$, and $a \in W$.

Then $a = \beta z + v_0$, for some $\beta \in \mathbb{K}$ and $v_0 \in V$. That is $v_0 = a - \beta z$.

Notice, we must have $\beta = 1$, or else $1 - \beta \neq 0$, and so

$$\frac{1}{1-\beta}a + \frac{-\beta}{1-\beta}z = \frac{1}{1-\beta}v_0 =: v_1 \in V.$$

Letting $t := \frac{1}{1-\beta} \in \mathbb{K}$, then $1 - t = \frac{-\beta}{1-\beta}$. So, we would have $v_1 = (1-t)z + ta \in V$. But

$$\|(1-t)z + ta - v_1\| \geq \eta > 0,$$

which is a contradiction. Therefore $\beta = 1$, and so, $a - z = v_0 \in V$.

As before, we define the vector subspace $W := \{v + \beta z : \beta \in \mathbb{K} \textit{ and } v \in V\}$, and the function $f : W \to \mathbb{K}$ by $f(v + \beta z) := \beta\eta$, for all $w = v + \beta z \in W$. Let $\Psi$ be the extension of $f$ as in Case 1.

Since $a = v_0 + z \in W$, then $\Psi(a) = f(a) = \eta$. It is clear that $\Psi(z) = f(z) = \eta$, and as before we can verify that $\|\Psi\|_{X^*} = 1$.

**Case 3.** Suppose $z \neq a$, and $a \notin W$.

We define the vector subspace $Q := \{\alpha a + \beta z + v : \alpha, \beta \in \mathbb{K} \textit{ and } v \in V\}$. We see that elements in $Q$ have unique representation: Let $q \in Q$ such that $q = \alpha a + \beta z + v = \alpha_1 a + \beta_1 z + v_1$, where $v, v_1 \in V$ and $\alpha, \alpha_1, \beta, \beta_1 \in \mathbb{K}$. Then

$$(\alpha - \alpha_1)a + (\beta - \beta_1)z = v_1 - v \in V.$$

In this case, we must have that $\alpha_1 = \alpha$, or else we can write $a = \frac{\beta_1 - \beta}{\alpha - \alpha_1}z + \frac{1}{\alpha - \alpha_1}(v_1 - v) \in W$. But $a \notin W$, by hypothesis of Case 3. Thus, $\alpha_1 = \alpha$, and therefore $(\beta - \beta_1)z = v_1 - v \in V$. But $z \notin V$, which implies $\beta = \beta_1$ and $v = v_1$.

So we can define the linear functional $g : Q \to \mathbb{K}$ by $g(q) := \alpha\eta + \beta\eta$ for all $q = \alpha a + \beta z + v \in Q$. Notice $g$ is well defined by uniqueness of representation of elements in $Q$.

Fix $q = \alpha a + \beta z + v \in Q$, then

$$|g(q)| = |\alpha + \beta|\eta = |\alpha + \beta| \inf\{\|(1-t)z + ta - u\| : t \in \mathbb{K} \textit{ and } u \in V\}.$$

If $\beta = -\alpha$, $|g(q)| = 0 \leq p(q)$. Suppose $\beta \neq -\alpha$, then $\alpha + \beta \neq 0$. Therefore

$$|g(q)| = \inf\{\|(\alpha+\beta)(1-t)z + (\alpha+\beta)ta - (\alpha+\beta)u : t \in \mathbb{K} \textit{ and } u \in V\}.$$

Let $t := \dfrac{\alpha}{\alpha+\beta} \in \mathbb{K}$, then $1-t = \dfrac{\beta}{\alpha+\beta}$. Also, let $u := \dfrac{-1}{\alpha+\beta} v \in V$. Then

$$|g(q)| \leq \|\beta z + \alpha a + v\| = \|q\|.$$

By Theorem A.0.1 there exists a linear functional $F : X \to \mathbb{K}$ such that $F|_Q = g$ and $|F(x)| \leq \|x\|$, for all $x \in X$.

This implies $F \in X^*$, $\|F\|_{X^*} \leq 1$ and $F|_V = 0$. Also, $F(z) = g(z) = \eta$ and $F(a) = g(a) = \eta$.

Further $\|F\|_{X^*} = 1$. To see this, we prove that $\|g\|_{Q^*} = 1$:

Fix $\epsilon > 0$. Then there exists $t_0 \in \mathbb{K}$ and $v_0 \in V$ such that

$$0 < \eta \leq \|(1-t_0)z + t_0 a - v_0\| < (1+\epsilon)\eta.$$

Let $q_0 := (1-t_0)z + t_0 a - v_0 \in Q$, and let $\tau_0 := \dfrac{q_0}{\|q_0\|}$. Then

$$|g(\tau_0)| = \frac{|g(q_0)|}{\|q_0\|} = \frac{|t_0\eta + (1-t_0)\eta|}{\|q_0\|} = \frac{\eta}{\|q_0\|}.$$

So, $|g(\tau_0)| \leq \dfrac{\eta}{\eta} = 1$, and $|g(\tau_0)| > \dfrac{\eta}{(1+\epsilon)\eta} = \dfrac{1}{1+\epsilon}$.

Therefore, $\|g\|_{Q^*} = \sup\limits_{\tau \in B_Q} |g(\tau)| = 1$. □

To show the existence of a Banach limit $\Psi$ in $L^\infty$, such that $\Psi$ is not Cesàro invariant, but preserves Cesàro convergence, we will first construct a special element $f \in L^\infty$. Later on, a constant times the function $Jf - f$ will play the role of $a$ when applying Theorem A.0.2, to obtain this desired Banach limit.

We construct $f \in L^\infty$ the following way: Choose $N_1 > 1$, choose $N_2$ such that $N_2 > 2N_1$, and choose $N_3$ such that $N_3 > 2N_2$, and such that $\frac{N_1}{N_3} < \frac{1}{2}$. For $k \in \mathbb{N}$, $k > 1$, inductively we choose $N_{2k}$ such that $N_{2k} > 2N_{2k-1}$, and $N_{2k+1}$ such that $N_{2k+1} > 2N_{2k}$, and such that $\dfrac{N_{2k-1}}{N_{2k+1}} < \dfrac{1}{2k}$ (e.g. define $N_j := (j+1)!$ for every $j \in \mathbb{N}$). Then, we have that

$$1 < N_1 < 2N_1 < N_2 < 2N_2 < N_3 < 2N_3 < N_4 < 2N_4 < N_5 < \ldots.$$

Define $f$ by

$$f(x) := \begin{cases} 1, & x \in (0, N_1] \cup (2N_1, N_3] \cup (2N_3, N_5] \cup \ldots \\ 0, & x \in (N_1, 2N_1] \cup (N_3, 2N_3] \cup (N_5, 2N_5] \cup \ldots. \end{cases}$$

We notice that, for any $x > 0$, $Jf(x) \leq 1$, since $\|f\|_\infty = 1$. Consequently,

$$\frac{1}{x}\int_0^x f(t)dt - f(x) = \frac{1}{x}\int_0^x f(t)dt - 1 \le 0, \text{ for all } x \in (N_{2k}, 2N_{2k}], \text{ for any } k \in \mathbb{N}.$$

For all $x \in (N_{2k+1}, 2N_{2k+1})$ for all $k \in \mathbb{N}$, we have that $x = N_{2k+1} + y$, for $y = x - N_{2k+1} \in (0, N_{2k+1})$. Hence

$$\begin{aligned}
\frac{1}{x}\int_0^x f(t)dt - f(x) &= \frac{N_{2k+1} - N_{2k-1} - \cdots - N_1}{N_{2k+1} + y} - 0 \\
&> \frac{N_{2k+1} - N_{2k-1} - \cdots - N_1}{2N_{2k+1}} \\
&> \frac{1}{2} - \frac{1}{2}\frac{N_{2k-1} + \cdots + N_1}{N_{2k+1}} \\
&> \frac{1}{2} - \frac{1}{2}\frac{kN_{2k-1}}{N_{2k+1}} \\
&> \frac{1}{4}.
\end{aligned}$$

Now, we apply Theorem A.0.2:

Let $g(x) := 4\left(\frac{1}{x}\int_0^x f(t)dt - f(x)\right) \in L^\infty$. Let $h(x) := \mathbb{1}(x) \in L^\infty$. Let $V := Ces_0 := \{f \in L^\infty : \lim_{x\to\infty} Jf(x) = 0\}$. Let $M := \{(1-t)h + tg : t \in \mathbb{R}\}$.

**Claim A.0.3.** *$d(M, V) := \inf\{\|(1-t)h + tg - j\|_\infty : t \in \mathbb{R}$ and $j \in Ces_0\} = 1$.*

*Proof.* First notice that, for the particular function $j(x) := 0$ for all $x > 0$, and for the particular value $t = 0$, we have that

$$d(M, V) := \inf\{\|(1-t)h + tg - j\|_\infty : t \in \mathbb{R} \text{ and } j \in Ces_0\} \le \|h\|_\infty = 1.$$

On the other hand, fix arbitrary $t \ge 0$ and $j \in Ces_0$. Also fix $k \in \mathbb{N}$ and $x \in (N_{2k+1}, 2N_{2k+1})$.

$$\begin{aligned}
\Delta := \|(1-t)h + tg - j\|_\infty &\ge |(1-t)h(x) + tg(x) - j(x)| \\
&\ge (1-t)\cdot 1 + t\cdot 4(Jf(x) - f(x)) - j(x) \\
&\ge 1 - t + t\cdot 1 - j(x) = 1 - j(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta = \frac{1}{N_{2k+1}}\int_{x=N_{2k+1}}^{x=2N_{2k+1}} \Delta dx &\ge \frac{1}{N_{2k+1}}\int_{x=N_{2k+1}}^{x=2N_{2k+1}} (1 - j(x))dx \\
&= 1 - \frac{1}{N_{2k+1}}\int_{x=N_{2k+1}}^{x=2N_{2k+1}} j(x)dx \\
&= 1 - \frac{1}{N_{2k+1}}\left(\int_{x=0}^{x=2N_{2k+1}} j(x)dx - \int_{x=0}^{x=N_{2k+1}} j(x)dx\right).
\end{aligned}$$

This implies

$$\Delta \geq 1 - 2Jj(2N_{2k+1}) + Jj(N_{2k+1}) \to 1 - 0 + 0, \text{ as } k \to \infty, \text{ since } j \in Ces_0.$$

Thus, for any $t \geq 0$, for any $j \in Ces_0$,

$$\|(1-t)h + tg - j\|_\infty \geq 1.$$

Next, fix $t < 0$ and $j \in Ces_0$. Also, fix $k \in \mathbb{N}$ and $x \in (N_{2k}, 2N_{2k})$.

$$\begin{aligned}\Delta := \|(1-t)h + tg - j\|_\infty &\geq (1-t)\cdot 1 + t \cdot 4(Jf(x) - f(x)) - j(x) \\ &\geq 1 - t - j(x) \geq 1 - j(x).\end{aligned}$$

Consequently, similarly to above, we get that

$$\Delta \geq 1 - 2Jj(2N_{2k}) + Jj(N_{2k}) \to 1, \text{ as } k \to \infty, \text{ since } j \in Ces_0.$$

Hence, for all $t < 0$ and for all $j \in Ces_0$,

$$\|(1-t)h + tg - j\|_\infty \geq 1.$$

So, we see that

$$\inf\{\|(1-t)h + tg - j\|_\infty : t \in \mathbb{R} \ and \ j \in Ces_0\} \geq 1.$$

This implies that $d(M, V) = 1$. □

Thus, by Theorem A.0.2, we know there exists $\Psi$ in $(L^\infty)^*$, with $\|\Psi\|_{(L^\infty)^*} = 1$, and: (i) $\Psi|_{Ces_0} = 0$, (ii) $\Psi(h) = \Psi(\mathbb{1}) = 1$, and (iii) $\Psi(g) = 1$.

Notice that (iii) implies $\Psi$ is not Cesàro invariant, since for the function $f$ constructed above, we have that

$$\Psi(Jf - f) = \tfrac{1}{4}\Psi(g) = \tfrac{1}{4} \neq 0.$$

On the other hand, (i) and (ii) imply that $\Psi$ is Cesàro convergence preserving, since for any $g \in Ces$, we have that $g = L\mathbb{1} + j$, where $L := \lim_{x\to\infty} Jg(x)$, and $j \in Ces_0$ (simply define $j(x) := g(x) - L\mathbb{1}(x)$, for all $x > 0$). Therefore

$$\Psi(g) = \Psi(L\mathbb{1} + j) = L.$$

Finally, to verify $\Psi$ is a Banach limit, first notice that for any $f \in BC_L$, we have that $f \in Ces$, and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} Jf(x)$. So we see that $\Psi$ preserves classical convergence. Also recall that $\|\Psi\|_{(L^\infty)^*} = 1$. We also have that $\Psi$ is left-shift invariant, since by Claim 3.1.2, we know that for any $r > 0$ we have $\{f - S_r f : f \in L^\infty\} \subseteq Ces_0$. Hence $\Psi(f - S_r f) = 0$, for any $r > 0$ and $f \in L^\infty$.

# Appendix B A different approach for defining the Cesàro averaging operator on $\mathcal{B}(\mathcal{H})$

As it was mentioned in Open Question 4.6.1, we do not know if the operator $h$, from definition 4.3.1, is such that $h : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. We suspect this is not the case, as numerical calculations suggest that for the elements $(C_n)_{n\in\mathbb{N}_0} \subset \mathcal{B}(\mathcal{H})$ in particular, the operator norm of $h(C_n)$ goes to infinity as $n$ goes to infinity, seemingly at the same rate as $\ln(\ln(40n))$. Here $C_n$ is the $n$ by $n$ matrix, corresponding to the Cesàro averaging operator on the finite dimensional space $\mathbb{C}^n$:

$$C_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & & \vdots \\ \vdots & & & & \ddots & \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$$

We turn our attention to trying a different way of defining the Cesàro averaging operator on $\mathcal{B}(\mathcal{H})$, such that the element $C \in \mathcal{B}(\mathcal{H})$ in particular, would have an output back in $\mathcal{B}(\mathcal{H})$, where $C$ is the Cesàro averaging operator on $\ell^\infty$, with matrix representation

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & & 0 & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & & \vdots & & \\ \vdots & & & & \ddots & & & \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & 0 & \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

**Definition B.0.1.** For arbitrary $S \in \mathcal{B}(\mathcal{H})$ with matrix representation $S := (s_{i,j})_{i,j\in\mathbb{N}_0}$ (acting on $\mathcal{H}$ with respect to the usual basis on $\mathcal{H} : e_0, e_1, e_2, \dots$), we define $\tilde{h}S$ by

$$\tilde{h}S := \begin{pmatrix} s_{0,0} & s_{0,1} & s_{0,2} & s_{0,3} & \dots \\ \frac{s_{1,0}}{2} & \frac{s_{0,0}+s_{1,1}}{2} & \frac{s_{0,1}+s_{1,2}}{2} & \frac{s_{0,2}+s_{1,3}}{2} & \\ \frac{s_{2,0}}{3} & \frac{s_{1,0}+s_{2,1}}{3} & \frac{s_{0,0}+s_{1,1}+s_{2,2}}{3} & \frac{s_{0,1}+s_{1,2}+s_{2,3}}{3} & \\ \vdots & & & & \ddots \end{pmatrix}$$

We next explain in more detail how the operator $\tilde{h}$ came to be defined:

Consider the rows of the matrix $S$, but write them as two sided sequences, by adding zeros in the entries indexed by negative integers, and follow this by shifting the entries of the $n$-th row, $n$ positions to the left (recall here $n = 0, 1, 2, \ldots$), as shown below:

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\quad \textit{0th -entry} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow \\
\vec{p^0} := (\ldots, 0, \ldots, 0, \quad 0, \quad 0\ , s_{0,0}, \ s_{0,1}, \ s_{0,2}, \ldots) \\
\vec{p^1} := (\ldots, 0, \ldots, 0, \quad 0, s_{1,0}, \ s_{1,1}, \ s_{1,2}, \ s_{1,3}, \ldots) \\
\vec{p^2} := (\ldots, 0, \ldots, 0, s_{2,0}, \ s_{2,1}, \ s_{2,2}, \ s_{2,3}, \ s_{2,4}, \ldots) \\
\qquad\qquad\qquad\qquad\qquad\quad \vdots
\end{array}
$$

Then we form the sequence of Cesàro averages of $(\vec{p^n})_{n \in \mathbb{N}_0}$ as follows:

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{0th -entry} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\vec{r^0} := (\ldots, 0, \ \ldots, \ 0, \ 0, \ 0, \ s_{0,0}, \ s_{0,1}, \ \ldots) \\
\vec{r^1} := (\ldots, 0, \ \ldots, \ 0, \ 0, \ \frac{s_{1,0}}{2}, \ \frac{s_{0,0}+s_{1,1}}{2}, \ \frac{s_{0,1}+s_{1,2}}{2}, \ \ldots) \\
\vec{r^2} := (\ldots, 0, \ \ldots, \ 0, \ \frac{s_{2,0}}{3}, \ \frac{s_{1,0}+s_{2,1}}{3}, \ \frac{s_{0,0}+s_{1,1}+s_{2,2}}{3}, \ \frac{s_{0,1}+s_{1,2}+s_{2,3}}{3}, \ \ldots) \\
\qquad\qquad\qquad\qquad\qquad\quad \vdots
\end{array}
$$

Next, from each two sided sequence $\vec{r^n}$, we obtain a one sided sequence, by simply dropping all entries before the position $-n$, for each $n \in \mathbb{N}_0$.

The resulted sequences form the rows of $\tilde{h}(S)$.

We are now going to consider a particular type of elements $S \in \mathcal{B}(\mathcal{H})$, with the following property:

$$\begin{aligned} s_{0,0} \geq s_{0,1} \geq & s_{0,2} \geq s_{0,3} \geq \ldots \geq 0 \qquad (*) \\ s_{1,0} \geq s_{1,1} \geq & s_{1,2} \geq s_{1,3} \geq \ldots \geq 0 \\ s_{2,0} \geq s_{2,1} \geq & s_{2,2} \geq s_{2,3} \geq \ldots \geq 0 \\ & \vdots \end{aligned}$$

For example, $S = C$. Here, we make the observation that even though $C \in \mathcal{B}(\mathcal{H})$, it is known that $C \notin \mathcal{K}(\mathcal{H})$.

We next introduce two definitions and a lemma, concerning the concept of decreasing rearrangements [11], that will help us prove that for any $S \in \mathcal{B}(\mathcal{H})$ with property (*), $\tilde{h}(S)$ is back again in $\mathcal{B}(\mathcal{H})$.

**Definition B.0.2.** For every $y \in c_0 = c_0(\mathbb{N}_0, \mathbb{C})$, define

$y_1^* := \max\{|y_n| : n \in \mathbb{N}_0\} = |y_{n_1}|,$ *for some* $n_1 \in \mathbb{N}_0,$ *the smallest integer possible.*

$y_2^* := \max\{|y_n| : n \in \mathbb{N}_0 \setminus \{n_1\}\} = |y_{n_2}|,$ *for some* $n_2 \in \mathbb{N}_0,$ *the smallest integer possible.*

$y_3^* := \max\{|y_n| : n \in \mathbb{N}_0 \setminus \{n_1, n_2\}\} = |y_{n_3}|,$ *for some* $n_3 \in \mathbb{N}_0,$ *the smallest possible.*

Continuing with this process, we obtain

$$y_1^* \geq y_2^* \geq y_3^* \geq \ldots \geq 0$$

**Definition B.0.3.** For $u, v \in c_0$, we write $u^* \preccurlyeq v^*$ whenever

$$\sum_{j=0}^{n} u_j^* \leq \sum_{j=0}^{n} v_j^*, \text{ for all } n \in \mathbb{N}_0.$$

**Lemma B.0.1.** *Let* $u, v, w \in c_0$, *such that* $u^* \preccurlyeq v^*$, *and* $w_0 \geq w_1 \geq w_2 \geq \ldots \geq 0$. *Then*

1. $\displaystyle\sum_{j=0}^{n} |u_j| w_j \leq \sum_{j=0}^{n} u_j^* w_j \leq \sum_{j=0}^{n} v_j^* w_j,$ *for all* $n \in \mathbb{N}_0,$ *and*
2. $\displaystyle\sum_{j=0}^{\infty} |u_j| w_j \leq \sum_{j=0}^{\infty} u_j^* w_j \leq \sum_{j=0}^{\infty} v_j^* w_j.$

**Claim B.0.2.** *For any* $S \in \mathcal{B}(\mathcal{H})$ *with property (*), we have that* $\tilde{h}(S)$ *is in* $\mathcal{B}(\mathcal{H})$.

*Proof.* Consider an arbitrary element $x = (x_0, x_1, x_2, \dots)^T \in \mathcal{H} := \ell^2(\mathbb{N}_0, \mathbb{C})$. Then for any $S \in \mathcal{B}(\mathcal{H})$ we have that

$$\begin{aligned}
\|\tilde{h}(S)x\|_2^2 &= |s_{0,0}x_0 + s_{0,1}x_1 + s_{0,2}x_2 + s_{0,3}x_3 + \dots|^2 \\
&+ \left|\frac{s_{1,0}}{2}x_0 + \frac{s_{0,0}+s_{1,1}}{2}x_1 + \frac{s_{0,1}+s_{1,2}}{2}x_2 + \frac{s_{0,2}+s_{1,3}}{2}x_3 + \dots\right|^2 \\
&+ \left|\frac{s_{2,0}}{3}x_0 + \frac{s_{1,0}+s_{2,1}}{3}x_1 + \frac{s_{0,0}+s_{1,1}+s_{2,2}}{3}x_2 + \frac{s_{0,1}+s_{1,2}+s_{2,3}}{3}x_3 + \dots\right|^2 \\
&+ \left|\frac{s_{3,0}}{4}x_0 + \frac{s_{2,0}+s_{3,1}}{4}x_1 + \frac{s_{1,0}+s_{2,1}+s_{3,2}}{4}x_2 + \frac{s_{0,0}+s_{1,1}+s_{2,2}+s_{3,3}}{4}x_3 + \dots\right|^2 \\
&+ \dots.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\tilde{h}(S)x\|_2^2 \\
&= |s_{0,0}x_0 + s_{0,1}x_1 + s_{0,2}x_2 + s_{0,3}x_3 + \dots|^2 \\
&+ \left|\frac{1}{2}\left(s_{0,0}x_1 + s_{0,1}x_2 + s_{0,2}x_3 + s_{0,3}x_4 + \dots\right) + \frac{1}{2}\left(s_{1,0}x_0 + s_{1,1}x_1 + s_{1,2}x_2 + s_{1,3}x_3 + \dots\right)\right|^2 \\
&+ \left|\frac{1}{3}\left(s_{0,0}x_2 + s_{0,1}x_3 + s_{0,2}x_4 + \dots\right) + \frac{1}{3}\left(s_{1,0}x_1 + s_{1,1}x_2 + s_{1,2}x_2 + \dots\right)\right. \\
&\qquad\qquad \left. + \frac{1}{3}\left(s_{2,0}x_0 + s_{2,1}x_1 + s_{2,2}x_2 + \dots\right)\right|^2 + \dots \\
&\le \left(\sum_{j=0}^{\infty} s_{0,j}|x_j|\right)^2 + \left[\frac{1}{2}\left(\sum_{j=0}^{\infty} s_{0,j}|x_{j+1}| + \sum_{j=0}^{\infty} s_{1,j}|x_j|\right)\right]^2 \\
&\qquad\qquad + \left[\frac{1}{3}\left(\sum_{j=0}^{\infty} s_{0,j}|x_{j+2}| + \sum_{j=0}^{\infty} s_{1,j}|x_{j+1}| + \sum_{j=0}^{\infty} s_{2,j}|x_j|\right)\right]^2 + \dots
\end{aligned}$$

Where the last inequality holds for any $S = (s_{i,j})_{i,j\in\mathbb{N}_0} \in \mathcal{B}(\mathcal{H})$, for which $s_{i,j} \ge 0$ for every $i, j \in \mathbb{N}_0$, e.g. $S \in \mathcal{B}(\mathcal{H})$ with property (*).

Using our hypothesis that $S$ has property (*), we are able to bound the last expression by $\|S\|_{op}^2\|x\|_2^2$ times a positive constant, as follows:

Notice that we clearly have $\sum_{j=0}^{\infty} s_{0,j}|x_j| \le \sum_{j=0}^{\infty} s_{0,j}x_j^*$.

Next, define $R_1x := (x_1, x_2, x_3, \dots)$ and $Q_1x := (0, x_1, x_2, x_3, \dots)$. Then

$$(R_1x)^* = (Q_1x)^*.$$

Also, pointwise we have that $|Q_1x| \leq |x|$. This implies $(R_1x)^* = (Q_1x)^* \preccurlyeq x^*$.

Now, using our hypothesis that $S$ has property (*), and Lemma B.0.1 part 2, we have that

$$\sum_{j=0}^{\infty} s_{0,j}|x_{j+1}| = \sum_{j=0}^{\infty} s_{0,j}|(R_1x)_j| \leq \sum_{j=0}^{\infty} s_{0,j}(R_1x)_j^* \leq \sum_{j=0}^{\infty} s_{0,j}x_j^*.$$

Also

$$\sum_{j=0}^{\infty} s_{1,j}|x_j| \leq \sum_{j=0}^{\infty} s_{1,j}x_j^*.$$

We also define $R_2x := (x_2, x_3, x_4, \ldots)$ and $Q_2x := (0, 0, x_2, x_3, x_4, \ldots)$. Then

$$(R_2x)^* = (Q_2x)^*.$$

Pointwise we have that $|Q_2x| \leq |x|$, which implies $(R_2x)^* = (Q_2x)^* \preccurlyeq x^*$.

By our hypothesis that $S$ has property (*), and Lemma B.0.1 part 2, we have that

$$\sum_{j=0}^{\infty} s_{0,j}|x_{j+2}| = \sum_{j=0}^{\infty} s_{0,j}|(R_2x)_j| \leq \sum_{j=0}^{\infty} s_{0,j}(R_2x)_j^* \leq \sum_{j=0}^{\infty} s_{0,j}x_j^*.$$

Also

$$\sum_{j=0}^{\infty} s_{1,j}|x_{j+1}| \leq \sum_{j=0}^{\infty} s_{1,j}x_j^*,$$

and

$$\sum_{j=0}^{\infty} s_{2,j}|x_j| \leq \sum_{j=0}^{\infty} s_{2,j}x_j^*.$$

Similarly,

$$\sum_{j=0}^{\infty} s_{0,j}|x_{j+3}| \leq \sum_{j=0}^{\infty} s_{0,j}x_j^*,$$
$$\sum_{j=0}^{\infty} s_{1,j}|x_{j+2}| \leq \sum_{j=0}^{\infty} s_{1,j}x_j^*,$$
$$\sum_{j=0}^{\infty} s_{2,j}|x_{j+1}| \leq \sum_{j=0}^{\infty} s_{2,j}x_j^* \text{ and}$$
$$\sum_{j=0}^{\infty} s_{3,j}|x_j| \leq \sum_{j=0}^{\infty} s_{3,j}x_j^*.$$

We continue with this process. So, we obtain

$$\|\tilde{h}(S)\|_2^2 \leq \left(\sum_{j=0}^{\infty} s_{0,j}x_j^*\right)^2 + \left[\frac{1}{2}\left(\sum_{j=0}^{\infty} s_{0,j}x_j^* + \sum_{j=0}^{\infty} s_{1,j}x_j^*\right)\right]^2 + \left[\frac{1}{3}\left(\sum_{j=0}^{\infty} s_{0,j}x_j^* + \sum_{j=0}^{\infty} s_{1,j}x_j^* + \sum_{j=0}^{\infty} s_{2,j}x_j^*\right)\right]^2 + \dots$$

Recall Hardy's Inequality [11]: For all sequences $u = (u_j)_{j\geq 0}$ such that each $u_j \in \mathbb{R}$ or $\mathbb{C}$, we have

$$\sum_{n=0}^{\infty}\left|\frac{1}{n+1}\sum_{m=0}^{n} u_m\right|^2 \leq 4\sum_{k=0}^{\infty}|u_k|^2.$$

Therefore, using $u_m := \sum_{j=0}^{\infty} s_{m,j}x_j^*$, for all $m \in \mathbb{N}_0$; we see that, for all $x = (x_j)_{j\geq 0} \in \mathcal{H}$,

$$\begin{aligned}\|\tilde{h}(S)x\|_2^2 &\leq 4\sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty} s_{k,j}x_j^*\right]^2 \\ &= 4\|Sx^*\|_2^2 \\ &\leq 4\left[\|S\|_{op}\|x^*\|_2\right]^2 \\ &= 4\|S\|_{op}^2\|x\|_2^2.\end{aligned}$$

Thus $\tilde{h}(S) \in \mathcal{B}(\mathcal{H})$ and $\|\tilde{h}(S)\|_{op} \leq 2\|S\|_{op}$. □

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