Post-Newtonian gravitational dynamics from effective field theory

by

Brian Pardo

B.A., Physics, Boston University, 2008

M.S. Physics, University of Pittsburgh, 2017

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This dissertation was presented

by

Brian Pardo

It was defended on

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and approved by

Adam Leibovich, Professor, University of Pittsburgh

Joseph Boudreau, Professor, University of Pittsburgh

Daniel Boyanovsky, Professor, University of Pittsburgh

Ira Rothstein, Professor, Carnegie Mellon University

Andrew Zentner, Professor, University of Pittsburgh

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Gravitational wave astronomy is rapidly maturing as a standard tool to study astrophysics, astronomy, and cosmology. However, to fully realize the promise of gravitational waves, accurate wave templates computed either numerically or using analytical approximations are essential, as exact solutions to the highly nonlinear Einstein field equations from which templates are derived are intractable for realistic systems.

In this thesis, we present progress made in computing observables from an effective field theory framework of gravity applicable to the early stages of binary mergers. We first tackle next-to-leading order spin dependent contributions to the equations of motion in the post-Newtonian expansion, and use these results to derive adiabatically conserved quantities and radiative flux-balance equations, including for energy and angular momentum. We then compute the accumulated phase for quasicircular, spin-aligned orbits, an important gravitational wave observable, including subleading spin-orbit and spin-spin effects. We also derive the linear momentum and center-of-mass frame corrections, and compute the kick velocity for spin-aligned systems. Using these results, we present a mapping to corresponding results computed with more traditional methods and confirm consistency for spin-orbit and spin-spin quantities through next-to-leading order in both approaches.

Finally, we present for the first time the second post-Newtonian corrections to the radiation-reaction equations of motion from the effective field theory approach. Matching between the near-field and far-field zones demonstrates the internal consistency of our results through 4.5 post-Newtonian order for nonspinning objects. Combined with the spin-dependent corrections, these results constitute important progress towards producing theoretical templates for generic orbits entirely within the effective field theory program at a precision needed for current and future gravitational wave detectors.

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Preface

Even during a normal period of history, the completion of a PhD requires a village both inside and outside of academia. But these were not normal times, and I am fortunate to have been supported by so many people who made this work possible.

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I. Introduction

In the years since the groundbreaking first detection of a merging black hole binary in 2015 by the Advanced Laser Interferometer Gravitational-Wave Observatory (LIGO), gravitational wave astronomy has emerged as a versatile and powerful new tool to study astrophysics, astronomy, and cosmology [1]. Analysis of the gravitational waveforms can give answers to important questions inaccessible using other means [2]. For example, they can be used to constrain alternative theories of gravity in the strong-field regime [3], applied as a probe of the stellar evolution and internal structure of binary constituents including neutron stars [4], and can even provide an independent measurement of the Hubble constant [5]. The challenge posed by extracting and interpreting the received signals, however, is daunting and touches an enormous range of scientific disciplines. First and foremost, the disturbances caused by gravitational waves are extremely weak, with typical strain on the order of 10^{-21}. This low amplitude waveform is convolved with a variety of noise sources, which must be modelled and subtracted effectively. Extracting the waveform from the background noise relies on a technique known as matched filtering, which relies on the existence of a large and accurate template bank consisting of theoretical models of gravitational wave (GW) signals from myriad sources [6].

Constructing templates involves solving Einstein's field equations in some numerical or analytic approximation for a particular GW source. While far from the only possible type, one particularly well-studied class of sources is compact object binaries consisting of pairs of black holes, neutron stars, or other dense stellar bodies in bound orbits. The orbiting bodies radiate energy and momentum through the emission of outgoing gravitational waves, which in turn causes the orbit to decay until the progenitor objects ultimately merge or collide (often violently, as in the case of neutron star binaries).

The evolution of compact object binaries can be broken into three distinct but somewhat overlapping phases. The latter stages of the dynamics can be broken into two parts known as the *merger* and the *ringdown*. During the merger phase, the relative motion of the objects becomes highly relativistic until the bodies merge into a single remnant. The full nonlinear structure of general relativity (GR) and sophisticated hydrodynamic description of the source are needed to properly model the dynamics through this process which encompasses the final few cycles of their orbit. Thus, computationally demanding numerical simulations are used through this stage to produce the waveform. Following merger, the remnant dissipates energy through vibrations with characteristic frequency profile known as quasinormal modes until settling into a final state, much like the fading ringing of a bell, for a black hole final state. This can be modeled using dynamical metric perturbations around a Kerr solution to efficiently produce the final stage of the template. For neutron-star–neutron-star mergers or more exotic scenarios, the final stage may be studied with hydrodynamic simulations or advanced numerical techniques, and is particularly sensitive to the neutron star equation of state [7].

This leaves the long initial *inspiral phase* in which the bodies are well-separated and their relative velocity is much less than the speed of light. This phase may cover 100s of cycles from when the binary enters the detector band around ~ 15 Hz through the start of the merger phase for low mass systems on the order of a few solar masses or less. For neutron star binaries in particular, the entire detected waveform consists of the inspiral phase as the merger occurs at frequencies above a few kilohertz, outside the band of sensitivity.[1] The length of this stage makes full GR numerical simulations prohibitively expensive, and thus analytical techniques are necessary. In this regime, techniques relying on a post-Newtonian (PN) approximation enable this analytical treatment of the dynamics, which use the small velocity relative to the speed of light v/c as the expansion parameter [8]. The leading order (LO) terms in the approximation are those pertaining to classical bound Keplerian orbits. Corrections to the leading orbital dynamics are computed order by order and pertain to increasingly subtle effects, including nonlinearities from spacetime curvature, tidal disruptions of the compact objects, spin effects, and emitted gravitational radiation. During this phase of the merger, even seemingly small effects may contribute significantly to the accumulated phase of the gravitational wave signal over many cycles. Thus, these effects must be known to a high order in the PN expansion to produce accurate templates.

[1] For more massive black hole binaries, ground-based detectors may miss much if not all of the inspiral phase as they merge at much lower frequencies closer to the limit of the detector sensitivity.

Traditional methods for computing these higher-order effects in the PN expansion such as direct integration of the field equations have been effective for producing waveforms up to 3.5PN order [9, 10, 11], and significant work has been done to push the envelope in both the spin-independent [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] and spin sector [30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. However, significant challenges arise in these approaches when the point particle approximation breaks down at the 3PN order due to dissipative and nonlocal-in-time effects [17]. Though progress has been made in overcoming these challenges, complementary approaches that corroborate and extend these results have been sought.

In this work, we take an alternative approach first developed by Rothstein and Goldberger [40], treating the system as an effective field theory (EFT) known as nonrelativistic GR (NRGR) and applying the machinery of quantum field theory to a classical system to simplify and systematize the calculations. We start with a point particle approximation coupled to full GR using a Lagrangian formalism guided by symmetry constraints. Higher dimensional operators in the theory may be constructed that conform to these symmetries that represent finite-size or other subleading effects. Power-counting to determine a quantity's PN order is manifest as fields have well-defined scaling within the expansion. Subleading effects such as spin dependence, radiation reaction, or background scattering can be computed from a subset of diagrams at a given order with a transparent physical interpretation.[2]

To this point, much work within the EFT framework has been done to compute effective potentials and radiative multipole moments for binary sources. The conservative dynamics were explored for nonspinning bodies in [40, 71, 72, 73, 74, 75, 76, 77, 78, 74, 79] and for spinning bodies in [80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93]. Additional work has been done in the radiative sector, including radiative multipole moments, tail contributions, absorption, and nonlocal effects, for nonspinning [94, 95, 96, 97, 98, 99, 100, 101, 78, 102] and spinning systems [38, 103, 104, 105, 106]. However, to produce templates it

[2] An alternative expansion known as post-Minkowskian theory approaches the problem relativistically, with the expansion parameter the Newtonian constant G. The post-Minkowskian approach has seen a boom due to the resurgence of scattering amplitude techniques which drastically simplify the computations, e.g. [41, 42]. In parallel with an extension of the EFT approach to the post-Minkowskian regime [43, 44, 45], these approaches have extended the knowledge of the binary dynamics in the conservative sector, both for nonspinning [46, 47, 48, 49, 50, 51, 44, 52, 53, 54, 55] and spinning objects [56, 57, 58, 59, 60, 61, 62, 63, 64, 65]. Radiation effects in the post-Minkowskian expansion have also been studied in, e.g., [66, 67, 68, 69, 70].

is necessary to derive the equations of motion, dynamical invariants, and flux equations used to derive waveforms. The purpose of this thesis is to make progress in both the spin sector and radiation sector on these quantities to produce observables useful to GW experiments, specifically at next-to-leading order (NLO) in the spin sector and next-to-next-to-leading order (NNLO) in radiation reaction. In the process of doing so, we make connection to traditional approaches directly at the level of the equations of motion, radiative multipole moments, and flux-balance laws and ultimately derive contributions to an observable, the accumulated orbital phase of generic compact object binaries.

This thesis is structured as follows, based largely on the work published in [107, 108, 109]. Chapter II provides an overview of the NRGR formalism, starting with scalar objects coupled to gravity and extending to spinning bodies. This section includes a demonstration of the computational machinery at work in computing the 1PN conservative Lagrangian and mass quadrupole moment. In chapter III, we obtain the equations of motion, adiabatic invariants, and flux-balance equations within the NRGR framework for NLO spin-orbit and spin-spin effects in compact object binaries. Using these results, we derive the 3PN accurate orbital phase for quasicircular orbits using an adiabatic approximation. Additionally, we demonstrate agreement with the literature for a broad range of these results through the use of appropriate spin transformations. In chapter IV we present the first calculation of the NNLO radiation-reaction equations of motion, a 4.5PN correction to the leading Newtonian acceleration. We begin the chapter with a review of the nonconservative classical mechanics, and show how this method allows the computation of the 2.5PN and 3.5PN radiation-reaction equations of motion as in [98] before continuing to compute the 4.5PN equations of motion piece by piece in the multipole expansion. Finally, we demonstrate the agreement of these results with the energy flux obtained in the far-zone and with the general-gauge results obtained in [110] in the quasicircular limit. In the appendices, we reproduce for the convenience of the reader the NRGR ingredients necessary for the calculations herein, including Feynman rules, multipole moments, and Lagrangians. The often lengthy coefficients of the results throughout the thesis have also been moved to the appendices for readability. Note that we make extensive use of the Mathematica package xAct [111] to make tractable the involved calculations.

Throughout this work, we use the mostly minus metric signature $\eta^{\alpha\beta} = (1, -1, -1, -1)$, and the Planck mass $m_{\text{pl}} \equiv \sqrt{1/32\pi G}$, where G is Newton's constant. We will employ natural units with $\hbar = c = 1$ throughout where appropriate. Additionally, we use the notation $(ab) \equiv \mathbf{a} \cdot \mathbf{b}$ for dot products and $(abc) \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ for triple-products where concision demands, particularly in the lengthy results sections of the appendices. Latin indices are contracted with the Euclidean metric δ^{ab}, while Greek indices are contracted with the full metric. We define relative coordinates: $\mathbf{x}^i \equiv \mathbf{x}_1^i - \mathbf{x}_2^i \equiv r\mathbf{n}^i$ as the relative position, $\mathbf{v}^i \equiv \mathbf{v}_1^i - \mathbf{v}_2^i$ the relative velocity, and $\mathbf{a}^i \equiv \mathbf{a}_1^i - \mathbf{a}_2^i$ the relative acceleration. We also define the total binary mass $m \equiv m_1 + m_2$, the symmetric mass ratio $\nu \equiv m_1 m_2/m^2$, and the mass differences $\delta m \equiv m_1 - m_2$ and $\delta \equiv \delta m/m$. For spinning objects, we use the combinations $\mathbf{S}^i \equiv \mathbf{S}_1^i + \mathbf{S}_2^i$ and $\mathbf{\Sigma}^i \equiv m(\mathbf{S}_2^i/m_2 - \mathbf{S}_1^i/m_1)$.

II. NRGR

A. Overview

Nonrelativistic general relativity is a Lagrangian EFT formalism for computing gravitational effects in a post-Newtonian expansion. When applied to the binary inspiral problem specifically, we can take advantage of the intrinsic symmetries of the system, namely general coordinate invariance, and the disparate length scales inherent in the problem to significantly simplify and systematize the calculation of quantities of interest to gravitational wave detectors. In this section, we cover the formalism briefly; see [40, 112, 113, 114] for comprehensive reviews of the EFT approach.

In the binary problem, we consider three distinct length scales that allow us to construct a hierarchy of EFTs. These are the compact object scale which we call r_s, the orbital separation we denote r, and the gravitational radiation wavelength λ. Importantly, these length scales are related to the relative velocity v of the binary constituents via the virial theorem by

$$\frac{r_s}{r} \sim v^2, \quad \lambda \sim \frac{r}{v}. \tag{1}$$

The relative velocity v is the power-counting parameter of effective theory which determines the order at which a particular effect enters in the approximation.

We treat the three relevant scales of the problem independently and then connect the scales via a matching procedure. The dynamical degrees of freedom at higher energies or shorter length scales only enter in longer-wavelength EFTs through the coefficients of the long-range theory. This drastically simplifies the computation of observables in the long-range theory while fully incorporating the shorter distance physics. In the problem of compact object binaries during the inspiral phase, the scales can be broken into the following, each with its own EFT description, organized from shortest to longest:

- **Compact object scale** - An isolated compact object coupled to full GR with radius of scale r_s. This incorporates all dynamics of the compact object including absorption, tidal dynamics, and finite-size effects.

- **Orbital scale** - Two point particles coupled to GR separated by radius r.
- **Radiation scale** - Gravitational radiation field coupled to the multipole moments of a single composite source with wavelength λ.

We define a metric $g_{\mu\nu}$ which we expand perturbatively around Minkowski spacetime as

$$g_{\mu\nu} \to \eta_{\mu\nu} + \frac{1}{m_{\rm pl}} h_{\mu\nu} + \cdots \,, \tag{2}$$

where $h_{\mu\nu}$ is the metric perturbation. However, this metric perturbation does not have consistent scaling in our PN parameter. We therefore further decompose this perturbation into two modes, a potential mode $\bar{h}_{\mu\nu}$ and a radiation mode $H_{\mu\nu}$. We can then write the perturbed metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}(x) + \bar{h}_{\mu\nu}(x), \tag{3}$$

The momenta of the potential modes scale as $\partial_0 H_{\mu\nu} \sim (\frac{v}{r}) H_{\mu\nu}$ and $\partial_i H_{\mu\nu} \sim (\frac{1}{r}) H_{\mu\nu}$ while derivatives of the radiation modes scale as $\partial_\alpha \bar{h}_{\mu\nu} \sim (\frac{v}{r})\bar{h}_{\mu\nu}$. This momentum scaling ensures that potential modes are off-shell and constrained to the bound system, while the radiation modes can be physical on-shell "particles" that propagate to infinity. With this definition, we are able to use field modes with consistent scaling within the PN expansion. It is sometimes, though not always, convenient to perform a partial Fourier transformation to make the distinction between the timelike and spacelike components' power counting clear, namely $H_{\mu\nu} = \int dx^0 \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{k}\mu\nu}(x^0)$, where we use the shorthand $\int_{\mathbf{k}} \equiv \int d^3\mathbf{k}/(2\pi)^3$. With this representation in mind, we can now proceed to define the tower of effective theories.

The shortest scale represents the full description of the individual component objects of the binary system. To fully describe the dynamics at this scale, we would need to choose a particular model of the object (a black hole, neutron star, or some other stellar object). Of course, the specifics of this model are of significant interest in practice, but for our purposes we remain model agnostic. The physical behavior of this system will be encoded in coefficients that appear at subsequent scales.

At the orbital scale, we can write the action of two point particles coupled to GR, written schematically as

$$S_{\rm eff} = S_{\rm pp} + S_{\rm EH} + S_{\rm gf}. \tag{4}$$

The point particle action $S_{\rm pp}$ is given by

$$S_{\rm pp} = -\sum_A \left\{ m_A \int d\tau_A + \sum_A c_R^{(A)} \int d\tau_A R(x_A) + \sum_A c_V^{(A)} \int d\tau_A R_{\mu\nu}(x_A) u_A^\mu u_A^\nu + \cdots \right\} \quad (5)$$

where $d\tau_A = \sqrt{g_{\mu\nu} dx_A^\mu dx_A^\nu}$, R is the Ricci scalar and $R_{\mu\nu}$ is the Ricci tensor. In principle, the nonminimal coupling terms with coefficients $c_{R,V}$ represent finite-size effects which could be determined from matching to a particular model of the compact object. However, it can be shown that these terms actually represent redundant operators which can be removed from the theory by an appropriately chosen redefinition of the metric.

The first higher-dimension operators that cannot be removed by a field redefinition are

$$C_{E^2} \int d\tau_A E^{\mu\nu} E_{\mu\nu} \quad \text{and} \quad C_{B^2} \int d\tau_A B^{\mu\nu} B_{\mu\nu}, \quad (6)$$

where $E_{\mu\nu} = C_{\mu\alpha\nu\beta} u^\alpha u^\beta$ and $B_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\alpha\beta\sigma} C^{\alpha\beta}{}_{\nu\rho} u^\sigma u^\rho$ are the electric and magnetic components of the Weyl tensor, respectively. The coefficients C_{E^2,B^2} are the mass and current quadrupole Love numbers which can be computed by matching to an underlying model of the compact object as discussed above. These are so-called Wilson coefficients of the effective theory. These operators contribute corrections at the 5PN order, as we discuss in section II.D, and thus will not be relevant for our purposes in this thesis.

The second term, the Einstein–Hilbert action, can be written as

$$S_{EH} = -2m_{\rm Pl}^2 \int d^4x \sqrt{g}\, R(x). \quad (7)$$

Finally, the gauge is a choice; we use linearized harmonic gauge, for which the gauge fixing term is

$$S_{\rm gf} = m_{\rm Pl}^2 \int d^4x \sqrt{\bar{g}}\, \Gamma_\mu \Gamma^\mu, \quad (8)$$

with $\Gamma_\mu = D_\alpha H_\mu^\alpha - \frac{1}{2} D_\mu H_\alpha^\alpha$, where D_μ is the covariant derivative derived from the background metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$. The choice of linearized harmonic gauge, in particular, is made to preserve symmetries across scales in the effective theory. Upon integrating out the potential modes to match onto the long range theory, the action remains invariant under general coordinate transformations of the background metric $\bar{h}$ in this gauge. We can interpret the field H as high-frequency potential modes propagating in a slowly varying background field $\bar{h}$.

At this point, we have the full Lagrangian for nonspinning compact object binaries coupled to GR at the orbital scale. We will tackle the additional complication of spin in section II.C. The next task is to integrate out the nonpropagating potential modes constrained to the orbital scale and match onto a radiation theory. We do this by integrating over the dynamical H modes as

$$e^{iS_{\rm rad}[x_a,\bar{h}]} = \int DH_{\mu\nu} e^{iS[\bar{h}+H,x_A]} e^{iS_{\rm gf}}. \tag{9}$$

Note that this is a classical description akin to solving for the potential modes perturbatively. This yields an effective theory with dynamical degrees of freedom consisting of radiation modes and worldline coordinates for the compact objects. In order to ensure a consistent power-counting in the radiation field, we must Taylor expand around a point, typically chosen as the binary's center of mass (COM) [115]. This localizes the interaction of the radiation field to a single worldline which we set at rest at the origin for simplicity. To linear order in the radiation field, the couplings in this theory will take the form of

$$S = -\frac{1}{2m_{\rm pl}} \int d^4x\, T^{\mu\nu} \bar{h}_{\mu\nu} = -\frac{1}{2m_{\rm pl}} \int dt \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3\mathbf{x} T^{\mu\nu}(t,\mathbf{x}) \mathbf{x}^N \left(\partial_N h_{\mu\nu}\right)(t,0). \tag{10}$$

The long-range theory can be written as a decomposition of the stress-energy pseudotensor $T^{\mu\nu}$ into multipole moments which then couple to the physical radiation modes. The details of this approach can be found in [94, 99], and we will simply present the results here.

The multipole expansion of the action, where the mass-type multipole moments I^L and current-type moments J^L couple to derivatives of the electric and magnetic components of the Weyl tensor, respectively, can be written as

$$\begin{aligned} S_{\rm source} &= -\frac{1}{2m_{\rm pl}} \int dt \left[M h_{00} + 2\mathbf{P}^i h_{0i} + M\mathbf{X}^i \partial_i h_{00} + \mathbf{L}^i \epsilon_{ijk} \partial_j h_{0k} \right] \\ &+ \int dt \sum_{\ell=2}^{\infty} \frac{1}{\ell!} I^L \partial_{L-2} E_{k_{\ell-1}k_\ell} - \int dt \sum_{\ell=2}^{\infty} \frac{2\ell}{(\ell+1)!} J^L \partial_{L-2} B_{k_{\ell-1}k_\ell}. \end{aligned} \tag{11}$$

The multipole moments act as Wilson coefficients of the effective theory, and can be determined through a matching procedure using equations (4) and (10). In practice, we do this by computing single radiation mode emission amplitudes from the orbital scale theory and match to corresponding amplitudes in the long-range theory. It is important to note that the multipole moments do not have uniform power counting, and so must be written term-by-term in the post-Newtonian expansion. For example, the mass quadrupole term can be written as $I^{ij} = I^{ij}_{0\mathrm{PN}} + I^{ij}_{1\mathrm{PN}} + I^{ij}_{1.5\mathrm{PN}} + \cdots$.

The multipole moments can be determined by comparing (10) with (11) through extensive use of integration-by-parts, the field equations of motion, and the Ward identity. Then we have the quantities

$$M = \int d^3\mathbf{x} T^{00}, \tag{12}$$

$$\mathbf{P}^i = \int d^3\mathbf{x} T^{0i}, \tag{13}$$

$$M\mathbf{X}^i = \int d^3\mathbf{x} T^{00}\mathbf{x}^i, \tag{14}$$

$$\mathbf{L}^i = -\int d^3\mathbf{x} \epsilon^{ijk} T^{0j}\mathbf{x}^k, \tag{15}$$

and the multipole moments can be computed via

$$\begin{aligned} I^L =& \sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(1+\frac{8p(\ell+p+1)}{(\ell+1)(\ell+2)}\right)\left[\int d^3\mathbf{x}\partial_0^{2p}T^{00}(t,\mathbf{x})\mathbf{x}^{2p}\mathbf{x}^L\right]_{\mathrm{STF}} \\ &+\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(1+\frac{4p}{(\ell+1)(\ell+2)}\right)\left[\int d^3\mathbf{x}\partial_0^{2p}T^{kk}(t,\mathbf{x})\mathbf{x}^{2p}\mathbf{x}^L\right]_{\mathrm{STF}} \\ &-\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(\frac{4}{\ell+1}\right)\left(1+\frac{2p}{\ell+2}\right)\left[\int d^3\mathbf{x}\partial_0^{2p+1}T^{0m}(t,\mathbf{x})\mathbf{x}^{2p}\mathbf{x}^{mL}\right]_{\mathrm{STF}} \\ &+\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(\frac{2}{(\ell+1)(\ell+2)}\right)\left[\int d^3\mathbf{x}\partial_0^{2p+2}T^{mn}(t,\mathbf{x})\mathbf{x}^{2p}\mathbf{x}^{mnL}\right]_{\mathrm{STF}} \end{aligned} \tag{16}$$

$$\begin{aligned} J^L =& \sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(1+\frac{2p}{\ell+2}\right)\left[\int d^3\mathbf{x}\epsilon^{k_\ell mn}\partial_0^{2p}T^{0m}(t,\mathbf{x})\mathbf{x}^{2p}\mathbf{x}^{nL-1}\right]_{\mathrm{STF}} \\ &-\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(\frac{1}{\ell+2}\right)\left[\int d^3\mathbf{x}\epsilon^{k_\ell mr}\partial_0^{2p+1}T^{mn}(t,\mathbf{x})\mathbf{x}^{2p}\mathbf{x}^{nrL-1}\right]_{\mathrm{STF}}. \end{aligned} \tag{17}$$

Finally, we can use equation (10) and the partial Fourier transform of the stress-energy pseudotensor

$$T^{\mu\nu}(t,\mathbf{k}) = \int d^3\mathbf{x} T^{\mu\nu}(t,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{18}$$

to extract the stress-energy pseudotensor moments. We take the long-wavelength limit $\mathbf{k} \to 0$ of equation (18) and find

$$T^{\mu\nu}(t,\mathbf{k}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\int d^3\mathbf{x} T^{\mu\nu}(t,\mathbf{x}) \mathbf{x}^{i_1} \cdots \mathbf{x}^{i_n} \right) \mathbf{k}_{i_1} \cdots \mathbf{k}_{i_n}, \tag{19}$$

which can be computed from one-point diagrams with a single radiation mode in the orbital-scale theory. We now turn to computing quantities of interest within the effective theory as a demonstration of the procedure.

B. Computing with NRGR

1. Effective Lagrangian

We will begin from the full point particle action defined in equation (4), which we expand to order v^2 to recover 1PN corrections. At this order, the corrections are conservative and are computed from terms containing only potential modes.

It is simplest to expand each term in the action separately. Starting with the point particle term, expanding the square root of the interval yields

$$\begin{aligned} L_{\rm pp} &= -\sum_a m_a \sqrt{1 - \mathbf{v}_a^2 + h_{00} + 2h_{0i}\mathbf{v}_{ai} + h_{ij}\mathbf{v}_a^i \mathbf{v}_a^j} \\ &\approx \sum_a m_a \big[\tfrac{1}{2}\mathbf{v}_a^2 + \tfrac{1}{8}\mathbf{v}_a^4 - \tfrac{1}{2}h_{00} - \tfrac{1}{4}\mathbf{v}_a^2 h_{00} + \tfrac{1}{8}{h_{00}}^2 - h_{0i}\mathbf{v}_a^i \\ &\qquad + \tfrac{1}{2}h_{00}h_{0i}\mathbf{v}_a^i - \tfrac{1}{2}h_{ij}\mathbf{v}_a^i\mathbf{v}_a^j + \tfrac{1}{4}h_{00}h_{ij}\ \mathbf{v}_a^i\mathbf{v}_a^j + \cdots \big] \end{aligned} \tag{20}$$

We can do the same thing with the other terms in the Lagrangian, the Einstein–Hilbert action and the gauge fixing term. The sum of the expanded Einstein–Hilbert action and the gauge fixing term determines the two-point function for the internal fields as well as the 3-point vertex. We find that the $\mathcal{O}(H^2)$ term is

$$L_{H^2} = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[k^2 H_{\mathbf{k}\mu\nu} H^{\mu\nu}_{-\mathbf{k}} - \frac{k^2}{2} H_{\mathbf{k}} H_{-\mathbf{k}} \right], \tag{21}$$

which importantly does not scale uniformly in velocity due to the combination of timelike and spacelike derivatives. The three-point vertex can be extracted by a similar expansion of the EH action and gauge-fixing term from the $\mathcal{O}(H^3)$ terms, though it is not reproduced here for concision but can be found in [40].

From the $\mathcal{O}(H^2)$ Lagrangian, we find that the propagator for the field is

$$\langle H_{\mathbf{k}\mu\nu}(x) H_{-\mathbf{k}\alpha\beta}(y)\rangle = P_{\mu\nu;\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik\cdot x}}{k^2 + i\epsilon} \tag{22}$$

where $P_{\mu\nu;\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right]$. Due to the scalings $k^0 \sim v/r$ and $\mathbf{k} \sim 1/r$, we can expand the propagator schematically with consistent power-counting to yield

$$\frac{i}{(k^0)^2 - \mathbf{k}^2} = -\frac{i}{\mathbf{k}^2}\left(1 + \frac{(k^0)^2}{\mathbf{k}^2} + \cdots\right) \tag{23}$$

which can be truncated at a given order in the PN expansion. Alternatively, the expansion can be done at the level of the $\mathcal{O}(H^2)$ action, with subleading H^2 operators entering through unique diagrams as kinetic corrections to the propagator. The LO propagator can then be written as

$$\langle H_{\mu\nu}(t,\mathbf{k}) H_{\alpha\beta}(t',\mathbf{q})\rangle = -i(2\pi)^3 P_{\mu\nu;\alpha\beta}\delta(t-t')\delta^3(\mathbf{k}+\mathbf{q})\frac{1}{\mathbf{k}^2}, \tag{24}$$

where the angle brackets represent a Green's function for the fields. We take the former approach here, with the latter approach taken in [40]. We now have all of the elements necessary to calculate the Feynman diagrams for the $\mathcal{O}(v^2)$ corrections corresponding to the Einstein–Infeld–Hoffmann (EIH) Lagrangian.

2. Calculation of the Newtonian and EIH Lagrangian

We start by computing the LO effective Lagrangian, which is simply the Newtonian potential. To determine the PN order in which a diagram enters, we will need an understanding of the scaling of objects in the EFT, discussed in detail in section II.D. The diagram we need to compute the leading potential comes from two insertions of the operator $-\frac{m_A}{2m_{\rm pl}}\int dt \int_{\mathbf{k}} H_{\mathbf{k}00}(x)$. This yields

$$i\mathcal{A} = -\frac{m_A m_B}{4m_{\rm pl}^2}\int dt_A \int dt_B \langle H_{\mathbf{k}00}(x_A) H_{\mathbf{q}00}(x_B)\rangle$$

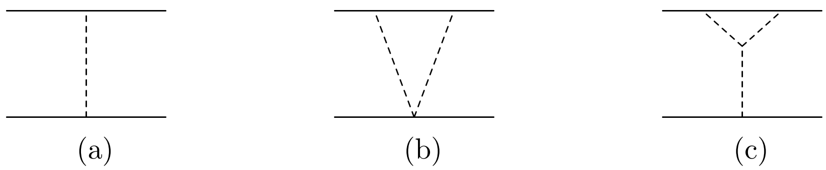


Figure 1: Topologies contributing to the 1PN conservative Lagrangian

$$= -\frac{m_A m_B}{4m_{\rm pl}^2}\int dt_A \int dt_B P_{00;00}\int_k \frac{ie^{-ik\cdot x_{AB}}}{k^2+i\epsilon}, \tag{25}$$

with $P_{00;00} = 1/2$, $x_{AB} \equiv x_A - x_B$, and $\int_k \equiv \int d^4k/(2\pi)^4$. Integrating over k_0 after expanding the denominator yields a delta function over the time difference $t_{AB} \equiv t_A - t_B$. Integrating over t_B and then performing the integral over $\mathbf{k}$ gives

$$i\mathcal{A} = \frac{im_A m_B}{8m_{\rm pl}^2}\int dt \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}_{AB}}}{\mathbf{k}^2} = i\int dt\, \frac{Gm_A m_B}{|\mathbf{x}_{AB}|}, \tag{26}$$

where we use the integral formula

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{(\mathbf{k}^2)^\alpha}e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)}\left(\frac{\mathbf{x}^2}{4}\right)^{\alpha-d/2} \tag{27}$$

to perform the integration over momentum $\mathbf{k}$. This is simply the Newtonian potential between two point masses.

We next calculate the $\mathcal{O}(v^2)$ corrections to the LO Newtonian potential, i.e., the EIH potential. There are 3 topologies that contribute at this order, shown in figure 1, though each topology may have multiple diagrams due to the power-counting at each vertex multiplicity.

The first topology we consider is shown in figure 1a, and has three associated diagrams. The first is the same as the Newtonian potential but with only the NLO term of the expanded propagator as in equation (23), which leads to

$$i\mathcal{A} = \frac{im_A m_B}{8m_{\rm pl}^2}\int dt_A \int dt_B \int_{k_0} k_0^2 e^{-ik_0 t_{AB}} \int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}_{AB}}}{\mathbf{k}^4}$$

$$= \frac{im_A m_B}{8m_{\rm pl}^2} \int dt_A \int dt_B \frac{d^2}{dt_A dt_B} \left(\int \frac{dk_0}{2\pi} e^{-ik_0 t_{AB}} \right) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}_{AB}}}{\mathbf{k}^4}. \tag{28}$$

The integral over k_0 yields a delta function over t_{AB}; we integrate by parts twice and complete the integral over the spatial momentum using equation (27). Then taking the time derivatives of the result and integrating over t_A we find

$$i\mathcal{A} = i \int dt \frac{Gm_A m_B}{2|\mathbf{x}_{AB}|} [\mathbf{v}_A \cdot \mathbf{v}_B - (\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB})] \tag{29}$$

The remaining two diagrams in this topology can be computed in a straightforward manner as in the derivation of the Newtonian gravitational potential above. The first of these, constructed from two insertions of the $\mathcal{O}(v^1)$ matter vertex yields

$$i\mathcal{A} = -\frac{m_A m_B}{m_{\rm pl}^2} \int dt_A \int dt_B \mathbf{v}_1^i \mathbf{v}_2^j \langle H_{\mathbf{k}0i}(x_A) H_{\mathbf{q}0j}(x_B)\rangle = -i \int dt\, (\mathbf{v}_1 \cdot \mathbf{v}_2) \frac{4Gm_A m_B}{|\mathbf{x}_{AB}|}. \tag{30}$$

The second remaining diagram is constructed from one LO matter vertex and one $\mathcal{O}(v^2)$ vertex, which gives the contribution

$$\begin{aligned} i\mathcal{A} &= -\frac{m_A m_B}{4m_{\rm pl}^2} \int dt_A \int dt_B \left[\mathbf{v}_A^i \mathbf{v}_A^j \langle H_{\mathbf{k}ij}(x_A) H_{\mathbf{q}00}(x_B)\rangle + \tfrac{1}{2}\mathbf{v}_A^2 \langle H_{\mathbf{k}00}(x_A) H_{\mathbf{q}00}(x_B)\rangle\right] \\ &= i \int dt\, \frac{3Gm_A m_B \mathbf{v}_A^2}{2|\mathbf{x}_{AB}|} \end{aligned} \tag{31}$$

Next we compute the "seagull" diagram in figure 1b, which consists of the leading nonlinear matter vertex and two insertions of the leading linear matter vertex. Then our amplitude is given by

$$\begin{aligned} \text{Fig. 3} &= \frac{im_A^2 m_B}{32m_{\rm pl}^4} \int dt_A \int dt_{A'} \int dt_B \langle H_{\mathbf{k}00}(x_A) H_{\mathbf{k}'\mu\nu}(x_B)\rangle \langle H_{\mathbf{q}00}(x_{A'}) H_{\mathbf{q}'\alpha\beta}(x_B)\rangle \\ &= \frac{im_A^2 m_B}{32m_{\rm pl}^4} \int dt_A \int dt_{A'} \int dt_B (P_{00;00})^2 \int_k \frac{e^{-ik\cdot x_{AB}}}{k^2 + i\epsilon} \int_q \frac{e^{-iq\cdot x_{A'B}}}{q^2 + i\epsilon} \\ &= \frac{i}{2} \int dt\, \frac{G^2 m_A^2 m_B}{|\mathbf{x}_{AB}|^2}, \end{aligned} \tag{32}$$

where we keep only the leading term in the expanded propagator.

The final diagram is shown in figure 1c and consists of a three-point gravitational vertex and three insertions of the LO linear matter vertex. The graviton vertex can be computed in a straightforward manner and is given by [78]

$$\langle T\{H^{00}_{\mathbf{k}_1}H^{00}_{\mathbf{k}_2}H^{00}_{\mathbf{k}_3}\}\rangle = -\frac{(2\pi)^3}{4m_{\rm pl}}\delta(t_2-t_1)\delta(t_3-t_1)\delta^3(\textstyle\sum_i \mathbf{k}_i)\frac{\sum_i \mathbf{k}_i^2}{\prod_i \mathbf{k}_i^2}. \tag{33}$$

With this in hand, we can now compute the diagram, where the vertices with subscripts A and A' lie on the same worldline, while the vertex with subscript B lies on the other. Thus, we have

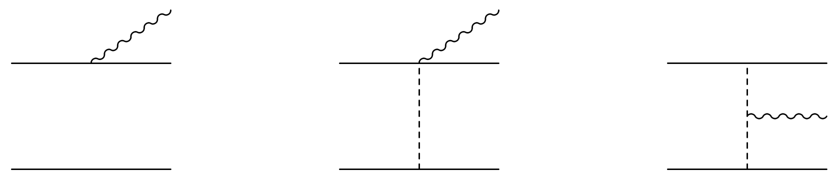
$$\begin{aligned}
i\mathcal{A} &= -\frac{im_A^2 m_B}{16m_{\rm pl}^3}\int dt_A\int dt_{A'}\int dt_B\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} e^{i\sum_i \mathbf{k}_i\cdot\mathbf{x}_i}\langle T\{H^{00}_{\mathbf{k}_1}H^{00}_{\mathbf{k}_2}H^{00}_{\mathbf{k}_3}\}\rangle \\
&= -\frac{im_A^2 m_B}{16m_{\rm pl}^3}\int_{t_{A,A',B}}\int_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} e^{i\sum_i \mathbf{k}_i\cdot\mathbf{x}_i}\left[-\frac{(2\pi)^3}{4m_{\rm pl}}\delta\left(t_B-t_A\right)\delta\left(t_C-t_A\right)\delta^3(\textstyle\sum_i \mathbf{k}_i)\frac{\sum_i \mathbf{k}_i^2}{\prod_i \mathbf{k}_i^2}\right] \\
&= -\frac{im_A^2 m_B}{64m_{\rm pl}^4}\int dt\left[\int_{\mathbf{k}_2\mathbf{k}_3}\frac{e^{-i(\mathbf{k}_2\cdot\mathbf{x}_{AA'}+\mathbf{k}_3\cdot\mathbf{x}_{AB})}}{\mathbf{k}_2^2\mathbf{k}_3^2}+\int_{\mathbf{k}_1\mathbf{k}_3}\frac{e^{-i(\mathbf{k}_1\cdot\mathbf{x}_{AA'}+\mathbf{k}_3\cdot\mathbf{x}_{A'B})}}{\mathbf{k}_1^2\mathbf{k}_3^2}+\int_{\mathbf{k}_1\mathbf{k}_2}\frac{e^{-i(\mathbf{k}_1\cdot\mathbf{x}_{AB}+\mathbf{k}_2\cdot\mathbf{x}_{A'B})}}{\mathbf{k}_1^2\mathbf{k}_2^2}\right] \\
&= -\frac{im_A^2 m_B}{1024\pi^2 m_{\rm pl}^4}\int dt\left[\frac{1}{|\mathbf{x}_{AA'}||\mathbf{x}_{AB}|}+\frac{1}{|\mathbf{x}_{AA'}||\mathbf{x}_{A'B}|}+\frac{1}{|\mathbf{x}_{AB}||\mathbf{x}_{A'B}|}\right]
\end{aligned} \tag{34}$$

The first two terms in the final line are linear divergences, which vanish in dimensional regularization, while the finite final contribution is, after relabeling, given by

$$i\mathcal{A} = -i\int dt\,\frac{G^2 m_A^2 m_B}{|\mathbf{x}_{AB}|^2}. \tag{35}$$

Combining our results at the 1PN order, we find that the full EIH Lagrangian is given by

$$L_{\rm EIH} = \sum_A \frac{m_A\mathbf{v}_A^4}{8}+\frac{Gm_Am_B}{2r}\left[3(\mathbf{v}_A^2+\mathbf{v}_B^2)-7(v_Av_B)-(v_An_{AB})(v_Bn_{AB})-\frac{Gm}{r}\right]. \tag{36}$$



(a) (b) (c)

Figure 2: Topologies needed to compute T_{00}, T_{0i}, and T_{kk} for the mass quadrupole correction through 1PN order.

3. Mass quadrupole through 1PN

We can extract the stress-energy pseudotensor $T^{\mu\nu}(x,t)$ from a matching calculation by integrating out potential modes in the full Lagrangian. The action can be written in the form

$$\Gamma(\bar{h}) = -\frac{1}{2m_{\rm pl}}\int dt\, T^{\mu\nu}(\mathbf{x},t)\bar{h}_{\mu\nu}(\mathbf{x},t). \tag{37}$$

Then using the expanded Fourier transform (19), we can read off the components of the stress-energy tensor needed to compute the multipole moments.

The LO mass quadrupole moment and the first diagram contributing to the 1PN mass quadrupole can be computed from a single insertion of the vertex

$$L_{\bar{h}} = -\frac{1}{2m_{\rm pl}}(1+\tfrac{1}{2}\mathbf{v}^2)\bar{h}_{00}.$$

Then the one-point function is given by

$$\mathcal{A} = -\frac{m_A}{2m_{\rm pl}}\int dt\,(1+\tfrac{1}{2}\mathbf{v}_A^2)e^{-i\mathbf{q}\cdot\mathbf{x}_A}\bar{h}_{00}(t,\mathbf{q})$$

and comparing with equation (37) we find the contribution to the stress-energy tensor is

$$T^{00}(t,\mathbf{q}) = \sum_{A\neq B} m_A(1+\tfrac{1}{2}\mathbf{v}_A^2)e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{38}$$

The second diagram is computed from the leading linear vertex and the leading quadratic vertex, given by

$$L_H = -\frac{1}{2m_{\rm pl}} H_{00},$$
$$L_{H\bar{h}} = \frac{m}{4m_{\rm pl}^2} H_{00}\bar{h}_{00}.$$

Then the amplitude is given by

$$\mathcal{A}_{S^{ij}} = -\frac{im_A m_B}{8m_{\rm pl}^2}\int dt_A \int dt_B \,\langle H_{\mathbf{k}00}\left(x_A\right) H_{\mathbf{q}00}\left(x_B\right)\rangle\, e^{-i\mathbf{q}\cdot\mathbf{x}_A}\bar{h}_{00}(t,\mathbf{q})$$
$$= \int dt\, \frac{Gm_A m_B}{|\mathbf{x}_{AB}|} e^{-i\mathbf{q}\cdot\mathbf{x}_A}\bar{h}_{00}(t,\mathbf{q}). \tag{39}$$

leading to the contribution to T^{00} of

$$T^{00}(t,\mathbf{q}) = \sum_{A\neq B} \frac{Gm_A m_B}{|\mathbf{x}_{AB}|} e^{-i\mathbf{q}\cdot\mathbf{x}_A} \tag{40}$$

The final diagram is computed from the gravitational 3-point vertex and 2 insertions of the leading linear matter vertex. The diagram is given by

$$\mathcal{A} = -\frac{im_A m_B}{4m_{\rm pl}^2}\int dt_A \int dt_B \int d^4y\, \langle T\{H^{00}(x_A)H^{00}(x_B)iS_{HH\bar{h}}(y)\}\rangle \tag{41}$$
$$= \frac{im_A m_B}{8m_{\rm pl}^3}\int dt\, e^{i\mathbf{q}\cdot\mathbf{x}_B}\int_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}_{AB}}F^{\langle H^{00}H^{00}\rangle}[\mathbf{k},\mathbf{q},\bar{h}^{00}]}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2} \tag{42}$$
$$= \frac{3im_A m_B}{32m_{\rm pl}^3}\int dt\, e^{i\mathbf{q}\cdot\mathbf{x}_B}\bar{h}_{00}\int_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}_{AB}}(\mathbf{k}^2+\mathbf{k}\cdot\mathbf{q})}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2}, \tag{43}$$

where we use the vertex function

$$F^{\langle H^{00}H^{00}\rangle}[\mathbf{k},\mathbf{q},\bar{h}^{00}] = \frac{3}{4}(\mathbf{k}^2+\mathbf{k}\cdot\mathbf{q})\,\bar{h}^{00}. \tag{44}$$

We can perform the integral over $\mathbf{k}$ by completing the square and discarding the trace term. We then integrate using the formula (27) to obtain

$$\mathcal{A} = \frac{1}{2m_{\rm pl}}\int dt\, \frac{3Gm_A m_B}{2|\mathbf{x}_{AB}|} e^{-i\mathbf{q}\cdot\mathbf{x}_A}\bar{h}_{00} \tag{45}$$

where we have taken $\mathbf{q} \to -\mathbf{q}$ in equation (43) for outgoing $\mathbf{q}$. Then

$$T^{00}(t, \mathbf{q}) = -\sum_{a \neq b} \frac{3G m_A m_B}{2|\mathbf{x}_{AB}|} e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{46}$$

We next compute the expression for T^{kk}, taking advantage of results from the previous calculation. The LO expression comes from a single insertion of the vertex

$$L = -\frac{m_A}{2m_{\rm pl}} \mathbf{v}_A^i \mathbf{v}_A^j \bar{h}^{ij}, \tag{47}$$

from which we find

$$T^{kk}(t, \mathbf{q}) = \sum_{A \neq B} m_A \mathbf{v}_A^2 e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{48}$$

The second diagram is easily computed by noting that the vertex function is given by

$$F^{\langle H^{00} H^{00} \rangle}[\mathbf{k}, \mathbf{q}, \bar{h}^{kk}] = \frac{1}{4}(\mathbf{k}^2 + \mathbf{k}\cdot\mathbf{q})\, \bar{h}^{kk} = \frac{1}{3} \times F^{\langle H^{00} H^{00} \rangle}[\mathbf{k}, \mathbf{q}, \bar{h}^{00}] \tag{49}$$

and thus the second contribution is

$$T^{kk}(t, \mathbf{q}) = -\sum_{a \neq b} \frac{G m_A m_B}{2|\mathbf{x}_{AB}|} e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{50}$$

Lastly, we need the expression for T^{0i} at leading order, which is computed from a single insertion of

$$L = \frac{m_A}{m_{\rm pl}} \mathbf{v}^i \bar{h}^{0i} \tag{51}$$

which gives the contribution

$$T^{0i}(t, \mathbf{q}) = \sum m_A \mathbf{v}^i e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{52}$$

Thus we have the following expressions for the components of the stress-energy pseudotensor through 1PN:

$$T^{00}(t, \mathbf{q}) = \sum_{A \neq B} m_A \left(1 + \frac{1}{2}\mathbf{v}_A^2 - \sum_{a \neq b} \frac{3G m_B}{2|\mathbf{x}_{AB}|}\right) e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{53}$$

$$T^{0i}(t, \mathbf{q}) = \sum_{A \neq B} m_A \mathbf{v}^i e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{54}$$

$$T^{kk}(t, \mathbf{q}) = \sum_{a \neq b} m_A \left(\mathbf{v}_A^2 - \frac{G m_B}{2|\mathbf{x}_{AB}|}\right) e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{55}$$

To compute the 1PN correction to the mass quadrupole, we need the moments derived from the expressions (16) and (19), and particularly the moments

$$\int d^3\mathbf{x}\, T^{00}\left[\mathbf{x}^i\mathbf{x}^j\right]^{\mathrm{TF}} = \sum_{A\neq B} m_A\left(1+\frac{1}{2}\mathbf{v}_A^2 - \sum_{a\neq b}\frac{3Gm_B}{2|\mathbf{x}_{AB}|}\right)\left[\mathbf{x}_A^i\mathbf{x}_A^j\right]^{\mathrm{TF}}, \tag{56}$$

$$\int d^3\mathbf{x}\, T^{kk}\left[\mathbf{x}^i\mathbf{x}^j\right]^{\mathrm{TF}} = \sum_{A\neq B} m_A\left(\mathbf{v}_A^2 - \frac{Gm_B}{2|\mathbf{x}_{AB}|}\right)\left[\mathbf{x}_A^i\mathbf{x}_A^j\right]^{\mathrm{TF}}, \tag{57}$$

$$\int d^3\mathbf{x}\, T^{0k}\mathbf{x}^k\left[\mathbf{x}^i\mathbf{x}^j\right]^{\mathrm{TF}} = \sum_{A} m_A(\mathbf{v}_A\cdot\mathbf{x}_A)\left[\mathbf{x}_A^i\mathbf{x}_A^j\right]^{\mathrm{TF}}, \tag{58}$$

$$\int d^3\mathbf{x}\, T^{00}\mathbf{x}^2\left[\mathbf{x}^i\mathbf{x}^j\right]^{\mathrm{TF}} = \sum_{A} m_A\left[\mathbf{x}_A^i\mathbf{x}_A^j\right]^{\mathrm{TF}}, \tag{59}$$

which produce the 1PN mass quadrupole moment given by

$$I^{ij} = \sum_{A\neq B} m_A\left(1+\frac{3}{2}\mathbf{v}_A^2 - \sum_{a\neq b}\frac{Gm_B}{|\mathbf{x}_{AB}|}\right)\left[\mathbf{x}_A^i\mathbf{x}_A^j\right]^{\mathrm{TF}} + \frac{11}{42}\sum_A m_A\frac{d^2}{dt^2}\left(\mathbf{x}_A^2\left[\mathbf{x}_A^i\mathbf{x}_A^j\right]^{\mathrm{TF}}\right)$$
$$-\frac{4}{3}\sum_A m_A\frac{d}{dt}\left((\mathbf{x}_A\cdot\mathbf{v}_A)\left[\mathbf{x}_A^i\mathbf{x}_A^j\right]^{\mathrm{TF}}\right) + \mathcal{O}(v^4). \tag{60}$$

C. Spin in NRGR

1. Point-particle action

The point-particle action in the nonspinning case can be modified to incorporate spin following the approach of Porto (see [80, 81, 82] for details). To introduce rotating degrees of freedom, we define a tetrad field e^I_μ in the corotating frame of the compact object, with

$$\eta^{IJ} = e^I_\mu e^J_\nu g^{\mu\nu}, \qquad \eta_{IJ}e^I_\mu e^J_\nu = g_{\mu\nu}. \tag{61}$$

where Latin indices transform under a residual Lorentz symmetry. Perturbing the metric, the tetrad can be written as

$$e^a_\mu = \delta^a_\mu + \tfrac{1}{2}\delta^\nu_a(h^\nu_\mu - \tfrac{1}{4}h^\nu_\alpha h^\alpha_\mu) + \cdots \tag{62}$$

which will allow the obtaining of the Feynman rules from the action. The point-particle action is given by

$$S_{\rm pp} = \sum_{A=\{1,2\}} \int \mathcal{R}_A d\sigma_A \,, \tag{63}$$

with σ_A an affine parameter, which for the case of spinning bodies is written in terms of a Routhian,

$$\mathcal{R}_A = -\left[m_A\sqrt{v_A^2} + \frac{1}{2}\omega_{\mu ab}S_A^{ab}v_A^\mu + \frac{1}{2m_A}R_{\nu cab}S_A^{ab}v_A^\nu S_A^{cd}v_{Ad} - \frac{C_{ES^2}^{(A)}}{2m_A}\frac{E_{ab}}{\sqrt{v_A^2}}S_A^{ac}S_{Ac}{}^{b} + \cdots \right], \tag{64}$$

displayed here to quadratic order in the spins as needed in this work, where $\omega_\mu^{ab} \equiv e_\nu^b D_\mu e^{a\nu}$ are the Ricci rotation coefficients with covariant derivative D. The spin tensor, $S^{\mu\nu}$, has been projected onto a frame that is locally-flat with respect to each compact body and is described by a tetrad field e_a^μ, such that $S^{ab} \equiv e_\mu^a e_\nu^b S^{\mu\nu}$.

There are also two curvature-dependent terms in the Routhian. The first ensures the preservation of the covariant spin supplementary conditions (SSC). The antisymmetric spin tensor $S^{\mu\nu}$ has six degrees of freedom, while the physical spin vector has only three. The SSC are additional constraints that reduce these nonphysical degrees of freedom. In this work, we use the covariant SSC defined by

$$S^{\mu\nu}p_\mu = 0, \tag{65}$$

which to the order we work here is simply

$$S_A^{ab}v_{Ab} = S_A^{ab}e_b^\mu v_{A\mu} = 0 + \mathcal{O}(S^3)\,. \tag{66}$$

The SSC and spin vector definitions will be discussed further in the subsequent section.

The second curvature-dependent term in the Routhian depending on the electric component of the Weyl tensor $E_{\mu\nu}$ encapsulates the spin-induced quadrupole moment of a rotating object. The $C_{ES^2}^{(A)}$ parameters are Wilson coefficients akin to those in equation (6) which carry information about the short-distance physics of the compact body. For example, C_{ES^2} is exactly 1 for Kerr black holes [84, 75]. The ellipsis contains an additional hierarchy of higher curvature effective operators that include other finite-size corrections as well as extra pieces required to enforce the SSC.

The equations of motion (EoM) for each particle can then be obtained via

$$\frac{\delta}{\delta x^\mu}\int \mathcal{R} d\sigma = 0, \quad \frac{dS^{ab}}{d\sigma} = \{S^{ab}, \mathcal{R}\}. \tag{67}$$

with the spin algebra

$$\{S_A^{ab}, S_A^{cd}\} = \eta^{ac} S_A^{bd} + \eta^{bd} S_A^{ac} - \eta^{ad} S_A^{bc} - \eta^{bc} S_A^{ad}. \tag{68}$$

After the dynamical equations are obtained, the SSC in (66) may be enforced, enabling us to rewrite the EoM in vectorial form. To the order we work in this paper, we have ($B \neq A$) [84, 75, 85]

$$S_A^{i0} = (\mathbf{v}_A \times \mathbf{S}_A)^i + \frac{2Gm_B}{r}\big(\mathbf{v} \times \mathbf{S}_A\big)^i + \frac{G}{r^3}(\mathbf{S}_B^i \mathbf{x} \cdot \mathbf{S}_A - \mathbf{x}^i \mathbf{S}_A \cdot \mathbf{S}_B) + \cdots \tag{69}$$

and the spin three-vector is defined as

$$\mathbf{S}_A^i \equiv \frac{1}{2}\epsilon^{ijk} S_A^{jk} \tag{70}$$

in the locally-flat frame. We will see in section II.D how to determine the PN order of spin effects in the EFT.

2. Spin definitions, transformations, and spin supplementary conditions

The antisymmetric spin tensor $S^{\mu\nu}$ has six independent degrees of freedom. To reduce these to the three physical degrees of freedom necessary to describe rotations, we must impose additional constraints known as spin supplementary conditions (SSC). Two common choices in the literature are the covariant SSC and the Newton–Wigner SSC, defined by

$$p^\mu S_{\mu\nu} = 0 \qquad \text{(Covariant)}, \tag{71}$$

$$S^{\mu 0} - S^{\mu j}\left(\frac{\tilde{\mathbf{p}}^j}{\tilde{p}_0 + m}\right) = 0 \qquad \text{(Newton–Wigner)}. \tag{72}$$

Note that the Newton–Wigner SSC is enforced in the local frame of the compact object and thus $\tilde{p}^a = e^a_\mu p^\mu$. The possibility of different spin definitions and SSCs is a consequence of the ambiguity in choosing a worldline COM for the individual compact objects [116, 117]. Choosing a different COM for the object induces a transfer of angular momentum between the orbital angular momentum and spin, though the total angular momentum is of course unchanged. Ultimately, the physical evolution of the system is agnostic to our definition, though we must be careful when performing calculations to be consistent in our choice. In this section, we will explore two choices of spin definitions and SSC, ultimately showing how to transform from one to another.

We proceed using the covariant SSC. A natural choice is to define the spin vector as

$$S_A^{\mu\nu} = -\frac{1}{m\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma} p^A_\rho S^A_\sigma, \tag{73}$$

where the $A = 1, 2$ is the binary constituent. This clearly preserves the covariant SSC for an arbitrary reference frame, and in the locally-flat frame reduces to

$$S_A^{ij} = -\frac{1}{m}\epsilon^{ij\rho\sigma} p_{A\rho} S_{A\sigma} = \epsilon^{ijk}\mathbf{S}^k_A. \tag{74}$$

However, the equations of motion derived from the Lagrangian with spin defined in the locally-flat frame differ in form from those in the PN frame of the binary. There will be terms that enter through the general coordinate transformation between frames which will ultimately change the equations of motion and observables. We can write the spin vector

S^{ab} in the locally-flat frame as a projection of the spin vector in the PN frame $\bar{S}^{\mu\nu}$ using the tetrads

$$S^{ab} = e^a_\mu e^b_\nu \bar{S}^{\mu\nu} \tag{75}$$

The tetrads can be written as

$$e^0_0(\mathbf{x}_1) = 1 - \frac{Gm_2}{r} + \cdots, \tag{76}$$

$$e^k_0(\mathbf{x}_1) = -2\frac{Gm_2}{r}\mathbf{v}^k_2 + \frac{G}{r^2}(\mathbf{n}\times\mathbf{S}_2)^k + \cdots, \tag{77}$$

$$e^i_j(\mathbf{x}_1) = \delta^i_j\left(1 + \frac{Gm_2}{r}\right) + \cdots, \tag{78}$$

and thus

$$S^{ij}_1 \approx \left(1 + 2\frac{Gm_2}{r}\right)\bar{S}^{ij}_1 + \mathcal{O}(v^4)\bar{S}^{ij}_1. \tag{79}$$

using equation (75) and the explicit expressions for the tetrads. We now need to use equation (73) to express this as a vector spin. To fix the timelike component of the spin 4-vector, we impose the condition

$$S^\mu p_\mu = 0 \tag{80}$$

as in [118]. This is clearly satisfied in the rest frame of the object for a spacelike vector S^μ. Then for an arbitrary frame, we have through the order we are working that

$$0 = g_{00}S^0p^0 + g_{ij}S^ip^j \approx g_{00}S^0u^0 + g_{ij}S^ju^j. \tag{81}$$

Using

$$u^0_A = 1 + \frac{\mathbf{v}^2_A}{2} + \frac{Gm_B}{r} - \mathcal{O}(v^4), \tag{82}$$

$$u^i_A = \mathbf{v}^i_A\left(1 + \frac{\mathbf{v}^2_A}{2} + \frac{Gm_B}{r}\right) + \mathcal{O}(v^5), \tag{83}$$

derived from $u^\mu = e^\mu_a v_\mu$, we find

$$S^0 = \mathbf{S}\cdot\mathbf{v}\left(1 + \frac{4Gm_B}{r}\right). \tag{84}$$

Then using equation (73), we have

$$\bar{S}^{ij}_A = \frac{1}{m_A\sqrt{-g}}(\epsilon^{ij0k}p^A_0\bar{S}^A_k + \epsilon^{ijk0}p^A_k\bar{S}^A_0) \approx -\frac{1}{\sqrt{-g}}\epsilon^{ijk}(u^A_0\bar{S}^A_k - u^A_k\bar{S}^A_0) \tag{85}$$

$$= \epsilon^{ijk} \left\{ \left(1 + \frac{\mathbf{v}_A^2}{2} - \frac{Gm_B}{r}\right) \bar{\mathbf{S}}_A^k - \mathbf{v}_A^k \bar{\mathbf{S}}_A \cdot \mathbf{v}_A \right\}. \tag{86}$$

Finally, using equation (79), we find

$$S^{ij} \approx \left(1 + \frac{2Gm_B}{r}\right) \bar{S}^{ij} \approx \epsilon^{ijk} \left\{ \left(1 + \frac{\mathbf{v}_A^2}{2} + \frac{Gm_B}{r}\right) \bar{\mathbf{S}}_A^k - \mathbf{v}_A^k \bar{\mathbf{S}}_A \cdot \mathbf{v}_A \right\}. \tag{87}$$

The origin of the transformation, then, is transparent. The spin vector was defined in the locally-flat frame on the world line of the particle. To go to the PN frame used in [30, 36], we simply express the spin vector in a way that manifestly enforces the covariant SSC in an arbitrary frame and then boost to the PN frame. These PN corrections must be considered when comparing with the results in the literature [30, 36] to ensure that the same spin vector is used.

This PN spin vector is not the only spin definition commonly found in the literature. In fact, an important spin vector to consider is one with constant magnitude. Using this definition of spin ensures that the spin dynamics takes a precession form, i.e.,

$$\frac{d\mathbf{S}_A}{dt} = \omega \times \mathbf{S}_A. \tag{88}$$

Though not necessary to use this definition in deriving the equations of motion and in general expressions of conserved quantities and fluxes, it is essential when deriving adiabatic expressions in the quasicircular limit.[1] The spin transformation that takes us from the locally-flat spin definition to constant magnitude spin was derived in [85], and is given by

$$\mathbf{S}^c = \left(1 - \frac{\mathbf{v}^2}{2}\right) \mathbf{S} + \frac{1}{2} \mathbf{v} (\mathbf{v} \cdot \mathbf{S}), \tag{89}$$

where $\mathbf{S}^c$ is the conserved norm spin vector. It is interesting to note that this is the spin vector of the Newton–Wigner SSC as defined above. However, to transform to the Newton–Wigner SSC and ensure a canonical spin algebra, it is necessary to perform a corresponding general coordinate transformation defined by

$$\bar{\mathbf{x}}_A^i \to \mathbf{x}_A^i - \frac{1}{2m_A} (\mathbf{v}_A \times \mathbf{S}_A)^i \tag{90}$$

through 1PN order.

[1] We must consider radiation reaction effects on spin magnitudes. However, these effects do not enter at the order we consider in this thesis and in fact the spin can be written in precession form through 4.5PN. See [119, 105, 106] for more discussion of radiative effects in the spin sector which complicate the adiabatic approximation.

$H_{\mathbf{k}}$	x^μ	$\mathbf{v}$	$\mathbf{k}$	k_0	$\frac{m}{m_{\rm pl}}$	r_s	C_{E^2,B^2}
$r^2\sqrt{v}$	r	v	$1/r$	v/r	$\sqrt{Lv}$	$v^2 r$	$m_{\rm pl}^2 r_s^5$

Table 1: Power-counting rules for the potential sector.

D. Power-counting

In this section, we revisit the power-counting for the EFT. We want to be able to assign consistent power-counting for every field and operator in our theory so that we can easily assign a PN order for a given contribution. We proceed along the lines of [114].

Let us begin with the fields and their momenta in the potential regime. We have already asserted in defining the scaling for derivatives, but we should check that these make sense as the proper scaling. The spatial momenta for potential modes scale as the inverse of the interaction distance, i.e., $\mathbf{k} \sim 1/r$ and thus must be treated nonperturbatively. The timelike momentum, on the other hand, scales like $k_0 \sim v/r$, down by one power of v relative to the spatial components

The scaling of the field operators themselves can be determined by looking at the LO propagator in (24). The delta functions scale as $\delta(t-t') \sim v/r$ and $\delta^3(\mathbf{k}-\mathbf{q}) \sim r^3$, while $P_{\mu\nu;\rho\sigma} \sim 1$. Then schematically we have

$$(H_{\mathbf{k}})^2 \sim (1) \times \left(\frac{v}{r}\right) \times r^3 \times \left(\frac{1}{r}\right)^{-2} \sim vr^4 \tag{91}$$

and thus $H_{\mathbf{k}} \sim v^{\frac{1}{2}} r^2$. Recall from the virial theorem that $G \sim v^2 r/m$ and by definition $m_{\rm pl} \sim G^{-\frac{1}{2}}$ such that

$$\frac{m}{m_{\rm pl}} \sim \sqrt{Lv}, \tag{92}$$

where $L \sim mvr$.

We now need to consider a power-counting schema for spin. The SSC gives us the relation

$$S_{0i} \sim S_{ij}\mathbf{v}^i, \tag{93}$$

S_{0i}	S_{ij}	$\mathbf{S}$	C_{S^2}
Lv^2	Lv	Lv	1

Table 2: Power-counting rules for the spin sector.

so the timelike component is suppressed by one power of v relative to the spatial spin tensor. We now turn to the spin vector; this requires a specific model of the spinning system. In general, the spin angular momentum of a compact body scales like $S \sim mv_{\text{rot}}r$. However, in this thesis we will limit ourselves to a *maximally rotating* body of radius r_s with rotational velocity $v_{\text{rot}} \sim 1$. Then using the scaling above we find

$$S \sim mr_s \sim Lv, \tag{94}$$

which importantly can be treated perturbatively since it is suppressed by a power of v.

There is one final piece that we need to determine a scaling for, which are the finite size operators for nonspinning objects (6) and for spinning objects (64). The spinning object coefficients are $\mathcal{O}(1)$, which can be determined by matching with a Kerr black hole solution. The mass and current quadrupole coefficients C_{E^2,B^2} are a little trickier to determine, but can be done schematically using an argument from [114]. We choose a particular process and perform a matching procedure to determine the scaling of the coefficients. Looking specifically at one-to-one gravitational scattering off the quadrupole moment, we find the diagram is schematically

$$\frac{C_{E^2,B^2}\partial^4 H^2}{m_{\text{pl}}^2} \sim C_{E^2,B^2}\frac{\omega^4}{m_{\text{pl}}^2}, \tag{95}$$

which gives a scattering cross-section

$$\sigma_{\text{EFT}} \sim \cdots C_{E^2,B^2}^2 \frac{\omega^8}{m_{\text{pl}}^4} \cdots . \tag{96}$$

Since the only scale in the problem is r_s, we expect a full calculation to have the form $\sigma_{\text{full}} \sim r_s^2 f(r_s\omega)$, where $f(z)$ is an analytic function. With $r_s\omega \ll 1$, we can expand this function to find

$$\sigma_{\text{full}} \sim r_s^2(\cdots (r_s\omega)^8 \cdots), \tag{97}$$

$\bar{h}$	k^μ
$v/r + \cdots$	v/r

Table 3: Power-counting rules for the radiation sector.

which allows us to identify the scaling of

$$C_{E^2,B^2} \sim r_s^5 m_{\rm pl}^2. \tag{98}$$

This gives us the tools to power count any diagram in the potential regime we need in this work.

In the radiation regime, we know that the radiation momentum scales like $k^\mu \sim v/r$. We should then be able to determine the scaling of the radiation modes using the full propagator

$$\langle \bar{h}_{\mu\nu}(x)\bar{h}_{\alpha\beta}(0)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{iP_{\mu\nu;\alpha\beta}e^{-ik\cdot x}}{k^2+i\epsilon}. \tag{99}$$

From this, it appears that fields should scale as $\bar{h} \sim v/r$. However, the exponential does not have a manifest power counting since $k \cdot x \sim v$, and so we must Taylor expand the radiation fields around some point, as in equation (10), to make the power counting manifest.

Now that we have a power-counting scheme, we can easily compute the PN order of a diagram. Let us, for example, perform the power-counting for the Newtonian potential. Schematically, the diagram is computed from two insertions of the LO mass vertex (305)

$$-\frac{m}{2m_{\rm pl}}\int dt \int \frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}H_{\mathbf{k}00} \sim \sqrt{Lv} \times \frac{r}{v} \times \left(\frac{1}{r}\right)^3 \times (\sqrt{v}r^2) \sim \sqrt{L}. \tag{100}$$

Then the Newtonian potential scales as the orbital angular momentum L.

The leading spin-orbit potential is computed from one insertion of the LO mass vertex as above, and one of the leading spin vertex (309) which can be power-counted as

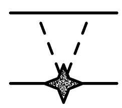
$$\frac{1}{2m_{\rm pl}}\int dt\, S^{ik}\partial_k \int \frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}H_{\mathbf{k}i0} \sim \frac{\sqrt{Lv}}{m} \times Lv \times \frac{1}{r} \times \frac{r}{v} \times \left(\frac{1}{r}\right)^3 \times (\sqrt{v}r^2) \sim \sqrt{L}v^3. \tag{101}$$

Then the leading spin-orbit potential should scale as $\sim Lv^3$, a 1.5PN correction to the Newtonian potential.

Let us now look at the leading finite-size potential correction, which arises from the diagram with two LO mass vertices and the leading finite size operator

$$\frac{C_{E^2}}{m_{\rm pl}^2}\int (\partial_i\partial_j H_{00})^2 dt \sim (v^2 r)^5 \times \left(\frac{1}{r}\right)^4 \times \frac{r}{v} \times \left(\frac{1}{r}\right)^6 \times (r^2\sqrt{v})^2 \sim v^{10} \tag{102}$$

such that



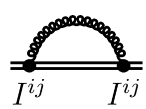
$$\sim \quad \sqrt{L}^2 \times v^{10} \sim Lv^{10}, \tag{103}$$

which confirms that finite-size operators don't contribute until the 5PN order for nonspinning objects.

In the radiation sector, the leading vertex comes from a coupling of the mass quadrupole and the electric component of the Weyl tensor, and is up to $\mathcal{O}(1)$ constants of the form

$$\frac{1}{m_{\rm pl}}\int dt\, I^{ij} E_{ij} \sim \frac{1}{m_{\rm pl}} \times \frac{r}{v} \times (mr^2) \times \left(\frac{v}{r}\right)^2 \times \frac{v}{r} \sim \sqrt{L} v^{5/2}. \tag{104}$$

The leading radiation-reaction diagram is computed from two insertions of this mass quadrupole vertex, and thus scales as



$$\sim \quad \sqrt{L} v^{5/2} \sim Lv^5, \tag{105}$$

a 2.5PN correction to the Newtonian potential. There are some subtleties to this calculation which we will discuss in chapter IV. Now that we have an understanding of how to determine the PN order of any effect systematically, we can now proceed to compute higher order effects in the PN expansion from the EFT.

III. NLO observables from NRGR

In this chapter, we demonstrate the computation of the NLO contributions to observables for both spin-orbit and spin-spin effects. We start by computing spin evolution equations from the Routhian, then proceed to obtain the equations of motion for the compact bodies. We then use these results as ingredients in the calculation of the adiabatically conserved quantities of the system, namely the energy, the orbital angular momentum, and the COM position. The latter in particular is used to define the COM frame of the system.

We then proceed to compute the radiative flux-balance laws associated with the emission of energy, momentum, and angular momentum in the form of gravitational waves. We then apply an adiabatic approximation for (quasi)circular orbits that enables a straightforward derivation of the subleading spin-orbit and spin-spin modifications to accumulated orbital phase of the binary, a key result of this work with direct application to current generation gravitational wave detectors. This chapter is based off of work published in [107, 108].

A. Equations of motion

1. Spin equations of motion

We first compute the expressions for the spin evolution given by $\dot{\mathbf{S}}_1$ and $\dot{\mathbf{S}}_2$. Using the Routhian approach, the spin dynamics can be computed from equation (67) using the spin algebra defined in equation (68). Note that the SSC is only imposed after obtaining the dynamics for the spin tensor using the covariant SSC definition.

The spin may evolve in two ways: it may precess or it may have variable magnitude. The LO expression for the spin-orbit spin evolution is given by

$$\frac{d\mathbf{S}_1}{dt} = 2\frac{m_2 G}{r^2}(\mathbf{n}\times\mathbf{v})\times\mathbf{S}_1 + \frac{m_2 G}{r^2}(\mathbf{S}_1\times\mathbf{n})\times\mathbf{v}_1 \tag{106}$$

while the leading spin-spin contribution is

$$\frac{d\mathbf{S}_1}{dt} = -\frac{G}{r^3}(\mathbf{S}_2 - 3\mathbf{n}\mathbf{S}_2\cdot\mathbf{n})\times\mathbf{S}_1 + 3C^{(1)}_{ES^2}\frac{m_2 G}{m_1 r^3}\mathbf{n}(\mathbf{S}_1\cdot\mathbf{n}). \tag{107}$$

The NLO spin equations of motion are computed using the potentials (320) and (322). The full expressions can be found written in the covariant SSC in Appendix D.

Note that these equations do not conserve the norm of the spin, i.e., $\mathbf{S}_1 \cdot (d\mathbf{S}_1/dt) \neq 0$. However, applying the spin transformation to constant magnitude spin vectors given by equation (89) allows the spin equations of motion to be written in precession form, i.e., $d\mathbf{S}_1/dt = \Omega_s \times \mathbf{S}_1$.[1]

2. Coordinate equations of motion

In the PN approximation, the acceleration of the constituents of the binary systems is given as a series of relativistic corrections to the dominant Newtonian gravitational acceleration. If we disregard radiation reaction, the acceleration can be presented as:[2]

$$\mathbf{a} = \mathbf{a}^{(0\mathrm{PN})} + \mathbf{a}^{(1\mathrm{PN})} + \mathbf{a}^{(1.5\mathrm{PN})}_{\mathrm{SO}} + \mathbf{a}^{(2\mathrm{PN})} + \mathbf{a}^{(2\mathrm{PN})}_{\mathrm{SS}} + \mathbf{a}^{(2.5\mathrm{PN})}_{\mathrm{SO}} + \mathbf{a}^{(3\mathrm{PN})} + \mathbf{a}^{(3\mathrm{PN})}_{\mathrm{SS}} + \cdots . \tag{108}$$

The expressions for the nonspin accelerations in the right-hand side of the equation above are given in Appendix D. The LO spin-orbit acceleration–a 1.5PN correction to the equation of motion–can be derived from the potential $V^{(1.5\mathrm{PN})}_{\mathrm{SO}}$ given in (319). Computing the Euler–Lagrange equations using that potential gives

$$\begin{aligned}(\mathbf{a}^i_1)^{V^{(1.5\mathrm{PN})}_{\mathrm{SO}}} = \frac{G}{r^3}\bigg\{ & \frac{m_2}{m_1}[(3\mathbf{n}\dot{r} - 2\mathbf{v}_1 + 3\mathbf{v}_2) \times \mathbf{S}_1]^i + [(6\mathbf{n}\dot{r} - 4\mathbf{v}_1 + 3\mathbf{v}_2) \times \mathbf{S}_2]^i \\ & - \left[\mathbf{x} \times \left(\frac{m_2}{m_1}\dot{\mathbf{S}}_1 + 2\dot{\mathbf{S}}_2\right)\right]^i + \left[S^{i0}_2 - \frac{m_2}{m_1}S^{i0}_1 + 3\mathbf{n}^i\mathbf{n}^j\left(\frac{m_2}{m_1}S^{j0}_1 - S^{j0}_2\right)\right]_{\mathrm{cov}} \\ & + 3\mathbf{n}^i\mathbf{n} \cdot \left(\frac{m_2}{m_1}\mathbf{v}_1 \times \mathbf{S}_1 - 2\frac{m_2}{m_1}\mathbf{v}_2 \times \mathbf{S}_1 + 2\mathbf{v}_1 \times \mathbf{S}_2 - \mathbf{v}_2 \times \mathbf{S}_2\right)\bigg\}. \end{aligned} \tag{109}$$

Notice that the expression above is given in a general form: it includes time derivatives of the spin vectors, which actually contribute only at orders higher than 1.5PN since $\dot{S} \sim \frac{v^3}{r}S$; it also shows the explicit dependence on the $S^{j0}_{1,2}$ variables, which will be removed by enforcing the covariant SSC defined in equation (71). Although we kept $S^{j0}_{1,2}$ variables to indicate that those terms will also contribute to orders higher than 1.5PN due to PN corrections in the

[1] The expression (89) is sufficient for the purposes of this work. However, to express the spin evolution in precession form at subleading order, we need post-Newtonian corrections to (89), derived in [85, 38, 103].

[2] Radiation reaction enters at 2.5PN order, and will be discussed in the subsequent chapter.

covariant SSC, the result in (109) is *only* valid in the covariant SSC and is not general to other choices of constraints.[3] The spin tensors can be written in terms of the spin vectors in the covariant SSC using equations (69) and (70). Therefore, after imposing the covariant SSC in (109) and keeping only terms entering at the lowest PN order, we write the definite order expression for the LO spin-orbit acceleration [85]:

$$(\mathbf{a}_1^i)_{\rm SO}^{(1.5{\rm PN})} = \frac{G}{r^3}\bigg\{3\frac{m_2}{m_1}[(\mathbf{S}_1\times\mathbf{v})^i - \dot{r}(\mathbf{S}_1\times\mathbf{n})^i - 2\mathbf{S}_1\cdot(\mathbf{v}\times\mathbf{n})\mathbf{n}^i] \\ + 4(\mathbf{S}_2\times\mathbf{v})^i - 6\dot{r}(\mathbf{S}_2\times\mathbf{n})^i - 6\mathbf{S}_2\cdot(\mathbf{v}\times\mathbf{n})\mathbf{n}^i\bigg\}. \tag{110}$$

The purpose of this section is to advance to the next step, namely, to obtain the equations of motion linear in the spins for the binary system at 1PN beyond equation (110), which is a 2.5PN correction to the Newtonian acceleration. The result for the NLO spin-orbit acceleration can be presented as the sum of two distinct contributions:

$$(\mathbf{a}_1^i)_{\rm SO}^{(2.5{\rm PN})} = (\mathbf{a}_1^i)^{V_{\rm SO}^{(2.5{\rm PN})}} + (\mathbf{a}_1^i)^{({\rm Red.})}. \tag{111}$$

The first term in the right-hand side of the equation above comes from computing the Euler–Lagrange equations of the NLO spin-orbit potential (320), which was obtained in [85]. The result for this contribution can be conveniently arranged as

$$(\mathbf{a}_1^i)^{V_{\rm SO}^{(2.5{\rm PN})}} = \frac{1}{m_1}\sum_{n=0}^{3}\bigg\{(-1)^{n+1}\left(\frac{d}{dt}\right)^n\frac{\partial}{\partial\mathbf{x}_1^{i(n)}}V_{\rm SO}^{(2.5{\rm PN})}\bigg\} = (\mathbf{A}_1^i)^{\rm cov}_{S^{i0}} + (\mathbf{A}_1^i)_{S^{ij}}, \tag{112}$$

where

$$(\mathbf{A}_1^i)^{\rm cov}_{S^{i0}} \equiv \frac{G}{r^3}\bigg\{\frac{m_2}{m_1}S_1^{j0}\bigg[\delta^{ij}\left(\frac{Gm_1}{r} + 2\frac{Gm_2}{r} + 2(vv_2) + \frac{1}{2}\mathbf{a}_2\cdot\mathbf{x} + \frac{3}{2}(v_2n)^2\right) \\ + \mathbf{v}_2^i(3(v_2n)\mathbf{n}^j - \mathbf{v}^j) + \frac{1}{2}\mathbf{a}_2^i\mathbf{x}^j - \frac{3}{2}r\mathbf{n}^i\mathbf{a}_2^j + 3(v_2n)\mathbf{n}^i\mathbf{v}^j \\ - \mathbf{n}^i\mathbf{n}^j\left(4\frac{Gm_1}{r} + 8\frac{Gm_2}{r} + 6(vv_2) + \frac{3}{2}(a_2x) + \frac{15}{2}(v_2n)^2\right)\bigg] \\ + S_2^{j0}\bigg[\delta^{ij}\left(-2\frac{Gm_1}{r} - \frac{Gm_2}{r} + 2(vv_1) + \frac{1}{2}(a_1x) - \frac{3}{2}(v_1n)^2\right) \\ - \mathbf{v}_1^i(3(v_1n)\mathbf{n}^j + \mathbf{v}^j) + \frac{1}{2}\mathbf{a}_1^i\mathbf{x}^j - \frac{3}{2}r\mathbf{n}^i\mathbf{a}_1^j + 3(v_1n)\mathbf{n}^i\mathbf{v}^j$$

[3] If we were working with the Newton–Wigner SSC, for instance, we would have to impose the constraint at the level of the potential before computing the Euler–Lagrange equations. See the discussion presented in Appendix E of [80] for more details.

$$+ \mathbf{n}^i\mathbf{n}^j\left(8\frac{Gm_1}{r} + 4\frac{Gm_2}{r} - 6(vv_1) - \frac{3}{2}(a_1x) + \frac{15}{2}(v_1n)^2\right)\bigg)\bigg]\bigg\}$$

$$- \frac{d}{dt}\left\{\frac{G}{r^2}\left[\frac{m_2}{m_1}S_1^{j0}(2\mathbf{v}_2^i\mathbf{n}^j - \delta^{ij}(v_2n))\right.\right.$$

$$\left.\left. + S_2^{j0}[2\mathbf{n}^j(2\mathbf{v}_1^i - \mathbf{v}_2^i) - \delta^{ij}(v_1n) - \mathbf{n}^i(\mathbf{v}^j + 3(v_1n)\mathbf{n}^j)]\right]\right\}$$

$$+ \frac{d^2}{dt^2}\left\{\frac{1}{2}\frac{G}{r}S_2^{j0}(3\delta^{ij} + \mathbf{n}^i\mathbf{n}^j)\right\}, \qquad (113)$$

and

$$(\mathbf{A}_1^i)_{S^{ij}} \equiv \frac{G}{r^3}\left\{\frac{m_2}{m_1}S_1^{ij}\left[-2(v_2x)\mathbf{a}_2^j - r^2\dot{\mathbf{a}}_2^j + \mathbf{v}_1^j\left(-\frac{Gm_1}{r} + \frac{1}{2}\frac{Gm_2}{r} + \frac{1}{2}(a_2x) + \frac{3}{2}(v_2n)^2\right)\right.\right.$$

$$\left. + \mathbf{v}_2^j\left(-\frac{5}{2}\frac{Gm_2}{r} - 2(vv_2) - (a_2x) - 3(v_2n)^2\right)\right]$$

$$+ S_2^{ij}\left[-2(v_1x)\mathbf{a}_1^j + r^2\dot{\mathbf{a}}_1^j + \mathbf{v}_1^j\left(\frac{5}{2}\frac{Gm_1}{r} - 2(vv_1) - (a_1x) + 3(v_1n)^2\right)\right.$$

$$\left. + \mathbf{v}_2^j\left(-\frac{1}{2}\frac{Gm_1}{r} + \frac{Gm_2}{r} + \frac{1}{2}(a_1x) - \frac{3}{2}(v_1n)^2\right)\right]$$

$$+ \mathbf{n}^i\left[\frac{m_2}{m_1}S_1^{kj}\left(\left(-4\frac{Gm_1}{r} + 2\frac{Gm_2}{r} + \frac{3}{2}(a_2x) + \frac{15}{2}(v_2n)^2\right)\mathbf{v}_1^k\mathbf{n}^j\right.\right.$$

$$- 3(v_2n)(\mathbf{v}_1^k\mathbf{v}_2^j + 2\mathbf{a}_2^k\mathbf{x}^j) + \frac{1}{2}r\mathbf{a}_2^k\mathbf{v}_1^j + r\mathbf{a}_2^k\mathbf{v}_2^j + r\mathbf{x}^k\dot{\mathbf{a}}_2^j$$

$$\left. - \left(10\frac{Gm_2}{r} + 6(vv_2) + 3(a_2x) + 15(v_2n)^2\right)\mathbf{v}_2^k\mathbf{n}^j\right)$$

$$+ S_2^{kj}\left(\left(10\frac{Gm_1}{r} - 6(vv_1) - 3(a_1x) + 15(v_1n)^2\right)\mathbf{v}_1^k\mathbf{n}^j\right.$$

$$- 3(v_1n)(\mathbf{v}_1^k\mathbf{v}_2^j + 2\mathbf{a}_1^k\mathbf{x}^j) + r\mathbf{a}_1^k\mathbf{v}_1^j + \frac{1}{2}r\mathbf{a}_1^k\mathbf{v}_2^j - r\mathbf{x}^k\dot{\mathbf{a}}_1^j$$

$$\left.\left. + \left(-2\frac{Gm_1}{r} + 4\frac{Gm_2}{r} + \frac{3}{2}(a_1x) - \frac{15}{2}(v_1n)^2\right)\mathbf{v}_2^k\mathbf{n}^j\right)\right]$$

$$+ \mathbf{v}_1^iS_2^{kj}[\mathbf{v}_1^k\mathbf{v}_2^j + 2\mathbf{a}_1^k\mathbf{x}^j + 3(v_1n)(\mathbf{v}_2^k - 2\mathbf{v}_1^k)\mathbf{n}^j]$$

$$+ \mathbf{a}_1^iS_2^{kj}[\mathbf{v}_1^k\mathbf{x}^j - \tfrac{1}{2}\mathbf{v}_2^k\mathbf{x}^j] + \mathbf{a}_2^i\frac{m_2}{m_1}S_1^{kj}[-\tfrac{1}{2}\mathbf{v}_1^k\mathbf{x}^j + \mathbf{v}_2^k\mathbf{x}^j]$$

$$\left. + \mathbf{v}_2^i\frac{m_2}{m_1}S_1^{kj}[\mathbf{v}_1^k\mathbf{v}_2^j + 2\mathbf{a}_2^k\mathbf{x}^j + 3(v_2n)(2\mathbf{v}_2^k - \mathbf{v}_1^k)\mathbf{n}^j]\right\}$$

$$- \frac{d}{dt}\left\{\frac{G}{r^2}\left\{\frac{m_2}{m_1}S_1^{ij}\left[\frac{1}{2}r\mathbf{a}_2^j + (v_2n)\mathbf{v}_2^j + \mathbf{n}^j\left(\frac{Gm_1}{r} - \frac{1}{2}\frac{Gm_2}{r} - \frac{1}{2}(a_2x) - \frac{3}{2}(v_2n)^2\right)\right]\right.\right.$$

$$+ S_2^{ij}\left[r\mathbf{a}_1^j + (v_1n)\mathbf{v}_2^j + \mathbf{n}^j\left(-\frac{5}{2}\frac{Gm_1}{r} + 2(vv_1) + (a_1x) - 3(v_1n)^2\right)\right]$$

$$+ 4\mathbf{v}_1^i S_2^{kj}\mathbf{v}_1^k\mathbf{n}^j + 2\mathbf{v}_2^i\bigg[\frac{m_2}{m_1}S_1^{kj}\mathbf{v}_2^k\mathbf{n}^j - S_2^{kj}\mathbf{v}_1^k\mathbf{n}^j\bigg]$$

$$+ \mathbf{n}^i S_2^{kj}[\mathbf{v}_1^k\mathbf{v}_2^j + 2\mathbf{a}_1^k\mathbf{x}^j + 3(v_1n)\mathbf{n}^j(\mathbf{v}_2^k - 2\mathbf{v}_1^k)]\bigg\}\bigg\}$$

$$+ \frac{d^2}{dt^2}\bigg\{\frac{G}{r}[S_2^{ij}(2(v_1n)\mathbf{n}^j - \mathbf{v}_1^j - \tfrac{1}{2}\mathbf{v}_2^j) + \mathbf{n}^i\mathbf{n}^j S_2^{kj}(\mathbf{v}_1^k - \tfrac{1}{2}\mathbf{v}_2^k)]\bigg\} + \frac{d^3}{dt^3}\{GS_2^{ij}\mathbf{n}^j\}. \quad (114)$$

The second term in the right hand side of (111) accounts for 2.5PN order terms coming from order reduction of lower PN order accelerations, which can be concisely presented as

$$(\mathbf{a}_1^i)^{(\mathrm{Red.})} = \bigg[\frac{1}{2}\frac{Gm_2}{r}\mathbf{a}_2\cdot\mathbf{n}\mathbf{n}^i - \mathbf{a}_1\cdot\mathbf{v}_1\mathbf{v}_1^i - \mathbf{a}_1^i\bigg(3\frac{Gm_2}{r} + \frac{1}{2}\mathbf{v}_1^2\bigg) + \frac{7}{2}\frac{Gm_2}{r}\mathbf{a}_2^i\bigg]_{\mathbf{a}_{\mathrm{SO}}^{(1.5\mathrm{PN})}}$$
$$+ \bigg[\frac{G}{r^3}\bigg(-\frac{m_2}{m_1}S_1^{i0} + 3\frac{m_2}{m_1}S_1^{j0}\mathbf{n}^j\mathbf{n}^i + S_2^{i0} - 3S_2^{j0}\mathbf{n}^j\mathbf{n}^i\bigg)\bigg]_{\mathrm{cov(1PN)}}$$
$$+ \bigg[-\frac{G}{r^3}\bigg(\frac{m_2}{m_1}\dot{S}_1^{ij}\mathbf{x}^j + 2\dot{S}_2^{ij}\mathbf{x}^j\bigg)\bigg]_{\dot{S}_{\mathrm{LO}}}. \quad (115)$$

The expression above includes three contributions from lower-order accelerations: reduced contributions from substituting the LO spin-orbit acceleration (110) in the acceleration terms present in the 1PN correction to the equation of motion (317); frame corrections from imposing the covariant SSC (69) in (109), and also, in that same equation, terms from reducing spin derivatives. At 2.5PN order, we only need the LO spin derivative term given by equation (106).

After imposing the covariant SSC and order reducing the accelerations in order to obtain a fixed order result at 2.5PN, (111) becomes

$$\mathbf{a}_1^i = \frac{G}{r^3}\bigg\{-\frac{\mathbf{n}^i}{m\nu r}\bigg[\frac{m_2}{m_1}(S_1L)\bigg(\frac{G}{r}(26m_1 + 22m_2) + 12(vv_2) + 3\mathbf{v}_2^2 + 3\mathbf{v}^2 + 15(v_2n)^2\bigg)$$
$$+ (S_2L)\bigg(\frac{G}{r}(\tfrac{61}{2}m_1 + 20m_2) + 6(vv_2) + 3\mathbf{v}_2^2 + 15(v_2n)^2\bigg)\bigg]$$
$$+ \mathbf{v}_1^i\bigg[-\frac{3m_2}{m_1}\bigg(\frac{1}{m\nu r}(S_1L)(2(v_2n) + \dot{r}) + \dot{r}\mathbf{S}_1\cdot(\mathbf{v}_2\times\mathbf{n}) + \mathbf{S}_1\cdot(\mathbf{v}\times\mathbf{v}_2)\bigg)$$
$$- 2\bigg(\frac{3}{m\nu r}(S_2L)((v_2n) + \dot{r}) + 3\dot{r}\mathbf{S}_2\cdot(\mathbf{v}_2\times\mathbf{n}) + 2\mathbf{S}_2\cdot(\mathbf{v}\times\mathbf{v}_2)\bigg)\bigg]$$
$$+ \mathbf{v}_2^i\bigg[\frac{6}{m\nu r}\bigg(\frac{m_2}{m_1}\mathbf{S}_1 + \mathbf{S}_2\bigg)\cdot\mathbf{L}((v_2n) + \dot{r})\bigg] - \frac{2}{m\nu r}\mathbf{L}^i\bigg[\frac{G}{r}\bigg(\frac{m_2^2}{m_1}(S_1n) + 2m_1(S_2n)\bigg)\bigg]$$
$$+ \frac{m_2}{m_1}(\mathbf{S}_1\times\mathbf{n})^i\bigg[\dot{r}\frac{G}{r}(14m_1 + 10m_2) + \frac{3}{2}\dot{r}\big(\mathbf{v}_1^2 + 5(v_2n)^2\big) - 3(vv_2)(v_2n)\bigg]$$

$$
\begin{aligned}
&+ (\mathbf{S}_2 \times \mathbf{n})^i \left[\dot{r}\frac{G}{r}(\tfrac{47}{2}m_1 + 16m_2) - 2(v_2 n)\left(\frac{Gm_1}{r} + 3(vv_2)\right) + 3\dot{r}\big(2(vv_2) + \mathbf{v}_2^2 + 5(v_2 n)^2\big)\right] \\
&- \frac{m_2}{m_1}(\mathbf{S}_1 \times \mathbf{v})^i \left[\frac{G}{r}(14m_1 + 10m_2) + 6(vv_2) + \frac{3}{2}\mathbf{v}_2^2 + \frac{3}{2}\mathbf{v}^2 + \frac{9}{2}(v_2 n)^2 - 3\dot{r}(v_2 n)\right] \\
&- (\mathbf{S}_2 \times \mathbf{v})^i \left[\frac{G}{r}(\tfrac{31}{2}m_1 + 12m_2) + 4(vv_2) + 2\mathbf{v}_2^2 + 6(v_2 n)^2 \right] \Bigg\}, \qquad (116)
\end{aligned}
$$

where the Newtonian angular momentum is $\mathbf{L} = m\nu\mathbf{v} \times \mathbf{x}$. Note that the spin vector used in this expression is defined in the locally flat frame; see section II.C for a discussion of alternative spin definitions.

We also present the NLO spin-orbit acceleration in the COM frame. In the latter, the expressions for $\mathbf{x}_1$ and $\mathbf{x}_2$ in terms of the relative coordinate $\mathbf{x}$ are given by

$$
\mathbf{x}_1 = \frac{m_2}{m}\mathbf{x} + \delta\mathbf{x}, \qquad (117)
$$

$$
\mathbf{x}_2 = -\frac{m_1}{m}\mathbf{x} + \delta\mathbf{x}, \qquad (118)
$$

where[4]

$$
\delta\mathbf{x} = \delta\mathbf{x}_{1\mathrm{PN}} + \delta\mathbf{x}_{\mathrm{SO}}^{1.5\mathrm{PN}} + \delta\mathbf{x}_{2\mathrm{PN}} + \delta\mathbf{x}_{\mathrm{SS}}^{2\mathrm{PN}} + \delta\mathbf{x}_{\mathrm{SO}}^{2.5\mathrm{PN}} + \delta\mathbf{x}_{\mathrm{SS}}^{3\mathrm{PN}} + \delta\mathbf{x}_{3\mathrm{PN}} + \mathcal{O}(v^7). \qquad (119)
$$

Considering only corrections necessary for the completion of the NLO spin effects, we need

$$
\delta\mathbf{x}_{1\mathrm{PN}} = \nu\frac{\delta m}{2m}\left(\mathbf{v}^2 - \frac{Gm}{r}\right)\mathbf{x}, \qquad (120)
$$

$$
\delta\mathbf{x}_{\mathrm{SO}}^{1.5\mathrm{PN}} = \frac{\nu}{m}\mathbf{v} \times \mathbf{\Sigma} \qquad (121)
$$

$$
\delta\mathbf{x}_{\mathrm{SS}}^{2\mathrm{PN}} = 0. \qquad (122)
$$

When these PN corrections to the COM frame are considered in the 1PN acceleration (317), they yield contributions to the equation of motion at the 2.5PN order. In principle, one must consider 1PN COM corrections in the LO spin-orbit acceleration (110) as well, but these vanish because this acceleration only depends on relative coordinates and velocities; this is also the reason why we do not need to consider 2.5PN spin-orbit correction to the

[4] Additionally, there will be radiative corrections to the COM which in principle enter at 2.5PN, though we will see that in our gauge that they in fact are 3.5PN effects in section IV.E. The 2.5PN spin-orbit and 3PN spin-spin corrections will be computed in section III.C.3.

COM in the Newtonian acceleration (316). Therefore, the final expression for the NLO spin-orbit acceleration in the COM frame comes solely from considering (117) and (118) in (116) and (317); the result is

$$
\begin{aligned}
(\mathbf{a}^i)^{(2.5\mathrm{PN})}_{\mathrm{SO}} =& \frac{G}{m\nu r^4}\Bigg\{\mathbf{n}^i\Bigg[\mathbf{S}\cdot\mathbf{L}\Bigg(-\frac{Gm}{r}(42+29\nu)+3(-1+10\nu)\mathbf{v}^2-30\nu\dot{r}^2\Bigg) \\
&-\delta\mathbf{\Sigma}\cdot\mathbf{L}\Bigg(\frac{Gm}{r}\Bigg(22+\frac{33}{2}\nu\Bigg)+3(1-5\nu)\mathbf{v}^2+15\nu\dot{r}^2\Bigg)\Bigg] \\
&+3\dot{r}\mathbf{v}^i\Bigg[3\mathbf{S}\cdot\mathbf{L}(-1+\nu)+\delta\mathbf{\Sigma}\cdot\mathbf{L}(-1+2\nu)\Bigg] \\
&-2\frac{Gm}{r}\mathbf{L}^i\Bigg[\mathbf{S}\cdot\mathbf{n}(1+2\nu)+\delta\mathbf{\Sigma}\cdot\mathbf{n}(1+\nu)\Bigg]\Bigg\} \\
&+\frac{G}{r^3}\Bigg\{(\mathbf{S}\times\mathbf{n})^i\dot{r}\Bigg[\frac{Gm}{r}(26+25\nu)+\frac{3}{2}(1-15\nu)\mathbf{v}^2+\frac{45}{2}\nu\dot{r}^2\Bigg] \\
&+\delta(\mathbf{\Sigma}\times\mathbf{n})^i\dot{r}\Bigg[\frac{Gm}{r}\Bigg(10+\frac{27}{2}\nu\Bigg)+\Bigg(\frac{3}{2}-12\nu\Bigg)\mathbf{v}^2+15\nu\dot{r}^2\Bigg] \\
&+(\mathbf{S}\times\mathbf{v})^i\Bigg[-\frac{Gm}{r}(22+15\nu)+\frac{3}{2}(-1+11\nu)\mathbf{v}^2-\frac{33}{2}\nu\dot{r}^2\Bigg] \\
&-\delta(\mathbf{\Sigma}\times\mathbf{v})^i\Bigg[\frac{Gm}{r}\Bigg(10+\frac{15}{2}\nu\Bigg)+\Bigg(\frac{3}{2}-8\nu\Bigg)\mathbf{v}^2+9\nu\dot{r}^2\Bigg]\Bigg\}. \qquad (123)
\end{aligned}
$$

This expression is valid for general orbits and for arbitrary spin orientations within the region of validity of the NRGR formalism.

We can follow a similar procedure for the LO spin-spin and NLO spin-spin equations of motion, which constitute 2PN and 3PN corrections to the Newtonian acceleration, respectively. We use the potentials previously computed in [84, 75], along with reduced contributions from the 1PN nonspinning acceleration and 1.5PN spin-orbit accelerations. This includes the 2PN acceleration inserted into the 1PN equations of motion as well as the leading spin-spin corrections to the spin evolution equation (107) which reduce spin derivatives in the leading spin-orbit acceleration. Finally, we must consider the leading spin-spin reference frame corrections to the SSC in (69) which generate a 3PN term from the leading spin-orbit acceleration. In principle, we expect COM corrections as well; however, the leading spin-spin correction vanishes (see section III.C.3.b).

The LO spin-spin acceleration in the COM frame is given by

$$
\mathbf{a}^i_{\text{LO-SS}} = \frac{G}{mr^4}\frac{3}{4}\Bigg[\Big((-8-4\kappa_+)(nS)+(-4\delta+2\kappa_--2\delta\kappa_+)(n\Sigma)\Big)\mathbf{S}^i
$$

$$+\Big[(-4\delta+2\kappa_- -2\delta\kappa_+)(nS)+\big(2\delta\kappa_- +8\nu+\kappa_+(-2+4\nu)\big)(n\Sigma)\Big]\mathbf{\Sigma}^i$$
$$+\mathbf{n}^i\Big[(20+10\kappa_+)(nS)^2+(20\delta-10\kappa_- +10\delta\kappa_+)(nS)(n\Sigma)$$
$$+\big(-5\delta\kappa_- +\kappa_+(5-10\nu)-20\nu\big)(n\Sigma)^2+(-4-2\kappa_+)\mathbf{S}^2$$
$$+(-4\delta+2\kappa_- -2\delta\kappa_+)(S\Sigma)+\big(\delta\kappa_- +4\nu+\kappa_+(-1+2\nu)\big)\mathbf{\Sigma}^2\Big]\Big], \tag{124}$$

where we use the finite size coefficient combinations $\kappa_\pm \equiv C^{(1)}_{ES^2} \pm C^{(2)}_{ES^2}$. The NLO correction can be expressed simply as

$$\mathbf{a}^i_{\text{NLO-SS}} = \frac{G}{mr^4}\left[\frac{1}{8}\mathbf{A}^i_0 + \frac{Gm}{r}\frac{1}{2}\mathbf{A}^i_1\right], \tag{125}$$

where the lengthy coefficients $\mathbf{A}^i_0$ and $\mathbf{A}^i_1$ can be found in Appendix D.

B. Quasicircular limit

Until this point, our results are valid for general orbits and arbitrary spin configurations. However, as is well known, the emission of angular momentum via gravitational waves tends to efficiently circularize orbits well before entering the observable frequency band of gravitational wave detectors [120].[5] We can then apply an adiabatic approximation in which orbits are approximately circular on an orbital timescale and orbit decay occurs on a radiation-reaction timescale. In this approximation, the expressions above can be expressed as coordinate-independent quantities as functions of a single orbital frequency ω, the orbital angular momentum L, and the spin vectors, and are gauge invariant under coordinate transformations.[6]

We proceed using the basis of vectors $\{\mathbf{n}, \lambda, \ell\}$, defined by

$$\ell \equiv \frac{\mathbf{x}\times\mathbf{v}}{|\mathbf{x}\times\mathbf{v}|}, \tag{126}$$

$$\lambda \equiv \ell \times \mathbf{n}. \tag{127}$$

[5] Alternative methods such as the dynamical renormalization group approach [121, 122] may be used to solve the equations of motion for more general systems.

[6] In our subsequent analysis, tail [97, 101] and radiation-reaction [96, 98] terms in the orbital frequency are omitted; these effects do not mix with our results and thus can be included independently later on.

Following the nonspinning case, we seek (quasi-)circular orbits obeying

$$\frac{d\mathbf{n}}{dt} = \omega(t)\,\lambda\,, \tag{128}$$
$$\frac{d\lambda}{dt} = -\omega(t)\,\mathbf{n}\,, \tag{129}$$
$$\frac{d\ell}{dt} = 0\,, \tag{130}$$

which are equivalent to

$$r\omega^2 = -\langle \mathbf{n}\cdot\mathbf{a}\rangle, \tag{131}$$
$$v = r\omega, \tag{132}$$
$$\dot{r} = 0. \tag{133}$$

The reader will immediately notice that these conditions cannot be fulfilled when spin effects are included, since only the total angular momentum is conserved.[7] Hence, we must enforce additional constraints in order to find (quasi)circular orbits in the case of spinning bodies. We demand that the spins conform to the simplifications

$$\mathbf{n}\cdot\mathbf{S}_A = 0\,, \tag{134}$$
$$\lambda\cdot\mathbf{S}_A = 0\,, \tag{135}$$
$$\mathbf{S}_1\times\mathbf{S}_2 = 0\,, \tag{136}$$

to ensure the orbit is confined to the plane. In order to guarantee these are valid throughout the entire evolution of the binary, their time derivatives must also be consistent with the equations of motion to the desired PN order. The above conditions then imply that the spin vectors must be aligned with the orbital angular momentum,

$$\mathbf{S}_A \equiv S_A\ell, \tag{137}$$

[7]When only spin-orbit corrections are included at leading order, orbits with $\dot{r} = 0$ are still possible but not restricted to the plane.

and remain constant in time ($\dot{S}_{1,2}=0$).[8] Moreover, because the radius is also constant, i.e., that $\dot{r}=0$, there must be a direct relationship between the latter and the orbital frequency, which also obeys $\dot{\omega}=0$ when neglecting radiative effects. Introducing the PN parameter $x\equiv(Gm\omega)^{2/3}$, which is formally $O(v^2)$, we solve for ω^2 using equation (131) and invert. This allows us to find an expression relating the gauge dependent separation r in terms of the gauge independent parameter x for nonspinning and spin-orbit contributions given by

$$\begin{aligned}\frac{Gm}{r} &= x + x^2\left[1-\frac{1}{3}\nu\right] + \frac{x^{5/2}}{Gm^2}\left[\delta\Sigma_\ell^c + \frac{5}{3}S_\ell^c\right] \\ &\quad + x^3\left[3-\frac{65}{12}\nu\right] + \frac{x^{7/2}}{Gm^2}\left[2\delta\Sigma_\ell^c + \left(\frac{10}{3}+\frac{8}{9}\nu\right)S_\ell^c\right],\end{aligned} \tag{138}$$

and for spin-spin contributions

$$\begin{aligned}\frac{Gm}{r} = \frac{x^3}{4G^2m^4}&\Big\{2(-2-\kappa_+)(S_\ell^c)^2 + 2(-2\delta+\kappa_- - \delta\kappa_+)S_\ell^c\Sigma_\ell^c \\ &+ \big(\delta\kappa_- - \kappa_+ + (4+2\kappa_+)\nu\big)(\Sigma_\ell^c)^2\Big\} \\ + \frac{x^4}{36G^2m^4}&\Big\{\big(74-42\delta\kappa_- - 24\kappa_+ + 24(-2-\kappa_+)\nu\big)(S_\ell^c)^2 \\ &+ [78\delta - 18\kappa_- + 18\delta\kappa_+ + 24(-2\delta+8\kappa_- - \delta\kappa_+)\nu]S_\ell^c\Sigma_\ell^c \\ &+ 3[3(4-\delta\kappa_- + \kappa_+) - 6(5-3\delta\kappa_- + 4\kappa_+)\nu + 8(2+\kappa_+)\nu^2](\Sigma_\ell^c)^2\Big\}.\end{aligned} \tag{139}$$

C. Adiabatic invariants

Using the results for the gravitational potential it is straightforward to compute the conserved quantities of the system, which do not evolve in time when radiation fluxes are turned off. Namely, these are associated with the symmetries of the asymptotically Minkowski spacetime of a distant observer. These are the objects, such as the binding energy and total angular momentum, that will be part of the balance equations in an adiabatic expansion, valid during the insipiral regime. It is somewhat convenient to express the values for these quantities in the COM frame.

[8] Notice that the covariant spin vector varies with time in generic orbits. However, since $\dot{\mathbf{v}}^2=0$ for (quasi)circular orbits when radiation-damping is omitted, the difference between $\mathbf{S}$ and $\mathbf{S}_c$ in (89) is simply an overall rescaling. Therefore, both vectors remain constant in time when initially aligned with the orbital angular momentum.

1. Binding Energy

From the gravitational potential we can derive the binding energy following the standard Euler–Lagrange procedure, yielding

$$E = E_{\rm N} + E_{\rm 1PN} + E_{\text{LO-SO}} + \Big[E_{\rm 2PN} + E_{\text{LO-SS}}\Big] + E_{\text{NLO-SO}} + \Big[E_{\rm 3PN} + E_{\text{NLO-SS}}\Big] + \cdots \,, \quad (140)$$

in a PN expansion.

The LO spin-orbit energy–a 1.5PN correction to the Newtonian binding energy–and the LO spin-spin energy–a 2PN correction–can be obtained from the potentials (319) and (321); they are given by

$$E_{\rm LO-SO} = \frac{G}{r^3}\mathbf{x}^i(m_2 S_1^{i0} - m_1 S_2^{i0})_{\rm cov} = \frac{Gm_2}{r^2}\mathbf{S}_1\cdot(\mathbf{n}\times\mathbf{v}_1) + 1\leftrightarrow 2, \quad (141)$$

$$E_{\rm LO-SS} = \frac{G\nu}{r^3}\frac{1}{4}e_4^0, \quad (142)$$

where the coefficient e_4^0 can be found in Appendix D. We have imposed the covariant SSC in the second expression of the first line. In this section, we obtain the 1PN correction to the LO spin-orbit and spin-spin binding energies:

$$E^{\rm NLO}_{\rm spin} = \sum_{A=1}^{2}\sum_{n=0}^{2}\mathbf{p}_{\mathbf{x}_A^{(n)}}\cdot\mathbf{x}_A^{(n+1)} + V^{\rm NLO}_{\rm spin} + E^{(\rm Red.)}, \quad (143)$$

$$\mathbf{p}_{q^{(n)}} = -\sum_{A=1}^{2}\sum_{k=n+1}^{3}\left(-\frac{d}{dt}\right)^{k-n-1}\frac{\partial V^{\rm NLO}_{\rm spin}}{\partial\mathbf{x}_A^{(k)}}, \quad (144)$$

where the notation $\mathbf{x}_A^{(k)}$ is a compact way to express $\frac{d^k\mathbf{x}_A}{dt^k}$. For the spin-orbit energy at the 2.5PN order, we have two contributions: one from the NLO spin-orbit potential (320), and another from frame corrections when applying the covariant SSC to the LO spin-orbit energy (141), which we represent by $E^{(\rm Red.)}$ in (143). The sum of the two contributions gives

$$\begin{aligned}E_{\text{NLO-SO}} = \frac{Gm_2}{r^2}\Bigg\{&\left(-2\mathbf{v}_2^2 + 3(v_1v_2) - \mathbf{v}_1^2 + 3((v_1n))^2 + 3(v_1n)(v_2n) + 3\frac{Gm_2}{r}\right)\mathbf{S}_1\cdot(\mathbf{v}_2\times\mathbf{n})\\ &+\left(\mathbf{v}_2^2 - (v_1v_2) - \frac{3}{2}((v_2n))^2 - 3(v_2n)(v_1n) + 2\frac{Gm_1}{r}\right)\mathbf{S}_1\cdot(\mathbf{v}_1\times\mathbf{n})\\ &+ ((v_1n) + 2(v_2n))\mathbf{S}_1\cdot(\mathbf{v}_1\times\mathbf{v}_2)\Bigg\} + 1\leftrightarrow 2. \end{aligned} \quad (145)$$

Transforming to the COM frame, we have

$$E_{\text{NLO-SO}} = \frac{Gm}{r}\left\{\left(2\nu\frac{Gm}{r} - 2\mathbf{v}^2 - \frac{3}{2}\nu\dot{r}^2\right)\frac{\mathbf{L}\cdot\mathbf{S}}{mr^2} + \delta\left(\frac{3}{2}\nu\frac{Gm}{r} - \frac{3}{2}\nu\mathbf{v}^2\right)\frac{\mathbf{L}\cdot\boldsymbol{\Sigma}}{mr^2}\right\}. \tag{146}$$

Using the results first obtained in [81, 82, 84], we next compute the contribution to the COM binding energy at 3PN quadratic in the spins. We find

$$E_{\text{NLO-SS}} = \frac{G\nu}{r^3}\left[\frac{1}{8}e_6^0 + \frac{Gm}{r}\frac{1}{4}e_6^1\right], \tag{147}$$

where the e_6^0, e_6^1 coefficients are displayed in Appendix D. After rewriting the answer in terms of the conserved-norm spin variable defined by equation (89),[9] the NLO result in (147) is completely equivalent to the derivation in [39]. This is expected, since the results in [39] for the binding energy are obtained after confirming the equivalence with the value of the gravitational potential first computed in [82, 84].

We next take the expressions for the energy for general orbits and confine them to those with spin vector parallel to the orbital angular momentum. We first take the circular limit of the expressions (146) and (147) defined by equations (131–133). Next, we substitute the expressions for Gm/r as a function of the PN parameter x given in equations (138) and (139) to write the binding energy as a function of the orbital frequency to find

$$\begin{aligned}(E)_{\text{SO}} = -\frac{1}{2}m\nu x\Big\{&1 + x(-\tfrac{3}{4} - \tfrac{1}{12}\nu) + \frac{x^{3/2}}{Gm^2}(2\delta\Sigma_\ell^c + \tfrac{14}{3}S_\ell^c) \\ &+ x^2(-\tfrac{27}{8} + \tfrac{19}{8}\nu - \tfrac{1}{24}\nu^2) + \frac{x^{5/2}}{Gm^2}[(3 - \tfrac{10}{3}\nu)\delta\Sigma_\ell^c + (11 - \tfrac{61}{9}\nu)S_\ell^c]\Big\},\end{aligned} \tag{148}$$

and

$$\begin{aligned}(E)_{\text{SS}} =& \frac{x^3\nu}{4G^2m^3}\big[(4 + 2\kappa_+)(S_\ell^c)^2 + (4\delta - 2\kappa_- + 2\delta\kappa_+)S_\ell^c\Sigma_\ell^c \\ &\quad + \big(-\delta\kappa_- + \kappa_+ + (-4 - 2\kappa_+)\nu\big)(\Sigma_\ell^c)^2\big] \\ &+ \frac{x^4\nu}{72G^2m^3}\big[\big(-200 + 60\delta\kappa_- + 150\kappa_+ + (-60 - 30\kappa_+)\nu\big)(S_\ell^c)^2 \\ &\quad + \big(-300\delta - 90\kappa_- + 90\delta\kappa_+ + (-60\delta - 210\kappa_- - 30\delta\kappa_+)\nu\big)S_\ell^c\Sigma_\ell^c \\ &\quad + \big(-180 - 45\delta\kappa_- + 45\kappa_+ + 45(8 - \delta\kappa_- - \kappa_+)\nu + 30(2 + \kappa_+)\nu^2\big)(\Sigma_\ell^c)^2\big],\end{aligned} \tag{149}$$

[9] We only quote the terms which are needed to match the multipoles. The reader should keep in mind, however, that higher-order corrections are necessary to achieve a precession form for the spin EoM; see, e.g. [85, 82, 84].

which agree with the results in [36] and [39], respectively. These expressions are useful in themselves as a coordinate-independent comparison with those in the literature derived in different gauges. We will see in section III.E, in addition, that these are important components needed to compute the accumulated orbital phase of the system.

2. Orbital angular momentum

The orbital angular momentum can be obtained either by matching the one-point function to the long-distance action through (15), or by obtaining the Noether current using the gravitational potential. Following the latter approach, we arrive at an expression that can likewise be PN expanded as

$$\mathbf{L}^i = \mathbf{L}^i_{\rm N} + \mathbf{L}^i_{\rm 1PN} + \mathbf{L}^i_{\text{LO-SO}} + \left[\mathbf{L}^i_{\rm 2PN} + \mathbf{L}^i_{\text{LO-SS}}\right] + \mathbf{L}^i_{\text{NLO-SO}} + \left[\mathbf{L}^i_{\rm 3PN} + \mathbf{L}^i_{\text{NLO-SS}}\right] + \cdots \,, \tag{150}$$

where

$$\mathbf{L}^i_{\rm N} = \nu m r (\mathbf{n}\times\mathbf{v})^i, \tag{151}$$

$$\mathbf{L}^i_{\rm 1PN} = \nu m r (\mathbf{n}\times\mathbf{v})^i \left(\frac{Gm}{r}(3+\nu) + \frac{1}{2}(1-3\nu)\mathbf{v}^2\right), \tag{152}$$

$$\mathbf{L}^i_{\text{LO-SO}} = \frac{Gm\nu}{r}\Big(\mathbf{n}^i\big(3(nS)+\delta(n\Sigma)\big) - 3\mathbf{S}^i - \delta\mathbf{\Sigma}^i\Big), \tag{153}$$

$$\mathbf{L}^i_{\text{LO-SS}} = 0. \tag{154}$$

At 2.5PN and 3PN order, respectively, we have

$$\mathbf{L}^i_{\text{NLO-SO}} = \frac{Gm\nu}{2r}\left[\ell^0_5 + \frac{Gm}{r}\ell^1_5\right], \tag{155}$$

$$\mathbf{L}^i_{\text{NLO-SS}} = \frac{Gm\nu}{2r^2}\ell^0_6. \tag{156}$$

The lengthy expression for the ℓ^0_5, ℓ^1_5 and ℓ^0_6 coefficients can be found in Appendix D. Notice that in the orbital angular momentum, there is no spin-spin contribution at 2PN order. The 3PN expression for the spin-spin orbital angular momentum is presented here for the first time. It can be confirmed explicitly that the evolution of the total angular momentum is conserved, i.e., that

$$\frac{d}{dt}(\mathbf{L}^i + \mathbf{S}^i) = 0, \tag{157}$$

using the EoM derived from the gravitational potential to 3PN order [80, 84, 85, 75, 81], which provides a nontrivial check of our result.

3. COM position

We now proceed to compute the COM correction at NLO for spin-orbit and spin-spin effects. But before proceeding to the details of its computation, notice that this correction should have, in principle, entered in the calculation of NLO spin-orbit and spin-spin quantities. This correction does not affect the result for the NLO spin-orbit and spin-spin acceleration since the Newtonian acceleration is naturally given in terms of relative coordinates. This argument does not hold for the Newtonian energy, but it turns out that the NLO contribution that would arise from it cancels due to its symmetry:

$$E^{(0\mathrm{PN})} = \frac{m_1 \mathbf{v}_1^2}{2} + \frac{m_2 \mathbf{v}_2^2}{2} - \frac{G m_1 m_2}{r} \rightarrow \frac{m_1}{2}\frac{m_2}{m}\mathbf{v}\cdot\delta\dot{\mathbf{x}} - \frac{m_2}{2}\frac{m_1}{m}\mathbf{v}\cdot\delta\dot{\mathbf{x}} = 0. \qquad (158)$$

The same happens to the LO mass quadrupole moment $I^{ij}_{0PN} = \sum_a m_a \, [\mathbf{x}^i_a \mathbf{x}^j_a]_{TF}$ when we try to extract its 2.5PN contribution going to the COM frame, and consequently the energy loss due to NLO spin-orbit effects is not affected by the correction to the COM at this order. Despite this, the NLO spin-orbit correction to the COM, which is an effect that enters at 2.5PN order, itself is a nonzero quantity and must be obtained, since it will lead to nonzero contributions in future computations at NNLO order. Below, we present how we proceed to obtain this quantity via the NRGR framework.

a. Spin-orbit

The COM position can be extracted using the expression (14) and is computed in a manner similar to the multipole moments as shown in section II.B.3. Specifically, the quantity in parentheses in the linear in momentum term

$$T^{00}(\mathbf{q},t) = \int_{\mathbf{q}} \left(\int T^{00}(\mathbf{x},t)\mathbf{x}^k \right) (-i\mathbf{q}^k) \qquad (159)$$

from the expansion of equation (19) is the COM integral. The diagrams that contribute to the NLO spin-orbit COM position are given in figure 3. Diagram 3a comes from a single

Figure 3: Diagrams contributing to the 1.5PN and 2.5PN spin-orbit COM correction.

insertion of the vertex (311). Imposing the covariant SSC gives a LO spin-orbit term and a 1PN correction given by

$$T^{00}_{3\mathrm{a}}(t,\mathbf{q}) = \sum_{A\neq B} S^{0i}_A (i\mathbf{q}^i) e^{-i\mathbf{q}\cdot\mathbf{x}_A} \xrightarrow{\text{(cov)}} \sum_{A\neq B} S^{ij}_A (i\mathbf{q}^j)\left(\mathbf{v}^i_A + \frac{2Gm_B}{r}\mathbf{v}^i\right) e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{160}$$

At the order we are working, figure 3b is composed of two different contributions, as we show next. Contracting (312) with (306), we find

$$T^{00}_{3\mathrm{b},1}(t,\mathbf{q}) = \sum_{A\neq B}\left[-\frac{2Gm_B}{r} S^{ij}_A \mathbf{v}^i_B (i\mathbf{q}^j)\right] e^{-i\mathbf{q}\cdot\mathbf{x}_A}, \tag{161}$$

and contracting (313) with (305) gives

$$T^{00}_{3\mathrm{b},2}(t,\mathbf{q}) = \sum_{A\neq B} \frac{Gm_B S^{0j}_A \mathbf{x}^j}{r^3} e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{162}$$

Figure 3c also accounts for two distinct contributions. Contracting (309) with (308), we have

$$T^{00}_{3\mathrm{c},1}(t,\mathbf{q}) = \sum_{A\neq B} \frac{2Gm_A S^{ij}_B \mathbf{v}^i_A \mathbf{x}^j}{r^3} e^{-i\mathbf{q}\cdot\mathbf{x}_A}, \tag{163}$$

and contracting (310) with (307) gives

$$T^{00}_{3\mathrm{c},2}(t,\mathbf{q}) = \sum_{A\neq B}\left[\frac{Gm_A}{r^3}\left(-S^{0i}_B \mathbf{x}^i - S^{ij}_B \mathbf{v}^i_B \mathbf{x}^j\right)\right] e^{i\mathbf{q}\cdot\mathbf{x}_A}. \tag{164}$$

Finally, figure 3d comes from three different contractions. The first contribution, constructed from (305) and (310) together with the LO 3-point vertex gives

$$T^{00}_{3\mathrm{d},1}(t,\mathbf{q}) = \sum_{A\neq B}\bigg[\frac{3}{2}\frac{Gm_A}{r^3}S^{0j}_B\mathbf{x}^j - \frac{3}{2}\frac{Gm_B}{r^3}S^{0j}_A\mathbf{x}^j - \frac{3}{2}\frac{Gm_B}{r}S^{0j}_A(i\mathbf{q}^j) + \frac{3}{2}\frac{Gm_A}{r^3}S^{ij}_B\mathbf{v}^i_B\mathbf{x}^j + \frac{1}{2}\frac{Gm_B}{r^3}S^{ij}_A\mathbf{v}^i_A\mathbf{x}^j + \frac{1}{2}\frac{Gm_B}{r}S^{ij}_A\mathbf{v}^i_A(i\mathbf{q}^j)\bigg]e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \quad (165)$$

The second, constructed from (306) and (309) with the LO 3-point vertex, is

$$T^{00}_{3\mathrm{d},2}(t,\mathbf{q}) = \sum_{A\neq B}\bigg[-\frac{Gm_A}{r^3}S^{ij}_B\mathbf{v}^i_A\mathbf{x}^j + \frac{Gm_B}{r^3}S^{ij}_A\mathbf{v}^i_B\mathbf{x}^j + \frac{Gm_B}{r}S^{ij}_A\mathbf{v}^i_B(i\mathbf{q}^j)\bigg]e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \quad (166)$$

The third, constructed from (305) and (309) with the 3-point vertex at $\mathcal{O}(v^1)$ reads

$$T^{00}_{3\mathrm{d},3}(t,\mathbf{q}) = \sum_{A\neq B}\bigg[\frac{Gm_A}{r}S^{ij}_B(\mathbf{v}^i_A+\mathbf{v}^i_B)(i\mathbf{q}^j) - \frac{Gm_A}{r^3}S^{ij}_B\mathbf{x}^i(i\mathbf{q}^j)\mathbf{x}\cdot(\mathbf{v}_A+\mathbf{v}_B)\bigg]e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \quad (167)$$

Now, putting all the contributions above together, we write the final expression for the 00-component of the stress-pseudo tensor accounting for NLO spin-orbit terms:

$$T^{00}_{\mathrm{SO}}(t,\mathbf{q}) = \sum_{A\neq B}\bigg\{S^{0i}_A(i\mathbf{q}^i) + \frac{G}{r^3}\bigg[\tfrac{1}{2}m_A S^{0j}_B\mathbf{x}^j - \tfrac{1}{2}m_B S^{0j}_A\mathbf{x}^j + \big(-\tfrac{3}{2}m_B S^{0j}_A + m_B(\tfrac{1}{2}\mathbf{v}^i_A - \mathbf{v}^i_B)S^{ij}_A + m_A(\mathbf{v}^k_A+\mathbf{v}^k_B)(\delta^{ik}-\mathbf{n}^i\mathbf{n}^k)S^{ij}_B\big)r^2(i\mathbf{q}^j) + m_B(\mathbf{v}^i_B+\tfrac{1}{2}\mathbf{v}^i_A)\mathbf{x}^j S^{ij}_A + m_A(\tfrac{1}{2}\mathbf{v}^i_B+\mathbf{v}^i_A)\mathbf{x}^j S^{ij}_B\bigg]\bigg\}e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \quad (168)$$

We can extract some information regarding the binary system from the expression above when we take the long-wavelength limit by Taylor expanding around $\mathbf{q}=0$. For instance, the zeroth order terms in the Taylor expansion give us the LO spin-orbit energy

$$E_{\text{LO-SO}} = \int d^3x\, T^{00}(\mathbf{x},t) = -\sum_{A\neq B}\frac{Gm_B}{r^3}S^{0j}_A\mathbf{x}^j, \quad (169)$$

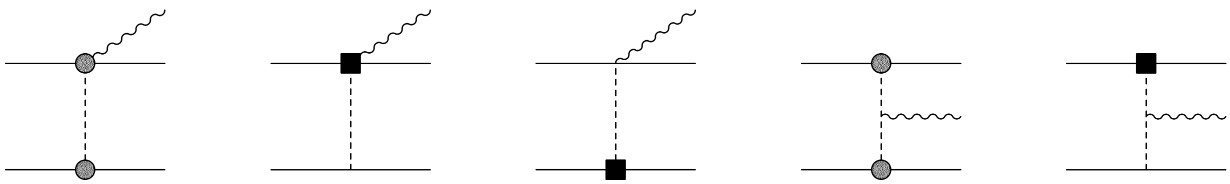


Figure 4: Diagrams contributing to the 2PN and 3PN spin-spin COM correction.

and this serves as a self-consistency check, since (169) agrees with equation (141), which we calculated from the LO spin-orbit potential. Next, the terms linear in $\mathbf{q}$ yield the COM position, which is also conveniently expressed through[10] $\mathbf{G} \equiv m\mathbf{X}_{\rm cm}$:

$$\mathbf{G}^k_{(1.5\rm PN)} = -\sum_{A=1}^{2} S_A^{0k} = -\sum_{A=1}^{2} S_A^{ik}\mathbf{v}_A^i, \tag{170}$$

$$\mathbf{G}^k_{(2.5\rm PN)} = \sum_{A\neq B} \frac{Gm_B}{r^3}\Big[S_A^{ij}\mathbf{x}^j(\mathbf{v}_B^i\mathbf{x}^k - \mathbf{v}_A^i\mathbf{x}_B^k) - S_A^{ik}\big(2r^2\mathbf{v}^i - \mathbf{x}^i\mathbf{x}\cdot(\mathbf{v}_A+\mathbf{v}_B)\big)\Big]. \tag{171}$$

b. Spin-spin

We now compute the NLO spin-spin COM, which requires the computation of the five diagrams shown in figure 4, as well as contributions emerging from applying the covariant SSC frame corrections to the LO result in equation (160).

The first diagram we compute is figure 4a, with the vertices (309) and (312) with a single graviton exchange, which yields

$$T^{00}_{(1)}(\mathbf{q},t) = \sum_{a\neq b} -\frac{G}{r^3} S_a^{ij} S_b^{jk}\mathbf{x}^k(i\mathbf{q}^i)e^{-i\mathbf{q}\cdot\mathbf{x}_a}. \tag{172}$$

[10]The expression for $\mathbf{G}$ can be expanded order by order as $\mathbf{G} = \mathbf{G}^{(0\rm PN)} + \mathbf{G}^{(1\rm PN)} + \mathbf{G}^{(1.5\rm PN)}_{\rm SO} + \mathbf{G}^{(2\rm PN)} + \mathbf{G}^{(2.5\rm PN)} + \cdots$; the LO and 1PN corrections can be found in [94], while the 2PN correction was computed in [78].

The second diagram in figure 4b, computed from the vertices (315) and (305), gives the contribution

$$T^{00}_{(2)}(\mathbf{q},t) = \sum_{a\neq b} C^{(a)}_{ES^2} \frac{Gm_b}{m_a r^3}\Big[\big(2S^{ik}_a S^{jk}_a \mathbf{x}^i + \tfrac{1}{2} S^{ik}_a S^{ik}_a \mathbf{x}^j\big)(i\mathbf{q}^j) \\ - \tfrac{1}{2} S^{ik}_a S^{ik}_a + \tfrac{3}{2}\mathbf{n}^i\mathbf{n}^j S^{ik}_a S^{jk}_a\Big] e^{-i\mathbf{q}\cdot\mathbf{x}_a}. \tag{173}$$

The last diagram with a single graviton exchange is shown in figure 4c, computed from the vertices (307) and (314). The contribution from this diagram is given by

$$T^{00}_{(3)}(\mathbf{q},t) = \sum_{a\neq b} \frac{C^{(b)}_{ES^2} G m_a}{2m_b r^3}\left(-S^{ik}_b S^{ik}_b + 3S^{ik}_b S^{jk}_b \mathbf{n}^i\mathbf{n}^j\right) e^{-i\mathbf{q}\cdot\mathbf{x}_a}. \tag{174}$$

The next two diagrams are constructed from two matter vertices and a 3-point graviton vertex. The first, shown in figure 4d, has two insertions of (309) and the three-point graviton vertex. We find from this diagram the expression

$$T^{00}_{(4)}(\mathbf{q},t) = \sum_{a\neq b} \frac{G}{2r^3}\left(S^{ik}_a S^{ik}_b - 3\mathbf{n}^l\mathbf{n}^k S^{ik}_a S^{il}_b - S^{il}_a S^{ik}_b \mathbf{x}^k (i\mathbf{q}^l)\right) e^{-i\mathbf{q}\cdot\mathbf{x}_a}. \tag{175}$$

The final diagram we need at 3PN order is shown in figure 4e, constructed from (314) and (305) and the three-point graviton vertex. Computing this final diagram yields

$$T^{00}_{(5)}(\mathbf{q},t) = \sum_{a\neq b} \frac{3G}{4r^3}\left[C^{(a)}_{ES^2}\frac{m_b}{m_a}\left(S^{ik}_a S^{ik}_a - (3\mathbf{n}^i\mathbf{n}^j - 2\mathbf{x}^j(i\mathbf{q}^i))S^{ik}_a S^{jk}_a\right) \\ + C^{(b)}_{ES^2}\frac{m_a}{m_b}\left(S^{ik}_b S^{ik}_b - 3S^{ik}_b S^{jk}_b \mathbf{n}^i\mathbf{n}^j\right)\right] e^{-i\mathbf{q}\cdot\mathbf{x}_a}. \tag{176}$$

Finally, we need the LO spin-orbit diagram shown in figure 3(a) and computed in equation (160) with the spin-spin piece of the SSC imposed, yielding

$$T^{00}_{(6)}(\mathbf{q},t) = \sum_{A\neq B} S^{0i}_a (i\mathbf{q}^i) e^{-i\mathbf{q}\cdot\mathbf{x}_a} \overset{\text{cov.}}{\to} \sum_{A\neq B} \frac{G}{r^3} S^{ik}_a S^{jk}_b \mathbf{x}^j (i\mathbf{q}^i) e^{-i\mathbf{q}\cdot\mathbf{x}_a}. \tag{177}$$

Summing over diagrams and taking the limit $\mathbf{q}\to 0$, we find the spin-spin energy

$$\int d^3x\, T^{00}(t,\mathbf{x}) = \frac{G}{2r^3}\sum_{a\neq b}\left[S^{ik}_a S^{jk}_b + 3C^{(a)}_{ES^2}\frac{m_b}{m_a} S^{ik}_a S^{jk}_a\right](\delta^{ij} - 3\mathbf{n}^i\mathbf{n}^j), \tag{178}$$

which agrees with our result derived from the 2PN spin-spin potential, an important check on our result.

Expanding to leading order in $\mathbf{q}$ we have

$$T^{00}_{(1)}(\mathbf{q},t) = \sum_{a\neq b} \frac{G}{r^3}\Big\{ \frac{3}{2} S^{ik}_a S^{jk}_b \mathbf{x}^j (i\mathbf{q}^i) - \tfrac{1}{2} S^{ik}_a S^{ik}_b (i\mathbf{q}\cdot\mathbf{x}_a) + \tfrac{3}{2} S^{ik}_a S^{jk}_b \mathbf{n}^i\mathbf{n}^j (i\mathbf{q}\cdot\mathbf{x}_a) + \frac{1}{2} C^{(a)}_{ES^2} \frac{m_b}{m_a} S^{ik}_a S^{jk}_a \big[\mathbf{x}^i (i\mathbf{q}^j) + \tfrac{1}{2}\delta^{ij}(i\mathbf{q}\cdot\mathbf{x}) - \tfrac{3}{2}\mathbf{n}^i\mathbf{n}^j (i\mathbf{q}\cdot(\mathbf{x}_a+\mathbf{x}_b))\big]\Big\} \tag{179}$$

Then the COM is found by using equation (159) and thus

$$\mathbf{G}^i_{(\text{2PN SS})} = 0 \tag{180}$$

$$\mathbf{G}^i_{(\text{3PN SS})} = \sum_{a\neq b} \frac{G}{r^3}\Big\{ -\frac{3}{2} S^{ik}_a S^{jk}_b \mathbf{x}^j + \frac{1}{2} S^{jk}_a S^{jk}_b \mathbf{x}^i_a - \frac{3}{2} S^{jk}_a S^{lk}_b \mathbf{n}^j\mathbf{n}^l \mathbf{x}^i_a - \frac{1}{2} C^{(a)}_{ES^2}\frac{m_b}{m_a}\Big[S^{jk}_a S^{ik}_a \mathbf{x}^j + \frac{1}{2} S^{jk}_a S^{jk}_a \mathbf{x}^i - \frac{3}{2} S^{jk}_a S^{lk}_a \mathbf{n}^j \mathbf{n}^l (\mathbf{x}_a+\mathbf{x}_b)^i\Big]\Big\}. \tag{181}$$

c. Coordinate shift to COM frame

Now, in order to extract its corrections, we put the COM at the origin, meaning $\mathbf{G}=0$, and iteratively solve for $\mathbf{x}_1$, $\mathbf{x}_2$. Using equations (117–119) we can determine PN corrections to the COM order by order. The corrections $\delta\mathbf{x}^{(1\text{PN})}$ and $\delta\mathbf{x}^{(1.5\text{PN})}_{\text{SO}}$ can be found in [94, 117] and are presented in equation (117), while the nonspin[11] $\delta\mathbf{x}^{(2\text{PN})}$ can be found in [78]. The NLO correction, with covariant SSC enforced, is

$$(\delta\mathbf{x}^i)_{\text{NLO}-\text{SO}} = \frac{\nu}{2m}\Big\{ \Big[\nu\mathbf{v}^2 - \frac{Gm}{r}(4+2\nu)\Big](\mathbf{\Sigma}\times\mathbf{v})^i + \delta\Big[\mathbf{v}^2 - \frac{Gm}{r}\Big](\mathbf{S}\times\mathbf{v})^i + \frac{2Gm}{r}\Big[\delta\mathbf{S}\cdot(\mathbf{v}\times\mathbf{n})\mathbf{n}^i + \frac{3\delta}{2}\dot{r}(\mathbf{S}\times\mathbf{n})^i + (1-4\nu)\dot{r}(\mathbf{\Sigma}\times\mathbf{n})^i\Big]\Big\}, \tag{182}$$

and

$$(\delta\mathbf{x}^i)_{\text{NLO}-\text{SS}} = \mathbf{S}^i\Big[12\kappa_-(nS) + (12+6\delta\kappa_- - 6\kappa_+)(n\Sigma)\Big] + \mathbf{\Sigma}^i\Big[(-4+6\delta\kappa_- - 6\kappa_+)(nS) + (-6\delta\kappa_+ + \kappa_-(6-12\nu))(n\Sigma)\Big] + \mathbf{n}^i\Big[(12\delta+6\delta\kappa_+)(nS)^2 + (12\delta^2 - 6\delta\kappa_- + 6\delta^2\kappa_+)(nS)(n\Sigma)$$

[11] There is no spin correction to the COM position at 2PN order.

$$+(-3\delta^2\kappa_- + \delta\kappa_+(3-6\nu) - 12\delta\nu)(n\Sigma)^2 + (-4\delta - 4\kappa_- - 2\delta\kappa_+)\mathbf{S}^2$$
$$+(-2\delta\kappa_- + 4(-3+4\nu) + \kappa_+(2+8\nu))(S\Sigma)$$
$$+(-\kappa_- + 4\delta\nu + \kappa_+(\delta + 2\delta\nu))\mathbf{\Sigma}^2\Big] \tag{183}$$

D. Flux-balance laws

1. Energy flux

The binary system's energy loss due to the emission of gravitational waves can be computed directly from the one-graviton emission amplitude in the effective theory (for a detailed discussion, see [97, 99]).

For the computation of the energy flux at the order we need in this section, we have, e.g., [97]

$$\frac{dE}{dt} = -\frac{G}{5}\left(I_{ij}^{(3)}\, I_{ij}^{(3)} + \frac{16}{9}\, J_{ij}^{(3)}\, J_{ij}^{(3)} + \frac{5}{189} I_{ijk}^{(4)} I_{ijk}^{(4)} + \frac{5}{84} J_{ijk}^{(4)}\, J_{ijk}^{(4)} + \cdots\right), \tag{184}$$

in terms of the source multipole moments. The explicit expression for the latter are given in Appendix C in the COM frame in equations (324–340). The time derivatives are order-reduced using the equations of motion, also found in Appendix C and in section III.A. The result can be expanded in the PN expansion as

$$\frac{dE}{dt} = \left(\frac{dE}{dt}\right)_{\rm N} + \left(\frac{dE}{dt}\right)_{\rm 1PN} + \left(\frac{dE}{dt}\right)_{\rm LO\text{-}SO} + \left[\left(\frac{dE}{dt}\right)_{\rm 2PN} + \left(\frac{dE}{dt}\right)_{\rm LO\text{-}SS}\right]$$
$$+ \left(\frac{dE}{dt}\right)_{\rm NLO\text{-}SO} + \left[\left(\frac{dE}{dt}\right)_{\rm 3PN} + \left(\frac{dE}{dt}\right)_{\rm NLO\text{-}SS}\right] + \cdots . \tag{185}$$

Through direct computation, we arrive at the NLO spin-orbit energy loss:

$$\left(\frac{dE}{dt}\right)_{\rm NLO\text{-}SO} = -\frac{2G^3m^3\nu}{105r^4}\left\{\frac{\mathbf{L}\cdot\mathbf{S}}{mr^2}\left[(3776+1560\nu)\frac{G^2m^2}{r^2} + (-12892+2024\nu)\frac{Gm}{r}\dot{r}^2\right.\right.$$
$$+ (15164-560\nu)\frac{Gm}{r}\mathbf{v}^2 + (-8976+12576\nu)\dot{r}^4$$
$$\left. + (13362-18252\nu)\dot{r}^2\mathbf{v}^2 + (-4226+5952\nu)\mathbf{v}^4\right]$$

$$
\begin{aligned}
&+\delta\frac{\mathbf{L}\cdot\mathbf{\Sigma}}{mr^2}\bigg[(-548+952\nu)\frac{G^2m^2}{r^2}+(-14654+4796\nu)\frac{Gm}{r}\dot{r}^2\\
&+(10718-1708\nu)\frac{Gm}{r}\mathbf{v}^2+(-7941+10704\nu)\dot{r}^4\\
&+(8742-13434\nu)\dot{r}^2\mathbf{v}^2+(-2001+3474\nu)\mathbf{v}^4\bigg]\bigg\}.
\end{aligned}
\tag{186}
$$

Keeping only spin-spin contributions to the leading order, we find

$$
\begin{aligned}
\left(\frac{dE}{dt}\right)_{\text{LO-SS}} = \frac{G^3m^2\nu^2}{105r^6}\bigg[&(696+348\kappa_+)(nS)(nv)(vS)\\
&+(348\delta-174\kappa_-+174\delta\kappa_+)(nv)(n\Sigma)(vS)+(-144-72\kappa_+)(vS)^2\\
&+\mathbf{S}^2\big((312+156\kappa_+)(nv)^2+(-288-144\kappa_+)\mathbf{v}^2\big)\\
&+(nS)^2\big((-1632-816\kappa_+)(nv)^2+(1008+504\kappa_+)\mathbf{v}^2\big)\\
&+(S\Sigma)\big((312\delta-156\kappa_-+156\delta\kappa_+)(nv)^2+(-288\delta+144\kappa_--144\delta\kappa_+)\mathbf{v}^2\big)\\
&+(nS)(n\Sigma)\big((-1632\delta+816\kappa_--816\delta\kappa_+)(nv)^2+(1008\delta-504\kappa_-+504\delta\kappa_+)\mathbf{v}^2\big)\\
&+(n\Sigma)^2\Big[\big(-9+408\delta\kappa_-+1632\nu+408\kappa_+(-1+2\nu)\big)(nv)^2\\
&+\big(-252\delta\kappa_-+\kappa_+(252-504\nu)-1008\nu\big)\mathbf{v}^2\Big]\\
&+(348\delta-174\kappa_-+174\delta\kappa_+)(nS)(nv)(v\Sigma)\\
&+\big(6-174\delta\kappa_-+\kappa_+(174-348\nu)-696\nu\big)(nv)(n\Sigma)(v\Sigma)\\
&+(-144\delta+72\kappa_--72\delta\kappa_+)(vS)(v\Sigma)\\
&+\big(-1+36\delta\kappa_-+144\nu+\kappa_+(-36+72\nu)\big)(v\Sigma)^2\\
&+\Big[\big(-9-78\delta\kappa_-+\kappa_+(78-156\nu)-312\nu\big)(nv)^2\\
&+\big(-3+72\delta\kappa_-+288\nu+72\kappa_+(-1+2\nu)\big)\mathbf{v}^2\Big]\mathbf{\Sigma}^2\bigg].
\end{aligned}
\tag{187}
$$

At 3PN order, we have

$$
\left(\frac{dE}{dt}\right)_{\text{NLO-SS}} = \frac{G^3m^2\nu^2}{105r^6}\left[f_6^0+\frac{Gm}{r}f_6^1+\frac{G^2m^2}{r^2}f_6^2\right],
\tag{188}
$$

with the value of the f_6^0, f_6^1 and f_6^2 coefficients given in Appendix D. After using the transformation in (89), we have checked that the above expressions are equivalent to those obtained in [36, 39].

We consider next the particular case of aligned-spin quasicircular orbits. Since radiation-reaction effects first enter at 2.5PN order beyond the leading effects, namely at 4PN and 4.5PN for spin-orbit and spin-spin contributions, respectively, we can safely ignore them here while working to 3PN order. Hence, we can consistently replace $\dot{r} = 0$ and $v^2 = r^2\omega^2$ in (187) and (188) to obtain the emitted radiation. However, we must still face a radiation-reaction force which may induce the precession of the orbital plane. Even though the precession effects might be small, $d\ell/dt \leq \mathcal{O}(v^8)$, these may accumulate over time. Nonadiabatic methods, such as dynamical renormalization group, offer an alternative approach that remains valid over a longer timescale by resumming secular effects [121, 122]. In this work, however, we restrict our results to time scales *shorter* than those induced by the secular evolution of the orbital plane. We find in terms of the conserved-norm spin vectors

$$(\dot{E}^c)_{\rm SO} = -\frac{32x^5\nu^2}{5G}\Big[1 + x(-\tfrac{1247}{336} - \tfrac{35}{12}\nu) + \frac{x^{3/2}}{Gm^2}(-\tfrac{5}{4}\delta\Sigma^c_\ell - 4S^c_\ell) + x^2(-\tfrac{44711}{9072} + \tfrac{9271}{504}\nu + \tfrac{65}{18}\nu^2) + \frac{x^{5/2}}{Gm^2}[(-\tfrac{13}{16} + \tfrac{43}{4}\nu)\delta\Sigma^c_\ell + (-\tfrac{9}{2} + \tfrac{272}{9}\nu)S^c_\ell]\Big]. \quad (189)$$

and

$$(\dot{E}^c)_{\rm SS} = -\frac{32x^7\nu^2}{5G^3m^4}\Big[(4 + 2\kappa_+)\mathbf{S}_{\rm c}^2 + (4\delta - 2\kappa_- + 2\delta\kappa_+)(S_{\rm c}\Sigma_{\rm c}) + \big(\tfrac{1}{16} - \delta\kappa_- + \kappa_+ + (-4 - 2\kappa_+)\nu\big)\mathbf{\Sigma}_{\rm c}^2\Big]$$
$$- \frac{32x^8\nu^2}{5G^3m^4}\Big[\big(-\tfrac{5239}{504} + \tfrac{41}{16}\delta\kappa_- - \tfrac{271}{112}\kappa_+ + (-\tfrac{43}{2} - \tfrac{43}{4}\kappa_+)\nu\big)\mathbf{S}_{\rm c}^2 + \big(-\tfrac{817}{56}\delta + \tfrac{279}{56}\kappa_- - \tfrac{279}{56}\delta\kappa_+ + (-\tfrac{43}{2}\delta + \tfrac{1}{2}\kappa_- - \tfrac{43}{4}\delta\kappa_+)\nu\big)(S_{\rm c}\Sigma_{\rm c})$$
$$+ \big(-\tfrac{25}{8} + \tfrac{279}{112}\delta\kappa_- - \tfrac{279}{112}\kappa_+ + (\tfrac{344}{21} + \tfrac{45}{16}\delta\kappa_- + \tfrac{243}{112}\kappa_+)\nu + (\tfrac{43}{2} + \tfrac{43}{4}\kappa_+)\nu^2\big)\mathbf{\Sigma}_{\rm c}^2\Big]. \quad (190)$$

Along with the expressions for the NLO spin accelerations and conserved energy, the above result is the final piece needed to compute the orbital phase evolution of the binary system in the quasicircular orbit approximation as in section III.E, accounting for NLO spin effects in the NRGR framework.

2. Angular momentum flux

The source multipoles allow us also to compute the radiated angular momentum via

$$\frac{d\mathbf{J}^i}{dt} = -G\epsilon^{iab}\left(\frac{2}{5}I^{(2)}_{aj}I^{(3)}_{bj} + \frac{32}{45}J^{(2)}_{aj}J^{(3)}_{bj} + \frac{1}{63}I^{(3)}_{ajk}I^{(4)}_{bjk} + \frac{1}{28}J^{(3)}_{ajk}J^{(4)}_{bjk} + \cdots\right), \tag{191}$$

which may be also expanded into PN contributions as

$$\begin{aligned}\frac{d\mathbf{J}^i}{dt} = \left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{N}} + \left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{1PN}} + \left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{LO-SO}} + \left[\left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{2PN}} + \left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{LO-SS}}\right] \\ + \left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{NLO-SO}} + \left[\left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{3PN}} + \left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{NLO-SS}}\right] + \cdots .\end{aligned} \tag{192}$$

For completeness, we include here both the spin-orbit and spin-spin terms to NLO in the PN expansion. The results can be written as

$$\left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{LO-SO}} = -\frac{G^2m^2\nu^2}{15r^3}\left[g_3^{0i} + 4\frac{Gm}{r}g_3^{1i} + 8\frac{G^2m^2}{r^2}g_3^{2i}\right], \tag{193}$$

$$\left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{LO-SS}} = -\frac{G^2m\nu^2}{5r^4}\left[g_4^{0i} + \frac{Gm}{r}g_4^{1i}\right], \tag{194}$$

$$\left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{NLO-SO}} = -\frac{G^2m^2\nu^2}{105r^3}\left[g_5^{0i} + \frac{Gm}{r}g_5^{1i} + \frac{G^2m^2}{r^2}g_5^{2i} + \frac{G^3m^3}{r^3}g_5^{3i}\right], \tag{195}$$

$$\left(\frac{d\mathbf{J}^i}{dt}\right)_{\text{NLO-SS}} = -\frac{G^2m\nu^2}{35r^4}\left[\frac{1}{2}g_6^{0i} + \frac{Gm}{r}\frac{1}{3}g_6^{1i} + \frac{G^2m^2}{r^2}\frac{1}{3}g_6^{2i}\right], \tag{196}$$

with the $g_3^{(0,1,2)i}, g_4^{(0,1)i}, g_5^{(0,1,2,3)i}$ and $g_6^{(0,1,2)i}$ coefficients displayed in Appendix D. The LO spin-orbit and spin-spin expressions agree with corresponding results in [117, 123]. It is straightforward to confirm that our results are consistent with

$$\frac{d\mathbf{J}^i}{dt} = \frac{1}{\omega}\frac{dE}{dt}\ell^i, \tag{197}$$

for spin effects to 3PN order for nonprecessing (quasi)circular orbits, a nontrivial check of our results.

3. Linear momentum flux, COM flux, and kick velocity

Finally, we can compute the flux associated with linear momentum and COM of the binary systems, which evolve due to the emission of GWs according to, e.g. [124],

$$\frac{d\mathbf{P}^i}{dt} = -G\left(\frac{2}{63}I^{(4)}_{ijk}I^{(3)}_{jk} + \frac{16}{45}\epsilon_{ijk}I^{(3)}_{jl}J^{(3)}_{kl} + \frac{1}{126}\epsilon_{ijk}I^{(4)}_{jlm}J^{(4)}_{klm} + \frac{4}{63}J^{(4)}_{ijk}J^{(3)}_{jk} + \cdots\right), \tag{198}$$

and (with $\mathbf{G}^i \equiv m\mathbf{X}^i$)

$$\frac{d\mathbf{G}^i}{dt} = \mathbf{P}^i - G\left(\frac{1}{21}\left(I^{(3)}_{ijk}I^{(3)}_{jk} - I^{(4)}_{ijk}I^{(2)}_{jk}\right) + \frac{2}{21}\left(J^{(3)}_{ijk}J^{(3)}_{jk} - J^{(4)}_{ijk}J^{(2)}_{jk}\right) + \cdots\right), \tag{199}$$

respectively. We can once again expand in various PN contributions,

$$\begin{aligned}\frac{d\mathbf{P}^i}{dt} &= \left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{1PN}} + \left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{LO-SO}} + \left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{2PN}} \\ &+ \left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{NLO-SO}} + \left[\left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{3PN}} + \left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{LO-SS}}\right] + \cdots\,,\end{aligned} \tag{200}$$

and likewise,

$$\frac{d\mathbf{G}^i}{dt} = \left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{1PN}} + \left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{2PN}} + \left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{LO-SO}} + \left[\left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{3PN}} + \left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{LO-SS}}\right] + \cdots\,. \tag{201}$$

Unfortunately, we do not have all the necessary source multipoles to complete the spin corrections to NLO in the COM, notably missing the NLO corrections to J_{ijk}, and therefore for the purpose of this paper we will keep spin-spin effects at leading order. We do, however, have all the information to compute the NLO spin-orbit contributions to the linear momentum. Inputting the source multipoles we find, in the COM frame,[12] the nonspinning piece

$$\begin{aligned}\left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{1PN}} = -\frac{8\delta G^3 m^4 \nu^2}{105 r^4}\bigg[&\mathbf{v}^i\left(-\frac{8Gm}{r} + 38(nv)^2 - 50\mathbf{v}^2\right) \\ &+ \mathbf{n}^i(nv)\left(\frac{12Gm}{r} - 45(nv)^2 + 55\mathbf{v}^2\right)\bigg],\end{aligned} \tag{202}$$

[12] The condition $\mathbf{X} = 0$ for the COM frame becomes more subtle once we allow for noninertial motion due to GW emission as in section IV.E. However, these effects can be ignored to 3PN order thanks to its nonsecular behavior, i.e., $\mathbf{X} = \mathcal{O}(x^{7/2})$, see e.g. [125].

and spinning pieces

$$\left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{LO-SO}} = -\frac{8G^3m^3\nu^2}{15r^5}\Big[4(nv)(\mathbf{v}\times\mathbf{\Sigma})^i - 2(\mathbf{n}\times\mathbf{\Sigma})^i\mathbf{v}^2 + \big(\mathbf{n}\times\mathbf{v}\big)^i(-3(nv)(n\Sigma) - 2(v\Sigma))\Big] \tag{203}$$

$$\left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{NLO-SO}} = -\frac{4G^3m^3\nu^2}{945r^5}\left[h_5^{0i} + \frac{Gm}{r}h_5^{1i} + \frac{G^2m^2}{r^2}h_5^{2i}\right], \tag{204}$$

$$\left(\frac{d\mathbf{P}^i}{dt}\right)_{\text{LO-SS}} = -\frac{2G^3m^2\nu^2}{105r^6}\left[h_6^{0i} + 4\frac{Gm}{r}h_6^{1i}\right], \tag{205}$$

whereas for the COM position we have

$$\left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{1PN}} = \frac{G^2m^3\nu^2\delta}{35r^2}\Bigg\{-\mathbf{v}^i(nv)\left[\frac{24Gm}{r} + \frac{4}{3}(15(nv)^2 - 29\mathbf{v}^2)\right] - \mathbf{n}^i\Bigg[-\frac{16G^2m^2}{r^2} + \frac{4Gm}{3r}(89(nv)^2 - 101\mathbf{v}^2) - \frac{2}{3}(225(nv)^4 - 366(nv)^2\mathbf{v}^2 + 113\mathbf{v}^4)\Bigg]\Bigg\} \tag{206}$$

$$\left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{LO-SO}} = -\frac{G^2m^2\nu^2}{105r^3}\left[k_5^{0i} + 2\frac{Gm}{r}k_5^{1i} + 8\frac{G^2m^2}{r^2}k_5^{2i}\right], \tag{207}$$

$$\left(\frac{d\mathbf{G}^i}{dt}\right)_{\text{LO-SS}} = -\frac{G^2m\nu^2}{105r^4}\left[k_6^{0i} + \frac{Gm}{r}k_6^{1i} + \frac{G^2m^2}{r^2}k_6^{2i}\right]. \tag{208}$$

The coefficients $h_5^{(0,1,2)i}, h_6^{(0,1)i}$ and $k_5^{(0,1,2)i}, k_6^{(0,1,2)i}$ are reproduced in Appendix D.

The calculation of the radiated linear momentum was also computed in [123], using the earlier results in [30, 36]. After transforming from the locally-flat to the PN frame, we find agreement for spin-orbit contributions to NLO in the PN expansion. However, we disagree on the spin-spin corrections for generic orbits. We have explicitly checked that the difference is due to the omission of the finite-size contributions to the current quadrupole and mass octupole moments in [123].

The above expressions take on a much simpler form for the case of (quasi)circular orbits. Using the conserved-norm spin vector, they become

$$\frac{d\mathbf{P}^i}{dt} = -\frac{\lambda^i\nu^2}{315G}\Bigg\{(-1392\delta x^{11/2} + \cdots) + \frac{336x^6}{Gm^2}\Sigma_\ell^{\rm c} + \frac{x^7}{Gm^2}[7520\delta S_\ell^{\rm c} + (1608 - 9888\nu)\Sigma_\ell^{\rm c}] + \frac{x^{15/2}}{G^2m^4}\Big[(-6960\delta - 240\kappa_- - 3480\delta\kappa_+)(S_\ell^{\rm c})^2 + \big(3240\delta\kappa_- + 120\kappa_+(-27+116\nu) + 48(-181+580\nu)\big)S_\ell^{\rm c}\Sigma_\ell^{\rm c}$$

$$+ 60\big(4\delta(-3+29\nu) + \delta\kappa_+(-27+58\nu) - \kappa_-(-27+112\nu)\big)(\Sigma_\ell^{\rm c})^2\Big]\Big\}, \tag{209}$$

for the linear momentum including leading spin-spin and up to NLO spin-orbit effects, whereas for the COM position we have

$$\begin{aligned}\frac{d\mathbf{G}^i}{dt} = -\frac{\mathbf{n}^i\nu^2 m}{640}\Big\{&(-4068\delta x^4 + \cdots) + \frac{x^{11/2}}{Gm^2}[21822\delta S_\ell^{\rm c} - 18(-439+1441\nu)\Sigma_\ell^{\rm c}]\\ &+ \frac{x^6}{G^2m^4}\big[9\big(-20\delta(4-113\nu) + \delta\kappa_+(-569+1130\nu) - \kappa_-(-569+2268\nu)\big)(\Sigma_\ell^{\rm c})^2\\ &\quad + 18\big(569\delta\kappa_- - 10(117-452\nu) + \kappa_+(-569+2260\nu)\big)S_\ell^{\rm c}\Sigma_\ell^{\rm c}\\ &\quad + (-20340\delta + 72\kappa_- - 10170\delta\kappa_+)(S_\ell^{\rm c})^2\big]\Big\}.\end{aligned} \tag{210}$$

a. Kick velocity

An important application of the above formula for the linear momentum flux is the derivation of the kick velocity, $\mathbf{V}^i_{\rm kick}$, obtained via the expression

$$\mathbf{V}^i_{\rm kick} = \frac{1}{m}\int_{-\infty}^t dt\,\frac{d\mathbf{P}^i}{dt}, \tag{211}$$

which we compute taking advantage of the circular-limit relation in (128). For instance, at the leading order we have

$$\frac{d\mathbf{P}^i}{dt} = \frac{464m\delta\nu^2x^4}{105}\frac{d\mathbf{n}^i}{dt} = \frac{d}{dt}\left[\frac{464m\delta\nu^2x^4}{105}\mathbf{n}^i\right] - \underbrace{\frac{464m\delta\nu^2}{105}\mathbf{n}^i\frac{d(x^4)}{dt}}_{\sim\mathcal{O}(x^8)}, \tag{212}$$

which allows us to neglect the last term through the order we are working. Performing the integration with the boundary condition $x=0$ at $t=-\infty$, we arrive at

$$\begin{aligned}(\mathbf{V}^i_{\rm kick}) = \frac{\nu^2}{Gm^2}\mathbf{n}^i\Big\{&-\frac{16}{15}x^{9/2}\Sigma_\ell^{\rm c} + \frac{8}{315}x^{11/2}[-940\delta S_\ell^{\rm c} + 3(-67+412\nu)\Sigma_\ell^{\rm c}]\\ &+ \frac{x^6}{21Gm^2}\Big[8\big(2\kappa_- + 29\delta(2+\kappa_+)\big)(S_\ell^{\rm c})^2\\ &\quad - \frac{8}{5}\big(-362 + 135\delta\kappa_- + 1160\nu + 5\kappa_+(-27+116\nu)\big)S_\ell^{\rm c}\Sigma_\ell^{\rm c}\\ &\quad + 4\big(12\delta + \delta\kappa_+(27-58\nu) - 116\delta\nu + \kappa_-(-27+112\nu)\big)(\Sigma_\ell^{\rm c})^2\Big]\Big\},\end{aligned} \tag{213}$$

where we have written the final expression in terms of the conserved-norm spin vectors. A similar result is given in [123], with full agreement in the spin-orbit sector. Yet, the disagreement for spin-spin contributions remains as expected due to the overlooked contributions in that work.

E. Accumulated orbital phase

For nonspinning objects, the procedure of computing the phase evolution of the binary system is unambiguous because the orientation of the orbital plane is constant in time; for spinning objects, the choice of spin vector is crucial because the spin vector may evolve by radiation reaction for an inappropriate choice. For spinning systems, we choose a spin vector with conserved norm; this allows us to work with orbit averaged spin vectors and to use energy balance arguments to compute the orbital phase [119, 33]. We first obtain a dimensionless adiabatic parameter (also called the orbital frequency evolution [117]) representing the orbital decay. To do this, we use the coordinate-independent expressions for the energy (148) and associated energy flux (189) and observe that

$$\frac{\dot{\omega}}{\omega} = \frac{3}{2x}\left(\frac{dE(x)}{dx}\right)^{-1}\frac{dE(x)}{dt}. \tag{214}$$

Then the dimensional adiabatic parameter $\dot{\omega}/\omega^2$ is explicitly given by

$$\begin{aligned}\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\nu x^{5/2}\Big\{&1 + x(-\tfrac{743}{336} - \tfrac{11}{4}\nu)\\ &+ \frac{x^{3/2}}{Gm^2}(-\tfrac{25}{4}\delta\Sigma^c_\ell - \tfrac{47}{3}S^c_\ell) + x^2(\tfrac{34103}{18144} + \tfrac{13661}{2016}\nu + \tfrac{59}{18}\nu^2)\\ &+ \frac{x^{5/2}}{Gm^2}[(-\tfrac{809}{84} + \tfrac{281}{8}\nu)\delta\Sigma^c_\ell + (-\tfrac{5861}{144} + \tfrac{1001}{12}\nu)S^c_\ell]\Big\}.\end{aligned} \tag{215}$$

for the nonspinning and spin-orbit terms, and

$$\begin{aligned}\frac{\dot{\omega}}{\omega^2} = &\frac{x^{9/2}}{G^2m^4}\Big[(192 + 96\kappa_+)\nu\mathbf{S}_c^2 + (192\delta - 96\kappa_- + 96\delta\kappa_+)\nu(S_c\Sigma_c)\\ &+ \big((\tfrac{6}{5} - 48\delta\kappa_- + 48\kappa_+)\nu + (-192 - 96\kappa_+)\nu^2\big)\mathbf{\Sigma}_c^2\Big]\\ &+ \frac{x^{11/2}}{G^2m^4}\Big[\big((\tfrac{102072}{35} + \tfrac{886}{5}\delta\kappa_- + \tfrac{10156}{35}\kappa_+)\nu + (-\tfrac{4128}{5} - \tfrac{2064}{5}\kappa_+)\nu^2\big)\mathbf{S}_c^2\end{aligned}$$

$$+\Big((\tfrac{789}{5}-\tfrac{1977}{35}\delta\kappa_-+\tfrac{1977}{35}\kappa_+)\nu+(-\tfrac{47269}{35}+\tfrac{146}{5}\delta\kappa_--\tfrac{4976}{35}\kappa_+)\nu^2+(\tfrac{4128}{5}+\tfrac{2064}{5}\kappa_+)\nu^3\Big)\boldsymbol{\Sigma}_{\rm c}^2$$

$$+\Big((\tfrac{14120}{7}\delta-\tfrac{3954}{35}\kappa_-+\tfrac{3954}{35}\delta\kappa_+)\nu+(-\tfrac{4128}{5}\delta-\tfrac{1864}{5}\kappa_--\tfrac{2064}{5}\delta\kappa_+)\nu^2\Big)(S_{\rm c}\Sigma_{\rm c})\Big]. \tag{216}$$

for the spin-spin terms. The orbital phase can then be computed in this adiabatic approximation, where the gravitational wave phase contains two contributions. The first comes from the evolution of the carrier phase, while the second arises due to the precession of the orbital plane due to spin effects. This can schematically be written $\Phi_{\rm GW}=\phi_{\rm GW}+\delta\phi$ using the notation of [36]. The carrier phase given by $\phi_{\rm GW}=2\phi$ can be computed using

$$\phi=\int dt\,\omega=\int d\omega\frac{\omega}{\dot{\omega}}. \tag{217}$$

In general, the carrier phase may be computed numerically for arbitrary spin alignments. However, for spins aligned or anti-aligned with the binary orbital angular momentum, this can be computed analytically using equation (215) to yield

$$\begin{aligned}\phi=-\frac{32}{\nu}\Big\{&x^{-5/2}+x^{-3/2}(\tfrac{3715}{1008}+\tfrac{55}{12}\nu)\\&+\frac{x^{-1}}{Gm^2}(\tfrac{125}{8}\delta\Sigma^c_\ell+\tfrac{235}{6}S^c_\ell)+x^{-1/2}(\tfrac{15293365}{1016064}+\tfrac{27145}{1008}\nu+\tfrac{3085}{144}\nu^2)\\&-\frac{\log x}{Gm^2}[(\tfrac{41745}{448}-\tfrac{15}{8}\nu)\delta\Sigma^c_\ell+(\tfrac{554345}{2016}+\tfrac{55}{8}\nu)S^c_\ell]\Big\},\end{aligned} \tag{218}$$

for nonspinning and spin-orbit pieces, and

$$\begin{aligned}\phi=\frac{x^{-5/2}}{32\nu G^2m^4}\Big\{&x^2\big[(-50-25\kappa_+)\mathbf{S}_{\rm c}^2+(-50\delta+25\kappa_--25\delta\kappa_+)(S_{\rm c}\Sigma_{\rm c})\\&+\big(-\tfrac{5}{16}+\tfrac{25}{2}\delta\kappa_--\tfrac{25}{2}\kappa_++(50+25\kappa_+)\nu\big)\boldsymbol{\Sigma}_{\rm c}^2\big]\\&+x^3\Big[\big(-\tfrac{31075}{126}+\tfrac{2215}{48}\delta\kappa_-+\tfrac{15635}{84}\kappa_++(60+30\kappa_+)\nu\big)\mathbf{S}_{\rm c}^2\\&+\big(-\tfrac{410825}{2688}-\tfrac{47035}{672}\delta\kappa_-+\tfrac{47035}{672}\kappa_++(\tfrac{23535}{112}-\tfrac{2935}{48}\delta\kappa_--\tfrac{4415}{56}\kappa_+)\nu+(-60-30\kappa_+)\nu^2\big)\boldsymbol{\Sigma}_{\rm c}^2\\&+\big(-\tfrac{9775}{42}\delta-\tfrac{47035}{336}\kappa_-+\tfrac{47035}{336}\delta\kappa_++(60\delta-\tfrac{2575}{12}\kappa_-+30\delta\kappa_+)\nu\big)(S_{\rm c}\Sigma_{\rm c})\Big]\Big\},\end{aligned} \tag{219}$$

for spin-spin components. These results are in perfect agreement with [36] and [39], respectively.

IV. Radiation Reaction in NRGR

Until this point, we have studied conservative systems which can be adequately described by a standard Lagrangian or Routhian approach. By variation of the Lagrangian or using Hamilton's principle, the equations of motion can be derived in a straightforward manner. In this chapter, we explore dissipative effects, in particular those associated with radiation reaction, also known as a self-force in which the system responds to its own radiation field. This effect is analogous to the much-studied radiation reaction seen in classical electrodynamics.

For nonspinning binaries, we have

$$a^i = a^i_0 + a^i_1 + a^i_2 + a^i_{2.5} + a^i_3 + a^i_{3.5} + a^i_4 + a^i_{4.5} + \cdots \,, \tag{220}$$

where the subscript denotes the PN order of the term. The leading term, a^i_0, is just the Newtonian acceleration. The 1PN correction is the EIH correction, which scales as $\mathcal{O}(v^2)$, while the 2PN correction scales as $\mathcal{O}(v^4)$. At 2.5PN order we have the radiation reaction, or Burke-Thorne, term. This is the first nonconservative piece of the acceleration. At 3PN order we again have a conservative correction. At 3.5PN we have the first correction to the Burke–Thorne term, while at 4PN we have a mix between conservative and nonconservative contributions, including the leading tail contribution. These contributions have all been calculated in traditional methods [126, 13, 15, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137] and using the EFT approach [73, 18, 138, 139, 98, 101]. Finally, at 4.5PN we get the 2PN correction to the Burke–Thorne acceleration, which is the objective of this chapter, as computed in [109].

A. Nonconservative dynamics

In this section, we will briefly present the tools we need from classical mechanics of nonconservative systems as introduced in [138]. One may find an extensive pedagogical presentation and several interesting applications in [139]. This approach has been used to compute the 2.5PN and 3.5PN radiation reaction [96, 98], gravitational tail effects in compact object binaries, and finite-size radiative effects in electrodynamics [140].

To calculate the nonconservative effects of radiation reaction using the action in (10), we need to formally double the number of degrees of freedom following the approach in [138, 139] (see Appendix A for a simple example of how to use this method). We take

$$\mathbf{x}_K \to (\mathbf{x}_{K(1)}, \mathbf{x}_{K(2)}), \quad \bar{h}_{\mu\nu} \to (\bar{h}^{(1)}_{\mu\nu}, \bar{h}^{(2)}_{\mu\nu}), \tag{221}$$

where the (1) and (2) are the different "history" labels of the coordinates and fields. The action is constructed from these degrees of freedom as

$$S[\mathbf{x}_{K(1)}, \mathbf{x}_{K(2)}, \bar{h}^{(1)}_{\mu\nu}, \bar{h}^{(2)}_{\mu\nu}] = S[\mathbf{x}_{K(1)}, \bar{h}^{(1)}_{\mu\nu}] - S[\mathbf{x}_{K(2)}, \bar{h}^{(2)}_{\mu\nu}], \tag{222}$$

where S includes both the worldline action (10) and the Einstein–Hilbert action, with appropriate gauge fixing.[1] By integrating out the long-wavelength gravitational modes, we obtain the effective action for the open dynamics of the binary inspiral, which can be written as

$$S_{\text{eff}}[\mathbf{x}_{K(1,2)}] = \int dt (L[\mathbf{x}_{K(1)}] - L[\mathbf{x}_{K(2)}] + R[\mathbf{x}_{K(1)}, \mathbf{x}_{K(2)}]), \tag{223}$$

where L is the usual Lagrangian accounting for the conservative interactions and R is the term containing nonconservative effects. To obtain the radiation-reaction force, we vary the effective action and then take the physical limit, in which the doubled variables are identified with the physical variables, i.e.,

$$\mathbf{x}_{K(1)}, \mathbf{x}_{K(2)} \to \mathbf{x}_K. \tag{224}$$

In this work, we make a convenient coordinate redefinition to "plus-minus" coordinates, defined by

$$\mathbf{x}_{K+} \equiv (\mathbf{x}_{K(1)} + \mathbf{x}_{K(2)})/2, \quad \mathbf{x}_{K-} \equiv \mathbf{x}_{K(1)} - \mathbf{x}_{K(2)}, \tag{225}$$

which simplifies the procedure for deriving the dynamics. We can then vary the action with respect to the minus degrees of freedom, and then take the physical limit by simply setting

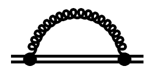
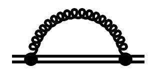
$$\mathbf{x}_{K+} \to \mathbf{x}_K, \quad \mathbf{x}_{K-} \to 0. \tag{226}$$

[1] We use linearized harmonic gauge in this work, see [40, 78] for details.

The nonconservative acceleration is thus obtained by the variation

$$a_K^i(t) = \frac{1}{m_K} \frac{\delta S_{\text{eff}}}{\delta \mathbf{x}_{Ki-}} \bigg|_{\substack{\mathbf{x}_{K-} \to 0 \\ \mathbf{x}_{K+} \to \mathbf{x}_K}} . \tag{227}$$

From the radiative Lagrangian in equation (11), we obtain the effective action by integrating out the self-interacting fields. It is clear that the topologies that we need to consider through the 4.5PN order can be represented diagrammatically as[2]

$$iS_{\text{eff}}[\mathbf{x}_K^{\pm}] \quad = iS_{\text{con}} \quad + \sum_{l \geq 2}$$  $I_A^L \quad I_B^L$ $+$  $J_A^L \quad J_B^L$. (228)

Cross-terms containing both mass- and current-type multipole moments vanish by parity arguments, or alternately by noting that we integrate over an odd power of momenta. We consider the contributions from the conservative action separately, focusing now on the dissipative terms. For this topology, we can find a general expression in terms of multipole moments and their derivatives given by [101]

$$\begin{aligned} \int dt\, L_{\text{eff}} = \sum_{\ell \geq 2} \frac{(-1)^{\ell+1}(\ell+2)G}{(\ell-1)} \int dt \, \Bigg[& \frac{2^{\ell}(\ell+1)}{\ell(2\ell+1)!} I_-^L(t) I_+^{L(2\ell+1)}(t) \\ & + \frac{2^{\ell+3}\ell}{(2\ell+2)!} J_-^L(t) J_+^{L(2\ell+1)}(t) \Bigg], \end{aligned} \tag{229}$$

where $I_-^L \equiv I_{(1)}^L - I_{(2)}^L$ and $I_+^L \equiv (I_{(1)}^L + I_{(2)}^L)/2$, and I_A^L for $A = (1), (2)$. We now turn to explicitly computing the nonconservative equations of motion order by order in the PN expansion, starting with the 2.5PN and 3.5PN Burke–Thorne forces.

[2]Note that we only have to consider diagrams to linear order in the radiation modes, neglecting nonlinear radiative effects that only contribute starting at the 5PN order. It is worth observing, however, that nonlinear gravitational effects enter through the multipole moments themselves within the potential regime. We also do not consider tail effects in equation (228), which enter at 4PN relative to the Newtonian order and are investigated in [101].

B. LO and NLO Burke–Thorne acceleration

From equation (229), the effective radiation-reaction term at leading order is given by

$$S_{\rm eff}^{2.5\rm PN} = -\frac{G}{5}\int dt\, I_-^{ij}(t) I_+^{ij(5)}(t), \tag{230}$$

from which we obtain the equations of motion using the variation (227) and taking the physical limit. At leading order, we only have a single contribution, the LO mass quadrupole term, as the mass octupole and current quadrupole terms are suppressed by $O(v^2)$. We need to calculate the moments I_- and I_+ in terms of the coordinates x_+ and x_-. But $I_0^{ij} = \mu x^i x^j$, and doubling the variables gives $I_+^{ij} = (I_1^{ij} + I_2^{ij})/2$, and $I_+^{ij} = I_1^{ij} - I_2^{ij}$. Writing these in terms of the coordinates, we have

$$I_+^{ij} = \frac{m}{2}\left(\mathbf{x}_+^i \mathbf{x}_+^j + \frac{\mathbf{x}_-^i \mathbf{x}_-^j}{4}\right), \quad I_-^{ij} = \sum_K m_K (\mathbf{x}_{+K}^i \mathbf{x}_{-K}^j + \mathbf{x}_{-K}^i \mathbf{x}_{+K}^j). \tag{231}$$

Then using these explicit expressions for the multipole moments and (227), we find the Burke–Thorne acceleration

$$\mathbf{a}_K^i = -\frac{2G}{5}\mathbf{x}_{Kj}\frac{d^5 I^{ij}(t)}{dt^5}. \tag{232}$$

The NLO Burke–Thorne acceleration was first computed from the EFT formalism in [98] and constitutes a 3.5PN correction to the Newtonian equations of motion. There are four distinct contributions: the NLO mass quadrupole, the leading mass octupole, the leading current quadrupole, and a reduced contribution from the leading Burke–Thorne acceleration inserted into the 1PN conservative equations of motion. The nonconservative action we consider at this order is then given by

$$S = -\frac{G}{5}\int dt\left[I_{0-}^{ij} I_{1+}^{ij(5)} + I_{1-}^{ij} I_{0+}^{ij(5)}\right] + \frac{G}{189}\int dt\, I_{0-}^{ijk} I_{0+}^{ijk(7)} - \frac{16G}{45}\int dt\, J_{0-}^{ij} J_{0+}^{ij(5)}, \tag{233}$$

where the numeric subscript on the multipole moments indicates the PN order.

The first term arising from the mass quadrupole coupling is simply the LO Burke–Thorne result with the 1PN mass quadrupole substituted for the LO mass quadrupole, i.e.,

$$\mathbf{a}_{\rm mq,1}^i = -\frac{2G}{5}\mathbf{x}^j I_1^{ij(5)} \tag{234}$$

which includes 1PN reduced accelerations when the time derivatives are applied to the LO mass quadrupole. The second term is more difficult. We write the 1PN mass quadrupole in terms of $+,-$ coordinates and perform the variation, which yields

$$
\begin{aligned}
\mathbf{a}^i_{\mathrm{mq},2} =& \frac{G}{105}(1-3\nu)(17\mathbf{x}^i\mathbf{x}^j - 11r^2\delta^{ij})\mathbf{x}^k I^{jk(7)} \\
&+ \frac{G}{5}\left[(3-9\nu)\Big(2\mathbf{v}^i\mathbf{x}^j\mathbf{v}^k - \frac{Gm}{r}\mathbf{n}^i\mathbf{n}^j\mathbf{x}^k - \mathbf{v}^2\mathbf{x}^k\delta^{ij}\Big) + \frac{Gm}{r}(1-2\nu)\mathbf{x}^k(2\delta^{ij} - \mathbf{n}^i\mathbf{n}^j)\right] I^{jk(5)} \\
&+ \frac{G}{15}(1-3\nu)(8\mathbf{x}^i\mathbf{x}^j\mathbf{v}^k - 8(xv)\mathbf{x}^k\delta^{ij} + 9\mathbf{v}^i\mathbf{x}^j\mathbf{x}^k)I^{jk(6)}
\end{aligned} \tag{235}
$$

The remaining two pieces come from the mass octupole and the current quadrupole couplings. From the mass octupole, we have

$$
\mathbf{a}^i_{\mathrm{mo}} = \frac{G}{63}\delta\nu m\mathbf{x}^j\mathbf{x}^k I^{ijk(7)}, \tag{236}
$$

while for the current quadrupole we have

$$
\mathbf{a}^i_{\mathrm{cq}} = \frac{16G}{45}\delta(2\mathbf{x}^j\mathbf{v}^k\epsilon^{ikl}J^{jl(5)} + \mathbf{x}^k\mathbf{v}^j\epsilon^{ikl}J^{jl(5)} - \mathbf{x}^j\mathbf{v}^k\epsilon^{kjl}J^{il(5)} + \mathbf{x}^j\mathbf{x}^k\epsilon^{ikl}J^{jl(6)}). \tag{237}
$$

The final contribution arising from the LO radiation-reaction acceleration inserted into the 1PN conservative acceleration is

$$
\mathbf{a}^i_{\mathrm{red.}} = \frac{2G}{5}\left[(3+\nu)\frac{Gm}{r}\mathbf{x}^k\delta^{ij} + \frac{Gm}{r}\nu\mathbf{n}^i\mathbf{n}^j\mathbf{x}^k + \frac{1}{2}(1-3\nu)\mathbf{v}^2\mathbf{x}^k\delta^{ij} + (1-3\nu)\mathbf{v}^i\mathbf{x}^j\mathbf{v}^k\right] I^{jk(5)}. \tag{238}
$$

Note that in principle, there could be a 2.5PN COM correction. However, this correction vanishes in our gauge in contrast to that in, e.g., [132]. Summing over all contributions, we find the NLO radiation-reaction acceleration given by

$$
\mathbf{a}^i_{3.5\mathrm{PN}} = \mathbf{a}^i_{\mathrm{mq},1} + \mathbf{a}^i_{\mathrm{mq},2} + \mathbf{a}^i_{\mathrm{mo}} + \mathbf{a}^i_{\mathrm{cq}} + \mathbf{a}^i_{\mathrm{red.}} \tag{239}
$$

$$
\begin{aligned}
=& -\frac{2G}{5}\mathbf{x}^j I_1^{ij(5)} + \frac{G}{105}(1-3\nu)(17\mathbf{x}^i\mathbf{x}^j - 11r^2\delta^{ij})\mathbf{x}^k I^{jk(7)} \\
&+ \frac{G}{15}(1-3\nu)(9\mathbf{v}^i\mathbf{x}^j + 8\mathbf{v}^j\mathbf{x}^i - 8(xv)\delta^{ij})\mathbf{x}^k I^{jk(6)} \\
&- \frac{2G}{5}\left[(1-3\nu)\mathbf{v}^2 - (4-\nu)\frac{Gm}{r}\right]\mathbf{x}^k I^{ik(5)} \\
&+ \frac{G}{5}\left[8(1-3\nu)\mathbf{v}^i\mathbf{v}^j - (4-13\nu)\frac{Gm}{r}\mathbf{n}^i\mathbf{n}^j\right]\mathbf{x}^k I^{jk(5)} - \frac{G}{63}\delta\mathbf{x}^j\mathbf{x}^k I^{ijk(7)} \\
&+ \frac{16G}{45}\delta\left(2\mathbf{x}^j\mathbf{v}^k\epsilon^{ikl}J^{jl(5)} + \mathbf{x}^k\mathbf{v}^j\epsilon^{ikl}J^{jl(5)} - \mathbf{x}^j\mathbf{v}^k\epsilon^{kjl}J^{il(5)} + \mathbf{x}^j\mathbf{x}^k\epsilon^{ikl}J^{jl(6)}\right).
\end{aligned} \tag{240}
$$

The full expression for the 3.5PN acceleration can be found in Appendix E.

C. NNLO Burke–Thorne acceleration

In this section, we compute the radiation reaction equations of motion at 4.5PN order. For simplicity, we break the calculation into distinct multipole terms as well as contributions from the conservative sector arising from order-reduced accelerations. Explicitly, through the 4.5PN order, the action that yields the radiation reaction acceleration can be written as

$$L_{\text{eff}} = -\frac{G}{5} I_-^{ij} I_+^{ij(5)} - \frac{16G}{45} J_-^{ij} J_+^{ij(5)} + \frac{G}{189} I_-^{ijk} I_+^{ijk(7)} + \frac{G}{84} J_-^{ijk} J_+^{ijk(7)} - \frac{G}{9072} I_-^{ijkl} I_+^{ijkl(9)} + \cdots, \tag{241}$$

where the first term enters at 2.5PN and the first three terms contribute at 3.5PN, as seen in the previous section. We now proceed to compute the 4.5PN acceleration term by term.

1. Mass quadrupole

The mass quadrupole can be expanded as[3]

$$\begin{aligned} I^{ij} =& I^{ij}_{\text{0PN}} + \epsilon I^{ij}_{\text{1PN}} + \epsilon^2 I^{ij}_{\text{2PN}} + \mathcal{O}(\epsilon^{2.5}) \\ =& \sum_{K\neq L} m_K \Bigg\{ \mathbf{x}_K^i \mathbf{x}_K^j + \Bigg(\frac{3}{2}\mathbf{v}_K^2 - \sum_{L\neq K} \frac{G m_L}{r} \Bigg) \mathbf{x}_K^i \mathbf{x}_K^j \\ &+ \frac{11}{42}\frac{d^2}{dt^2}(\mathbf{x}_K^2 \mathbf{x}_K^i \mathbf{x}_K^j) - \frac{4}{3}\frac{d}{dt}(\mathbf{x}_K \cdot \mathbf{v}_K \mathbf{x}_K^i \mathbf{x}_K^j) \Bigg\}_{\text{TF}} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{242}$$

where ϵ counts the PN order and the $\mathcal{O}(\epsilon^2)$ expression can be found in [97]. Then through 4.5PN, the mass quadrupole component of the action can be written as

$$\begin{aligned} S_{\text{mq}} =& -\frac{G}{5}\int dt\, [I^{ij}_{0-} I^{ij(5)}_{0+} + \epsilon (I^{ij}_{0-} I^{ij(5)}_{1+} + I^{ij}_{1-} I^{ij(5)}_{0+}) \\ &+ \epsilon^2 (I^{ij}_{0-} I^{ij(5)}_{2+} + I^{ij}_{2-} I^{ij(5)}_{0+} + I^{ij}_{1-} I^{ij(5)}_{1+}) + \cdots], \end{aligned} \tag{243}$$

where the numeric subscript on the multipole moments is its PN order. The $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ terms correspond to the 2.5PN and 3.5PN radiation-reaction mass quadrupole contributions, respectively, as computed in the previous section. However, upon variation, these terms also

[3] We neglect spin dependence in the multipole moments, which enter here at 1.5PN for spin-orbit couplings [103] and 2PN for spin-spin couplings [108, 104, 38].

contribute to the 4.5PN acceleration through order reduction of accelerations. Looking specifically at the terms that depend on I^{ij}_{0-}, after varying the action, we find

$$\mathbf{a}^i_K = -\frac{2G}{5}\mathbf{x}^j_K(I^{ij(5)}_0 + \epsilon I^{ij(5)}_1 + \epsilon^2 I^{ij(5)}_2). \tag{244}$$

Each of these terms contributes at 4.5PN, as each of these terms is itself dependent on accelerations and higher time derivatives that are not of definite PN order. The first term receives corrections through order reduction using the 2PN conservative acceleration or two order-reduced 1PN accelerations. The second term has corrections from the 1PN acceleration. The final term only requires order reduction using the Newtonian acceleration. We must of course also vary the terms that depend upon I^{ij}_{1-} and I^{ij}_{2-}. These are more complicated, but follow similarly to the above. For instance, for the $I^{ij}_{1-}I^{ij(5)}_{0+}$ term, one will again need the order-reduced 1PN acceleration.

In this general frame, this completes the 4.5PN mass quadrupole contribution. However, we must consider additional terms that arise when making the coordinate transformation to the COM frame, in which we compute the relative acceleration $\mathbf{a}^i = \mathbf{a}^i_1 - \mathbf{a}^i_2$. The coordinate shift can be written as in equation (117), although at this point we require higher-order corrections than were necessary to NLO in the spin-orbit and spin-spin case. The term $\delta\mathbf{x}^i_{2\mathrm{PN}}$ was recently computed using EFT methods in [78], while $\delta\mathbf{x}^i_{2.5\mathrm{PN}}$ vanishes in our gauge.[4] The 3PN COM shift has yet to be computed in our gauge, but is beyond the scope of this paper, as it does not contribute at the 4.5PN order in the dissipative sector. The 2PN correction, which we reproduce here for convenience, is

$$\begin{aligned}\delta\mathbf{x}^i_{2\mathrm{PN}} = \frac{\nu\delta}{2}\Bigg\{\mathbf{x}^i\Bigg[\left(\frac{3}{4} - 3\nu\right)\mathbf{v}^4 + \frac{Gm}{r}\Bigg(\left(\frac{19}{4} + 3\nu\right)\mathbf{v}^2\Bigg) \\ + \left(-\frac{1}{4} + \frac{3\nu}{2}\right)\dot{r}^2 + \left(\frac{7}{2} - \nu\right)\frac{Gm}{r}\Bigg] - \mathbf{v}^i\left[\frac{7}{2}Gm\dot{r}\right]\Bigg\}.\end{aligned} \tag{245}$$

Note that $\delta\mathbf{x}^i_{3.5\mathrm{PN}}$ is nonvanishing in our gauge (see section IV.E), and will be discussed in section IV.C.4 as it contributes at 4.5PN when shifting to the COM frame within the conservative sector.

[4]Contrast this with the nonzero COM correction at 2.5PN computed in [141] using harmonic coordinates.

We apply COM coordinate transformations to the 2.5PN and 3.5PN mass quadrupole terms in the acceleration in relative coordinates. Working with the multipole moments in the COM frame, we have a contribution given by

$$\mathbf{a}^i_{\mathrm{COM}} = -\frac{2G}{5}\mathbf{x}^j \frac{d^5}{dt^5}(m\delta\mathbf{x}^i_{1\mathrm{PN}}\delta\mathbf{x}^j_{1\mathrm{PN}} - \frac{1}{3}m\delta\mathbf{x}^2_{1\mathrm{PN}}\delta^{ij}), \tag{246}$$

where the 2PN shift does not contribute due to the symmetry of the mass quadrupole moment. Additionally, we find that there will be COM corrections to expressions containing the 1PN multipole moment when applied after variation with respect to the minus coordinates. Adding these corrections to our result yields a final expression for the COM frame mass quadrupole contribution, which can be found in Appendix E.

2. Current quadrupole

The current quadrupole can be written as

$$J^{ij} = \epsilon^{0.5} J^{ij}_0 + \epsilon^{1.5} J^{ij}_1 + \mathcal{O}(\epsilon^{2.5}) \tag{247}$$

$$\begin{aligned} =&\sum_A m_A\left(1+\frac{\mathbf{v}_A^2}{2}\right)\left[(\mathbf{x}_A\times\mathbf{v}_A)^i\mathbf{x}^j_A\right]_{\mathrm{STF}} \\ &+\sum_{A\neq B}\frac{Gm_Am_B}{r}\Bigg[2(\mathbf{x}_A\times\mathbf{v}_A)^i\mathbf{x}^j_A - \frac{11}{4}(\mathbf{x}_B\times\mathbf{v}_A)^i\mathbf{x}^j_B - \frac{3}{4}(\mathbf{x}_B\times\mathbf{v}_A)^i\mathbf{x}^j_A \\ &\quad+(\mathbf{x}_A\times\mathbf{v}_A)^i\mathbf{x}^j_B + \frac{7}{4}(\mathbf{x}_A\times\mathbf{x}_B)^i\mathbf{v}^j_A + \frac{\mathbf{v}_A\cdot\mathbf{x}}{4r^2}(\mathbf{x}_A\times\mathbf{x}_B)^i(\mathbf{x}^j_A+\mathbf{x}^j_B)\Bigg]_{\mathrm{STF}} \\ &+\frac{1}{28}\frac{d}{dt}\Bigg[\sum_A m_A(\mathbf{x}_A\times\mathbf{v}_A)^i(3\mathbf{x}^2_A\mathbf{v}^j_A - \mathbf{x}_A\cdot\mathbf{v}_A\mathbf{x}^j_A) \\ &\quad+\sum_{A\neq B}\frac{Gm_Am_B}{2r^3}\mathbf{x}^i_A(\mathbf{x}_A\times\mathbf{x}_B)^j(6\mathbf{x}^2_A - 7\mathbf{x}_A\cdot\mathbf{x}_B + 7\mathbf{x}^2_B)\Bigg]_{\mathrm{STF}} + \mathcal{O}(\epsilon^{2.5}), \end{aligned} \tag{248}$$

and thus the term in the action that contributes through 4.5PN is given by

$$S_{\mathrm{cq}} = -\frac{16G}{45}\int dt\,[\epsilon J^{ij}_{0-}J^{ij(5)}_{0+} + \epsilon^2(J^{ij}_{1-}J^{ij(5)}_{0+} + J^{ij}_{0-}J^{ij(5)}_{1+}) + \cdots]. \tag{249}$$

The current quadrupole term first enters the acceleration at 3.5PN through the first term in equation (249). The 4.5PN acceleration contains three contributions: two pieces from the 1PN current quadrupole in either the plus or minus coordinates, and a third from reducing accelerations in the 3.5PN acceleration with the 1PN conservative acceleration. Upon shifting to the frame of the COM, we find an additional contribution from the 1PN coordinate correction in the 3.5PN acceleration. The full result for the current quadrupole contribution can be found in Appendix E.

3. Mass octupole

The mass octupole can be written as

$$I^{ijk} = I_0^{ijk} + \epsilon I_1^{ijk} + \mathcal{O}(\epsilon^2) \tag{250}$$

$$= \sum_{A\neq B} m_A \Bigg\{ \Bigg[\Bigg(1 + \frac{3}{2}\mathbf{v}_A^2 - \sum_{B\neq A} \frac{Gm_B}{r} \Bigg) \mathbf{x}_A^i \mathbf{x}_A^j \mathbf{x}_A^k + \frac{1}{18}\frac{d^2}{dt^2}(\mathbf{x}_A^2 \mathbf{x}_A^i \mathbf{x}_A^j \mathbf{x}_A^k) \Bigg]_{\mathrm{STF}} - \frac{7}{9}\frac{d}{dt}\big[(\mathbf{x}_A^i \mathbf{x}_A^j \mathbf{x}_A^k \mathbf{x}_A^l)_{\mathrm{STF}} \mathbf{v}^l \big] \Bigg\} + \mathcal{O}(\epsilon^2), \tag{251}$$

again neglecting spin. Then the octupole contributions to the action, using Eq. (241), are

$$S_{\mathrm{mo}} = \frac{G}{189} \int dt\, [\epsilon I_{0-}^{ijk} I_{0+}^{ijk(7)} + \epsilon^2 (I_{1-}^{ijk} I_{0+}^{ij(7)} + I_{0-}^{ijk} I_{1+}^{ij(7)}) + \cdots]. \tag{252}$$

The $\mathcal{O}(\epsilon)$ term contributes to the 3.5PN radiation-reaction acceleration, as well as to the 4.5PN acceleration upon reducing accelerations using the 1PN acceleration. As an example, after varying the third term, proportional to $I_{0-}^{ijk} I_{1+}^{ij(7)}$, we find that

$$\mathbf{a}_1^i = \frac{G}{63} \mathbf{x}_1^j \mathbf{x}_1^k I_1^{ijk(7)}. \tag{253}$$

We additionally must consider the first term in equation (252) with 1PN acceleration reductions and the second term with the Newtonian acceleration, which together with the above yield the 4.5PN acceleration. This completes the octupole term in the original frame. When transforming to the COM frame, we pick up an additional 4.5PN piece from the 1PN COM shift in the 3.5PN term. The sum of the mass octupole contributions to the 4.5PN acceleration can be found in Appendix E.

4. Current octupole

The current octupole term contributes first at the 4.5PN order as well and is given by

$$S_{\rm co} = \frac{G}{84}\int dt\, \epsilon^2 J^{ijk}_{0-} J^{ijk(7)}_{0+} + \mathcal{O}(\epsilon^3). \tag{254}$$

Thus, we only need the LO expression of the current octupole, given by

$$J^{ijk}_0 = \sum_A m_A [\epsilon^{ilm}\mathbf{x}^j_A \mathbf{x}^k_A \mathbf{x}^l_A \mathbf{v}^m_A]_{\rm STF} = m\nu(1-3\nu)[\epsilon^{ilm}\mathbf{x}^j\mathbf{x}^k\mathbf{x}^l\mathbf{v}^m]_{\rm STF}. \tag{255}$$

Upon variation, we find that the contribution to the acceleration from the current octupole is

$$\mathbf{a}^i_{1\rm co} = \frac{G}{84}\mathbf{x}^j_1\mathbf{x}^k_1[2(\epsilon^{ilm}J^{jkm(7)}_0 + \epsilon^{ijm}J^{klm(7)}_0 + \epsilon^{jlm}J^{ikm(7)}_0)\mathbf{v}^l_1 + \epsilon^{ijm}\mathbf{x}^l_1 J^{klm(8)}_0], \tag{256}$$

and in relative coordinates

$$\mathbf{a}^i_{\rm co} = \frac{G}{84}(1-3\nu)\mathbf{x}^j\mathbf{x}^k[2(\epsilon^{ilm}J^{jkm(7)}_0 + \epsilon^{ijm}J^{klm(7)}_0 + \epsilon^{jlm}J^{ikm(7)}_0)\mathbf{v}^l + \epsilon^{ijm}\mathbf{x}^l J^{klm(8)}_0]. \tag{257}$$

The full result for the current octupole contribution can be found in Appendix E.

5. Mass hexadecapole

The mass hexadecapole term contributes first at the 4.5PN order and is given by

$$S_{\rm mh} = -\frac{G}{9072}\int dt\, \epsilon^2 I^{ijkl}_{0-} I^{ijkl(9)}_{0+} + \mathcal{O}(\epsilon^3). \tag{258}$$

Thus, we only need the LO expression of the mass hexadecapole, given by

$$I^{ijkl}_0 = \sum_A m_A[\mathbf{x}^i_A\mathbf{x}^j_A\mathbf{x}^k_A\mathbf{x}^l_A]_{\rm STF}. \tag{259}$$

Upon variation, we find that

$$\mathbf{a}^i_{1\rm mh} = -\frac{G}{2268}\mathbf{x}^j_1\mathbf{x}^k_1\mathbf{x}^l_1 I^{ijkl(9)}_0, \tag{260}$$

and in the COM frame

$$\mathbf{a}^i_{\rm mh} = -\frac{G}{2268}(1-3\nu)\mathbf{x}^j\mathbf{x}^k\mathbf{x}^l I^{ijkl(9)}_0. \tag{261}$$

The mass hexadecapole expression can be found in Appendix E.

6. Conservative acceleration reductions

In this section, we discuss 4.5PN terms that result from corrections to the conservative accelerations at lower orders. Variation of the conservative Lagrangian yields accelerations that are themselves acceleration dependent. Order reducing these conservative terms with nonconservative accelerations yields additional nonconservative corrections at 4.5PN. Additionally, since the COM is no longer conserved at 3.5PN, there arise nonconservative corrections at 4.5PN in relative coordinates when shifting to the COM frame.

In the 1PN acceleration, we obtain a reduced contribution from inserting the 3.5PN acceleration as

$$\mathbf{a}_1^{(\text{red})} = \left[\frac{1}{2}\frac{Gm_2}{r}(\mathbf{a}_2\cdot\mathbf{n})\mathbf{n}^i - (\mathbf{a}_1\cdot\mathbf{v}_1)\mathbf{v}_1^i - \mathbf{a}_1^i\left(3\frac{Gm_2}{r} + \frac{1}{2}\mathbf{v}_1^2\right) + \frac{7}{2}\frac{Gm_2}{r}\mathbf{a}_2^i\right]_{\mathbf{a}_{3.5\text{PN}}}. \tag{262}$$

where the full 3.5PN acceleration can be found in section IV.B. There is an additional piece resulting from order reducing accelerations and higher coordinate derivatives in the 2PN acceleration using the LO Burke–Thorne acceleration at 2.5PN, yielding a 4.5PN correction. This concludes the calculation of the 4.5PN acceleration in the original coordinate system.

When shifting to relative coordinates in the COM frame, we must consider one additional contribution. Naively, we would expect to have a 2.5PN COM correction to the 2PN acceleration; however, this vanishes because there is no 2.5PN COM shift in our gauge. However, there is a nonzero 3.5PN correction, as discussed in section IV.E. Applying this to the 1PN acceleration yields

$$\begin{aligned}\mathbf{a}^{\text{i}}_{\text{COM}} = \frac{\delta m}{m}\Bigg[&\left(-\frac{G}{2r} + \frac{1}{2}\mathbf{v}^2\right)\delta\ddot{\mathbf{x}}^i_{3.5\text{PN}} - \frac{Gm}{2r^3}(\mathbf{x}\cdot\delta\ddot{\mathbf{x}}_{3.5\text{PN}})\mathbf{x}^i + (\mathbf{v}\cdot\delta\ddot{r}_{3.5\text{PN}})\mathbf{v}^i \\ &- \frac{Gm}{r^3}\left(2(\mathbf{x}\cdot\delta\dot{\mathbf{x}}_{3.5\text{PN}})\mathbf{v}^i + (\mathbf{v}\cdot\delta\dot{\mathbf{x}}_{3.5\text{PN}})\mathbf{x}^i\right)\Bigg],\end{aligned} \tag{263}$$

where $\delta\mathbf{x}_{3.5\text{PN}} = (\mathbf{n}\cdot\delta\mathbf{x}_{3.5\text{PN}})$ and

$$\begin{aligned}\delta\mathbf{x}^i_{3.5\text{PN}} = \frac{\delta m}{m}\Bigg\{&\left[\frac{212G^3m^3\nu^2}{105r^3}\dot{r} + \frac{G^2m^2\nu^2}{r^2}\left(\frac{78}{7}\dot{r}\mathbf{v}^2 - \frac{26}{5}\dot{r}^3\right)\right]\mathbf{x}^i \\ &+\left[-\frac{8}{35}Gm\mathbf{v}^4\nu^2 - \frac{48G^3m^3\nu^2}{35r^2} - \frac{G^2m^2\nu^2}{r}\left(\frac{58}{105}\mathbf{v}^2 + \frac{398}{105}\dot{r}^2\right)\right]\mathbf{v}^i\Bigg\}\end{aligned} \tag{264}$$

is the 3.5PN COM correction. The result for all reduced contributions arising from the conservative sector can be found in Appendix E.

7. Full result

We now arrive at a full 4.5PN result by summing over all contributions as described above. The full result in relative coordinates can be written as

$$\mathbf{a}^i_{4.5\mathrm{PN}} = \mathcal{A}\mathbf{x}^i + \mathcal{B}\mathbf{v}^i, \tag{265}$$

where the coefficients $\mathcal{A}$ and $\mathcal{B}$ can be written as

$$\begin{aligned}
\mathcal{A} = {} & \frac{G^5 m^5 \nu \dot{r}}{r^7}\left(-\frac{73178}{135} - \frac{56416}{105}\nu - \frac{3040}{21}\nu^2\right) \\
& + \frac{G^4 m^4 \nu \dot{r}}{r^6}\left[\left(\frac{493214}{315} + \frac{207052}{105}\nu + \frac{9932}{35}\nu^2\right)\mathbf{v}^2 - \left(\frac{2177438}{945} + \frac{1028644}{189}\nu + \frac{96436}{105}\nu^2\right)\dot{r}^2\right] \\
& + \frac{G^3 m^3 \nu \dot{r}}{r^5}\left[\left(-\frac{20747}{105} - 1774\nu + \frac{3928}{3}\nu^2\right)\mathbf{v}^4 + \left(-\frac{19151}{105} + \frac{108773}{15}\nu - \frac{19756}{7}\nu^2\right)\mathbf{v}^2\dot{r}^2 \right.\\
& \qquad \left. + \left(\frac{21508}{21} - \frac{716027}{105}\nu + \frac{83414}{105}\nu^2\right)\dot{r}^4\right] \\
& + \frac{G^2 m^2 \nu \dot{r}}{r^4}\left[\left(-\frac{46372}{105} + \frac{74759}{35}\nu - \frac{51532}{35}\nu^2\right)\mathbf{v}^6 + \left(\frac{54085}{21} - \frac{258268}{21}\nu + \frac{186635}{21}\nu^2\right)\mathbf{v}^4\dot{r}^2 \right.\\
& \qquad \left. + \left(-\frac{12274}{3} + \frac{56683}{3}\nu - \frac{40526}{3}\nu^2\right)\mathbf{v}^2\dot{r}^4 + \left(1980 - 8658\nu + 5823\nu^2\right)\dot{r}^6\right],
\end{aligned} \tag{266}$$

and

$$\begin{aligned}
\mathcal{B} = {} & \frac{G^5 m^5 \nu}{r^6}\left(\frac{927826}{2835} + \frac{73772}{105}\nu + \frac{3008}{35}\nu^2\right) \\
& + \frac{G^4 m^4 \nu}{r^5}\left[\left(-\frac{89926}{315} - \frac{305576}{315}\nu - \frac{17972}{105}\nu^2\right)\mathbf{v}^2 + \left(\frac{33926}{45} + \frac{13504}{3}\nu + \frac{4380}{7}\nu^2\right)\dot{r}^2\right] \\
& + \frac{G^3 m^3 \nu}{r^4}\left[\left(-\frac{673}{21} + \frac{173872}{315}\nu - \frac{2376}{35}\nu^2\right)\mathbf{v}^4 + \left(\frac{75211}{105} - \frac{1332301}{315}\nu - \frac{79052}{105}\nu^2\right)\mathbf{v}^2\dot{r}^2 \right.\\
& \qquad \left. + \left(-\frac{46224}{35} + \frac{177399}{35}\nu + \frac{140618}{105}\nu^2\right)\dot{r}^4\right] \\
& + \frac{G^2 m^2 \nu}{r^3}\left[\left(\frac{18124}{315} - \frac{9621}{35}\nu + \frac{5004}{35}\nu^2\right)\mathbf{v}^6 + \left(-\frac{23053}{35} + \frac{98632}{35}\nu - \frac{39359}{35}\nu^2\right)\mathbf{v}^4\dot{r}^2 \right.\\
& \qquad \left. + \left(\frac{28274}{21} - \frac{104015}{21}\nu + \frac{19774}{21}\nu^2\right)\mathbf{v}^2\dot{r}^4 + \left(-\frac{2324}{3} + \frac{7070}{3}\nu + \frac{833}{3}\nu^2\right)\dot{r}^6\right].
\end{aligned} \tag{267}$$

D. Consistency checks on results

1. Quasicircular limit

In this section, we consider the special case of quasicircular orbits. In particular, we will use the acceleration through 4.5PN order to compute the expression $\dot{\omega}/\omega^2$, where ω is a well-defined orbital frequency for the special case of quasicircular orbits. This expression, when written as a function of ω, is gauge independent under a large class of gauge transformations and will allow comparison with the general results derived in [110]. Additionally, one can use flux-balance arguments to compute $\dot{\omega}/\omega^2$ exclusively from the far-field, allowing a direct comparison between the near-field and far-field EFT regimes.

We follow the approach of [110]. The quasicircular orbit limit is defined by the relations in equations (131–133). Using these relations, and using $\gamma \equiv Gm/r$, we can write

$$\frac{v^2}{r^2} = \omega^2 = \gamma\left[1 + \gamma(-3+\nu) + \gamma^2\left(\frac{41}{4}\nu + \nu^2\right) + \mathcal{O}(v^5)\right] \tag{268}$$

using the conservative equations of motion through 2PN, which can be inverted and written in terms of either ω or v as

$$\gamma = (Gm\omega)^{2/3}\left[1 + \left(1 - \frac{\nu}{3}\right)(Gm\omega)^{2/3} + \left(3 - \frac{65\nu}{12}\right)(Gm\omega)^{4/3} + \mathcal{O}(v^5)\right], \tag{269}$$

$$= v^2\left[1 + (3-\nu)v^2 + \left(18 - \frac{89}{4}\nu + \nu^2\right)v^4 + \mathcal{O}(v^5)\right]. \tag{270}$$

Taking the circular limit of the radiation-reaction acceleration in equation (265) and using equation (268), we have

$$(\mathbf{a}^i)^{\rm RR}_{\rm circ} = \frac{32\gamma^4\nu\mathbf{v}^i}{5m^3}\left[1 + \left(-\frac{3431}{336} + \frac{5}{4}\nu\right)\gamma + \left(\frac{659217}{18144} + \frac{26095}{2016}\nu - \frac{7}{4}\nu^2\right)\gamma^2 + \mathcal{O}(v^{10})\right]. \tag{271}$$

Taking a time derivative of equation (270) and solving for $\dot{r}$, reducing the acceleration terms using the conservative equations of motion through 2PN and nonconservative equations of motion in equation (271), we find

$$\dot{r} = -\frac{64}{5}\nu\gamma^3\left[1 - \left(\frac{1751}{336} + \frac{7\nu}{4}\right)\gamma + \left(\frac{230879}{18144} + \frac{40981\eta}{2016} + \frac{\eta^2}{2}\right)\gamma^2 + \mathcal{O}(v^{10})\right]. \tag{272}$$

Finally, taking a time derivative of Eq. (268) and solving for $\dot{\omega}/\omega^2$ as a function of ω, we find

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\nu(Gm\omega)^{5/3}\left[1 - \left(\frac{743}{336} + \frac{11}{4}\nu\right)(Gm\omega)^{2/3} + \left(\frac{34103}{18144} + \frac{13661}{2016}\nu + \frac{59}{18}\nu^2\right)(Gm\omega)^{4/3}\right]. \tag{273}$$

This exactly reproduces the near-field results of [110] and far-field expression in [132] and our equation (215), a nontrivial check on our results.

E. COM at 3.5PN

With the radiation of gravitational waves, the binary system can receive a velocity kick as discussed in section III.D.3.a. This leads to a radiative frame correction which must be considered when transforming to the COM frame. We computed conservative corrections to the COM frame in section III.C.3 and the radiative flux-balance equations in section III.D.3. We will use these flux-balance arguments to extract the radiative momentum and COM, which appear as Schott-like total derivative terms in the flux-balance equations for momentum and COM much like those that appear in electromagnetism [142].

The conservative definition of the COM vector $\mathbf{G}^i$ relates to the linear momentum $\mathbf{P}^i$ through the Noetherian integral $\mathbf{K}^i = \mathbf{G}^i - t\mathbf{P}^i$. The invariance of the conservative Lagrangian under the Lorentz boost leads to the conservation of $\mathbf{K}^i$, which implies that [141]

$$\frac{d\mathbf{G}^i}{dt} = \mathbf{P}^i, \tag{274}$$

where $\mathbf{P}^i$ remains constant, i.e.,

$$\frac{d\mathbf{P}^i}{dt} = 0 \tag{275}$$

for all conservative PN orders. This relation no longer holds upon the inclusion of dissipative effects. With radiation reaction considered in the binary dynamics, the flux-balance equations for the energy, angular momentum, and linear momentum can be combined to compute the averaged secular evolution of the binaries. In this section, we focus on solving for a COM position $\mathbf{G}^i$ at 3.5PN that is consistent with the flux-balance equation results in [143].

At the leading 2.5PN order, the balance equation for total linear momentum including the net force can be written as

$$\sum_A m_A \mathbf{a}^i_{A,2.5\mathrm{PN}} + \left.\frac{d\mathbf{P}^i_{2.5\mathrm{PN}}}{dt}\right|_{\mathbf{a}_{0\mathrm{PN}}} = -\frac{2G}{5} I^j I^{ij(5)}, \tag{276}$$

where the mass-type dipole moment $I^i = \int d^3\mathbf{x} T^{00}\mathbf{x}^i$ corresponds to the conserved COM $\mathbf{G}^i$. The $\mathbf{P}^i_{2.5\mathrm{PN}}$ is a possible linear momentum term at 2.5PN that can be solved by the equation above, similar to the "Schott" term in electromagnetism [142]. The Schott-like momentum depends on the expression of the radiation-reaction force on the right-hand side of Eq. (276). We can rewrite some of the time derivatives on the multipole moments with the addition of a total time derivative that can be absorbed into the left-hand side, equivalent to a gauge transformation.

Using the Burke–Thorne acceleration (232) and the leading mass dipole $I^i = \sum_A m_A \mathbf{x}^i_A$, it can be shown from equation (276) that there is no net radiation effects on the linear momentum at 2.5PN related by some gauge transformation, i.e., $\mathbf{P}^i_{2.5\mathrm{PN}} = 0$. With the 3.5PN radiation reaction included, the balance equation for linear momentum is given by [143]

$$\frac{d\mathbf{P}^i}{dt} = -\frac{2G}{5} I_j I^{(5)}_{ij} - \left(\frac{2G}{63} I^{(4)}_{ijk} I^{(3)}_{jk} + \frac{16G}{45}\epsilon_{ijk} I^{(3)}_{jm} J^{(3)}_{km}\right) + \mathcal{O}\left(\epsilon^{4.5}\right), \tag{277}$$

where the net force at 3.5PN contains contributions from

$$\left.\frac{d\mathbf{P}^i}{dt}\right|_{3.5\mathrm{PN}} = \sum_A m_A \mathbf{a}^i_{A,3.5\mathrm{PN}} + \left.\frac{d\mathbf{P}^i_{1\mathrm{PN}}}{dt}\right|_{\mathbf{a}_{2.5\mathrm{PN}}} + \left.\frac{d\mathbf{P}^i_{3.5\mathrm{PN}}}{dt}\right|_{\mathbf{a}_{0\mathrm{PN}}}, \tag{278}$$

with $\mathbf{P}^i_{3.5\mathrm{PN}}$ a possible Schott term modification to the linear momentum at 3.5PN. Equating equations (277) and (278) to solve for $\mathbf{P}^i_{3.5\mathrm{PN}}$ leads to an explicit expression for the secularly evolving linear momentum consistent with the flux-balance equations. At 3.5PN order, equation (277) includes

$$\left.\frac{d\mathbf{P}^i}{dt}\right|_{3.5\mathrm{PN}} = -\frac{2G}{5} I_{j,0} I^{(5)}_{ij,1} - \frac{2G}{5} I_{j,1} I^{(5)}_{ij,0} - \left.\frac{2G}{5} I_{j,0} I^{(5)}_{ij,0}\right|_{\mathbf{a}_{1\mathrm{PN}}} - \frac{2G}{63} I^{(4)}_{ijk} I^{(3)}_{jk} - \frac{16G}{45}\epsilon_{ijk} I^{(3)}_{jm} J^{(3)}_{km}, \tag{279}$$

with the LO mass octupoles and current quadrupoles. For the terms in Eq. (278), the radiation-reaction force $\sum_A m_A \mathbf{a}^i_{A,3.5\text{PN}}$ taken from [98] includes

$$\sum_A m_A \mathbf{a}^i_{A,3.5\text{PN}} = \sum_A \frac{\delta}{\delta \mathbf{x}^i_{A-}} \left(-\frac{G}{5} I^{ij}_{0-} I^{ij(5)}_{1+} - \frac{G}{5} I^{ij}_{1-} I^{ij(5)}_{0+} - \frac{16G}{45} J^{ij}_{0-} J^{ij(5)}_{0+} + \frac{G}{189} I^{ijk}_{0-} I^{ijk(7)}_{0+} \right)_{\text{PL}}$$
$$- \frac{2G}{5} m_A \mathbf{x}^j_A I^{(5)}_{ij} \bigg|_{\mathbf{a}_{1\text{PN}}} + \left(\frac{\partial \mathcal{L}_{1\text{PN}}}{\partial \mathbf{x}^i_A} - \frac{d}{dt} \frac{\partial \mathcal{L}_{1\text{PN}}}{\partial \mathbf{v}^i_A} \right) \bigg|_{\mathbf{a}_{2.5\text{PN}}}, \tag{280}$$

where $L_{1\text{PN}}$ is the 1PN conservative Lagrangian, and the 1PN momentum can be derived from

$$\frac{d\mathbf{P}^i_{1\text{PN}}}{dt} \bigg|_{\mathbf{a}_{2.5\text{PN}}} = \frac{d}{dt} \left(\sum_A \frac{\partial \mathcal{L}_{1\text{PN}}}{\partial \mathbf{v}^i_A} \right) \bigg|_{\mathbf{a}_{2.5\text{PN}}}, \tag{281}$$

with $\sum_A \partial \mathcal{L}_{1\text{PN}} / \partial \mathbf{x}^i_A = 0$. Cancellations from Eq. (278) and Eq. (279) give

$$\sum_A \frac{\delta}{\delta \mathbf{x}^i_{A,-}} \left(-\frac{G}{5} I^{ij}_{1-} I^{ij(5)}_{0+} - \frac{16G}{45} J^{ij}_{0-} J^{ij(5)}_{0+} + \frac{G}{189} I^{ijk}_{0-} I^{ijk(7)}_{0+} \right)_{\text{PL}} + \frac{d\mathbf{P}^i_{3.5\text{PN}}}{dt}$$
$$= - \frac{2G}{5} I^j_1 I^{ij(5)}_0 - \frac{2G}{63} I^{ijk(4)}_0 I^{jk(3)}_0 - \frac{16G}{45} \epsilon^{ijk} I^{jm(3)}_0 J^{km(3)}_0. \tag{282}$$

Integrating by parts liberally and performing the variation of the minus coordinates, we find

$$\frac{d\mathbf{P}^i_{3.5\text{PN}}}{dt} = \frac{d}{dt} \Bigg\{ \frac{G}{63} (-I^{(6)}_{ijk} I_{jk} + I^{(5)}_{ijk} I^{(1)}_{jk} - I^{(4)}_{ijk} I^{(2)}_{jk} - I^{(3)}_{ijk} I^{(3)}_{jk} + I^{(2)}_{ijk} I^{(4)}_{jk})$$
$$+ \frac{8G}{45} \epsilon_{ijk} (2 I_{jl} J^{(5)}_{kl} + I^{(1)}_{jl} J^{(4)}_{kl} - I^{(2)}_{jl} J^{(3)}_{kl} - I^{(3)}_{jl} J^{(2)}_{kl} + I^{(4)}_{jl} J^{(1)}_{kl}) + \frac{8G}{15} J_j J^{(4)}_{ij}$$
$$+ \frac{G}{105} \sum_A m_A \Big[(11 \mathbf{x}^2_A \mathbf{x}^j_A \delta^{ik} - 17 \mathbf{x}^i_A \mathbf{x}^j_A \mathbf{x}^k_A) I^{(6)}_{jk}$$
$$+ \big(34 (\mathbf{x}_A \cdot \mathbf{v}_A) \mathbf{x}^j_A \delta^{ik} - 11 \mathbf{x}^2_A \mathbf{v}^j_A \delta^{ik} - 46 \mathbf{v}^i_A \mathbf{x}^j_A \mathbf{x}^k_A - 22 \mathbf{x}^i_A \mathbf{x}^j_A \mathbf{v}^k_A \big) I^{(5)}_{jk} \Big] \Bigg\}$$
$$- \sum_A \frac{8G}{15} m_A \epsilon_{jkl} \mathbf{x}^j_A \mathbf{a}^k_A J^{(4)}_{il} - I^{(5)}_{jk} \Bigg[\frac{G}{63} I^{(2)}_{ijk} + \frac{8G}{45} \epsilon_{ijl} J^{(1)}_{kl}$$
$$+ \frac{G}{105} \sum_A m_A \Bigg(-22 \mathbf{x}^i_A \mathbf{v}^j_A \mathbf{v}^k_A - 22 \mathbf{x}^i_A \mathbf{x}^j_A \mathbf{a}^k_A + 12 \mathbf{v}^i_A \mathbf{x}^j_A \mathbf{v}^k_A + 17 \mathbf{a}^i_A \mathbf{x}^j_A \mathbf{x}^k_A$$
$$+ \delta^{ik} \left(8 \mathbf{v}^2_A \mathbf{x}^j_A + 34 (\mathbf{x}_A \cdot \mathbf{a}_A) \mathbf{x}^j_A + 12 (\mathbf{x}_A \cdot \mathbf{v}_A) \mathbf{v}^j_A - 11 \mathbf{x}^2_A \mathbf{a}^j_A + \frac{21 G m_B \mathbf{x}^j_A}{r} \right) \Bigg) \Bigg], \tag{283}$$

in which the terms outside the total time derivative vanish after substituting the Newtonian equations of motion and the LO multipole moments. Therefore an explicit linear momentum $\mathbf{P}^i$ that obeys the flux-balance equation at the 3.5PN order is given by

$$\begin{aligned}
\mathbf{P}^i_{3.5\mathrm{PN}} =& \frac{G}{63}(-I^{(6)}_{ijk}I_{jk} + I^{(5)}_{ijk}I^{(1)}_{jk} - I^{(4)}_{ijk}I^{(2)}_{jk} - I^{(3)}_{ijk}I^{(3)}_{jk} + I^{(2)}_{ijk}I^{(4)}_{jk}) \\
&+ \frac{8G}{45}\epsilon_{ijk}(2I_{jl}J^{(5)}_{kl} + I^{(1)}_{jl}J^{(4)}_{kl} - I^{(2)}_{jl}J^{(3)}_{kl} - I^{(3)}_{jl}J^{(2)}_{kl} + I^{(4)}_{jl}J^{(1)}_{kl}) + \frac{8G}{15}J_j J^{(4)}_{ij} \\
&+ \frac{G}{105}\sum_A m_A\Big[(11\mathbf{x}^2_A\mathbf{x}^j_A\delta^{ik} - 17\mathbf{x}^i_A\mathbf{x}^j_A\mathbf{x}^k_A)I^{(6)}_{jk} \\
&\quad + \big(34(\mathbf{x}_A\cdot\mathbf{v}_A)\mathbf{x}^j_A\delta^{ik} - 11\mathbf{x}^2_A\mathbf{v}^j_A\delta^{ik} - 46\mathbf{v}^i_A\mathbf{x}^j_A\mathbf{x}^k_A - 22\mathbf{x}^i_A\mathbf{x}^j_A\mathbf{v}^k_A\big)I^{(5)}_{jk}\Big]. \qquad (284)
\end{aligned}$$

Next, the 3.5PN COM position $\mathbf{G}^i$ is related to $\mathbf{P}^i_{3.5\mathrm{PN}}$ by [144]

$$\frac{d\mathbf{G}^i}{dt} = \mathbf{P}^i - \frac{2G}{21}I^{ijk(3)}I^{jk(3)}. \qquad (285)$$

We equate Eq. (285) with the total 3.5PN expansion of the flux of the COM position, which can be constructed by some total time derivatives,

$$\begin{aligned}
\frac{d\mathbf{G}^i}{dt} =& \frac{d\mathbf{G}^i_{3.5\mathrm{PN}}}{dt} + \frac{d\mathbf{G}^i_{1\mathrm{PN}}}{dt}\bigg|_{\mathbf{a}_{2.5\mathrm{PN}}} \\
=& \frac{d}{dt}\bigg\{\frac{G}{63}(-I_{jk}I^{(5)}_{ijk} + 2I^{(1)}_{jk}I^{(4)}_{ijk} - 3I^{(2)}_{jk}I^{(3)}_{ijk} - 4I^{(3)}_{jk}I^{(2)}_{ijk} + 5I^{(4)}_{jk}I^{(1)}_{ijk}) \\
&+ \frac{8G}{45}\epsilon^{ijk}(2I_{jm}J^{(4)}_{km} - I^{(1)}_{jm}J^{(3)}_{km} - I^{(3)}_{jm}J^{(1)}_{km} + 2I^{(4)}_{jm}J_{km}) + \frac{8G}{15}J^k J^{ik(3)} \\
&+ \frac{G}{105}I^{(5)}_{kl}\sum_A m_A(11\delta^{ik}\mathbf{x}^2_A\mathbf{x}^l_A - 17\mathbf{x}^i_A\mathbf{x}^l_A\mathbf{x}^k_A)\bigg\} - \frac{8G}{15}\epsilon^{jkl}\sum_A m_A\mathbf{x}^k_A\mathbf{a}^l_A J^{ij(3)}, \qquad (286)
\end{aligned}$$

where all multipole moments are their LO expressions. The last term outside the total derivative vanishes after substituting the Newtonian equations of motion. Therefore, the COM position $\mathbf{G}^i$ at 3.5PN can be determined as the Schott terms inside the time derivative of Eq. (286), which contributes to a 4.5PN piece of corrections to the 1PN acceleration.

F. Energy flux-balance equation through NNLO

In this section, we use the energy flux-balance equations as a consistency check on the radiative equations of motion. We expect the locally-induced power loss to be equivalent to the energy flux at infinity up to a total derivative that time-averages to zero. This total time derivative amounts to a redefinition of the local conserved energy, akin to a "Schott" term in electrodynamics, which vanishes in the far-field regime. Through NNLO, the energy flux-balance equation is given by

$$\frac{dE}{dt} = -\left(\frac{G}{5} I_{ij}^{(3)} I_{ij}^{(3)} + \frac{G}{189} I_{ijk}^{(4)} I_{ijk}^{(4)} + \frac{16G}{45} J_{ij}^{(3)} J_{ij}^{(3)} + \frac{G}{84} J_{ijk}^{(4)} J_{ijk}^{(4)} + \frac{G}{9072} I_{ijkl}^{(5)} I_{ijkl}^{(5)} + \dots\right). \tag{287}$$

We first show that the LO radiation-reaction acceleration is consistent with the energy flux-balance equation. With the LO radiation reaction $\mathbf{a}_K^i = -2G/5\mathbf{x}_K^j I_0^{ij(5)}$, the 2.5PN energy flux-balance equation is given by

$$\frac{dE}{dt} = -\frac{2G}{5} \sum_A m_A \mathbf{x}_A^i \mathbf{v}_A^j I_0^{ij(5)} = -\frac{G}{5} I_0^{ij(1)} I_0^{ij(5)}, \tag{288}$$

which agrees with the mass quadrupole term in Eq. (287) modulo a total time-derivative.

At NLO, the acceleration, as derived from the nonconservative Lagrangian, is given by

$$\begin{aligned} \mathbf{a}_A^i = &-\frac{16G}{45}\epsilon^{ikl}\left(\mathbf{x}_A^j \mathbf{x}_A^k J_0^{jl(6)} + 3\mathbf{x}_A^j \mathbf{v}_A^k J_0^{jl(5)}\right) + \frac{G}{63}\mathbf{x}_A^j \mathbf{x}_A^k I_0^{ijk(7)} \\ &- \frac{2G}{5}\mathbf{x}_A^j I_1^{ij(5)} - \frac{G}{5}\left(\frac{\partial I_{1-}^{ij}}{\partial \mathbf{x}_{A-}^i} I_{0+}^{ij(5)} - \frac{d}{dt}\frac{\partial I_{1-}^{ij}}{\partial \mathbf{v}_{A-}^i} I_{0+}^{ij(5)}\right)_{\mathrm{PL}}. \end{aligned} \tag{289}$$

We do not need to consider the order-reduced accelerations arising from substituting the 2.5PN acceleration in the 1PN conservative acceleration; these do not contribute to energy loss due to energy conservation. Integrating by parts liberally, for example with $I_{ij,0}^{(6)}$ and $I_{ij,0}^{(7)}$, the NLO energy flux becomes

$$\frac{dE}{dt} = \frac{16G}{45} J_{jl,0} J_{ij,0}^{(6)} + \frac{G}{189} I_{ijk,0}^{(1)} I_{ijk,0}^{(7)} - \frac{G}{5} I_{ij,0}^{(1)} I_{ij,1}^{(5)} - \frac{G}{5} I_{ij,1}^{(1)} I_{ij,0}^{(5)}, \tag{290}$$

which again agrees with the energy-flux balance equation, Eq. (287), at NLO modulo a total time derivative.

Similarly, given the NNLO accelerations in sections IV.C.1–IV.C.4 (again neglecting the reduced conservative equations of motion) we find that

$$\begin{aligned}\frac{dE}{dt} &= -\frac{G}{84}\sum_A m_A \epsilon^{ijk}\mathbf{v}_A^i\mathbf{x}_A^k\mathbf{x}_A^l\mathbf{x}_A^m J_0^{jlm(8)} - \frac{G}{2268}\sum_A m_A \mathbf{v}_A^i\mathbf{x}_A^j\mathbf{x}_A^k\mathbf{x}_A^l I_0^{ijkl(9)} \\ &+ \frac{16G}{45}J_0^{jl}J_1^{ij(6)} + \frac{G}{189}I_0^{ijk(1)}I_1^{ijk(7)} - \frac{G}{5}I_0^{ij(1)}I_2^{ij(5)} - \frac{G}{5}I_1^{ij(1)}I_1^{ij(5)} \\ &+ \sum_A \mathbf{v}_A^i\Bigg[-\frac{G}{5}\left(\frac{\partial I_{2-}^{jk}}{\partial \mathbf{x}_{A-}^i}I_{0+}^{jk(5)} - \frac{d}{dt}\frac{\partial I_{2-}^{ij}}{\partial \mathbf{v}_{A-}^i}I_{0+}^{ij(5)}\right) - \frac{16G}{45}\left(\frac{\partial J_{1-}^{ij}}{\partial \mathbf{x}_{A-}^i}J_{0+}^{ij(5)} - \frac{d}{dt}\frac{\partial J_{1-}^{ij}}{\partial \mathbf{v}_{A-}^i}J_{0+}^{ij(5)}\right) \\ &\quad + \frac{G}{189}\left(\frac{\partial I_{1,-}^{ijk}}{\partial \mathbf{x}_{A-}^i}I_{0+}^{ijk(7)} - \frac{d}{dt}\frac{\partial I_{1-}^{ijk}}{\partial \mathbf{v}_{A-}^i}I_{0+}^{ijk(7)}\right)\Bigg]_{\mathrm{PL}}. \qquad (291)\end{aligned}$$

To show that this is consistent with the right-hand side of Eq. (287), we must simplify this expression. We collect the higher time-derivative terms, explicitly calculate the variations and rewrite them in terms of multipole moments. This is a lengthy but straightforward process; it can be shown that the left-hand side of the NNLO energy flux then becomes

$$\begin{aligned}\frac{dE}{dt} &= -\frac{G}{84}J_{ijk,0}J_{ijk,0}^{(8)} - \frac{G}{9072}I_{ijkl,0}^{(1)}I_{ijkl,0}^{(9)} + \frac{16G}{45}\left(J_{jl,0}J_{ij,1}^{(6)} + J_{jl,1}J_{ij,0}^{(6)}\right) \\ &+ \frac{G}{189}\left(I_{ijk,0}^{(1)}I_{ijk,1}^{(7)} + I_{ijk,1}^{(1)}I_{ijk,0}^{(7)}\right) - \frac{G}{5}\left(I_{ij,0}^{(1)}I_{ij,2}^{(5)} + I_{ij,1}^{(1)}I_{ij,1}^{(5)} + I_{ij,2}^{(1)}I_{ij,0}^{(5)}\right), \qquad (292)\end{aligned}$$

which again agrees with the energy flux-balance equation (287), at NNLO modulo a total time derivative. This check helps establish the validity of the multipole moments in use and the nonconservative action approach for the derivation of the radiation-reaction effects.

V. Conclusion

In this thesis, we presented advances towards the generation of usable templates from the effective field theory of gravity, particularly in computing the dynamics from existing potentials and multipole moments. Building upon spin-independent, linear-in-spin, and quadratic-in-spin conservative and dissipative results obtained using the EFT approach in [81, 82, 84, 85, 38, 103] to NLO in the PN expansion, we have completed the derivation of the equations of motion, adiabatic invariants and associated flux-balance equations for the energy, angular momentum, momentum, and COM of binary systems with spinning compact objects. We then used these results to compute the evolution of the orbital frequency and accumulated phase for (quasi)circular orbits to NLO in spins, including finite-size effects, finding agreement with corresponding expressions presented in [36, 39]. This is not surprising since, after all, the gravitational potential computed in [39] was shown to be equivalent to the ones previously obtained in the EFT [81, 82, 75] and ADM [56, 57] approaches; and moreover, as we demonstrated here, the source multipole moments in [39] are in complete agreement with those derived before in [38], after the latter are expressed in terms of conserved-norm spin variables.

Our results here include also the angular momentum flux for the first time, which is necessary to compute the phase evolution in elliptic-like orbits, allowing us to incorporate spin effects in the waveforms for generic configurations [145, 146], thus extending the validity of the PN approximation towards higher frequencies. This will be crucial for proper data analysis with spinning binaries, since eccentricity can rapidly deteriorate the accuracy of waveforms relying on quasicircular approximations [147]. Using the source multipoles and equations of motion we have also computed the radiated flux of linear momentum, with which we obtained the kick velocity for quasicircular orbits, including linear and bilinear spin effects to NLO and LO, respectively. While perfect agreement is found with an earlier derivation in [123] for spin-orbit effects, unfortunately we disagree in the spin-spin sector even for the case of black holes (with $\kappa_\pm = 1$). We trace the discrepancy to finite-size contributions in the current quadrupole and mass octupole moments which were not included in the derivation in [123]. As it turns out, only the quadratic-in-spin correction to J_{ikl} at NLO is missing to complete the value of the kick velocity at the same order. Moreover, building on the recent rederivation of the spin-independent radiated fluxes at 2PN in [78], it is straightforward to continue pushing forward at the next order for spin effects in the EFT approach. The derivation of GW observables to NNLO was completed in [104].

In the penultimate chapter of this thesis, we computed for the first time the NNLO radiation-reaction equations of motion completely within the EFT framework using ingredients previously derived in [78]. This result pushes the known equations of motion to 4.5PN beyond the leading order Newtonian acceleration. We have performed consistency checks on our result to confirm that it agrees with energy flux-balance results in the far-field regime through 2PN beyond the leading quadrupole result. Additionally, we take the circular limit of our result in a gauge-invariant way, finding agreement with the generic-gauge equations of motion from the literature [8]. This is an important step towards completing the equations of motion through 5PN for use in producing templates for gravitational-wave detectors now and in the future.

Appendix A Nonconservative classical mechanics

Lagrangian mechanics applies specifically to conservative systems as a consequence of the imposition of boundary conditions in time. For nonconservative systems, on the other hand, one should impose *initial conditions* and let the dynamics evolve causally forward in time. In order to accomplish the latter from an action principle, we use the approach of [138]. The approach is as follows:

- Double the degrees of freedom.
- Introduce a variation with the initial conditions specified at t_i.
- Impose an equality condition such that the final conditions are not fixed but match at t_f.
- Derive the equations of motion by ensuring that the integrand is zero for arbitrary variations.
- Take the physical limit by setting the two histories equal to each other.

We will now demonstrate how this works in practice using an example from [138] of two coupled simple harmonic oscillators.

Suppose we have coupled simple harmonic oscillators defined by the action

$$S[q,Q] = \int_{t_i}^{t_f} dt \left\{ \frac{m}{2}(\dot{q}^2 - \omega^2 q^2) + \lambda q Q + \frac{M}{2}(\dot{Q}^2 - \Omega^2 Q^2) \right\}, \tag{293}$$

one with coordinate $q(t)$, frequency ω, and mass m and the other with coordinate $Q(t)$, frequency Ω, and mass M with coupling λ. Integrating out Q by solving for its equations of motion and substituting back into the action above, we obtain an effective action

$$S_{\text{eff}}[q] = \int_{t_i}^{t_f} dt \left\{ \frac{m}{2}(\dot{q}^2 - \omega^2 q^2) + \lambda q Q^{(h)}(t) + \frac{\lambda^2}{2M} \int_{t_i}^{t_f} dt'\, q(t) G_{\text{ret}}(t-t') q(t') \right\} \tag{294}$$

where $Q^{(h)}(t)$ is the homogeneous solution for Q, and $G_{\text{ret}}(t-t')$ is the retarded Green's function for Q. Note that the last term can be written as

$$\frac{\lambda^2}{2M} \int_{t_i}^{t_f} dt\, dt'\, q(t) \left[\frac{G_{\text{ret}}(t-t') + G_{\text{adv}}(t-t')}{2} \right] q(t'), \tag{295}$$

and upon variation with respect to q yields the equations of motion

$$m\ddot{q} + m\omega^2 q = \lambda Q^{(h)} + \frac{\lambda^2}{2M}\int_{t_i}^{t_f} dt' \left[\frac{G_{\rm ret}(t-t') + G_{\rm adv}(t-t')}{2}\right] q(t'). \tag{296}$$

The second term is time-symmetric and couples to the advanced Green's function, which demonstrates that we are describing conservative interactions between q and Q and that we are specifying boundary conditions in time. This is inconsistent with our desired description of a dissipative system specifying initial conditions alone.

To resolve this, we can double the variables q and Q as

$$q(t) \to (q_1(t), q_2(t)), \quad Q(t) \to (Q_1(t), Q_2(t)), \tag{297}$$

where the index represents different histories, and rewrite the action as

$$S[q_{1,2}, Q_{1,2}] = \int_{t_i}^{t_f} dt\, L[q_1, Q_1] + \int_{t_f}^{t_i} dt\, L[q_2, Q_2] = \int_{t_i}^{t_f} dt\, (L[q_1, Q_1] - L[q_2, Q_2]). \tag{298}$$

We can rewrite the variables $1, 2$ in terms of $+, -$ variables as

$$q_+ = q_1 + q_2, \quad q_- = \frac{q_1 - q_2}{2}, \tag{299}$$

and the same for $Q_{+,-}$; the physical limit corresponds to $q_1 \to q, q_2 \to q$, or equivalently $q_+ \to q$ and $q_- \to 0$, and the same for Q. This yields the action

$$S[q_\pm, Q_\pm] = \int_{t_i}^{t_f} dt \Big\{ m(\dot{q}_- \dot{q}_+ - \omega^2 q_- q_+) + \lambda q_- Q_+ + \lambda q_+ Q_- + M(\dot{Q}_- \dot{Q}_+ - \Omega^2) Q_- Q_+ \Big\}. \tag{300}$$

Varying with respect to $Q_{+,-}$ and taking the physical limit, we obtain the solutions

$$Q_+(t) = Q^{(h)}(t) + \frac{\lambda}{M}\int_{t_i}^{t_f} dt' G_{\rm ret}(t-t') q_+(t') \tag{301}$$

$$Q_-(t) = \frac{\lambda}{M}\int_{t_i}^{t_f} dt' G_{\rm adv}(t-t') q_-(t'). \tag{302}$$

Note that the equation of motion for Q_+ evolves forward in time from initial data, whereas the equation of motion for Q_- evolves backwards in time from final data, and in fact is unphysical. Integrating out the Q_i from (300) yields the effective action

$$\mathcal{S}_{\rm eff}[q_\pm] = \int_{t_i}^{t_f} dt \Big\{ m(\dot{q}_- \dot{q}_+ - \omega^2 q_- q_+) + \lambda q_- Q^{(h)} + \frac{\lambda^2}{M}\int_{t_i}^{t_f} dt' q_-(t) G_{\rm ret}(t-t') q_+(t') \Big\}. \tag{303}$$

We now have an action that is no longer symmetric in time and couples to the full retarded Green's function. At this point, we can vary with respect to the minus coordinates and take the physical limit to obtain the correct equations of motion. We find that

$$m\ddot{q} + m\omega^2 q = \lambda Q^{(h)}(t) + \frac{\lambda^2}{M}\int_{t_i}^{t_f} dt' G_{\text{ret}}(t-t')q(t') \tag{304}$$

after taking the physical limit $q_+ \to q, q_- \to 0$.

Appendix B NRGR vertices

The vertices needed to compute the 2.5PN and 3PN COM corrections [80, 112] are

$$S_H^{v^0} = -\sum_A \frac{m_A}{2m_{\rm pl}} \int dt_A \, H_{00} \tag{305}$$

$$S_H^{v^1} = -\sum_A \frac{m_A}{m_{\rm pl}} \int dt_A \, v_A^i H_{0i} \tag{306}$$

$$S_{H\bar{h}_{00}}^{v^0} = \sum_A \frac{m_A}{4m_{\rm pl}^2} \int dt_A \, H_{00}\bar{h}_{00} \tag{307}$$

$$S_{H\bar{h}_{00}}^{v^1} = \sum_A \frac{m_A}{2m_{\rm pl}^2} \int dt_A \, v_A^i H_{0i}\bar{h}_{00} \tag{308}$$

$$S_H^{Sv^0} = \sum_A \frac{1}{2m_{\rm pl}} \int dt_A \, H_{i0,k} S_A^{ik} \tag{309}$$

$$S_H^{Sv^1} = \sum_A \frac{1}{2m_{\rm pl}} \int dt_A \left[H_{ij,k} S_A^{ik} v_A^j + H_{00,k} S_A^{0k}\right] \tag{310}$$

$$S_{\bar{h}_{00}}^{Sv^1} = \sum_A \frac{1}{2m_{\rm pl}} \int dt_A \, \bar{h}_{00,k} S_A^{0k} \tag{311}$$

$$S_{H\bar{h}_{00}}^{Sv^0} = \sum_A \frac{1}{4m_{\rm pl}^2} \int dt_A \, S_A^{ij} H_j{}^0 \bar{h}_{00,i} \tag{312}$$

$$S_{H\bar{h}_{00}}^{Sv^1} = \sum_A \frac{1}{4m_{\rm pl}^2} \int dt_A \, S_A^{i0}\left[H_{00}\bar{h}_{00,i} + \bar{h}_{00} H_{00,i} + H^l{}_i \bar{h}_{00,l}\right], \tag{313}$$

$$S_H^{S^2v^0} = -\sum_A \frac{C_{ES^2}}{4m_A m_{\rm pl}^2} \int dt_A \, H_{00,ij} S^{ik} S^{jk} \tag{314}$$

$$S_{H\bar{h}_{00}}^{S^2v^0} = \sum_A \frac{C_{ES^2}}{8m_A m_{\rm pl}^2} \int dt_A \, S^{ik} S^{jk} [2H_{00,i}\bar{h}_{00,j} + \bar{h}_{00,l}(H_{ij,l} - 2H_{il,j}) + \bar{h}_{00} H_{00,ij}] \tag{315}$$

Vertices are expressed using the Minkowski metric.

Appendix C Ingredients

A. Accelerations

1. Nonspin accelerations

The PN corrections to the Newtonian acceleration of one of the bodies–let us choose body 1–in the binary system are given below. In the EFT formalism, the 1PN correction to the LO gravitational acceleration

$$(\mathbf{a}_1^i)^{(0\mathrm{PN})} = -\frac{Gm_2}{r^2}\mathbf{n}^i, \tag{316}$$

can be derived from the Lagrangian obtained in [40], and it reads as

$$\begin{aligned}(\mathbf{a}_1^i)^{(1\mathrm{PN})} =& \frac{Gm_2}{2r^2}\Big\{\mathbf{n}^i\left[\frac{2Gm}{r} - 3(\mathbf{v}_1^2+\mathbf{v}_2^2) + 7\mathbf{v}_1\cdot\mathbf{v}_2 + 3\mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2\cdot\mathbf{n}\right] \\ &- \mathbf{v}_2\cdot\mathbf{n}\mathbf{v}_1^i - \mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2^i + \dot{r}(6\mathbf{v}_1^i - 7\mathbf{v}_2^i - \mathbf{n}^i\mathbf{v}_2\cdot\mathbf{n}) \\ &- 6r\mathbf{a}_1^i + 7r\mathbf{a}_2^i + (\mathbf{v}^i - \mathbf{n}^i\dot{r})\mathbf{v}_2\cdot\mathbf{n} + r\mathbf{a}_2\cdot\mathbf{n}\mathbf{n}^i + \mathbf{n}^i\big(\mathbf{v}_2\cdot(\mathbf{v}-\mathbf{n}\dot{r})\big)\Big\} \\ &- \frac{1}{2}\mathbf{a}_1^i\mathbf{v}_1^2 - \mathbf{v}_1^i\mathbf{v}_1\cdot\mathbf{a}_1. \end{aligned}\tag{317}$$

The second PN correction to the gravitational acceleration was derived in [78] considering the EFT theory in the linearized harmonic gauge, and it is given as follows:

$$\begin{aligned}(\mathbf{a}_1^i)^{(2\mathrm{PN})} =& \frac{1}{8}\frac{Gm_2}{r^3}\mathbf{x}^i\Big\{\frac{G^2}{r^2}(-2m_1^2 - 20m_1m_2 + 16m_2^2) + \frac{G}{r}\Big[(18m_1+56m_2)\mathbf{v}_1^2 \\ &- (84m_1+128m_2)\mathbf{v}_1\cdot\mathbf{v}_2 + (58m_1+64m_2)\mathbf{v}_2^2 + 30m_1\mathbf{a}_1\cdot\mathbf{x} - 12m\mathbf{a}_2\cdot\mathbf{x} \\ &+ \frac{28}{r^2}(m_1-4m_2)\mathbf{v}_1\cdot\mathbf{x}(\mathbf{v}_1\cdot\mathbf{x} - 2\mathbf{v}_2\cdot\mathbf{x}) - \frac{1}{r^2}(56m_1+176m_2)(\mathbf{v}_2\cdot\mathbf{x})^2\Big] \\ &+ 2\mathbf{v}_1^4 - 16(\mathbf{v}_1\cdot\mathbf{v}_2)^2 - 16\mathbf{v}_2^4 + 32\mathbf{v}_1\cdot\mathbf{v}_2\mathbf{v}_2^2 - 2\mathbf{v}_1^2\mathbf{a}_2\cdot\mathbf{x} - 2\mathbf{v}_2^2\mathbf{a}_2\cdot\mathbf{x} \\ &- 4\mathbf{a}_2\cdot\mathbf{v}_2\mathbf{v}_2\cdot\mathbf{x} + \frac{(\mathbf{v}_2\cdot\mathbf{x})^2}{r^2}(12\mathbf{v}_1^2 - 48\mathbf{v}_1\cdot\mathbf{v}_2 + 36\mathbf{v}_2^2) - 15\frac{(\mathbf{v}_2\cdot\mathbf{x})^4}{r^4}\Big\} \\ &+ \frac{1}{4}\frac{Gm_2}{r^3}\mathbf{v}_1^i\Big\{\frac{G}{r}\Big[(48m_2-15m_1)\mathbf{v}_1\cdot\mathbf{x} + (23m_1-40m_2)\mathbf{v}_2\cdot\mathbf{x}\Big]\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{v}_2\cdot\mathbf{x}(4\mathbf{v}_1^2+16\mathbf{v}_1\cdot\mathbf{v}_2-20\mathbf{v}_2^2)-24\frac{\mathbf{v}_1\cdot\mathbf{x}(\mathbf{v}_2\cdot\mathbf{x})^2}{r^2}+18\frac{(\mathbf{v}_2\cdot\mathbf{x})^3}{r^2}\\
&+\mathbf{v}_1\cdot\mathbf{x}(8\mathbf{v}_1^2-16\mathbf{v}_1\cdot\mathbf{v}_2+16\mathbf{v}_2^2-2\mathbf{a}_2\cdot\mathbf{x})+2r^2(12\mathbf{a}_1-7\mathbf{a}_2)\cdot\mathbf{v}_1\Bigg\}\\
&+2\mathbf{a}_1\cdot\mathbf{v}_1\mathbf{v}_1^2\mathbf{v}_1^i+\frac{1}{4}\mathbf{a}_1^i\Bigg(49\frac{G^2m_1m_2}{r^2}+36\frac{G^2m_2^2}{r^2}+12\frac{Gm_2}{r}\mathbf{v}_1^2+\mathbf{v}_1^4\Bigg)\\
&+\frac{1}{4}\frac{Gm_2}{r^3}\mathbf{v}_2^i\Bigg\{\frac{G}{r}\Bigg[(31m_1-24m_2)\mathbf{v}_1\cdot\mathbf{x}+(40m_2-9m_1)\mathbf{v}_2\cdot\mathbf{x}\Bigg]+24\frac{\mathbf{v}_1\cdot\mathbf{x}(\mathbf{v}_2\cdot\mathbf{x})^2}{r^2}\\
&+\mathbf{v}_2\cdot\mathbf{x}(-4\mathbf{v}_1^2-16\mathbf{v}_1\cdot\mathbf{v}_2+20\mathbf{v}_2^2)-18\frac{(\mathbf{v}_2\cdot\mathbf{x})^3}{r^2}-14r^2\mathbf{a}_2\cdot\mathbf{v}_2\\
&+\mathbf{v}_1\cdot\mathbf{x}(16\mathbf{v}_1\cdot\mathbf{v}_2-16\mathbf{v}_2^2)\Bigg\}-\frac{7}{4}\frac{Gm_2}{r}\mathbf{a}_2^i\Bigg(6\frac{Gm}{r}+\mathbf{v}_1^2+\mathbf{v}_2^2\Bigg). \qquad (318)
\end{aligned}
$$

2. Spin potentials

The LO and NLO spin-orbit potentials [80, 85]–from which the LO and NLO spin-orbit accelerations and binding energies are computed–read, respectively, as

$$
V_{\rm SO}^{(1.5\rm PN)}=\frac{G\mathbf{x}^j}{r^3}\Bigg\{m_2(S_1^{j0}+S_1^{jk}\mathbf{v}_1^k-2S_1^{jk}\mathbf{v}_2^k)-m_1(S_2^{j0}+S_2^{jk}\mathbf{v}_2^k-2S_2^{jk}\mathbf{v}_1^k)\Bigg\}, \qquad (319)
$$

$$
\begin{aligned}
V_{\rm SO}^{2.5\rm PN}=\frac{Gm_2}{r^3}\Bigg\{&\Bigg[S_1^{i0}\Bigg(2\mathbf{v}_2^2-2\mathbf{v}_1\cdot\mathbf{v}_2-\frac{3}{2r^2}(\mathbf{v}_2\cdot\mathbf{x})^2-\frac{1}{2}\mathbf{a}_2\cdot\mathbf{x}\Bigg)\\
&+\Bigg(2\mathbf{v}_1\cdot\mathbf{v}_2+\frac{3(\mathbf{v}_2\cdot\mathbf{x})^2}{r^2}-2\mathbf{v}_2^2+\mathbf{a}_2\cdot\mathbf{x}\Bigg)S_1^{ij}\mathbf{v}_2^j\\
&-\Bigg(\frac{3}{2r^2}(\mathbf{v}_2\cdot\mathbf{x})^2+\frac{1}{2}\mathbf{a}_2\cdot\mathbf{x}\Bigg)S_1^{ij}\mathbf{v}_1^j+2S_1^{ij}\mathbf{a}_2^j\mathbf{v}_2\cdot\mathbf{x}+r^2S_1^{ij}\mathbf{a}_2^j\Bigg]\mathbf{x}^i\\
&+S_1^{i0}\Bigg((\mathbf{v}_1-\mathbf{v}_2)^i\mathbf{v}_2\cdot\mathbf{x}-\frac{3}{2}\mathbf{a}_2^ir^2\Bigg)+S_1^{ij}\Bigg(\mathbf{v}_2^i\mathbf{v}_1^j\mathbf{v}_2\cdot\mathbf{x}-r^2\mathbf{a}_2^j\mathbf{v}_2^i-\frac{1}{2}r^2\mathbf{a}_2^j\mathbf{v}_1^i\Bigg)\Bigg\}\\
+\frac{G^2m_2}{r^4}\mathbf{x}^i&\Bigg[-(m_1+2m_2)S_1^{i0}+\Bigg(m_1-\frac{m_2}{2}\Bigg)S_1^{ij}\mathbf{v}_1^j+\frac{5m_2}{2}S_1^{ij}\mathbf{v}_2^j\Bigg]+1\leftrightarrow 2. \qquad (320)
\end{aligned}
$$

The spin-spin potential at LO is given by [84, 81, 75]

$$
V_{\rm LO}^{\rm SS}=\frac{G}{r^3}S_1^{jk}S_2^{ji}\left(\delta^{ki}-3n^kn^i\right)-\Bigg\{C_{ES^2}^{(1)}\frac{m_2G}{2m_1r^3}\left(\mathbf{S}_1^2-3\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2\right)+1\leftrightarrow 2\Bigg\} \qquad (321)
$$

and at NLO is

$$
\begin{aligned}
V_{\rm NLO}^{\rm SS}=-\frac{G}{r^3}\Bigg[&\left(\delta^{ij}-3n^in^j\right)\Bigg(S_1^{i0}S_2^{j0}+\frac{1}{2}\mathbf{v}_1\cdot\mathbf{v}_2S_1^{ik}S_2^{jk}+v_1^mv_2^kS_1^{ik}S_2^{jm}-v_1^kv_2^mS_1^{ik}S_2^{jm}\\
&+S_1^{i0}S_2^{jk}\left(v_2^k-v_1^k\right)+S_1^{ik}S_2^{j0}\left(v_1^k-v_2^k\right)\Bigg)+\frac{1}{2}S_1^{ki}S_2^{kj}\left(3\mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2\cdot\mathbf{n}\left(\delta^{ij}-5n^in^j\right)\right.
\end{aligned}
$$

$$+3\mathbf{v}_1\cdot\mathbf{n}\left(v_2^jn^i+v_2^in^j\right)+3\mathbf{v}_2\cdot\mathbf{n}\left(v_1^jn^i+v_1^in^j\right)-v_1^iv_2^j-v_2^iv_1^j\right)$$

$$+\left(3n^l\mathbf{v}_2\cdot\mathbf{n}-v_2^l\right)S_1^{0k}S_2^{kl}+\left(3n^l\mathbf{v}_1\cdot\mathbf{n}-v_1^l\right)S_2^{0k}S_1^{kl}\Big]-\frac{3MG^2}{r^4}S_1^{jk}S_2^{ji}\left(\delta^{ki}-3n^kn^i\right)$$

$$+\Bigg\{C_{ES^2}^{(1)}\frac{Gm_2}{2m_1r^3}\left[S_1^{j0}S_1^{i0}\left(3n^in^j-\delta^{ij}\right)-2S_1^{k0}\left(\left(\mathbf{v}_1\times\mathbf{S}_1\right)^k-3\left(\mathbf{n}\cdot\mathbf{v}_1\right)\left(\mathbf{n}\times\mathbf{S}_1\right)^k\right)\right]$$

$$+C_{ES^2}^{(1)}\frac{Gm_2}{2m_1r^3}\Bigg[\mathbf{S}_1^2\left(6\left(\mathbf{n}\cdot\mathbf{v}_1\right)^2-\frac{15}{2}\mathbf{n}\cdot\mathbf{v}_1\mathbf{n}\cdot\mathbf{v}_2+\frac{13}{2}\mathbf{v}_1\cdot\mathbf{v}_2-\frac{3}{2}\mathbf{v}_2^2-\frac{7}{2}\mathbf{v}_1^2-2\mathbf{a}_1\cdot\mathbf{x}\right)$$

$$+\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2\left(\frac{9}{2}\left(\mathbf{v}_1^2+\mathbf{v}_2^2\right)-\frac{21}{2}\mathbf{v}_1\cdot\mathbf{v}_2-\frac{15}{2}\mathbf{n}\cdot\mathbf{v}_1\mathbf{n}\cdot\mathbf{v}_2\right)+2\mathbf{v}_1\cdot\mathbf{S}_1\mathbf{v}_1\cdot\mathbf{S}_1$$

$$-3\mathbf{v}_1\cdot\mathbf{S}_1\mathbf{v}_2\cdot\mathbf{S}_1-6\mathbf{n}\cdot\mathbf{v}_1\mathbf{n}\cdot\mathbf{S}_1\mathbf{v}_1\cdot\mathbf{S}_1+9\mathbf{n}\cdot\mathbf{v}_2\mathbf{n}\cdot\mathbf{S}_1\mathbf{v}_1\cdot\mathbf{S}_1+3\mathbf{n}\cdot\mathbf{v}_1\mathbf{n}\cdot\mathbf{S}_1\mathbf{v}_2\cdot\mathbf{S}_1\Bigg]$$

$$+C_{ES^2}^{(1)}\frac{m_2G^2}{2r^4}\left(1+\frac{4m_2}{m_1}\right)\left(\mathbf{S}_1^2-3\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2\right)$$

$$-\frac{G^2m_2}{r^4}\left(\mathbf{S}_1\cdot\mathbf{n}\right)^2+\left(\tilde{\mathbf{a}}_{1(1)}^{\rm so}\right)^lS_1^{0l}+\mathbf{v}_1\times\mathbf{S}_1\cdot\tilde{\mathbf{a}}_{1(1)}^{\rm so}+1\leftrightarrow2\Bigg\},\qquad(322)$$

where $\tilde{\mathbf{a}}_{1(1)}^{\rm so}$ is the spin-orbit acceleration in the local frame given by

$$\tilde{\mathbf{a}}_{1(1)}^{\rm so}=\frac{m_2G_N}{m_1r^3}\left[-3\mathbf{v}\times\mathbf{S}_1+6\mathbf{n}\left(\mathbf{v}\times\mathbf{S}_1\right)\cdot\mathbf{n}+3\mathbf{n}\cdot\mathbf{v}\left(\mathbf{n}\times\mathbf{S}_1\right)\right].\qquad(323)$$

B. Multipole moments

The multipole moments needed to obtain the spin-orbit and spin-spin flux-balance results through NLO were obtained in [97, 38, 103]. We present them here, written in the CM frame with the covariant SSC imposed and spin vector defined in the locally-flat frame. Note that we need to take the symmetric, trace-free part of the following expressions. The mass quadrupole moments are

$$I^{ij}_{(0\rm PN)}=m\nu\{\mathbf{x}^i\mathbf{x}^j\},\qquad(324)$$

$$I^{ij}_{(1\rm PN)}=\frac{m\nu}{42}\Bigg\{\left[(-30+48\nu)\frac{Gm}{r}+(29-58\nu)\mathbf{v}^2\right]\mathbf{x}^i\mathbf{x}^j$$
$$+(22-66\nu)r^2\mathbf{v}^i\mathbf{v}^j+(-24+72\nu)r\dot{r}\mathbf{v}^j\mathbf{x}^i\Bigg\},\qquad(325)$$

$$I^{ij}_{(1.5\rm PN)}=\frac{4\nu}{3}\{2(\mathbf{v}\times\mathbf{S})^i\mathbf{x}^j-(\mathbf{x}\times\mathbf{S})^i\mathbf{v}^j+2\delta(\mathbf{v}\times\mathbf{\Sigma})^i\mathbf{x}^j-\delta(\mathbf{x}\times\mathbf{\Sigma})^i\mathbf{v}^j\},\qquad(326)$$

$$I^{ij}_{(2\rm PN)}=\frac{1}{2m}\{(-\delta\kappa_--\kappa_+)\mathbf{S}^i\mathbf{S}^j+4\kappa_-\nu\mathbf{S}^i\mathbf{\Sigma}^j+(\delta\kappa_-\nu-\kappa_+\nu)\mathbf{\Sigma}^i\mathbf{\Sigma}^j\},\qquad(327)$$

$$
\begin{aligned}
I^{ij}_{(2.5\mathrm{PN})} = \frac{\nu}{21}\Big\{&[(5-15\nu)\mathbf{v}\cdot(\mathbf{x}\times\mathbf{S})+(5+12\nu)\delta\mathbf{v}\cdot(\mathbf{x}\times\boldsymbol{\Sigma})]\mathbf{v}^i\mathbf{v}^j \\
&+[(-52+30\nu)\mathbf{v}\cdot(\mathbf{n}\times\mathbf{S})+(-62+54\nu)\delta\mathbf{v}\cdot(\mathbf{n}\times\boldsymbol{\Sigma})]\frac{Gm}{r}\mathbf{n}^i\mathbf{x}^j \\
&+\Big[(19+167\nu)\frac{Gm}{r}+(-2+6\nu)\mathbf{v}^2\Big](\mathbf{v}\times\mathbf{S})^i\mathbf{x}^j \\
&+\Big[(-7+140\nu)\frac{Gm}{r}+(-2-60\nu)\mathbf{v}^2\Big]\delta(\mathbf{v}\times\boldsymbol{\Sigma})^i\mathbf{x}^j \\
&+\Big[(-154-70\nu)\frac{Gm}{r}+(-4+12\nu)\mathbf{v}^2\Big](\mathbf{x}\times\mathbf{S})^i\mathbf{v}^j \\
&+\Big[(-56-34\nu)\frac{Gm}{r}+(-4+36\nu)\mathbf{v}^2\Big]\delta(\mathbf{x}\times\boldsymbol{\Sigma})^i\mathbf{v}^j \\
&+[(56-112\nu)\mathbf{S}\cdot\mathbf{n}+(56-56\nu)\delta\boldsymbol{\Sigma}\cdot\mathbf{n}]\frac{Gm}{r}(\mathbf{v}\times\mathbf{n})^i\mathbf{x}^j \\
&+(10-30\nu)r\dot{r}(\mathbf{v}\times\mathbf{S})^j\mathbf{v}^i+(10-8\nu)\delta r\dot{r}(\mathbf{v}\times\boldsymbol{\Sigma})^i\mathbf{v}^j \\
&+(31+19\nu)\frac{Gm}{r}\dot{r}(\mathbf{n}\times\mathbf{S})^j\mathbf{x}^i+(35+6\nu)\frac{Gm}{r}\delta\dot{r}(\mathbf{n}\times\boldsymbol{\Sigma})^i\mathbf{x}^j\Big\}. \qquad (328)
\end{aligned}
$$

$$
\begin{aligned}
I^{ij}_{(3\mathrm{PN})} = \frac{\nu}{84m}\Big\{&12\mathbf{S}^i\mathbf{n}^j\frac{Gm}{r}\Big[(nS)[2\delta\kappa_-+\kappa_+(-13+5\nu)+(-22+10\nu)] \\
&+(n\Sigma)[20\delta(-1+3\nu)+3\kappa_-(30-26\nu)+3\delta\kappa_+(-30+10\nu)]\Big] \\
&+\boldsymbol{\Sigma}^i\mathbf{n}^j\frac{Gm}{r}\Big[(nS)[3\kappa_-(30-26\nu)+3\delta\kappa_+(-30-10\nu)+4\delta(-61+15\nu)] \\
&+3(n\Sigma)[\delta\kappa_-(30-18\nu)+4\nu(22-10\nu)+\kappa_+(-30+78\nu-20\nu^2)]\Big] \\
&+24\mathbf{v}^i\mathbf{S}^j\Big[(-\delta\kappa_-+\kappa_+)(vS)+[\delta\kappa_++\kappa_-(-1+2\nu)](v\Sigma)\Big] \\
&+24\mathbf{v}^i\boldsymbol{\Sigma}^j\Big[[\delta\kappa_++\kappa_-(-1+2\nu)](vS)+[\kappa_+(1-3\nu)+\delta\kappa_-(-1+\nu)-7\nu](v\Sigma)\Big] \\
&+\mathbf{S}^i\mathbf{S}^j\big(\frac{Gm}{r}(34\delta\kappa_-+218\kappa_+)+13(-\delta\kappa_-+\kappa_+)\mathbf{v}^2\big) \\
&+\boldsymbol{\Sigma}^i\boldsymbol{\Sigma}^j\Big[\frac{Gm}{r}[112\nu-2\delta\kappa_-(46+17\nu)-2\kappa_+(-46+75\nu)] \\
&+[\kappa_+(13-39\nu)+13\delta\kappa_-(-1+\nu)+140\nu]\mathbf{v}^2\Big] \\
&+2\boldsymbol{\Sigma}^i\mathbf{S}^j\Big[\frac{Gm}{r}[92\delta\kappa_+-4\kappa_-(23+17\nu)]+13[\delta\kappa_++\kappa_-(-1+2\nu)]\mathbf{v}^2\Big] \\
&+\mathbf{v}^i\mathbf{v}^j\Big[22(\delta\kappa_--\kappa_+)\mathbf{S}^2+44[-\delta\kappa_++\kappa_-(1-2\nu)](S\Sigma) \\
&+[-22\delta\kappa_-(-1+\nu)+28\nu+\kappa_+(-22+66\nu)]\boldsymbol{\Sigma}^2\Big] \\
&+\mathbf{n}^i\mathbf{n}^j\frac{Gm}{r}\Big[6(nS)^2[7\delta\kappa_--4(-9+20\nu)-2\kappa_+(-9+20\nu)] \\
&+6(nS)(n\Sigma)[\kappa_-(-11+12\nu)-4\delta(-9+20\nu)-\delta\kappa_+(-11+40\nu)]
\end{aligned}
$$

$$+ 3(n\Sigma)^2[8\nu(-9+20\nu) + \delta\kappa_-(-11+26\nu) + \kappa_+(11-48\nu+80\nu^2)]$$
$$+ \mathbf{S}^2[-22\delta\kappa_- + 24(-4+5\nu) + 4\kappa_+(4+15\nu)]$$
$$+ 2(S\Sigma)[12\delta(-4+5\nu) + \kappa_-(-19+14\nu) + \delta\kappa_+(19+30\nu)]$$
$$+ \mathbf{\Sigma}^2[-\delta\kappa_-(19+8\nu) - 8\nu(2+15\nu) - \kappa_+(-19+30\nu+60\nu^2)]\Big]\Bigg\} \quad (329)$$

The mass octupole moments are

$$I^{ijk}_{(0\mathrm{PN})} = -\nu m\delta\{\mathbf{x}^i\mathbf{x}^j\mathbf{x}^k\}, \quad (330)$$
$$I^{ijk}_{(1\mathrm{PN})} = -\frac{\nu m\delta}{6}\Bigg\{\Big[(-5+13)\frac{Gm}{r} + (5-19\nu)v^2\Big]\mathbf{x}^i\mathbf{x}^j\mathbf{x}^k$$
$$+ 6(-1+2\nu)r\dot{r}\mathbf{x}^i\mathbf{x}^j\mathbf{v}^k + 6(1-2\nu)r^2\mathbf{x}^i\mathbf{v}^j\mathbf{v}^k\Bigg\}, \quad (331)$$
$$I^{ijk}_{(1.5\mathrm{PN})} = \frac{3\nu}{2}\{-9\delta(\mathbf{v}\times\mathbf{S})^i\mathbf{x}^j\mathbf{x}^k + (-3+11\nu)(\mathbf{v}\times\mathbf{\Sigma})^i\mathbf{x}^j\mathbf{x}^k$$
$$+ 2\delta(\mathbf{x}\times\mathbf{S})^i\mathbf{x}^j\mathbf{v}^k + 2(1-3\nu)(\mathbf{x}\times\mathbf{\Sigma})^i\mathbf{x}^j\mathbf{v}^k\}, \quad (332)$$
$$I^{ijk}_{(2\mathrm{PN})} = \frac{3\nu}{2m}\{[\delta\kappa_+ + \kappa_-(-1+2\nu)]\mathbf{\Sigma}^i\mathbf{\Sigma}^j\mathbf{r}^k - 2\kappa_-\mathbf{S}^i\mathbf{S}^j\mathbf{r}^k + 2(-\delta\kappa_- + \kappa_+)\mathbf{S}^i\mathbf{\Sigma}^j\mathbf{r}^k\} \quad (333)$$

The current quadrupoles moments are

$$J^{ij}_{(0\mathrm{PN})} = \nu m\delta\{(\mathbf{v}\times\mathbf{x})^i\mathbf{x}^j\}, \quad (334)$$
$$J^{ij}_{(0.5\mathrm{PN})} = -\frac{3}{2}\nu\{\mathbf{\Sigma}^i\mathbf{x}^j\}, \quad (335)$$
$$J^{ij}_{(1\mathrm{PN})} = \frac{\nu m\delta}{28}\Bigg\{\Big(\Big[(54+60\nu)\frac{Gm}{r} + (13-68\nu)\mathbf{v}^2\Big]\mathbf{x}^i + (5-10\nu)r\dot{r}\mathbf{v}^i\Big)(\mathbf{v}\times\mathbf{x})^j\Bigg\}, \quad (336)$$
$$J^{ij}_{(1.5\mathrm{PN})} = \frac{\nu}{28}\Bigg\{\Big[(61-71\nu)\frac{Gm}{r} + 8(-1+10\nu)\mathbf{v}^2\Big]\mathbf{\Sigma}^i\mathbf{x}^j + \Big[40\frac{Gm}{r} + 13\mathbf{v}^2\Big]\delta\mathbf{S}^i\mathbf{x}^j$$
$$+ [-58\delta\mathbf{S}\cdot\mathbf{n} + 2(-8+31\nu)\mathbf{\Sigma}\cdot\mathbf{n}]\frac{Gm}{r}\mathbf{n}^i\mathbf{x}^j + [12\delta\mathbf{S}\cdot\mathbf{v} + 4(3-23\nu)\mathbf{\Sigma}\cdot\mathbf{v}]\mathbf{v}^i\mathbf{x}^j$$
$$+ [-22\delta\mathbf{S}\cdot\mathbf{x} + 2(-11+47\nu)\mathbf{\Sigma}\cdot\mathbf{x}]\mathbf{v}^i\mathbf{v}^j + 12\delta r\dot{r}\mathbf{S}^j\mathbf{v}^i + 4(3-16\nu)r\dot{r}\mathbf{\Sigma}^i\mathbf{v}^j\Bigg\}. \quad (337)$$
$$J^{ij}_{(2\mathrm{PN})} = \frac{\nu}{2m}\{2\kappa_-(\mathbf{v}\times\mathbf{S})^i\mathbf{S}^j + (3+\delta\kappa_- - \kappa_+)(\mathbf{v}\times\mathbf{\Sigma})^i\mathbf{S}^j$$
$$+ (\delta\kappa_- - \kappa_+)(\mathbf{v}\times\mathbf{S})^i\mathbf{\Sigma}^j + [-\delta\kappa_+ + \kappa_-(1-2\nu)](\mathbf{v}\times\mathbf{\Sigma})^i\mathbf{\Sigma}^j\}. \quad (338)$$

The current octupole moments are

$$J^{ijk}_{(0\mathrm{PN})} = -m\nu(1-3\nu)\{(\mathbf{v}\times\mathbf{x})^i\mathbf{x}^j\mathbf{x}^k\}, \quad (339)$$
$$J^{ijk}_{(0.5\mathrm{PN})} = 2\nu\{\mathbf{S}^i\mathbf{x}^j\mathbf{x}^k + \delta\mathbf{\Sigma}^i\mathbf{x}^j\mathbf{x}^k\}. \quad (340)$$

Appendix D Complete results

A. Equations of motion

1. Spin dynamics

The NLO spin contributions to the spin equations of motion for one compact body are given by

$$
\begin{aligned}
\frac{d\mathbf{S}_1^i}{dt} = \Bigg[& -\frac{\delta G^2 m m_2 \nu \mathbf{n}^i (nS_1)(nv)}{2r^3} + \frac{G^2 m_1 m_2^2 (nv)\mathbf{S}_1^i}{mr^3} + \frac{\delta G^2 m m_2 \nu (nv)\mathbf{S}_1^i}{r^3} \\
& + \frac{3G m_1^2 m_2^2 (nv)^3 \mathbf{S}_1^i}{2m^3 r^2} - \frac{G^2 m_1^2 m_2 (nS_1)\mathbf{v}^i}{mr^3} - \frac{3G^2 m_1 m_2^2 (nS_1)\mathbf{v}^i}{2mr^3} - \frac{G^2 m_2^3 (nS_1)\mathbf{v}^i}{2mr^3} \\
& - \frac{3G m_1^3 m_2 (nS_1)(nv)^2 \mathbf{v}^i}{m^3 r^2} - \frac{3G m_1^2 m_2^2 (nS_1)(nv)^2 \mathbf{v}^i}{m^3 r^2} - \frac{G m_1^2 m_2^2 (nv)\mathbf{S}_1^i \mathbf{v}^2}{m^3 r^2} \\
& - \frac{G m_1 m_2^3 (nv)\mathbf{S}_1^i \mathbf{v}^2}{m^3 r^2} - \frac{\delta G m_2 \nu (nv)\mathbf{S}_1^i \mathbf{v}^2}{2r^2} + \frac{2G m_1^3 m_2 (nS_1)\mathbf{v}^i \mathbf{v}^2}{m^3 r^2} + \frac{4G m_1^2 m_2^2 (nS_1)\mathbf{v}^i \mathbf{v}^2}{m^3 r^2} \\
& + \frac{2G m_1 m_2^3 (nS_1)\mathbf{v}^i \mathbf{v}^2}{m^3 r^2} + \frac{G^2 m_1^2 m_2 \mathbf{n}^i (vS_1)}{mr^3} \\
& + \frac{G^2 m_1 m_2^2 \mathbf{n}^i (vS_1)}{2mr^3} + \frac{G^2 m_2^3 \mathbf{n}^i (vS_1)}{2mr^3} - \frac{\delta G^2 m m_2 \nu \mathbf{n}^i (vS_1)}{2r^3} + \frac{3G m_1^3 m_2 \mathbf{n}^i (nv)^2 (vS_1)}{m^3 r^2} \\
& + \frac{3G m_1^2 m_2^2 \mathbf{n}^i (nv)^2 (vS_1)}{2m^3 r^2} - \frac{G m_1^2 m_2^2 (nv)\mathbf{v}^i (vS_1)}{m^3 r^2} - \frac{G m_1 m_2^3 (nv)\mathbf{v}^i (vS_1)}{m^3 r^2} \\
& - \frac{2G m_1^3 m_2 \mathbf{n}^i \mathbf{v}^2 (vS_1)}{m^3 r^2} - \frac{2G m_1^2 m_2^2 \mathbf{n}^i \mathbf{v}^2 (vS_1)}{m^3 r^2} + \frac{\delta G m_2 \nu \mathbf{n}^i \mathbf{v}^2 (vS_1)}{2r^2} \Bigg]_{\text{NLO SO}} \\
& + \Bigg[-\frac{G^2 m_2 \mathbf{n}^i (nS_1S_2)}{r^4} + \frac{G^2 m_1 \nu \mathbf{n}^i (nS_1S_2)}{r^4} + \frac{G^2 m_2 \nu \mathbf{n}^i (nS_1S_2)}{r^4} \\
& - \frac{2G^2 m_2 (\mathbf{n}\times\mathbf{S}_1)^i (nS_1)}{r^4} - \frac{3C_{ES^2}^{(1)} G^2 m_2 (\mathbf{n}\times\mathbf{S}_1)^i (nS_1)}{r^4} \\
& - \frac{12C_{ES^2}^{(1)} G^2 m_2^2 (\mathbf{n}\times\mathbf{S}_1)^i (nS_1)}{m_1 r^4} - \frac{9G^2 m (\mathbf{n}\times\mathbf{S}_1)^i (nS_2)}{r^4} \\
& + \frac{G^2 m_2 (\mathbf{n}\times\mathbf{S}_1)^i (nS_2)}{r^4} + \frac{15C_{ES^2}^{(1)} G m_2^2 (\mathbf{n}\times\mathbf{S}_1)^i (nS_1)(nv)^2}{2m^2 r^3} \\
& + \frac{15G m_1 m_2 (\mathbf{n}\times\mathbf{S}_1)^i (nS_2)(nv)^2}{2m^2 r^3} - \frac{3G^2 m (\mathbf{S}_1\times\mathbf{S}_2)^i}{r^4} + \frac{G^2 m_2 (\mathbf{S}_1\times\mathbf{S}_2)^i}{r^4} \\
& - \frac{3G m_1^2 (nv)^2 (\mathbf{S}_1\times\mathbf{S}_2)^i}{m^2 r^3} - \frac{3G m_1 m_2 (nv)^2 (\mathbf{S}_1\times\mathbf{S}_2)^i}{2m^2 r^3} + \frac{3G m_2^2 (nv)\mathbf{S}_1^i (S_1 nv)}{m m_1 r^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3Gm_1m_2(nv)\mathbf{S}_1^i(S_2nv)}{m^2r^3}+\frac{3Gm_2^2(nv)\mathbf{S}_1^i(S_2nv)}{m^2r^3}+\frac{3Gm_1m_2(nS_1S_2)(nv)\mathbf{v}^i}{m^2r^3}\\
&+\frac{3C^{(1)}_{ES^2}Gm_2^3(nS_1)(S_1nv)\mathbf{v}^i}{m^2m_1r^3}-\frac{6Gm_1m_2(nS_1)(S_2nv)\mathbf{v}^i}{m^2r^3}-\frac{3Gm_2^2(nS_1)(S_2nv)\mathbf{v}^i}{m^2r^3}\\
&+\frac{9C^{(1)}_{ES^2}Gm_1m_2(\mathbf{n}\times\mathbf{S}_1)^i(nS_1)\mathbf{v}^2}{2m^2r^3}+\frac{21C^{(1)}_{ES^2}Gm_2^2(\mathbf{n}\times\mathbf{S}_1)^i(nS_1)\mathbf{v}^2}{2m^2r^3}\\
&+\frac{9C^{(1)}_{ES^2}Gm_2^3(\mathbf{n}\times\mathbf{S}_1)^i(nS_1)\mathbf{v}^2}{2m^2m_1r^3}+\frac{3Gm_1^2(\mathbf{n}\times\mathbf{S}_1)^i(nS_2)\mathbf{v}^2}{m^2r^3}\\
&+\frac{9Gm_1m_2(\mathbf{n}\times\mathbf{S}_1)^i(nS_2)\mathbf{v}^2}{2m^2r^3}+\frac{2Gm_1^2(\mathbf{S}_1\times\mathbf{S}_2)^i\mathbf{v}^2}{m^2r^3}+\frac{5Gm_1m_2(\mathbf{S}_1\times\mathbf{S}_2)^i\mathbf{v}^2}{2m^2r^3}\\
&-\frac{6C^{(1)}_{ES^2}Gm_2^2(nS_1)(nv)(\mathbf{v}\times\mathbf{S}_1)^i}{m^2r^3}-\frac{6C^{(1)}_{ES^2}Gm_2^3(nS_1)(nv)(\mathbf{v}\times\mathbf{S}_1)^i}{m^2m_1r^3}\\
&-\frac{3Gm_1^2(nS_2)(nv)(\mathbf{v}\times\mathbf{S}_1)^i}{m^2r^3}-\frac{3Gm_1m_2(nS_2)(nv)(\mathbf{v}\times\mathbf{S}_1)^i}{m^2r^3}-\frac{3Gm_1m_2\mathbf{v}^i(vS_1S_2)}{m^2r^3}\\
&-\frac{Gm_2^2\mathbf{v}^i(vS_1S_2)}{m^2r^3}-\frac{6C^{(1)}_{ES^2}Gm_2^2(\mathbf{n}\times\mathbf{S}_1)^i(nv)(vS_1)}{m^2r^3}-\frac{3Gm_2^2(\mathbf{n}\times\mathbf{S}_1)^i(nv)(vS_1)}{mm_1r^3}\\
&-\frac{3C^{(1)}_{ES^2}Gm_2^3(\mathbf{n}\times\mathbf{S}_1)^i(nv)(vS_1)}{m^2m_1r^3}-\frac{6Gm_2^2\mathbf{n}^i(S_1nv)(vS_1)}{mm_1r^3}\\
&+\frac{3C^{(1)}_{ES^2}Gm_2^2(\mathbf{v}\times\mathbf{S}_1)^i(vS_1)}{m^2r^3}+\frac{3Gm_2^2(\mathbf{v}\times\mathbf{S}_1)^i(vS_1)}{mm_1r^3}+\frac{3C^{(1)}_{ES^2}Gm_2^3(\mathbf{v}\times\mathbf{S}_1)^i(vS_1)}{m^2m_1r^3}\\
&-\frac{3Gm_1^2(\mathbf{n}\times\mathbf{S}_1)^i(nv)(vS_2)}{m^2r^3}-\frac{6Gm_1m_2(\mathbf{n}\times\mathbf{S}_1)^i(nv)(vS_2)}{m^2r^3}+\frac{2Gm_1^2(\mathbf{v}\times\mathbf{S}_1)^i(vS_2)}{m^2r^3}\\
&+\left.\frac{2Gm_1m_2(\mathbf{v}\times\mathbf{S}_1)^i(vS_2)}{m^2r^3}\right]_{\text{NLO SS}}, \qquad (341)
\end{aligned}
$$

and similarly for particle 2 after exchanging $1\leftrightarrow 2$.

2. Acceleration

The coefficients for the spin-spin acceleration at 3PN in equation (125) are given by

$$
\begin{aligned}
\mathbf{A}_0^i=\mathbf{S}^i\Big\{&(72+36\delta\kappa_--36\kappa_+)(nv)(vS)+(nS)\Big[\big(-60\delta\kappa_-+120\nu+60\kappa_+(1+\nu)\big)(nv)^2\\
&+\big(-24\kappa_+(1+4\nu)-24(1+8\nu)\big)\mathbf{v}^2\Big]+(n\Sigma)\Big[\big(60\delta(-2+\nu)+30\delta\kappa_+(2+\nu)\\
&+\kappa_-(-60+90\nu)\big)(nv)^2+\big(-12\delta\kappa_+(1+4\nu)-12\delta(-1+8\nu)+\kappa_-(12+48\nu)\big)\mathbf{v}^2\Big]\\
&+\big(120\delta-36\delta\kappa_++\kappa_-(36-72\nu)\big)(nv)(v\Sigma)\Big\}\\
&+\boldsymbol{\Sigma}^i\Big\{\big(48\delta-36\delta\kappa_++\kappa_-(36-72\nu)\big)(nv)(vS)
\end{aligned}
$$

$$+ (nS)\Big[\big(60\delta\nu + 30\delta\kappa_+(2+\nu) + \kappa_-(-60+90\nu)\big)(nv)^2$$
$$+ \big(-12\delta\kappa_+(1+4\nu) - 12\delta(1+8\nu) + \kappa_-(12+48\nu)\big)\mathbf{v}^2\Big]$$
$$+ (n\Sigma)\Big[\big(30\delta\kappa_-(-2+\nu) - 120(-1+\nu)^2$$
$$- 30\kappa_+(-2+5\nu+2\nu^2)\big)(nv)^2 + \big(12\delta\kappa_-(1+4\nu)$$
$$+ 12\kappa_+(-1+2\nu)(1+4\nu) + 24(1-\nu+8\nu^2)\big)\mathbf{v}^2\Big]$$
$$+ \big(96 - 36\delta\kappa_-(-1+\nu) - 264\nu + 36\kappa_+(-1+3\nu)\big)(nv)(v\Sigma)\Big\}$$
$$+ \mathbf{v}^i\Big\{\big(120\kappa_+(-2+\nu) + 240\nu\big)(nS)^2(nv)$$
$$+ \big(-120\kappa_-(-2+\nu) + 120\delta\kappa_+(-2+\nu) + 240\delta(1+\nu)\big)(nS)(nv)(n\Sigma)$$
$$+ \big(-60\delta\kappa_-(-2+\nu) - 60\kappa_+(-2+\nu)(-1+2\nu) - 240(-1+2\nu+\nu^2)\big)(nv)(n\Sigma)^2$$
$$+ \big(-12\delta\kappa_- + \kappa_+(60-24\nu) - 48(1+\nu)\big)(nv)\mathbf{S}^2$$
$$+ \big(-24\delta\kappa_+(-3+\nu) + 72\kappa_-(-1+\nu) - 48\delta(3+\nu)\big)(nv)(S\Sigma)$$
$$+ \big(72 + 12\delta\kappa_- + \kappa_+(84-24\nu) - 48\nu\big)(nS)(vS)$$
$$+ \big(-12\delta\kappa_+(-3+\nu) - 24\delta\nu - 12\kappa_-(3+\nu)\big)(n\Sigma)(vS)$$
$$+ \big(-12\delta\kappa_+(-3+\nu) - 24\delta(-1+\nu) - 12\kappa_-(3+\nu)\big)(nS)(v\Sigma)$$
$$+ \big(-36\delta\kappa_- + 12\kappa_+(3-6\nu+2\nu^2) + 24(-2+\nu+2\nu^2)\big)(n\Sigma)(v\Sigma)$$
$$+ \big(12\delta\kappa_-(-3+2\nu) + 48(-2+5\nu+\nu^2) + 12\kappa_+(3-8\nu+2\nu^2)\big)(nv)\boldsymbol{\Sigma}^2\Big\}$$
$$+ \mathbf{n}^i\Big\{(-360 - 60\delta\kappa_- + 60\kappa_+)(nS)(nv)(vS)$$
$$+ \big(-240\delta + 60\delta\kappa_+ + 60\kappa_-(-1+2\nu)\big)(nv)(n\Sigma)(vS)$$
$$+ \big(12\delta\kappa_- + 12\kappa_+(-1+\nu) + 24(3+\nu)\big)(vS)^2$$
$$+ (nS)(n\Sigma)\Big[(-420\delta\nu + 210\kappa_-\nu - 210\delta\kappa_+\nu)(nv)^2$$
$$+ \big(-360\kappa_-\nu + 240\delta\kappa_+\nu + 60\delta(1+8\nu)\big)\mathbf{v}^2\Big]$$
$$+ (nS)^2\Big[(-420\nu - 210\kappa_+\nu)(nv)^2 + \big(30\delta\kappa_- + 60(1+8\nu) + 30\kappa_+(1+8\nu)\big)\mathbf{v}^2\Big]$$
$$+ (n\Sigma)^2\Big[\big(105\delta\kappa_-\nu + 420\nu^2 + 105\kappa_+\nu(-1+2\nu)\big)(nv)^2$$
$$+ \big(-150\delta\kappa_-\nu - 30\kappa_+\nu(-5+8\nu) - 60\nu(1+8\nu)\big)\mathbf{v}^2\Big]$$

$$
\begin{aligned}
&+ \mathbf{S}^2\Big[\big(30\delta\kappa_- + 30\kappa_+(-1+\nu) + 60(2+\nu)\big)(nv)^2 + \big(-12\delta\kappa_- - 48\kappa_+\nu - 12(3+8\nu)\big)\mathbf{v}^2\Big] \\
&+ (S\Sigma)\Big[\big(\kappa_-(60-150\nu) + 30\delta\kappa_+(-2+\nu) + 60\delta(4+\nu)\big)(nv)^2 \\
&+ \big(12\delta^3\kappa_+ - 12\delta(5+8\nu) + \kappa_-(-12+96\nu)\big)\mathbf{v}^2\Big] \\
&+ \big(-360\delta + 60\delta\kappa_+ + 60\kappa_-(-1+2\nu)\big)(nS)(nv)(v\Sigma) \\
&+ \big(\kappa_+(60-180\nu) + 60\delta\kappa_-(-1+\nu) + 120(-2+7\nu)\big)(nv)(n\Sigma)(v\Sigma) \\
&+ \big(\kappa_-(24-60\nu) + 12\delta\kappa_+(-2+\nu) + 24\delta(5+\nu)\big)(vS)(v\Sigma) \\
&+ \big(-6\delta\kappa_-(-2+3\nu) - 24(-2+7\nu+\nu^2) - 6\kappa_+(2-7\nu+2\nu^2)\big)(v\Sigma)^2 \\
&+ \Big[\big(-15\delta\kappa_-(-2+3\nu) - 60(-2+6\nu+\nu^2) - 15\kappa_+(2-7\nu+2\nu^2)\big)(nv)^2 \\
&+ \big(6\delta\kappa_-(-1+6\nu) + 6\kappa_+(1-8\nu+8\nu^2) + 12(-2+7\nu+8\nu^2)\big)\mathbf{v}^2\Big]\mathbf{\Sigma}^2\bigg\}, \qquad (342a)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}_1^i &= \mathbf{S}^i\Big[\big(82 - 12\delta\kappa_- + 36\nu + 18\kappa_+(2+\nu)\big)(nS) \\
&+ \big(18\delta(2+\nu) + 3\delta\kappa_+(8+3\nu) + 3\kappa_-(-8+5\nu)\big)(n\Sigma)\Big] \\
&+ \mathbf{\Sigma}^i\Big[\big(6\delta(7+3\nu) + 3\delta\kappa_+(8+3\nu) + 3\kappa_-(-8+5\nu)\big)(nS) \\
&+ \big(3\delta\kappa_-(-8+\nu) - 2\nu(37+18\nu) - 3\kappa_+(-8+17\nu+6\nu^2)\big)(n\Sigma)\Big] \\
&+ \mathbf{n}^i\Big[\big(24\delta\kappa_- - 48\kappa_+(2+\nu) - 6(35+16\nu)\big)(nS)^2 \\
&+ \big(\kappa_-(120-48\nu) - 24\delta\kappa_+(5+2\nu) - 6\delta(33+16\nu)\big)(nS)(n\Sigma) \\
&+ \big(60\delta\kappa_- + 6\nu(31+16\nu) + 12\kappa_+(-5+10\nu+4\nu^2)\big)(n\Sigma)^2 \\
&+ \big(-4\delta\kappa_- + 10\kappa_+(2+\nu) + 4(9+5\nu)\big)\mathbf{S}^2 \\
&+ \big(6\kappa_-(-4+\nu) + 4\delta(9+5\nu) + 2\delta\kappa_+(12+5\nu)\big)(S\Sigma) \\
&+ \big(-\delta\kappa_-(12+\nu) - 4\nu(9+5\nu) + \kappa_+(12-23\nu-10\nu^2)\big)\mathbf{\Sigma}^2\Big]. \qquad (342b)
\end{aligned}
$$

B. Adiabatic invariants

1. Binding energy

The coefficients of the binding energy in the COM in (147) are given by

$$
\begin{aligned}
e_4^0 &= (12+6\kappa_+)(nS)^2 + (12\delta - 6\kappa_- + 6\delta\kappa_+)(nS)(n\Sigma) + \big(-3\delta\kappa_- + \kappa_+(3-6\nu) - 12\nu\big)(n\Sigma)^2 \\
&\quad + (-4-2\kappa_+)\mathbf{S}^2 + (-4\delta + 2\kappa_- - 2\delta\kappa_+)(S\Sigma) + \big(\delta\kappa_- + 4\nu + \kappa_+(-1+2\nu)\big)\mathbf{\Sigma}^2 \qquad (343)
\end{aligned}
$$

$$
\begin{aligned}
e_6^0 &= (-72\delta - 24\kappa_- + 24\delta\kappa_+)(nv)(n\Sigma)(vS) + 48\kappa_-\nu(nv)(n\Sigma)(vS) \\
&\quad + \big(8\delta\kappa_- + 4\kappa_+(-2+\nu) + 8(3+\nu)\big)(vS)^2 + \big(30\kappa_- + 6\delta\kappa_+(-5+\nu) \\
&\quad + 12\delta(8+\nu)\big)(nS)(n\Sigma)\mathbf{v}^2 + (nS)^2\Big[(-60\nu - 30\kappa_+\nu)(nv)^2 \\
&\quad + \big(6\delta\kappa_- + 6\kappa_+(-4+\nu) + 12(4+\nu)\big)\mathbf{v}^2\Big] + \mathbf{S}^2\Big[\big(6\delta\kappa_- + 6\kappa_+(-1+\nu) + 12(2+\nu)\big)(nv)^2 \\
&\quad + \big(-4\delta\kappa_- - 2\kappa_+(-5+\nu) - 4(6+\nu)\big)\mathbf{v}^2\Big] + (S\Sigma)\Big[\big(\kappa_-(12-30\nu) + 6\delta\kappa_+(-2+\nu) \\
&\quad + 12\delta(4+\nu)\big)(nv)^2 + \big(-2\delta\kappa_+(-7+\nu) - 4\delta(12+\nu) + 2\kappa_-(-7+9\nu)\big)\mathbf{v}^2\Big] \\
&\quad + (n\Sigma)^2\Big[\big(15\delta\kappa_-\nu + 60\nu^2 + 15\kappa_+\nu(-1+2\nu)\big)(nv)^2 \\
&\quad + \big(-3\delta\kappa_-(-5+3\nu) + \kappa_+(-15+39\nu-6\nu^2) - 12(-4+12\nu+\nu^2)\big)\mathbf{v}^2\Big] \\
&\quad + \big(\kappa_+(24-72\nu) + 24\delta\kappa_-(-1+\nu) + 72(-1+3\nu)\big)(nv)(n\Sigma)(v\Sigma) \\
&\quad + \big(\kappa_-(16-36\nu) + 4\delta\kappa_+(-4+\nu) + 8\delta(6+\nu)\big)(vS)(v\Sigma) \\
&\quad + \big(-2\delta\kappa_-(-4+5\nu) + \kappa_+(-8+26\nu-4\nu^2) - 8(-3+9\nu+\nu^2)\big)(v\Sigma)^2 \\
&\quad + (nS)\Big[(-72 - 24\delta\kappa_- + 24\kappa_+)(nv)(vS) + (n\Sigma)\big((-60\delta\nu + 30\kappa_-\nu \\
&\quad - 30\delta\kappa_+\nu)(nv)^2 - 30\kappa_-\nu\mathbf{v}^2\big) + \big(-72\delta + 24\delta\kappa_+ + \kappa_-(-24+48\nu)\big)(nv)(v\Sigma)\Big] \\
&\quad + \Big[\big(\delta\kappa_-(6-9\nu) + \kappa_+(-6+21\nu-6\nu^2) - 12(-2+6\nu+\nu^2)\big)(nv)^2 \\
&\quad + \big(\delta\kappa_-(-7+5\nu) + 4(-6+18\nu+\nu^2) + \kappa_+(7-19\nu+2\nu^2)\big)\mathbf{v}^2\Big]\mathbf{\Sigma}^2, \qquad (344)
\end{aligned}
$$

$$
\begin{aligned}
e_6^1 &= (-36 + 9\delta\kappa_- - 15\kappa_+)(nS)^2 + \big(-32\delta - 24\delta\kappa_+ + \kappa_-(24-36\nu)\big)(nS)(n\Sigma) \\
&\quad + \big(30\nu - 3\delta\kappa_-(-4+3\nu) + \kappa_+(-12+33\nu)\big)(n\Sigma)^2 + (8 - 3\delta\kappa_- + 5\kappa_+)\mathbf{S}^2 \\
&\quad + \big(8\delta + 8\delta\kappa_+ + \kappa_-(-8+12\nu)\big)(S\Sigma) + \big(\kappa_+(4-11\nu) - 10\nu + \delta\kappa_-(-4+3\nu)\big)\mathbf{\Sigma}^2. \qquad (345)
\end{aligned}
$$

2. Orbital angular momentum

The coefficients of the angular momentum in the COM in (155) and (156) are given by

$$\ell_5^0 = \mathbf{S}^i\big((4+5\nu)(nv)^2 - 6\mathbf{v}^2\big) + \mathbf{v}^i\big((-6+12\nu)(nS)(nv) + 4\delta\nu(nv)(n\Sigma) + 4(vS) + 2\delta\nu(v\Sigma)\big)$$
$$+ \mathbf{n}^i\big((-2-8\nu)(nv)(vS) + (nS)(-9\nu(nv)^2 + 6\mathbf{v}^2)$$
$$+ (n\Sigma)(-6\delta\nu(nv)^2 + 3\delta\nu\mathbf{v}^2) - 6\delta\nu(nv)(v\Sigma)\big), \tag{346}$$

$$\ell_5^1 = \mathbf{n}^i\big((-1-4\nu)(nS) - (\delta+3\delta\nu)(n\Sigma)\big) + \mathbf{S}^i(1+4\nu) + \mathbf{\Sigma}^i(\delta+3\delta\nu) \tag{347}$$

$$\ell_6^0 = \Big[\big(6+\delta\kappa_- + \kappa_+(-1+2\nu)\big)(nS)$$
$$+ \big(6\delta + \kappa_-(1-3\nu) + \delta\kappa_+(-1+\nu) - 2\nu\big)(n\Sigma)\Big](\mathbf{v}\times\mathbf{S})^i$$
$$- 6\delta(Sn\Sigma)\mathbf{v}^i + \Big[\big(\kappa_-(1-3\nu) + \delta\kappa_+(-1+\nu) + 2(3\delta+\nu)\big)(nS)$$
$$+ \big(6 - 16\nu + \kappa_-(\delta - 2\delta\nu) + \kappa_+(-1+4\nu-2\nu^2)\big)(n\Sigma)\Big](\mathbf{v}\times\mathbf{\Sigma})^i$$
$$+ \mathbf{S}^i\Big[\big(-\delta\kappa_- + \kappa_+(1-2\nu) + 6(2+\nu)\big)(Snv)$$
$$+ \big(\kappa_-(1-3\nu) + \delta\kappa_+(-1+\nu) - \nu - 3\delta(2+\nu)\big)(vn\Sigma)\Big]$$
$$+ \mathbf{n}^i\Big\{6\delta(nv)(Sn\Sigma) + (n\Sigma)\Big[\big(3\nu - 3\delta(6+\nu)\big)(Snv) - 6(-2+7\nu+\nu^2)(vn\Sigma)\Big]$$
$$+ (nS)\Big[-6(3+\nu)(Snv) + 3\big(\nu + \delta(4+\nu)\big)(vn\Sigma)\Big]\Big\}$$
$$+ (\mathbf{n}\times\mathbf{S})^i\Big\{\big(-3\delta\kappa_- + 6(-3+\nu) + \kappa_+(3+6\nu)\big)(nS)(nv)$$
$$+ \Big[3\kappa_-(-1+\nu) + 3\delta\kappa_+(1+\nu) + 3\big(\delta(-6+\nu)+\nu\big)\Big](nv)(n\Sigma)$$
$$+ \big(14 + 2\delta\kappa_- + 6\nu - 2\kappa_+(1+\nu)\big)(vS)$$
$$+ \big(\kappa_-(2-3\nu) - \nu - \delta\kappa_+(2+\nu) + 3\delta(4+\nu)\big)(v\Sigma)\Big\}$$
$$+ (\mathbf{n}\times\mathbf{\Sigma})^i\Big[\big(3\delta(-6+\nu) + 3\kappa_-(-1+\nu) - 3\nu + 3\delta\kappa_+(1+\nu)\big)(nS)(nv)$$
$$+ \big(-3\delta\kappa_- + \kappa_+(3-6\nu-6\nu^2) - 6(3-9\nu+\nu^2)\big)(nv)(n\Sigma)$$
$$+ \big(\kappa_-(2-3\nu) + \nu - \delta\kappa_+(2+\nu) + \delta(14+3\nu)\big)(vS)$$
$$+ \big(12 - \delta\kappa_-(-2+\nu) - 40\nu - 6\nu^2 + \kappa_+(-2+5\nu+2\nu^2)\big)(v\Sigma)\Big]$$
$$+ \Big[\big(-\nu + 3\delta(4+\nu) + \kappa_-(-1+3\nu) + \kappa_+(\delta-\delta\nu)\big)(Snv)$$

$$
\begin{aligned}
&+\big(-6+26\nu+6\nu^2+\kappa_-(\delta-2\delta\nu)+\kappa_+(-1+4\nu-2\nu^2)\big)(vn\Sigma)\Big]\boldsymbol{\Sigma}^i\\
&+\big(\mathbf{n}\times\mathbf{v}\big)^i\Big[\big(24-3\kappa_+(3+\nu)\big)(nS)^2+\big(48\delta+3\kappa_-(3+\nu)-3\delta\kappa_+(3+\nu)\big)(nS)(n\Sigma)\\
&+\big(24-72\nu+\tfrac{3}{2}\delta\kappa_-(3+\nu)+\tfrac{3}{2}\kappa_+(3+\nu)(-1+2\nu)\big)(n\Sigma)^2+\big(3\kappa_+(1+\nu)-2(7+2\nu)\big)\mathbf{S}^2\\
&+\big(-3\kappa_-(1+\nu)+3\delta\kappa_+(1+\nu)-2\delta(13+2\nu)\big)(S\Sigma)\\
&+\big(-\tfrac{3}{2}\delta\kappa_-(1+\nu)-\tfrac{3}{2}\kappa_+(1+\nu)(-1+2\nu)+4(-3+10\nu+\nu^2)\big)\boldsymbol{\Sigma}^2\Big]. \qquad (348)
\end{aligned}
$$

C. Radiation sector

1. Energy flux

The coefficients of the radiated energy in (188) are given by

$$
\begin{aligned}
f_6^0 =\,& (vS)^2\Big[\big(-8688\delta\kappa_-+\kappa_+(11424-9720\nu)-6(529+3240\nu)\big)(nv)^2\\
&+\big(4462+2676\delta\kappa_-+3792\nu+\kappa_+(-684+1896\nu)\big)\mathbf{v}^2\Big]\\
&+(nS)(vS)\Big[\big(25464\delta\kappa_-+3384\kappa_+(-17+12\nu)+48(-581+1692\nu)\big)(nv)^3\\
&+\big(4800-11832\delta\kappa_-+\kappa_+(22200-8712\nu)-17424\nu\big)(nv)\mathbf{v}^2\Big]\\
&+(n\Sigma)(vS)\Big[\big(\kappa_-(41496-71232\nu)+24\delta\kappa_+(-1729+846\nu)+12\delta(1273+3384\nu)\big)(nv)^3\\
&+\big(-66\delta(161+132\nu)-12\delta\kappa_+(-1418+363\nu)+\kappa_-(-17016+28020\nu)\big)(nv)\mathbf{v}^2\Big]\\
&+\mathbf{S}^2\Big[\big(-35436-2274\delta\kappa_-+28224\nu+6\kappa_+(-2153+2352\nu)\big)(nv)^4\\
&+\big(49800+3684\delta\kappa_-+\kappa_+(16452-15744\nu)-31488\nu\big)(nv)^2\mathbf{v}^2\\
&+\big(-16260-1374\delta\kappa_-+5136\nu+6\kappa_+(-671+428\nu)\big)\mathbf{v}^4\Big]\\
&+(nS)(n\Sigma)\Big[\big(-60\delta\kappa_+(-1725+1192\nu)-120\delta(-591+1192\nu)\\
&+60\kappa_-(-1725+2054\nu)\big)(nv)^4\\
&+\big(-2448\kappa_-(-32+25\nu)+288\delta\kappa_+(-272+179\nu)+6\delta(-10043+17184\nu)\big)(nv)^2\mathbf{v}^2\\
&+\big(-12\kappa_-(691+438\nu)-12\delta\kappa_+(-691+548\nu)-24\delta(-67+548\nu)\big)\mathbf{v}^4\Big]\\
&+(nS)^2\Big[\big(81570-12930\delta\kappa_-+\kappa_+(90570-71520\nu)-143040\nu\big)(nv)^4
\end{aligned}
$$

$$
\begin{aligned}
&+\big(2412\delta\kappa_- + 12(-4483+8592\nu)+\kappa_+(-75924+51552\nu)\big)(nv)^2\mathbf{v}^2\\
&+\big(2958\delta\kappa_- + \kappa_+(11250-6576\nu) - 6(85+2192\nu)\big)\mathbf{v}^4\Big]\\
&+(S\Sigma)\Big[\big(\kappa_-(10644-5016\nu)+252\delta(-201+112\nu)+12\delta\kappa_+(-887+1176\nu)\big)(nv)^4\\
&+\big(336\kappa_-(-38+3\nu)-96\delta\kappa_+(-133+164\nu)-6\delta(-11269+5248\nu)\big)(nv)^2\mathbf{v}^2\\
&+\big(12\delta\kappa_+(-221+214\nu)+2\delta(-9377+2568\nu)+\kappa_-(2652+2928\nu)\big)\mathbf{v}^4\Big]\\
&+(n\Sigma)^2\Big[\big(90\delta\kappa_-(-575+541\nu)\\
&+30\kappa_+(1725-5073\nu+2384\nu^2)+60(-307-1016\nu+2384\nu^2)\big)(nv)^4\\
&+\big(8238+67806\nu-103104\nu^2-36\delta\kappa_-(-1088+783\nu)\\
&-36\kappa_+(1088-2959\nu+1432\nu^2)\big)(nv)^2\mathbf{v}^2\\
&+\big(6\delta\kappa_-(-691+55\nu)+6(-515-638\nu+2192\nu^2)+\kappa_+(4146-8622\nu+6576\nu^2)\big)\mathbf{v}^4\Big]\\
&+(vS)\Big[\big(-24\delta\kappa_+(-838+405\nu)-12\delta(1543+1620\nu)+24\kappa_-(-838+1853\nu)\big)(nv)^2\\
&+\big(-840\kappa_-(-4+15\nu)+24\delta\kappa_+(-140+79\nu)+2\delta(3065+1896\nu)\big)\mathbf{v}^2\Big](v\Sigma)\\
&+(nS)\Big[\big(\kappa_-(41496-71232\nu)+24\delta\kappa_+(-1729+846\nu)+48\delta(-233+846\nu)\big)(nv)^3\\
&+\big(-12\delta\kappa_+(-1418+363\nu)-6\delta(-2023+1452\nu)+\kappa_-(-17016+28020\nu)\big)(nv)\mathbf{v}^2\Big](v\Sigma)\\
&+(n\Sigma)\Big[\big(-24\delta\kappa_-(-1729+1907\nu)-192(-174+193\nu+423\nu^2)\\
&-24\kappa_+(1729-5365\nu+1692\nu^2)\big)(nv)^3+\big(12\delta\kappa_-(-1418+1349\nu)\\
&+12\kappa_+(1418-4185\nu+726\nu^2)+12(-652+275\nu+1452\nu^2)\big)(nv)\mathbf{v}^2\Big](v\Sigma)\\
&+\Big[\big(12\delta\kappa_-(-838+1129\nu)+48(-296+668\nu+405\nu^2)\\
&+12\kappa_+(838-2805\nu+810\nu^2)\big)(nv)^2+\big(2200-7646\nu-3792\nu^2\\
&-24\delta\kappa_-(-70+151\nu)-24\kappa_+(70-291\nu+79\nu^2)\big)\mathbf{v}^2\Big](v\Sigma)^2\\
&+\Big[\big(-9714+64788\nu-28224\nu^2-6\delta\kappa_-(-887+797\nu)\\
&-6\kappa_+(887-2571\nu+2352\nu^2)\big)(nv)^4\\
&+\big(12\delta\kappa_-(-532+349\nu)+12\kappa_+(532-1413\nu+1312\nu^2)\\
&+6(2402-14085\nu+5248\nu^2)\big)(nv)^2\mathbf{v}^2+\big(-3754+21194\nu-5136\nu^2\\
&+6\delta\kappa_-(221+15\nu)-6\kappa_+(221-427\nu+428\nu^2)\big)\mathbf{v}^4\Big]\mathbf{\Sigma}^2, \qquad (349)
\end{aligned}
$$

$$
\begin{aligned}
f_6^1 = {} & \big(20472\delta\kappa_- - 24(5125 + 161\nu) - 12\kappa_+(5338 + 161\nu)\big)(nS)(nv)(vS) \\
& + \big(\kappa_-(42264 - 39978\nu) - 6\delta\kappa_+(7044 + 161\nu) - 4\delta(11608 + 483\nu)\big)(nv)(n\Sigma)(vS) \\
& + \big(27240 - 4664\delta\kappa_- + 336\nu + \kappa_+(14652 + 168\nu)\big)(vS)^2 \\
& + \mathbf{S}^2\Big[\big(2772\delta\kappa_- + 8(-6109 + 330\nu) + \kappa_+(-23844 + 1320\nu)\big)(nv)^2 \\
& + \big(-2572\delta\kappa_- + \kappa_+(21028 - 720\nu) - 72(-581 + 20\nu)\big)\mathbf{v}^2\Big] \\
& + (nS)^2\Big[\big(300528 - 28788\delta\kappa_- + \kappa_+(135588 - 2028\nu) - 4056\nu\big)(nv)^2 \\
& + \big(-182752 + 12380\delta\kappa_- + 3984\nu + 24\kappa_+(-3239 + 83\nu)\big)\mathbf{v}^2\Big] \\
& + (S\Sigma)\Big[\big(\kappa_-(26616 - 12408\nu) + 264\delta(-209 + 10\nu) + 24\delta\kappa_+(-1109 + 55\nu)\big)(nv)^2 \\
& + \big(-80\delta\kappa_+(-295 + 9\nu) - 8\delta(-6109 + 180\nu) + 16\kappa_-(-1475 + 688\nu)\big)\mathbf{v}^2\Big] \\
& + (nS)(n\Sigma)\Big[\big(-12\delta\kappa_+(-13698 + 169\nu) - 12\delta(-23537 + 338\nu) \\
& + 108\kappa_-(-1522 + 1085\nu)\big)(nv)^2 \\
& + \big(\kappa_-(90116 - 51512\nu) + 4\delta\kappa_+(-22529 + 498\nu) + 4\delta(-45929 + 996\nu)\big)\mathbf{v}^2\Big] \\
& + (n\Sigma)^2\Big[\big(6\delta\kappa_-(-13698 + 4967\nu) + 6\kappa_+(13698 - 32363\nu + 338\nu^2) \\
& + 12(467 - 22056\nu + 338\nu^2)\big)(nv)^2 + \big(-2\delta\kappa_-(-22529 + 6688\nu) \\
& - 2\kappa_+(22529 - 51746\nu + 996\nu^2) - 4(2425 - 46383\nu + 996\nu^2)\big)\mathbf{v}^2\Big] \\
& + \big(\kappa_-(42264 - 39978\nu) - 6\delta\kappa_+(7044 + 161\nu) - 4\delta(12686 + 483\nu)\big)(nS)(nv)(v\Sigma) \\
& + \big(9808 + 69804\nu + 3864\nu^2 - 6\delta\kappa_-(-7044 + 3251\nu) \\
& + 6\kappa_+(-7044 + 17339\nu + 322\nu^2)\big)(nv)(n\Sigma)(v\Sigma) + \big(336\delta(55 + \nu) + 4\delta\kappa_+(4829 + 42\nu) \\
& + 4\kappa_-(-4829 + 4622\nu)\big)(vS)(v\Sigma) \\
& + \big(2\delta\kappa_-(-4829 + 2290\nu) + \kappa_+(9658 - 23896\nu - 168\nu^2) - 16(299 + 613\nu + 21\nu^2)\big)(v\Sigma)^2 \\
& + \Big[\big(-12\delta\kappa_-(-1109 + 286\nu) - 12\kappa_+(1109 - 2504\nu + 110\nu^2) \\
& - 8(151 - 7717\nu + 330\nu^2)\big)(nv)^2 + \big(4\delta\kappa_-(-2950 + 733\nu) + 8(551 - 7036\nu + 180\nu^2) \\
& + 4\kappa_+(2950 - 6633\nu + 180\nu^2)\big)\mathbf{v}^2\Big]\boldsymbol{\Sigma}^2, \qquad (350)
\end{aligned}
$$

$$
\begin{aligned}
f_6^2 = {} & (936 + 48\delta\kappa_- + 432\delta^2\kappa_+ - 3456\nu)(nS)^2 \\
& + \big(-384\delta^2\kappa_- - 192\delta\kappa_+(-2 + 9\nu) - 16\delta(-49 + 216\nu)\big)(nS)(n\Sigma)
\end{aligned}
$$

$$+\big(-48\delta^2\kappa_+(-4+9\nu)+48\delta\kappa_-(-4+17\nu)-16\delta^2(-1+54\nu)\big)(n\Sigma)^2$$
$$+\big(-16\delta\kappa_- -144\delta^2\kappa_+ +16(-23+72\nu)\big)\mathbf{S}^2$$
$$+\big(128\delta^2\kappa_- +64\delta\kappa_+(-2+9\nu)+32\delta(-7+36\nu)\big)(S\Sigma)$$
$$+\big(16\delta^2\kappa_+(-4+9\nu)+24\delta^2(-1+12\nu)-16\delta\kappa_-(-4+17\nu)\big)\mathbf{\Sigma}^2. \quad (351)$$

2. Angular momentum flux

The coefficients for the radiated angular momentum in (193)–(196) are given by

$$g_3^{0i}=\mathbf{S}^i(-120(nv)^4+264(nv)^2\mathbf{v}^2-160\mathbf{v}^4)$$
$$+\mathbf{v}^i\big((vS)(-348(nv)^2+160\mathbf{v}^2)+(nS)(780(nv)^3-444(nv)\mathbf{v}^2)$$
$$+(n\Sigma)(420\delta(nv)^3-228\delta(nv)\mathbf{v}^2)+(-204\delta(nv)^2+88\delta\mathbf{v}^2)(v\Sigma)\big)$$
$$+\mathbf{n}^i\big((vS)(120(nv)^3+84(nv)\mathbf{v}^2)+(nS)(-780(nv)^2\mathbf{v}^2+444\mathbf{v}^4)$$
$$+(n\Sigma)(-420\delta(nv)^2\mathbf{v}^2+228\delta\mathbf{v}^4)+(120\delta(nv)^3+12\delta(nv)\mathbf{v}^2)(v\Sigma)\big)$$
$$+(-120\delta(nv)^4+192\delta(nv)^2\mathbf{v}^2-88\delta\mathbf{v}^4)\mathbf{\Sigma}^i, \quad (352)$$
$$g_3^{1i}=\mathbf{S}^i(76(nv)^2-72\mathbf{v}^2)+\mathbf{v}^i(-175(nS)(nv)-76\delta(nv)(n\Sigma)+113(vS)+44\delta(v\Sigma))$$
$$+\mathbf{n}^i\big(-109(nv)(vS)+(nS)(45(nv)^2+122\mathbf{v}^2)+(n\Sigma)(27\delta(nv)^2+47\delta\mathbf{v}^2)$$
$$-46\delta(nv)(v\Sigma)\big)+(26\delta(nv)^2-22\delta\mathbf{v}^2)\mathbf{\Sigma}^i, \quad (353)$$
$$g_3^{2i}=\mathbf{n}^i(2(nS)-\delta(n\Sigma))-2\mathbf{S}^i+\delta\mathbf{\Sigma}^i, \quad (354)$$
$$g_4^{0i}=(\mathbf{v}\times\mathbf{S})^i\Big[(-24-12\kappa_+)(nv)(vS)+(nS)\big((120+60\kappa_+)(nv)^2+(-72-36\kappa_+)\mathbf{v}^2\big)$$
$$+(n\Sigma)\big((60\delta-30\kappa_-+30\delta\kappa_+)(nv)^2+(-36\delta+18\kappa_--18\delta\kappa_+)\mathbf{v}^2\big)$$
$$+(-12\delta+6\kappa_--6\delta\kappa_+)(nv)(v\Sigma)\Big]$$
$$+(\mathbf{v}\times\mathbf{\Sigma})^i\Big\{(-12\delta+6\kappa_--6\delta\kappa_+)(nv)(vS)+(nS)\big((60\delta-30\kappa_-+30\delta\kappa_+)(nv)^2$$
$$+(-36\delta+18\kappa_--18\delta\kappa_+)\mathbf{v}^2\big)+(n\Sigma)\Big[\big(-30\delta\kappa_-+\kappa_+(30-60\nu)-120\nu\big)(nv)^2$$
$$+\big(18\delta\kappa_-+72\nu+\kappa_+(-18+36\nu)\big)\mathbf{v}^2\Big]+\big(6\delta\kappa_-+24\nu+\kappa_+(-6+12\nu)\big)(nv)(v\Sigma)\Big\}$$
$$+\big(\mathbf{n}\times\mathbf{v}\big)^i\Big\{(-360-180\kappa_+)(nS)(nv)(vS)+(-180\delta+90\kappa_--90\delta\kappa_+)(nv)(n\Sigma)(vS)$$
$$+(24+12\kappa_+)(vS)^2+(nS)^2\big((840+420\kappa_+)(nv)^2+(-240-120\kappa_+)\mathbf{v}^2\big)$$

$$
\begin{aligned}
&+\mathbf{S}^2\big((-120-60\kappa_+)(nv)^2+(48+24\kappa_+)\mathbf{v}^2\big)\\
&+(nS)(n\Sigma)\big((840\delta-420\kappa_-+420\delta\kappa_+)(nv)^2+(-240\delta+120\kappa_--120\delta\kappa_+)\mathbf{v}^2\big)\\
&+(S\Sigma)\big((-120\delta+60\kappa_--60\delta\kappa_+)(nv)^2+(48\delta-24\kappa_-+24\delta\kappa_+)\mathbf{v}^2\big)\\
&+(n\Sigma)^2\Big[\big(-210\delta\kappa_-+\kappa_+(210-420\nu)-840\nu\big)(nv)^2\\
&+\big(60\delta\kappa_-+240\nu+60\kappa_+(-1+2\nu)\big)\mathbf{v}^2\Big]+(-180\delta+90\kappa_--90\delta\kappa_+)(nS)(nv)(v\Sigma)\\
&+\big(90\delta\kappa_-+360\nu+90\kappa_+(-1+2\nu)\big)(nv)(n\Sigma)(v\Sigma)+(24\delta-12\kappa_-+12\delta\kappa_+)(vS)(v\Sigma)\\
&+\big(-6\delta\kappa_-+\kappa_+(6-12\nu)-24\nu\big)(v\Sigma)^2+\Big[\big(30\delta\kappa_-+120\nu+\kappa_+(-30+60\nu)\big)(nv)^2\\
&+\big(-12\delta\kappa_-+\kappa_+(12-24\nu)-48\nu\big)\mathbf{v}^2\Big]\mathbf{\Sigma}^2\Big\}, \qquad (355)
\end{aligned}
$$

$$
\begin{aligned}
g_4^{1i}&=\big((-48-24\kappa_+)(nS)+(-24\delta+12\kappa_--12\delta\kappa_+)(n\Sigma)\big)(\mathbf{v}\times\mathbf{S})^i\\
&+\Big[(-24\delta+12\kappa_--12\delta\kappa_+)(nS)+\big(-2+12\delta\kappa_-+48\nu+\kappa_+(-12+24\nu)\big)(n\Sigma)\Big](\mathbf{v}\times\mathbf{\Sigma})^i\\
&+(\mathbf{n}\times\mathbf{S})^i\big((-24-12\kappa_+)(nS)(nv)+(-12\delta+6\kappa_--6\delta\kappa_+)(nv)(n\Sigma)\\
&+(24+12\kappa_+)(vS)+(12\delta-6\kappa_-+6\delta\kappa_+)(v\Sigma)\big)\\
&+(\mathbf{n}\times\mathbf{\Sigma})^i\Big[(-12\delta+6\kappa_--6\delta\kappa_+)(nS)(nv)+\big(6\delta\kappa_-+24\nu+\kappa_+(-6+12\nu)\big)(nv)(n\Sigma)\\
&+(12\delta-6\kappa_-+6\delta\kappa_+)(vS)+\big(2-6\delta\kappa_-+\kappa_+(6-12\nu)-24\nu\big)(v\Sigma)\Big]\\
&+\big(\mathbf{n}\times\mathbf{v}\big)^i\Big[(-360-180\kappa_+)(nS)^2+(-360\delta+180\kappa_--180\delta\kappa_+)(nS)(n\Sigma)\\
&+\big(90\delta\kappa_-+360\nu+90\kappa_+(-1+2\nu)\big)(n\Sigma)^2\\
&+(96+48\kappa_+)\mathbf{S}^2+(96\delta-48\kappa_-+48\delta\kappa_+)(S\Sigma)\\
&+\big(2-24\delta\kappa_-+\kappa_+(24-48\nu)-96\nu\big)\mathbf{\Sigma}^2\Big], \qquad (356)
\end{aligned}
$$

$$
\begin{aligned}
g_5^{0i}&=\mathbf{S}^i\big(-105(1+40\nu)(nv)^6+15(-43+716\nu)(nv)^4\mathbf{v}^2\\
&+(2589-10608\nu)(nv)^2\mathbf{v}^4+(-1607+4004\nu)\mathbf{v}^6\big)\\
&+\mathbf{v}^i\Big[(vS)\big((2145-7740\nu)(nv)^4+30(-65+334\nu)(nv)^2\mathbf{v}^2+(1607-4004\nu)\mathbf{v}^4\big)\\
&+(n\Sigma)\big(210\delta(-4+35\nu)(nv)^5-30\delta(-49+388\nu)(nv)^3\mathbf{v}^2+6\delta(-217+787\nu)(nv)\mathbf{v}^4\big)\\
&+(nS)\big(1260(-1+6\nu)(nv)^5-30(-53+496\nu)(nv)^3\mathbf{v}^2+(-2874+8376\nu)(nv)\mathbf{v}^4\big)\\
&+\big(-15\delta(-79+455\nu)(nv)^4+6\delta(-147+1427\nu)(nv)^2\mathbf{v}^2+\delta(563-2477\nu)\mathbf{v}^4\big)(v\Sigma)\Big]\\
&+\mathbf{n}^i\Big[(vS)\big(105(1+40\nu)(nv)^5-1500(1+2\nu)(nv)^3\mathbf{v}^2+(-639+588\nu)(nv)\mathbf{v}^4\big)
\end{aligned}
$$

$$+ (nS)\big(-1260(-1+6\nu)(nv)^4\mathbf{v}^2 + 30(-53+496\nu)(nv)^2\mathbf{v}^4 + (2874-8376\nu)\mathbf{v}^6\big)$$

$$+ (n\Sigma)\big(-210\delta(-4+35\nu)(nv)^4\mathbf{v}^2 + 30\delta(-49+388\nu)(nv)^2\mathbf{v}^4 - 6\delta(-217+787\nu)\mathbf{v}^6\big)$$

$$+ \big(525\delta(5+13\nu)(nv)^5 - 270\delta(20+29\nu)(nv)^3\mathbf{v}^2 + 3\delta(559+571\nu)(nv)\mathbf{v}^4\big)(v\Sigma)\Big]$$

$$+ \mathbf{\Sigma}^i\big(-525\delta(5+13\nu)(nv)^6 + 15\delta(281+977\nu)(nv)^4\mathbf{v}^2$$

$$- 15\delta(53+685\nu)(nv)^2\mathbf{v}^4 + \delta(-563+2477\nu)\mathbf{v}^6\big), \tag{357}$$

$$g_5^{1i} = \mathbf{S}^i\big((-8577+7320\nu)(nv)^4 + (7022-15068\nu)(nv)^2\mathbf{v}^2 + 7(-155+972\nu)\mathbf{v}^4\big)$$

$$+ \mathbf{v}^i\Big[(vS)\big((-588+10982\nu)(nv)^2 + (3138-8238\nu)\mathbf{v}^2\big)$$

$$+ (n\Sigma)\big(-9\delta(1519+216\nu)(nv)^3 + \delta(4789+1994\nu)(nv)\mathbf{v}^2\big)$$

$$+ (nS)\big(-6(2750+1699\nu)(nv)^3 + (3542+6802\nu)(nv)\mathbf{v}^2\big)$$

$$+ \big(\delta(4313+1486\nu)(nv)^2 - 17\delta(39+112\nu)\mathbf{v}^2\big)(v\Sigma)\Big]$$

$$+ \mathbf{n}^i\Big[(vS)\big((12570-10914\nu)(nv)^3 + (-12836+9162\nu)(nv)\mathbf{v}^2\big)$$

$$+ (nS)\big(-3(1381+574\nu)(nv)^4 + 6(3919+2760\nu)(nv)^2\mathbf{v}^2 + (-6057-11494\nu)\mathbf{v}^4\big)$$

$$+ (n\Sigma)\big(3\delta(-807+\nu)(nv)^4 + 3\delta(5909+1242\nu)(nv)^2\mathbf{v}^2 - \delta(6364+3679\nu)\mathbf{v}^4\big)$$

$$+ \big(-3\delta(-759+296\nu)(nv)^3 + 3\delta(-1729+674\nu)(nv)\mathbf{v}^2\big)(v\Sigma)\Big]$$

$$+ \mathbf{\Sigma}^i\big(-3\delta(-179+361\nu)(nv)^4 - 10\delta(336+85\nu)(nv)^2\mathbf{v}^2 + \delta(2023+1117\nu)\mathbf{v}^4\big), \tag{358}$$

$$g_5^{2i} = \mathbf{S}^i\big(-2(7711+272\nu)(nv)^2 + (16218+2296\nu)\mathbf{v}^2\big)$$

$$+ \mathbf{v}^i\big(8(6227+386\nu)(nS)(nv) + 36\delta(685+36\nu)(nv)(n\Sigma)$$

$$- 2(14701+728\nu)(vS) - 10\delta(1501+54\nu)(v\Sigma)\big)$$

$$+ \mathbf{n}^i\Big[(27954+536\nu)(nv)(vS) + (n\Sigma)\big(-2\delta(5157+847\nu)(nv)^2 - 2\delta(7151+163\nu)\mathbf{v}^2\big)$$

$$+ (nS)\big(-4(5115+413\nu)(nv)^2 - 12(2392+189\nu)\mathbf{v}^2\big) - 2\delta(-7561+96\nu)(nv)(v\Sigma)\Big]$$

$$+ \mathbf{\Sigma}^i\big(2\delta(-4040+541\nu)(nv)^2 + 2\delta(3962+187\nu)\mathbf{v}^2\big), \tag{359}$$

$$g_5^{3i} = \mathbf{n}^i\big(-8(265+31\nu)(nS) - 20\delta(-12+7\nu)(n\Sigma)\big) + 8(265+31\nu)\mathbf{S}^i + 20\delta(-12+7\nu)\mathbf{\Sigma}^i, \tag{360}$$

$$g_6^{0i} = (\mathbf{v}\times\mathbf{S})^i\Big\{(vS)\Big[\big(-3020\delta\kappa_- - 280\kappa_+(1+\nu) - 80(-52+7\nu)\big)(nv)^3$$

$$+ \big(1916\delta\kappa_- + \kappa_+(680-252\nu) - 56(68+9\nu)\big)(nv)\mathbf{v}^2\Big]$$

$$
\begin{aligned}
&+ (n\Sigma)\Big[\big(140\delta\kappa_+(-23+2\nu) + 280\delta(-1+2\nu) - 140\kappa_-(-23+88\nu)\big)(nv)^4\\
&+ \big(-10\delta\kappa_+(-258+19\nu) - 20\delta(-169+19\nu) + 30\kappa_-(-86+349\nu)\big)(nv)^2\mathbf{v}^2\\
&+ \big(\kappa_-(504-954\nu) + 14\delta\kappa_+(-36+19\nu) + 4\delta(-337+133\nu)\big)\mathbf{v}^4\Big]\\
&+ (nS)\Big[\big(6020\delta\kappa_- + 1120(-11+\nu) + 140\kappa_+(-3+4\nu)\big)(nv)^4\\
&+ \big(16280 - 5140\delta\kappa_- + \kappa_+(20-380\nu) - 760\nu\big)(nv)^2\mathbf{v}^2\\
&+ \big(344\delta\kappa_- + 8(-449+133\nu) + \kappa_+(-664+532\nu)\big)\mathbf{v}^4\Big]\\
&+ \Big[\big(-10\delta\kappa_+(-137+14\nu) - 20\delta(57+14\nu) + \kappa_-(-1370+6180\nu)\big)(nv)^3\\
&+ \big(\kappa_-(618-3706\nu) - 252\delta\nu - 6\delta\kappa_+(103+21\nu)\big)(nv)\mathbf{v}^2\Big](v\Sigma)\Big\}\\
&+ (\mathbf{v}\times\boldsymbol{\Sigma})^i\Big\{(vS)\Big[\big(-20\delta(-223+14\nu) - 10\delta\kappa_+(-137+14\nu)\\
&+ \kappa_-(-1370+6180\nu)\big)(nv)^3\\
&+ \big(\kappa_-(618-3706\nu) - 84\delta(46+3\nu) - 6\delta\kappa_+(103+21\nu)\big)(nv)\mathbf{v}^2\Big]\\
&+ (nS)\Big[\big(560\delta(-25+\nu) + 140\delta\kappa_+(-23+2\nu) - 140\kappa_-(-23+88\nu)\big)(nv)^4\\
&+ \big(-20\delta(-813+19\nu) - 10\delta\kappa_+(-258+19\nu) + 30\kappa_-(-86+349\nu)\big)(nv)^2\mathbf{v}^2\\
&+ \big(\kappa_-(504-954\nu) + 14\delta\kappa_+(-36+19\nu) + 4\delta(-687+133\nu)\big)\mathbf{v}^4\Big]\\
&+ (n\Sigma)\Big[\big(-140\delta\kappa_-(-23+45\nu) - 140\kappa_+(23-91\nu+4\nu^2) - 280(7-58\nu+4\nu^2)\big)(nv)^4\\
&+ \big(10\delta\kappa_-(-258+533\nu) + 40(84-568\nu+19\nu^2) + 10\kappa_+(258-1049\nu+38\nu^2)\big)(nv)^2\mathbf{v}^2\\
&+ \big(\delta\kappa_-(504-610\nu) + \kappa_+(-504+1618\nu-532\nu^2) - 8(63-568\nu+133\nu^2)\big)\mathbf{v}^4\Big]\\
&+ \Big[\big(10\delta\kappa_-(-137+316\nu) + 40(-21-83\nu+14\nu^2) + 10\kappa_+(137-590\nu+28\nu^2)\big)(nv)^3\\
&+ \big(-2\delta\kappa_-(-309+895\nu) + 56(-1+77\nu+9\nu^2)\\
&+ \kappa_+(-618+3026\nu+252\nu^2)\big)(nv)\mathbf{v}^2\Big](v\Sigma)\Big\}\\
&+ (\mathbf{n}\times\mathbf{v})^i\Big\{(vS)^2\Big[\big(2280\delta\kappa_- + 40(-3+142\nu) + \kappa_+(-2100+2840\nu)\big)(nv)^2\\
&+ \big(-512\delta\kappa_- + 8(-99+19\nu) + \kappa_+(-100+76\nu)\big)\mathbf{v}^2\Big]\\
&+ (n\Sigma)(vS)\Big[\big(-70\delta\kappa_+(-113+128\nu) - 140\delta(1+128\nu)\\
&+ 70\kappa_-(-113+412\nu)\big)(nv)^3 + \big(\kappa_-(1730-10090\nu) + 10\delta\kappa_+(-173+157\nu)
\end{aligned}
$$

$$+ 20\delta(36 + 157\nu)\big)(nv)\mathbf{v}^2\Big]$$

$$+ (nS)(vS)\Big[\big(7560 - 9940\delta\kappa_- - 35840\nu - 280\kappa_+(-21 + 64\nu)\big)(nv)^3$$

$$+ \big(40 + 4260\delta\kappa_- + 6280\nu + \kappa_+(800 + 3140\nu)\big)(nv)\mathbf{v}^2\Big]$$

$$+ \mathbf{S}^2\Big[\big(4060 + 1400\delta\kappa_- - 8400\nu - 70\kappa_+(-23 + 60\nu)\big)(nv)^4$$

$$+ \big(-1420\delta\kappa_- + 40(-90 + 181\nu) + 20\kappa_+(-69 + 181\nu)\big)(nv)^2\mathbf{v}^2$$

$$+ \big(1068 + 248\delta\kappa_- + \kappa_+(486 - 616\nu) - 1232\nu\big)\mathbf{v}^4\Big]$$

$$+ (nS)^2\Big[\big(8820\delta\kappa_- + 1260(-13 + 40\nu) + 630\kappa_+(-13 + 40\nu)\big)(nv)^4 + \big(-4900\delta\kappa_-$$

$$- 420\kappa_+(-7 + 29\nu) - 280(-49 + 87\nu)\big)(nv)^2\mathbf{v}^2$$

$$+ \big(-4660 - 140\delta\kappa_- + 3600\nu + 30\kappa_+(-59 + 60\nu)\big)\mathbf{v}^4\Big]$$

$$+ (nS)(n\Sigma)\Big[\big(-1890\kappa_-(-9 + 32\nu) + 630\delta\kappa_+(-27 + 40\nu) + 1260\delta(-13 + 40\nu)\big)(nv)^4$$

$$+ \big(-840\delta(-21 + 29\nu) - 140\delta\kappa_+(-56 + 87\nu) + 140\kappa_-(-56 + 227\nu)\big)(nv)^2\mathbf{v}^2$$

$$+ \big(\kappa_-(1630 - 1240\nu) + 180\delta(-29 + 20\nu) + 10\delta\kappa_+(-163 + 180\nu)\big)\mathbf{v}^4\Big]$$

$$+ (S\Sigma)\Big[\big(-420\delta(-19 + 20\nu) - 210\delta\kappa_+(-1 + 20\nu) - 70\kappa_-(3 + 20\nu)\big)(nv)^4$$

$$+ \big(40\delta(-174 + 181\nu) + 20\delta\kappa_+(2 + 181\nu) + \kappa_-(-40 + 2060\nu)\big)(nv)^2\mathbf{v}^2$$

$$+ \big(\kappa_-(-238 - 376\nu) - 14\delta\kappa_+(-17 + 44\nu) - 4\delta(-351 + 308\nu)\big)\mathbf{v}^4\Big]$$

$$+ (n\Sigma)^2\Big[\big(-1260\nu(-13 + 40\nu) - 315\delta\kappa_-(-27 + 68\nu) - 315\kappa_+(27 - 122\nu + 80\nu^2)\big)(nv)^4$$

$$+ \big(70\delta\kappa_-(-56 + 157\nu) + 280(14 - 77\nu + 87\nu^2) + 70\kappa_+(56 - 269\nu + 174\nu^2)\big)(nv)^2\mathbf{v}^2$$

$$+ \big(\delta\kappa_-(815 - 760\nu) - 20(28 - 289\nu + 180\nu^2) - 5\kappa_+(163 - 478\nu + 360\nu^2)\big)\mathbf{v}^4\Big]$$

$$+ (vS)\Big[\big(\kappa_-(4380 - 11960\nu) + 80\delta(23 + 71\nu) + 20\delta\kappa_+(-219 + 142\nu)\big)(nv)^2$$

$$+ \big(8\delta(-134 + 19\nu) + 4\delta\kappa_+(103 + 19\nu) + \kappa_-(-412 + 1972\nu)\big)\mathbf{v}^2\Big](v\Sigma)$$

$$+ (nS)\Big[\big(-70\delta\kappa_+(-113 + 128\nu) - 140\delta(-13 + 128\nu) + 70\kappa_-(-113 + 412\nu)\big)(nv)^3$$

$$+ \big(\kappa_-(1730 - 10090\nu) + 10\delta\kappa_+(-173 + 157\nu) + 20\delta(8 + 157\nu)\big)(nv)\mathbf{v}^2\Big](v\Sigma)$$

$$+ (n\Sigma)\Big[\big(70\delta\kappa_-(-113 + 270\nu) + 280(-21 + 15\nu + 128\nu^2)$$

$$+ 70\kappa_+(113 - 496\nu + 256\nu^2)\big)(nv)^3 + \big(-10\delta\kappa_-(-173 + 583\nu) - 40(-21 + 43\nu + 157\nu^2)$$

$$
\begin{aligned}
&- 10\kappa_+(173 - 929\nu + 314\nu^2)\big)(nv)\mathbf{v}^2\Big](v\Sigma) \\
&+ \Big[\big(-10\delta\kappa_-(-219 + 370\nu) - 40(-49 + 88\nu + 142\nu^2) - 10\kappa_+(219 - 808\nu + 284\nu^2)\big)(nv)^2 \\
&+ \big(2\delta\kappa_-(-103 + 237\nu) + \kappa_+(206 - 886\nu - 76\nu^2) - 8(35 - 162\nu + 19\nu^2)\big)\mathbf{v}^2\Big](v\Sigma)^2 \\
&+ \Big[\big(35\delta\kappa_-(-3 + 20\nu) + 140(28 - 85\nu + 60\nu^2) + 35\kappa_+(3 - 26\nu + 120\nu^2)\big)(nv)^4 \\
&+ \big(-10\delta\kappa_-(2 + 39\nu) + \kappa_+(20 + 350\nu - 3620\nu^2) - 40(84 - 251\nu + 181\nu^2)\big)(nv)^2\mathbf{v}^2 \\
&+ \big(\delta\kappa_-(-119 + 60\nu) + 4(84 - 421\nu + 308\nu^2) + \kappa_+(119 - 298\nu + 616\nu^2)\big)\mathbf{v}^4\Big]\mathbf{\Sigma}^2\Big\} \quad (361)
\end{aligned}
$$

$$
\begin{aligned}
g_6^{1i} &= (\mathbf{S}\times\mathbf{\Sigma})^i(-792\delta(nv)^3 + 904\delta(nv)\mathbf{v}^2) \\
&+ (\mathbf{v}\times\mathbf{S})^i\Big\{\big(6648 - 1492\delta\kappa_- + 48\nu + \kappa_+(-92 + 24\nu)\big)(nv)(vS) \\
&+ (nS)\Big[\big(5772\delta\kappa_- - 144\kappa_+(54 + 11\nu) - 36(647 + 88\nu)\big)(nv)^2 \\
&+ \big(-2824\delta\kappa_- + 4(2530 + 279\nu) + \kappa_+(3838 + 558\nu)\big)\mathbf{v}^2\Big] \\
&+ (n\Sigma)\Big[\big(\kappa_-(6774 - 10752\nu) - 12\delta(877 + 132\nu) - 6\delta\kappa_+(1129 + 132\nu)\big)(nv)^2 \\
&+ \big(\delta\kappa_+(3331 + 279\nu) + \delta(3779 + 558\nu) + \kappa_-(-3331 + 5369\nu)\big)\mathbf{v}^2\Big] \\
&+ \big(4\delta\kappa_+(175 + 3\nu) + 8\delta(536 + 3\nu) + \kappa_-(-700 + 2972\nu)\big)(nv)(v\Sigma)\Big\} \\
&+ (\mathbf{v}\times\mathbf{\Sigma})^i\Big\{\big(4\delta\kappa_+(175 + 3\nu) + 2\delta(1963 + 12\nu) + \kappa_-(-700 + 2972\nu)\big)(nv)(vS) \\
&+ (nS)\Big[\big(\kappa_-(6774 - 10752\nu) - 72\delta(235 + 22\nu) - 6\delta\kappa_+(1129 + 132\nu)\big)(nv)^2 \\
&+ \big(\delta\kappa_+(3331 + 279\nu) + \delta(9419 + 558\nu) + \kappa_-(-3331 + 5369\nu)\big)\mathbf{v}^2\Big] \\
&+ (n\Sigma)\Big[\big(-6\delta\kappa_-(-1129 + 830\nu) + 6\kappa_+(-1129 + 3088\nu + 264\nu^2) + 6(-447 + 5230\nu \\
&+ 528\nu^2)\big)(nv)^2 + \big(\delta\kappa_-(-3331 + 2545\nu) + \kappa_+(3331 - 9207\nu - 558\nu^2) \\
&- 4(-381 + 3974\nu + 279\nu^2)\big)\mathbf{v}^2\Big] + \big(20\delta\kappa_-(-35 + 74\nu) - 4\kappa_+(-175 + 720\nu + 6\nu^2) \\
&- 6(-205 + 1695\nu + 8\nu^2)\big)(nv)(v\Sigma)\Big\} + (\mathbf{n}\times\mathbf{S})^i\Big\{(vS)\Big[\big(-8016 + 2238\delta\kappa_- \\
&+ 1884\nu + \kappa_+(-966 + 942\nu)\big)(nv)^2 + \big(8012 - 862\delta\kappa_- + 168\nu + \kappa_+(448 + 84\nu)\big)\mathbf{v}^2\Big] \\
&+ (n\Sigma)\Big[\big(6\delta(399 + 103\nu) + 3\delta\kappa_+(1135 + 103\nu) \\
&+ \kappa_-(-3405 + 8943\nu)\big)(nv)^3 + \big(\kappa_-(2295 - 5601\nu)
\end{aligned}
$$

$$
\begin{aligned}
&- 15\delta\kappa_+(153+65\nu) - 6\delta(782+325\nu)\big)(nv)\mathbf{v}^2\Big] + (nS)\Big[\big(13632 - 4626\delta\kappa_- \\
&+ 1236\nu + \kappa_+(2184+618\nu)\big)(nv)^3 \\
&+ \big(3288\delta\kappa_- - 6\kappa_+(217+325\nu) - 12(1371+325\nu)\big)(nv)\mathbf{v}^2\Big] + \Big[\big(\kappa_-(1602-4947\nu) \\
&+ 3\delta\kappa_+(-534+157\nu) + 6\delta(-448+157\nu)\big)(nv)^2 + \big(3\delta(1133+28\nu) \\
&+ \delta\kappa_+(655+42\nu) + \kappa_-(-655+1682\nu)\big)\mathbf{v}^2\Big](v\Sigma)\Big\} \\
&+ (\mathbf{n}\times\mathbf{\Sigma})^i\Big\{(vS)\Big[\big(\kappa_-(1602-4947\nu) + 6\delta(-940+157\nu) + 3\delta\kappa_+(-534+157\nu)\big)(nv)^2 \\
&+ \big(\delta\kappa_+(655+42\nu) + \delta(4769+84\nu) + \kappa_-(-655+1682\nu)\big)\mathbf{v}^2\Big] \\
&+ (nS)\Big[\big(3\delta\kappa_+(1135+103\nu) + 6\delta(2493+103\nu) + \kappa_-(-3405+8943\nu)\big)(nv)^3 \\
&+ \big(\kappa_-(2295-5601\nu) - 15\delta\kappa_+(153+65\nu) \\
&- 6\delta(2468+325\nu)\big)(nv)\mathbf{v}^2\Big] + (n\Sigma)\Big[\big(3\delta\kappa_-(-1135+1439\nu) \\
&+ \kappa_+(3405-11127\nu-618\nu^2) - 12(-380+1779\nu+103\nu^2)\big)(nv)^3 \\
&+ \big(-9\delta\kappa_-(-255+257\nu) + 12(-394+1839\nu+325\nu^2) \\
&+ 3\kappa_+(-765+2301\nu+650\nu^2)\big)(nv)\mathbf{v}^2\Big] \\
&+ \Big[\big(-9\delta\kappa_-(-178+301\nu) + \kappa_+(-1602+5913\nu-942\nu^2) \\
&- 6(269-1702\nu+314\nu^2)\big)(nv)^2 + \big(5\delta\kappa_-(-131+164\nu) + \kappa_+(655-2130\nu-84\nu^2) \\
&- 2(-855+4373\nu+84\nu^2)\big)\mathbf{v}^2\Big](v\Sigma)\Big\} + \big(\mathbf{n}\times\mathbf{v}\big)^i\Big\{\big(-11784\delta\kappa_- + 36\kappa_+(767+243\nu) \\
&+ 24(3610+729\nu)\big)(nS)(nv)(vS) + \big(42\kappa_-(-469+457\nu) + 12\delta(3049+729\nu) \\
&+ 6\delta\kappa_+(3283+729\nu)\big)(nv)(n\Sigma)(vS) + \big(1290\delta\kappa_- - 6\kappa_+(598+333\nu) \\
&- 4(2587+999\nu)\big)(vS)^2 + (S\Sigma)\Big[\big(60\delta\kappa_+(135+53\nu) + 24\delta(574+265\nu) \\
&+ \kappa_-(-8100+5316\nu)\big)(nv)^2 + \big(\kappa_-(1476-2656\nu) - 36\delta\kappa_+(41+50\nu) \\
&- 50\delta(115+72\nu)\big)\mathbf{v}^2\Big] + \mathbf{S}^2\Big[\big(-2124\delta\kappa_- + 24(543+265\nu) + \kappa_+(5976+3180\nu)\big)(nv)^2 \\
&+ \big(1114\delta\kappa_- - 2\kappa_+(181+900\nu) - 4(521+900\nu)\big)\mathbf{v}^2\Big] + (nS)(n\Sigma)\Big[\big(\kappa_-(78072-78042\nu) \\
&- 6\delta\kappa_+(13012+2797\nu) - 12\delta(13956+2797\nu)\big)(nv)^2 + \big(6\delta\kappa_+(2653+1102\nu) \\
&+ 12\delta(4023+1102\nu) + 6\kappa_-(-2653+3714\nu)\big)\mathbf{v}^2\Big]
\end{aligned}
$$

$$+ (nS)^2\Big[\big(23706\delta\kappa_- - 6\kappa_+(9061+2797\nu) - 6(29683+5594\nu)\big)(nv)^2$$

$$+ \big(-7224\delta\kappa_- + 6(7003+2204\nu) + \kappa_+(8694+6612\nu)\big)\mathbf{v}^2\Big]$$

$$+ (n\Sigma)^2\Big[\big(-3\delta\kappa_-(-13012+5105\nu)$$

$$+ 6(-1869+26316\nu+5594\nu^2) + \kappa_+(-39036+93387\nu+16782\nu^2)\big)(nv)^2$$

$$+ \big(3\delta\kappa_-(-2653+1306\nu)$$

$$+ \kappa_+(7959-19836\nu-6612\nu^2) - 6(-1519+9138\nu+2204\nu^2)\big)\mathbf{v}^2\Big]$$

$$+ \big(36\delta(923+243\nu) + 42\kappa_-(-469+457\nu) + 6\delta\kappa_+(3283+729\nu)\big)(nS)(nv)(v\Sigma)$$

$$+ \big(6\delta\kappa_-(-3283+1235\nu) - 24(29+2235\nu+729\nu^2)$$

$$- 6\kappa_+(-3283+7801\nu+1458\nu^2)\big)(nv)(n\Sigma)(v\Sigma)$$

$$+ \big(\kappa_-(4878-3162\nu) - 18\delta\kappa_+(271+111\nu) - 2\delta(2533+1998\nu)\big)(vS)(v\Sigma)$$

$$+ \big(1922-38\nu+3996\nu^2 - 3\delta\kappa_-(-813+97\nu) + 3\kappa_+(-813+1723\nu+666\nu^2)\big)(v\Sigma)^2$$

$$+ \Big[\big(786-15180\nu-6360\nu^2 + 6\delta\kappa_-(-675+89\nu) - 6\kappa_+(-675+1439\nu+530\nu^2)\big)(nv)^2$$

$$+ \big(\delta\kappa_-(738-214\nu) + 24(-95+401\nu+150\nu^2) + 2\kappa_+(-369+845\nu+900\nu^2)\big)\mathbf{v}^2\Big]\mathbf{\Sigma}^2\Big\}, \tag{362}$$

$$g_6^{2i} = (\mathbf{v}\times\mathbf{S})^i\Big[\big(-1268\delta\kappa_- + 12(941+30\nu) + \kappa_+(6122+180\nu)\big)(nS)$$

$$+ \big(5\delta\kappa_+(739+18\nu) + 2\delta(2543+90\nu) + \kappa_-(-3695+2446\nu)\big)(n\Sigma)\Big]$$

$$+ (\mathbf{v}\times\mathbf{\Sigma})^i\Big[\big(60\delta(104+3\nu) + 5\delta\kappa_+(739+18\nu) + \kappa_-(-3695+2446\nu)\big)(nS)$$

$$+ \big(\delta\kappa_-(-3695+1178\nu) + \kappa_+(3695-8568\nu-180\nu^2) - 4(-54+2831\nu+90\nu^2)\big)(n\Sigma)\Big]$$

$$+ (\mathbf{n}\times\mathbf{S})^i\Big[\big(-2522\delta\kappa_- + 8\kappa_+(703+90\nu) + 8(1303+180\nu)\big)(nS)(nv) + \big(18\delta(159+40\nu)$$

$$+ \delta\kappa_+(4073+360\nu) + \kappa_-(-4073+4684\nu)\big)(nv)(n\Sigma) + \big(1318\delta\kappa_-$$

$$+ 8(-926+27\nu) + 4\kappa_+(-901+27\nu)\big)(vS)$$

$$+ \big(\kappa_-(2461-2690\nu) + \delta\kappa_+(-2461+54\nu) + 2\delta(-1111+54\nu)\big)(v\Sigma)\Big]$$

$$+ (\mathbf{n}\times\mathbf{\Sigma})^i\Big[\big(24\delta(187+30\nu) + \delta\kappa_+(4073+360\nu) + \kappa_-(-4073+4684\nu)\big)(nS)(nv)$$

$$+ \big(\delta\kappa_-(-4073+2162\nu)$$

$$+ \kappa_+(4073-10308\nu-720\nu^2) - 6(195+779\nu+240\nu^2)\big)(nv)(n\Sigma) + \big(\kappa_-(2461-2690\nu)$$

$$+ \delta\kappa_+(-2461 + 54\nu) + 2\delta(-1661 + 54\nu)\big)(vS)$$

$$+ \big(954 + \delta\kappa_-(2461 - 1372\nu) + 4322\nu - 216\nu^2 + \kappa_+(-2461 + 6294\nu - 108\nu^2)\big)(v\Sigma)\Big]$$

$$+ \big(\mathbf{n} \times \mathbf{v}\big)^i \Big[\big(98500 - 6486\delta\kappa_- + 72\nu + 6\kappa_+(7469 + 6\nu)\big)(nS)^2$$

$$+ \big(36\delta\kappa_+(1425 + \nu) + 2\delta(48905 + 36\nu) + \kappa_-(-51300 + 25908\nu)\big)(nS)(n\Sigma)$$

$$+ \big(2558 - 96542\nu - 72\nu^2 + 6\delta\kappa_-(-4275 + 1078\nu) - 6\kappa_+(-4275 + 9628\nu + 6\nu^2)\big)(n\Sigma)^2$$

$$+ \big(1300\delta\kappa_- + 4\kappa_+(-2924 + 3\nu) + 8(-2891 + 3\nu)\big)\mathbf{S}^2 + \big(\kappa_-(12996 - 5212\nu)$$

$$+ 12\delta\kappa_+(-1083 + \nu) + 8\delta(-3124 + 3\nu)\big)(S\Sigma) + \big(-2\delta\kappa_-(-3249 + 653\nu)$$

$$- 6(201 - 4355\nu + 4\nu^2) - 2\kappa_+(3249 - 7151\nu + 6\nu^2)\big)\mathbf{\Sigma}^2\Big]. \tag{363}$$

3. Linear momentum and COM fluxes

The coefficients for the expressions in section III.D.3 are given by

$$h_5^{0i} = (\mathbf{v} \times \mathbf{S})^i(-669\delta(nv)^3 + 251\delta(nv)\mathbf{v}^2)$$

$$+ (\mathbf{n} \times \mathbf{S})^i(2940\delta(nv)^4 - 5307\delta(nv)^2\mathbf{v}^2 + 2659\delta\mathbf{v}^4)$$

$$+ (\mathbf{v} \times \mathbf{\Sigma})^i\big(-5331(nv)^3 + 2513(nv)\mathbf{v}^2 + \nu(5952(nv)^3 - 3464(nv)\mathbf{v}^2)\big)$$

$$+ (\mathbf{n} \times \mathbf{\Sigma})^i\big(9195(nv)^4 - 9987(nv)^2\mathbf{v}^2 + 2260\mathbf{v}^4$$

$$+ \nu(-5442(nv)^4 + 8610(nv)^2\mathbf{v}^2 - 4594\mathbf{v}^4)\big) + \big(\mathbf{n} \times \mathbf{v}\big)^i\big(-12(nv)^3(1541\delta(nS) + 431(n\Sigma))$$

$$+ 3(nv)(5287\delta(nS) + 1087(n\Sigma))\mathbf{v}^2 - 2\mathbf{v}^2(2252\delta(vS) + 1481(v\Sigma))$$

$$+ (nv)^2(6348\delta(vS) + 6024(v\Sigma))$$

$$+ \nu(32658(nv)^3(n\Sigma) - 27615(nv)(n\Sigma)\mathbf{v}^2 - 15717(nv)^2(v\Sigma) + 9247\mathbf{v}^2(v\Sigma))\big)$$

$$+ \mathbf{v}^i\big(-21(nv)^2(463\delta(Snv) + 274(\Sigma nv)) + \mathbf{v}^2(9049\delta(Snv)$$

$$+ 4780(\Sigma nv)) + \nu(22089(nv)^2(\Sigma nv) - 18265\mathbf{v}^2(\Sigma nv))\big)$$

$$+ \mathbf{n}^i\big(-1056(nv)\mathbf{v}^2(13\delta(Snv) + 7(\Sigma nv)) + (nv)^3(14604\delta(Snv)$$

$$+ 8304(\Sigma nv)) + \nu(-31146(nv)^3(\Sigma nv) + 27534(nv)\mathbf{v}^2(\Sigma nv))\big), \tag{364}$$

$$h_5^{1i} = 496\delta(nv)(\mathbf{v} \times \mathbf{S})^i + (-6422(nv) - 40\nu(nv))(\mathbf{v} \times \mathbf{\Sigma})^i$$

$$+ (\mathbf{n} \times \mathbf{\Sigma})^i\big(3557(nv)^2 + \nu(4618(nv)^2 - 3372\mathbf{v}^2) + 1389\mathbf{v}^2\big)$$

$$
\begin{aligned}
&+ (\mathbf{n}\times\mathbf{S})^i(-2092\delta(nv)^2 + 1056\delta\mathbf{v}^2) \\
&+ \big(\mathbf{n}\times\mathbf{v}\big)^i\big((nv)(-1886\delta(nS) + 4930(n\Sigma)) + \nu(2807(nv)(n\Sigma) - 759(v\Sigma)) \\
&- 120(\delta(vS) - 5(v\Sigma))\big) + \mathbf{v}^i(2568\delta(Snv) + 669(\Sigma nv) - 2391\nu(\Sigma nv)) \\
&+ \mathbf{n}^i\big((nv)(-3460\delta(Snv) - 1537(\Sigma nv)) + 5563\nu(nv)(\Sigma nv)\big), \qquad (365)
\end{aligned}
$$

$$
h_5^{2i} = -72\delta(\mathbf{n}\times\mathbf{S})^i, \qquad (366)
$$

$$
\begin{aligned}
h_6^{0i} &= \mathbf{S}^i\Big\{(nv)^3\Big[(1920\delta + 4032\kappa_- + 960\delta\kappa_+)(nS) \\
&+ \big(1536\delta\kappa_- - 384\kappa_+(4+5\nu) - 60(-45+64\nu)\big)(n\Sigma)\Big] \\
&+ (vS)(228\delta\mathbf{v}^2 + 1104\kappa_-\mathbf{v}^2 + 114\delta\kappa_+\mathbf{v}^2) \\
&+ (nv)\Big[(nS)(-1776\delta\mathbf{v}^2 - 3192\kappa_-\mathbf{v}^2 - 888\delta\kappa_+\mathbf{v}^2) \\
&+ (n\Sigma)\big(-1152\delta\kappa_-\mathbf{v}^2 + 48\kappa_+(24+37\nu)\mathbf{v}^2 + 12(-103+296\nu)\mathbf{v}^2\big)\Big] \\
&+ \big(495\delta\kappa_-\mathbf{v}^2 - 6(-329+76\nu)\mathbf{v}^2 - 3\kappa_+(165+76\nu)\mathbf{v}^2\big)(v\Sigma) \\
&+ (nv)^2\Big[(-312\delta - 1962\kappa_- - 156\delta\kappa_+)(vS) \\
&+ \big(-903\delta\kappa_- + 6(-549+104\nu) + 3\kappa_+(301+104\nu)\big)(v\Sigma)\Big]\Big\} \\
&+ \Big\{(nv)^3\Big[\big(1536\delta\kappa_- - 384\kappa_+(4+5\nu) - 12(251+320\nu)\big)(nS) \\
&+ \big(-192\delta\kappa_+(8+5\nu) - 192\kappa_-(-8+11\nu) - 60\delta(19+32\nu)\big)(n\Sigma)\Big] \\
&+ (vS)\big(495\delta\kappa_-\mathbf{v}^2 - 8(70+57\nu)\mathbf{v}^2 - 3\kappa_+(165+76\nu)\mathbf{v}^2\big) \\
&+ (nv)\Big[(n\Sigma)\big(24\delta\kappa_+(48+37\nu)\mathbf{v}^2 + 24\kappa_-(-48+59\nu)\mathbf{v}^2 + 12\delta(131+148\nu)\mathbf{v}^2\big) \\
&+ (nS)\big(-1152\delta\kappa_-\mathbf{v}^2 + 48\kappa_+(24+37\nu)\mathbf{v}^2 + 12(219+296\nu)\mathbf{v}^2\big)\Big] \\
&+ \big(-3\delta\kappa_+(165+38\nu)\mathbf{v}^2 - 2\delta(-299+114\nu)\mathbf{v}^2 - 3\kappa_-(-165+292\nu)\mathbf{v}^2\big)(v\Sigma) \\
&+ (nv)^2\Big[\big(-903\delta\kappa_- + 156(5+4\nu) + 3\kappa_+(301+104\nu)\big)(vS) \\
&+ \big(6\delta(-185+52\nu) + 3\delta\kappa_+(301+52\nu) + 3\kappa_-(-301+550\nu)\big)(v\Sigma)\Big]\Big\}\mathbf{\Sigma}^i \\
&+ \mathbf{v}^i\Big((-888\delta - 588\kappa_- - 444\delta\kappa_+)(vS)^2 + \mathbf{S}^2(-1248\delta\mathbf{v}^2 - 120\kappa_-\mathbf{v}^2 - 624\delta\kappa_+\mathbf{v}^2) \\
&+ (nS)^2(4404\delta\mathbf{v}^2 - 156\kappa_-\mathbf{v}^2 + 2202\delta\kappa_+\mathbf{v}^2) \\
&+ (S\Sigma)\big(504\delta\kappa_-\mathbf{v}^2 + 24\kappa_+(-21+104\nu)\mathbf{v}^2 + 4(-449+1248\nu)\mathbf{v}^2\big)
\end{aligned}
$$

$$+ (n\Sigma)^2\big(-3\delta\kappa_+(-393+734\nu)\mathbf{v}^2 - 6\delta(-71+734\nu)\mathbf{v}^2 + 3\kappa_-(-393+1520\nu)\mathbf{v}^2\big)$$
$$+ (nS)(n\Sigma)\big(-2358\delta\kappa_-\mathbf{v}^2 - 6\kappa_+(-393+1468\nu)\mathbf{v}^2 - 6(-875+2936\nu)\mathbf{v}^2\big)$$
$$+ \big(-144\delta\kappa_- + 48\kappa_+(3+37\nu) + 2(-953+1776\nu)\big)(vS)(v\Sigma)$$
$$+ \big(-12\kappa_-(6+25\nu) + 12\delta\kappa_+(6+37\nu) + 2\delta(-257+444\nu)\big)(v\Sigma)^2$$
$$+ (nv)\bigg\{\Big[(5268\delta + 2190\kappa_- + 2634\delta\kappa_+)(nS)$$
$$+ \big(-222\delta\kappa_- - 6\kappa_+(-37+878\nu) - 6(-127+1756\nu)\big)(n\Sigma)\Big](vS)$$
$$+ \Big[\big(-222\delta\kappa_- - 24(-93+439\nu) - 6\kappa_+(-37+878\nu)\big)(nS)$$
$$+ \big(-6\delta\kappa_+(-37+439\nu) + 6\kappa_-(-37+513\nu) - 6\delta(183+878\nu)\big)(n\Sigma)\Big](v\Sigma)\bigg\}$$
$$+ \big(12\delta\kappa_+(-21+52\nu)\mathbf{v}^2 - 12\kappa_-(-21+94\nu)\mathbf{v}^2 + 8\delta(-37+156\nu)\mathbf{v}^2\big)\mathbf{\Sigma}^2$$
$$+ (nv)^2\Big[(-8988\delta - 1632\kappa_- - 4494\delta\kappa_+)(nS)^2 + \big(2862\delta\kappa_-$$
$$+ 6\kappa_+(-477+2996\nu) + 6(-989+5992\nu)\big)(nS)(n\Sigma)$$
$$+ \big(3\delta\kappa_+(-477+1498\nu) + 6\delta(243+1498\nu) - 3\kappa_-(-477+2452\nu)\big)(n\Sigma)^2$$
$$+ (1344\delta + 468\kappa_- + 672\delta\kappa_+)\mathbf{S}^2$$
$$+ \big(-204\delta\kappa_- - 12\kappa_+(-17+224\nu) - 12(-211+448\nu)\big)(S\Sigma)$$
$$+ \big(-24\delta(-25+56\nu) - 6\delta\kappa_+(-17+112\nu) + 6\kappa_-(-17+146\nu)\big)\mathbf{\Sigma}^2\Big]\Bigg)$$
$$+ \mathbf{n}^i\Bigg((vS)\Big[(nS)(1992\delta\mathbf{v}^2 - 564\kappa_-\mathbf{v}^2 + 996\delta\kappa_+\mathbf{v}^2)$$
$$+ (n\Sigma)\big(-780\delta\kappa_-\mathbf{v}^2 - 24(-89+166\nu)\mathbf{v}^2 - 12\kappa_+(-65+166\nu)\mathbf{v}^2\big)\Big]$$
$$+ \Big[(n\Sigma)\big(-12\delta\kappa_+(-65+83\nu)\mathbf{v}^2 - 12\delta(69+166\nu)\mathbf{v}^2 + 12\kappa_-(-65+213\nu)\mathbf{v}^2\big)$$
$$+ (nS)\big(-780\delta\kappa_-\mathbf{v}^2 - 12\kappa_+(-65+166\nu)\mathbf{v}^2 - 12(151+332\nu)\mathbf{v}^2\big)\Big](v\Sigma)$$
$$+ (nv)^2\bigg\{\Big[(-8472\delta - 2172\kappa_- - 4236\delta\kappa_+)(nS) + \big(1032\delta\kappa_- + 24\kappa_+(-43+353\nu)$$
$$+ 6(-1023+2824\nu)\big)(n\Sigma)\Big](vS)$$
$$+ \Big[\big(1032\delta\kappa_- + 24\kappa_+(-43+353\nu) + 12(-151+1412\nu)\big)(nS)$$
$$+ \big(12\delta\kappa_+(-86+353\nu) - 12\kappa_-(-86+525\nu) + 6\delta(31+1412\nu)\big)(n\Sigma)\Big](v\Sigma)\bigg\}$$

$$
\begin{aligned}
&+(nv)^3\Big[(12600\delta-1230\kappa_-+6300\delta\kappa_+)(nS)^2+\big(-7530\delta\kappa_- -90(-143+560\nu)\\
&-30\kappa_+(-251+840\nu)\big)(nS)(n\Sigma)+\big(-90\delta(-3+140\nu)-15\delta\kappa_+(-251+420\nu)\\
&+15\kappa_-(-251+922\nu)\big)(n\Sigma)^2+(-2016\delta-210\kappa_- -1008\delta\kappa_+)\mathbf{S}^2\\
&+\big(798\delta\kappa_- +42\kappa_+(-19+96\nu)+48(-25+168\nu)\big)(S\Sigma)\\
&+\big(21\delta\kappa_+(-19+48\nu)+36\delta(11+56\nu)-21\kappa_-(-19+86\nu)\big)\mathbf{\Sigma}^2\Big]\\
&+(nv)\Big[(1464\delta+1200\kappa_-+732\delta\kappa_+)(vS)^2+(nS)^2(-7512\delta\mathbf{v}^2+2934\kappa_-\mathbf{v}^2\\
&-3756\delta\kappa_+\mathbf{v}^2)+\mathbf{S}^2(1944\delta\mathbf{v}^2-126\kappa_-\mathbf{v}^2+972\delta\kappa_+\mathbf{v}^2)+(S\Sigma)\big(-1098\delta\kappa_-\mathbf{v}^2\\
&-72(-7+108\nu)\mathbf{v}^2-18\kappa_+(-61+216\nu)\mathbf{v}^2\big)+(n\Sigma)^2\big(3\delta\kappa_+(-1115+1252\nu)\mathbf{v}^2\\
&+6\delta(-233+1252\nu)\mathbf{v}^2-3\kappa_-(-1115+3482\nu)\mathbf{v}^2\big)\\
&+(nS)(n\Sigma)\big(6690\delta\kappa_-\mathbf{v}^2+6\kappa_+(-1115+2504\nu)\mathbf{v}^2+6(-1709+5008\nu)\mathbf{v}^2\big)\\
&+\big(468\delta\kappa_- -12\kappa_+(39+244\nu)-6(-751+976\nu)\big)(vS)(v\Sigma)\\
&+\big(6\kappa_-(39+44\nu)-6\delta\kappa_+(39+122\nu)-6\delta(-255+244\nu)\big)(v\Sigma)^2\\
&+\big(-36\delta(19+54\nu)\mathbf{v}^2-9\delta\kappa_+(-61+108\nu)\mathbf{v}^2+9\kappa_-(-61+230\nu)\mathbf{v}^2\big)\mathbf{\Sigma}^2\Big]\bigg), \qquad (367)
\end{aligned}
$$

$$
\begin{aligned}
h_6^{1i}=&\ \mathbf{S}^i\bigg\{(nv)\Big[(60\delta+32\kappa_-+30\delta\kappa_+)(nS)+\big(\delta\kappa_- +\kappa_+(-1-60\nu)-5(1+24\nu)\big)(n\Sigma)\Big]\\
&+(-12\delta-44\kappa_- -6\delta\kappa_+)(vS)+\big(59-19\delta\kappa_- +24\nu+\kappa_+(19+12\nu)\big)(v\Sigma)\bigg\}\\
&+\bigg\{(nv)\Big[\big(79+\delta\kappa_- +\kappa_+(-1-60\nu)-120\nu\big)(nS)+\big(\kappa_-(1+28\nu)-\delta\kappa_+(1+30\nu)\\
&-\delta(7+60\nu)\big)(n\Sigma)\Big]+\big(-67-19\delta\kappa_- +24\nu+\kappa_+(19+12\nu)\big)(vS)\\
&+\big(-3\delta^3+\delta\kappa_+(19+6\nu)+\kappa_-(-19+32\nu)\big)(v\Sigma)\bigg\}\mathbf{\Sigma}^i+\mathbf{v}^i\Big[(192\delta-16\kappa_- +96\delta\kappa_+)(nS)^2\\
&+\big(-112\delta\kappa_- -16\kappa_+(-7+24\nu)-4(-49+192\nu)\big)(nS)(n\Sigma)\\
&+\big(-8\delta\kappa_+(-7+12\nu)+8\kappa_-(-7+26\nu)-3\delta(1+64\nu)\big)(n\Sigma)^2\\
&+(-60\delta+20\kappa_- -30\delta\kappa_+)\mathbf{S}^2+\big(50\delta\kappa_- +10\kappa_+(-5+12\nu)+5(-13+48\nu)\big)(S\Sigma)\\
&+\big(5\delta\kappa_+(-5+6\nu)-5\kappa_-(-5+16\nu)+2\delta(1+30\nu)\big)\mathbf{\Sigma}^2\Big]\\
&+\mathbf{n}^i\bigg\{\Big[(24\delta+52\kappa_- +12\delta\kappa_+)(nS)+\big(20\delta\kappa_- -4\kappa_+(5+6\nu)-3(-17+16\nu)\big)(n\Sigma)\Big](vS)\\
&+\Big[\big(20\delta\kappa_- -4\kappa_+(5+6\nu)-2(13+24\nu)\big)(nS)
\end{aligned}
$$

$$+\big(-4\delta\kappa_+(5+3\nu)-4\kappa_-(-5+7\nu)-\delta(-1+24\nu)\big)(n\Sigma)\Big](v\Sigma)$$
$$+(nv)\Big[(-282\delta-27\kappa_--141\delta\kappa_+)(nS)^2$$
$$+\big(114\delta\kappa_-+6\kappa_+(-19+94\nu)+3(-99+376\nu)\big)(nS)(n\Sigma)$$
$$+\big(3\delta\kappa_+(-19+47\nu)+6\delta(1+47\nu)-3\kappa_-(-19+85\nu)\big)(n\Sigma)^2+(66\delta-19\kappa_-+33\delta\kappa_+)\mathbf{S}^2$$
$$+\big(59-52\delta\kappa_--264\nu-4\kappa_+(-13+33\nu)\big)(S\Sigma)$$
$$+\big(-66\delta\nu-\delta\kappa_+(-26+33\nu)+\kappa_-(-26+85\nu)\big)\mathbf{\Sigma}^2\Big]\Bigg\}. \quad (368)$$

$$k_5^{0i}=(\mathbf{v}\times\mathbf{S})^i(-2475\delta(nv)^4+3726\delta(nv)^2\mathbf{v}^2-1135\delta\mathbf{v}^4)$$
$$+(\mathbf{v}\times\mathbf{\Sigma})^i\big(75(-17+71\nu)(nv)^4-6(-313+1315\nu)(nv)^2\mathbf{v}^2+(-559+2325\nu)\mathbf{v}^4\big)$$
$$+(\mathbf{n}\times\mathbf{S})^i(3675\delta(nv)^5-6210\delta(nv)^3\mathbf{v}^2+2451\delta(nv)\mathbf{v}^4)$$
$$+(\mathbf{n}\times\mathbf{\Sigma})^i\big(-525(-3+13\nu)(nv)^5+30(-87+379\nu)(nv)^3\mathbf{v}^2-3(-341+1483\nu)(nv)\mathbf{v}^4\big)$$
$$+\mathbf{v}^i\Big[(nv)^3\big(-1680\delta(Snv)+60(-14+47\nu)(\Sigma nv)\big)$$
$$+(nv)\big(1716\delta(Snv)\mathbf{v}^2-12(-73+253\nu)\mathbf{v}^2(\Sigma nv)\big)\Big]$$
$$+\mathbf{n}^i\Big[1878\delta(Snv)\mathbf{v}^4-6(-167+645\nu)\mathbf{v}^4(\Sigma nv)$$
$$+(nv)^4\big(5250\delta(Snv)-1050(-3+11\nu)(\Sigma nv)\big)$$
$$+(nv)^2\big(-7320\delta(Snv)\mathbf{v}^2+60(-70+261\nu)\mathbf{v}^2(\Sigma nv)\big)\Big] \quad (369)$$
$$k_5^{1i}=(\mathbf{v}\times\mathbf{S})^i(792\delta(nv)^2-778\delta\mathbf{v}^2)$$
$$+(\mathbf{v}\times\mathbf{\Sigma})^i\big(-3(-136+567\nu)(nv)^2+3(-134+567\nu)\mathbf{v}^2\big)$$
$$+(\mathbf{n}\times\mathbf{S})^i(-920\delta(nv)^3+806\delta(nv)\mathbf{v}^2)$$
$$+(\mathbf{n}\times\mathbf{\Sigma})^i\big((-404+1763\nu)(nv)^3-(-354+1559\nu)(nv)\mathbf{v}^2\big)$$
$$+\big(\mathbf{n}\times\mathbf{v}\big)^i\Big[(nv)^2\big(-16\delta(nS)-2(-7+13\nu)(n\Sigma)\big)+12\delta(nS)\mathbf{v}^2+2(-7+15\nu)(n\Sigma)\mathbf{v}^2$$
$$+(nv)(4\delta(vS)-4\nu(v\Sigma))\Big]+(nv)\mathbf{v}^i\big(-48\delta(Snv)+4(-21+116\nu)(\Sigma nv)\big)$$
$$+\mathbf{n}^i\Big[2025\delta(Snv)\mathbf{v}^2-5(-175+662\nu)\mathbf{v}^2(\Sigma nv)$$
$$+(nv)^2\big(-1955\delta(Snv)+(-833+2978\nu)(\Sigma nv)\big)\Big], \quad (370)$$
$$k_5^{2i}=59\delta(\mathbf{n}\times\mathbf{S})^i(nv)+(21-95\nu)(\mathbf{n}\times\mathbf{\Sigma})^i(nv)+\big(\mathbf{n}\times\mathbf{v}\big)^i(\delta(nS)-\nu(n\Sigma))-35\delta(\mathbf{v}\times\mathbf{S})^i$$

$$
\begin{aligned}
&+5(-3+13\nu)(\mathbf{v}\times\mathbf{\Sigma})^i+\mathbf{n}^i\big(115\delta(Snv)-2(-12+41\nu)(\Sigma nv)\big), \qquad (371)\\
k_6^{0i}&=\mathbf{S}^i\bigg\{(nv)^4\big((-2100\delta-1050\delta\kappa_+)(nS)+(-1050\delta^2+525\delta\kappa_--525\delta^2\kappa_+)(n\Sigma)\big)\\
&+(nS)(-852\delta\mathbf{v}^4-12\kappa_-\mathbf{v}^4-426\delta\kappa_+\mathbf{v}^4)+(n\Sigma)\big(-426\delta^2\mathbf{v}^4+207\delta\kappa_-\mathbf{v}^4\\
&+3\kappa_+(-69+284\nu)\mathbf{v}^4\big)+(nv)^2\Big[(nS)(3000\delta\mathbf{v}^2+60\kappa_-\mathbf{v}^2+1500\delta\kappa_+\mathbf{v}^2)\\
&+(n\Sigma)\big(1500\delta^2\mathbf{v}^2-720\delta\kappa_-\mathbf{v}^2-120\kappa_+(-6+25\nu)\mathbf{v}^2\big)\Big]\\
&+(nv)^3\Big[(600\delta-300\kappa_-+300\delta\kappa_+)(vS)+\big(300\delta^2-300\delta\kappa_--300\kappa_+(-1+2\nu)\big)(v\Sigma)\Big]\\
&+(nv)\Big[(vS)(-576\delta\mathbf{v}^2+156\kappa_-\mathbf{v}^2-288\delta\kappa_+\mathbf{v}^2)+\big(-288\delta^2\mathbf{v}^2+222\delta\kappa_-\mathbf{v}^2\\
&+6\kappa_+(-37+96\nu)\mathbf{v}^2\big)(v\Sigma)\Big]\bigg\}+\bigg\{(nv)^4\Big[(-1050\delta^2+525\delta\kappa_--525\delta^2\kappa_+)(nS)\\
&+\big(525\delta^2\kappa_-+2100\delta\nu+525\delta\kappa_+(-1+2\nu)\big)(n\Sigma)\Big]+(n\Sigma)\big(852\delta\nu\mathbf{v}^4\\
&+3\delta\kappa_+(-69+142\nu)\mathbf{v}^4-3\kappa_-(-69+280\nu)\mathbf{v}^4\big)+(nS)\big(-426\delta^2\mathbf{v}^4+207\delta\kappa_-\mathbf{v}^4\\
&+3\kappa_+(-69+284\nu)\mathbf{v}^4\big)+(nv)^2\Big[(nS)\big(1500\delta^2\mathbf{v}^2-720\delta\kappa_-\mathbf{v}^2-120\kappa_+(-6+25\nu)\mathbf{v}^2\big)\\
&+(n\Sigma)\big(-3000\delta\nu\mathbf{v}^2-60\delta\kappa_+(-12+25\nu)\mathbf{v}^2+60\kappa_-(-12+49\nu)\mathbf{v}^2\big)\Big]\\
&+(nv)^3\Big[\big(300\delta^2-300\delta\kappa_--300\kappa_+(-1+2\nu)\big)(vS)+\big(-300\delta\kappa_+(-1+\nu)\\
&-600\delta\nu+300\kappa_-(-1+3\nu)\big)(v\Sigma)\Big]+(nv)\Big[(vS)\big(-288\delta^2\mathbf{v}^2+222\delta\kappa_-\mathbf{v}^2\\
&+6\kappa_+(-37+96\nu)\mathbf{v}^2\big)\\
&+\big(576\delta\nu\mathbf{v}^2+6\delta\kappa_+(-37+48\nu)\mathbf{v}^2-6\kappa_-(-37+122\nu)\mathbf{v}^2\big)(v\Sigma)\Big]\bigg\}\mathbf{\Sigma}^i\\
&+\mathbf{v}^i\Bigg((vS)\Big[(nS)(-984\delta\mathbf{v}^2-24\kappa_-\mathbf{v}^2-492\delta\kappa_+\mathbf{v}^2)+(n\Sigma)\big(-492\delta^2\mathbf{v}^2\\
&+234\delta\kappa_-\mathbf{v}^2+6\kappa_+(-39+164\nu)\mathbf{v}^2\big)\Big]+\Big[(n\Sigma)\big(984\delta\nu\mathbf{v}^2\\
&+6\delta\kappa_+(-39+82\nu)\mathbf{v}^2-6\kappa_-(-39+160\nu)\mathbf{v}^2\big)+(nS)\big(-492\delta^2\mathbf{v}^2+234\delta\kappa_-\mathbf{v}^2\\
&+6\kappa_+(-39+164\nu)\mathbf{v}^2\big)\Big](v\Sigma)+(nv)^2\bigg\{\Big[(2880\delta+120\kappa_-+1440\delta\kappa_+)(nS)\\
&+\big(1440\delta^2-660\delta\kappa_--60\kappa_+(-11+48\nu)\big)(n\Sigma)\Big](vS)\\
&+\Big[\big(1440\delta^2-660\delta\kappa_--60\kappa_+(-11+48\nu)\big)(nS)\\
&+\big(-2880\delta\nu-60\delta\kappa_+(-11+24\nu)+60\kappa_-(-11+46\nu)\big)(n\Sigma)\Big](v\Sigma)\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ (nv)^3\Big[(-5460\delta - 2730\delta\kappa_+)(nS)^2 + (-5460\delta^2 + 2730\delta\kappa_- - 2730\delta^2\kappa_+)(nS)(n\Sigma) \\
&+ \big(1365\delta^2\kappa_- + 5460\delta\nu + 1365\delta\kappa_+(-1+2\nu)\big)(n\Sigma)^2 + (660\delta + 60\kappa_- + 330\delta\kappa_+)\mathbf{S}^2 \\
&+ \big(660\delta^2 - 270\delta\kappa_- - 30\kappa_+(-9+44\nu)\big)(S\Sigma) + \big(-660\delta\nu \\
&- 15\delta\kappa_+(-9+22\nu) + 15\kappa_-(-9+40\nu)\big)\mathbf{\Sigma}^2\Big] + (nv)\Big[(-336\delta + 12\kappa_- - 168\delta\kappa_+)(vS)^2 \\
&+ \mathbf{S}^2(-708\delta\mathbf{v}^2 - 48\kappa_-\mathbf{v}^2 - 354\delta\kappa_+\mathbf{v}^2) + (nS)^2(4020\delta\mathbf{v}^2 + 2010\delta\kappa_+\mathbf{v}^2) \\
&+ (nS)(n\Sigma)(4020\delta^2\mathbf{v}^2 - 2010\delta\kappa_-\mathbf{v}^2 + 2010\delta^2\kappa_+\mathbf{v}^2) + (n\Sigma)^2\big(-1005\delta^2\kappa_-\mathbf{v}^2 \\
&- 4020\delta\nu\mathbf{v}^2 - 1005\delta\kappa_+(-1+2\nu)\mathbf{v}^2\big) \\
&+ (S\Sigma)\big(-708\delta^2\mathbf{v}^2 + 306\delta\kappa_-\mathbf{v}^2 + 6\kappa_+(-51+236\nu)\mathbf{v}^2\big) \\
&+ \big(-336\delta^2 + 180\delta\kappa_- + 12\kappa_+(-15+56\nu)\big)(vS)(v\Sigma) \\
&+ \big(336\delta\nu + 6\delta\kappa_+(-15+28\nu) - 6\kappa_-(-15+58\nu)\big)(v\Sigma)^2 \\
&+ \big(708\delta\nu\mathbf{v}^2 + 3\delta\kappa_+(-51+118\nu)\mathbf{v}^2 - 3\kappa_-(-51+220\nu)\mathbf{v}^2\big)\mathbf{\Sigma}^2\Big]\Bigg) \\
&+ \mathbf{n}^i\Bigg((vS)^2(-636\delta\mathbf{v}^2 + 30\kappa_-\mathbf{v}^2 - 318\delta\kappa_+\mathbf{v}^2) + \mathbf{S}^2(-714\delta\mathbf{v}^4 - 6\kappa_-\mathbf{v}^4 \\
&- 357\delta\kappa_+\mathbf{v}^4) + (nS)^2(3630\delta\mathbf{v}^4 + 1815\delta\kappa_+\mathbf{v}^4) + (nS)(n\Sigma)(3630\delta^2\mathbf{v}^4 - 1815\delta\kappa_-\mathbf{v}^4 \\
&+ 1815\delta^2\kappa_+\mathbf{v}^4) + (n\Sigma)^2\big(-\tfrac{1815}{2}\delta^2\kappa_-\mathbf{v}^4 - 3630\delta\nu\mathbf{v}^4 \\
&- \tfrac{1815}{2}\delta\kappa_+(-1+2\nu)\mathbf{v}^4\big) + (S\Sigma)\big(-714\delta^2\mathbf{v}^4 + 351\delta\kappa_-\mathbf{v}^4 + 3\kappa_+(-117+476\nu)\mathbf{v}^4\big) \\
&+ (vS)\big(-636\delta^2\mathbf{v}^2 + 348\delta\kappa_-\mathbf{v}^2 + 12\kappa_+(-29+106\nu)\mathbf{v}^2\big)(v\Sigma) + \big(636\delta\nu\mathbf{v}^2 \\
&+ 6\delta\kappa_+(-29+53\nu)\mathbf{v}^2 - 6\kappa_-(-29+111\nu)\mathbf{v}^2\big)(v\Sigma)^2 + (nv)^3\Big\{\big((-16800\delta - 8400\delta\kappa_+)(nS) \\
&+ (-8400\delta^2 + 4200\delta\kappa_- - 4200\delta^2\kappa_+)(n\Sigma)\big)(vS) + \Big[(-8400\delta^2 + 4200\delta\kappa_- - 4200\delta^2\kappa_+)(nS) \\
&+ \big(4200\delta^2\kappa_- + 16800\delta\nu + 4200\delta\kappa_+(-1+2\nu)\big)(n\Sigma)\Big](v\Sigma)\Big\} \\
&+ (nv)\Big\{(vS)\big((nS)(10680\delta\mathbf{v}^2 + 5340\delta\kappa_+\mathbf{v}^2) + (n\Sigma)(5340\delta^2\mathbf{v}^2 - 2670\delta\kappa_-\mathbf{v}^2 \\
&+ 2670\delta^2\kappa_+\mathbf{v}^2)\big) + \Big[(nS)(5340\delta^2\mathbf{v}^2 - 2670\delta\kappa_-\mathbf{v}^2 + 2670\delta^2\kappa_+\mathbf{v}^2) \\
&+ (n\Sigma)\big(-2670\delta^2\kappa_-\mathbf{v}^2 - 10680\delta\nu\mathbf{v}^2 - 2670\delta\kappa_+(-1+2\nu)\mathbf{v}^2\big)\Big](v\Sigma)\Big\} \\
&+ \big(714\delta\nu\mathbf{v}^4 + \tfrac{3}{2}\delta\kappa_+(-117+238\nu)\mathbf{v}^4 - \tfrac{3}{2}\kappa_-(-117+472\nu)\mathbf{v}^4\big)\mathbf{\Sigma}^2 \\
&+ (nv)^4\Big[(28350\delta + 14175\delta\kappa_+)(nS)^2 + (28350\delta^2 - 14175\delta\kappa_- + 14175\delta^2\kappa_+)(nS)(n\Sigma)
\end{aligned}
$$

$$
\begin{aligned}
&+\big(-\tfrac{14175}{2}\delta^2\kappa_- - 28350\delta\nu - \tfrac{14175}{2}\delta\kappa_+(-1+2\nu)\big)(n\Sigma)^2 + (-3150\delta - 1575\delta\kappa_+)\mathbf{S}^2 \\
&+ (-3150\delta^2 + 1575\delta\kappa_- - 1575\delta^2\kappa_+)(S\Sigma) + \big(\tfrac{1575}{2}\delta^2\kappa_- + 3150\delta\nu \\
&+ \frac{1575}{2}\delta\kappa_+(-1+2\nu)\big)\mathbf{\Sigma}^2\Big] + (nv)^2\Big[(2100\delta - 150\kappa_- + 1050\delta\kappa_+)(vS)^2 \\
&+ (nS)^2(-27300\delta\mathbf{v}^2 - 13650\delta\kappa_+\mathbf{v}^2) + \mathbf{S}^2(3840\delta\mathbf{v}^2 + 30\kappa_-\mathbf{v}^2 + 1920\delta\kappa_+\mathbf{v}^2) \\
&+ (nS)(n\Sigma)(-27300\delta^2\mathbf{v}^2 + 13650\delta\kappa_-\mathbf{v}^2 - 13650\delta^2\kappa_+\mathbf{v}^2) + (n\Sigma)^2\big(6825\delta^2\kappa_-\mathbf{v}^2 \\
&+ 27300\delta\nu\mathbf{v}^2 + 6825\delta\kappa_+(-1+2\nu)\mathbf{v}^2\big) + (S\Sigma)\big(3840\delta^2\mathbf{v}^2 - 1890\delta\kappa_-\mathbf{v}^2 \\
&- 30\kappa_+(-63+256\nu)\mathbf{v}^2\big) + \big(2100\delta^2 - 1200\delta\kappa_- - 600\kappa_+(-2+7\nu)\big)(vS)(v\Sigma) \\
&+ \big(-2100\delta\nu - 150\delta\kappa_+(-4+7\nu) + 150\kappa_-(-4+15\nu)\big)(v\Sigma)^2 + \big(-3840\delta\nu\mathbf{v}^2 \\
&- 15\delta\kappa_+(-63+128\nu)\mathbf{v}^2 + 15\kappa_-(-63+254\nu)\mathbf{v}^2\big)\mathbf{\Sigma}^2\Big]\Bigg), \qquad (372)
\end{aligned}
$$

$$
\begin{aligned}
k_6^{1i} &= \mathbf{S}^i\Big\{(nv)^2\Big[(1212\delta + 108\kappa_- + 606\delta\kappa_+)(nS) \\
&+ \big(-249\delta\kappa_- - 6(-121+404\nu) - 3\kappa_+(-83+404\nu)\big)(n\Sigma)\Big] \\
&+ (nS)(-1092\delta\mathbf{v}^2 - 80\kappa_-\mathbf{v}^2 - 546\delta\kappa_+\mathbf{v}^2) + (n\Sigma)\big(233\delta\kappa_-\mathbf{v}^2 + 26(-25+84\nu)\mathbf{v}^2 \\
&+ \kappa_+(-233+1092\nu)\mathbf{v}^2\big) + (nv)\big(68\kappa_-(vS) + (-16+34\delta\kappa_- - 34\kappa_+)(v\Sigma)\big)\Big\} \\
&+ \Big\{(nv)^2\Big[\big(-249\delta\kappa_- - 6(-93+404\nu) - 3\kappa_+(-83+404\nu)\big)(nS) + \big(-12\delta(-6+101\nu) \\
&- 3\delta\kappa_+(-83+202\nu) + 3\kappa_-(-83+368\nu)\big)(n\Sigma)\Big] \\
&+ (n\Sigma)\big(4\delta(-17+273\nu)\mathbf{v}^2 + \delta\kappa_+(-233+546\nu)\mathbf{v}^2 - \kappa_-(-233+1012\nu)\mathbf{v}^2\big) \\
&+ (nS)\big(233\delta\kappa_-\mathbf{v}^2 + 6(-85+364\nu)\mathbf{v}^2 + \kappa_+(-233+1092\nu)\mathbf{v}^2\big) \\
&+ (nv)\Big[(12+34\delta\kappa_- - 34\kappa_+)(vS) + \big(-4\delta - 34\delta\kappa_+ - 34\kappa_-(-1+2\nu)\big)(v\Sigma)\Big]\Big\}\mathbf{\Sigma}^i \\
&+ \mathbf{v}^i\Big\{\Big[(-480\delta - 8\kappa_- - 240\delta\kappa_+)(nS) \\
&+ \big(116\delta\kappa_- + 8(-37+120\nu) + 4\kappa_+(-29+120\nu)\big)(n\Sigma)\Big](vS) \\
&+ \Big[\big(116\delta\kappa_- + 64(-2+15\nu) + 4\kappa_+(-29+120\nu)\big)(nS) + \big(4\delta\kappa_+(-29+60\nu) \\
&+ 8\delta(7+60\nu) - 4\kappa_-(-29+118\nu)\big)(n\Sigma)\Big](v\Sigma) + (nv)\Big[(-168\delta - 78\kappa_- - 84\delta\kappa_+)(nS)^2 \\
&+ \big(6\delta\kappa_- + 12(-19+56\nu) + 6\kappa_+(-1+56\nu)\big)(nS)(n\Sigma)
\end{aligned}
$$

$$
\begin{aligned}
&+\big(12\delta(-5+14\nu)+3\delta\kappa_+(-1+28\nu)\\
&-3\kappa_-(-1+30\nu)\big)(n\Sigma)^2+(216\delta+6\kappa_-+108\delta\kappa_+)\mathbf{S}^2\\
&+\big(-102\delta\kappa_--6\kappa_+(-17+72\nu)-8(-25+108\nu)\big)(S\Sigma)\\
&+\big(-8\delta(2+27\nu)-3\delta\kappa_+(-17+36\nu)+3\kappa_-(-17+70\nu)\big)\mathbf{\Sigma}^2\Big]\Bigg\}\\
&+\mathbf{n}^i\Bigg(\big(-1176\delta-4\kappa_--588\delta\kappa_+)(vS)^2+\mathbf{S}^2(-2460\delta\mathbf{v}^2+30\kappa_-\mathbf{v}^2-1230\delta\kappa_+\mathbf{v}^2)\\
&+(nS)^2(9648\delta\mathbf{v}^2-6\kappa_-\mathbf{v}^2+4824\delta\kappa_+\mathbf{v}^2)+(S\Sigma)\big(1260\delta\kappa_-\mathbf{v}^2\\
&+60\kappa_+(-21+82\nu)\mathbf{v}^2+4(-641+2460\nu)\mathbf{v}^2\big)\\
&+(nS)(n\Sigma)\big(-4830\delta\kappa_-\mathbf{v}^2-6\kappa_+(-805+3216\nu)\mathbf{v}^2\\
&-12(-803+3216\nu)\mathbf{v}^2\big)+(n\Sigma)^2\big(-12\delta(1+804\nu)\mathbf{v}^2-3\delta\kappa_+(-805+1608\nu)\mathbf{v}^2\\
&+3\kappa_-(-805+3218\nu)\mathbf{v}^2\big)+\big(584\delta\kappa_-+112(-11+42\nu)+8\kappa_+(-73+294\nu)\big)(vS)(v\Sigma)\\
&+\big(56\delta(-1+21\nu)+4\delta\kappa_+(-73+147\nu)-4\kappa_-(-73+293\nu)\big)(v\Sigma)^2\\
&+(nv)\Bigg\{\Big[(7080\delta+96\kappa_-+3540\delta\kappa_+)(nS)+\big(-1722\delta\kappa_--48(-76+295\nu)\\
&-6\kappa_+(-287+1180\nu)\big)(n\Sigma)\Big](vS)\\
&+\Big[\big(-1722\delta\kappa_--120(-29+118\nu)-6\kappa_+(-287+1180\nu)\big)(nS)\\
&+\big(-24\delta(-2+295\nu)-6\delta\kappa_+(-287+590\nu)+6\kappa_-(-287+1164\nu)\big)(n\Sigma)\Big](v\Sigma)\Bigg\}\\
&+\big(30\delta\kappa_+(-21+41\nu)\mathbf{v}^2-30\kappa_-(-21+83\nu)\mathbf{v}^2+4\delta(-26+615\nu)\mathbf{v}^2\big)\mathbf{\Sigma}^2\\
&+(nv)^2\Big[(-16104\delta+12\kappa_--8052\delta\kappa_+)(nS)^2+\big(8064\delta\kappa_-+48\kappa_+(-168+671\nu)\\
&+48(-335+1342\nu)\big)(nS)(n\Sigma)+\big(12\delta\kappa_+(-336+671\nu)+24\delta(1+671\nu)\\
&-12\kappa_-(-336+1343\nu)\big)(n\Sigma)^2+(2604\delta-72\kappa_-+1302\delta\kappa_+)\mathbf{S}^2\\
&+\big(-1374\delta\kappa_--6\kappa_+(-229+868\nu)-12(-227+868\nu)\big)(S\Sigma)\\
&+\big(-12\delta(-10+217\nu)-3\delta\kappa_+(-229+434\nu)+3\kappa_-(-229+892\nu)\big)\mathbf{\Sigma}^2\Big]\Bigg),
\end{aligned}
\tag{373}
$$

$$
\begin{aligned}
k_6^{2i}=&\Big[(-288\delta+32\kappa_--144\delta\kappa_+)(nS)\\
&+\big(88\delta\kappa_-+32(-5+18\nu)+8\kappa_+(-11+36\nu)\big)(n\Sigma)\Big]\mathbf{S}^i\\
&+\Big[\big(88\delta\kappa_-+8\kappa_+(-11+36\nu)+12(-11+48\nu)\big)(nS)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(8\delta\kappa_+(-11+18\nu)-8\kappa_-(-11+40\nu)+4\delta(-1+72\nu)\Big)(n\Sigma)\Big]\boldsymbol{\Sigma}^i \\
&+\mathbf{n}^i\Big[(1368\delta+4\kappa_-+684\delta\kappa_+)(nS)^2 \\
&+\Big(-680\delta\kappa_- -8\kappa_+(-85+342\nu)-4(-341+1368\nu)\Big)(nS)(n\Sigma) \\
&+\Big(-4\delta\kappa_+(-85+171\nu)+4\kappa_-(-85+341\nu)-4\delta(1+342\nu)\Big)(n\Sigma)^2 \\
&+(-360\delta-12\kappa_- -180\delta\kappa_+)\mathbf{S}^2 \\
&+\Big(168\delta\kappa_- +24\kappa_+(-7+30\nu)+8(-47+180\nu)\Big)(S\Sigma)+\Big(12\delta\kappa_+(-7+15\nu) \\
&-12\kappa_-(-7+29\nu)+8\delta(-2+45\nu)\Big)\boldsymbol{\Sigma}^2\Big]. \qquad (374)
\end{aligned}
$$

Appendix E Intermediate results for NNLO radiation reaction

In this section, we present the partial results used to compute the NNLO radiation reaction acceleration for the convenience of the reader.

A. 3.5PN radiation-reaction acceleration

The full 3.5PN acceleration in the COM frame is given by

$$\begin{aligned}\mathbf{a}^{3.5\mathrm{PN}} =&\Bigg\{ \frac{G^4m^4\dot{r}\nu}{r^6}\left[\tfrac{256}{7}+\tfrac{88}{3}\nu\right]+\frac{G^3m^3\dot{r}\nu}{r^5}\left[\mathbf{v}^2(-\tfrac{10758}{35}-\tfrac{1088}{15}\nu)+\dot{r}^2(\tfrac{2706}{5}+\tfrac{1064}{5}\nu)\right]\\ &+\frac{G^2m^2\dot{r}\nu}{r^4}\left[\mathbf{v}^4(\tfrac{348}{35}-\tfrac{384}{5}\nu)+\dot{r}^2\mathbf{v}^2(20+100\nu)+\dot{r}^4(-28+28\nu)\right]\Bigg\}\mathbf{x}^i\\ &+\Bigg\{\frac{G^4m^4\nu}{r^5}\left[-\tfrac{6208}{105}-\tfrac{88}{3}\nu\right]+\frac{G^3m^3\nu}{r^4}\left[\mathbf{v}^2(\tfrac{9722}{105}+\tfrac{464}{15}\nu)+\dot{r}^2(-\tfrac{5182}{15}-\tfrac{776}{5}\nu)\right]\\ &+\frac{G^2m^2\nu}{r^3}\left[\tfrac{108}{35}\mathbf{v}^4+\dot{r}^2\mathbf{v}^2(-\tfrac{156}{5}+\tfrac{444}{5}\nu)+\dot{r}^4(20-140\nu)\right]\Bigg\}\mathbf{v}^i. \end{aligned} \tag{375}$$

B. Mass quadrupole

The mass quadrupole contribution is given by

$$\begin{aligned}\mathbf{a}^{4.5\mathrm{PN}}_{\mathrm{mq}} =&\Bigg\{\frac{G^5m^5\dot{r}\nu}{r^7}\left[-\tfrac{114934}{245}-\tfrac{541648}{2205}\nu-\tfrac{255896}{2205}\nu^2\right]\\ &+\frac{G^4m^4\dot{r}\nu}{r^6}\left[\mathbf{v}^2(\tfrac{29732}{147}+\tfrac{766580}{147}\nu-\tfrac{180968}{245}\nu^2)+\dot{r}^2(\tfrac{117436}{6615}-\tfrac{67308818}{6615}\nu+\tfrac{7553204}{6615}\nu^2)\right]\\ &+\frac{G^3m^3\dot{r}\nu}{r^5}\Bigg[\mathbf{v}^4(\tfrac{133848}{245}-\tfrac{1920036}{245}\nu+\tfrac{2270218}{735}\nu^2)\\ &\quad+\dot{r}^2\mathbf{v}^2(-\tfrac{2537642}{735}+\tfrac{1615640}{49}\nu-\tfrac{7423448}{735}\nu^2)+\dot{r}^4(\tfrac{418454}{105}-\tfrac{3013168}{105}\nu+\tfrac{771914}{105}\nu^2)\Bigg]\\ &+\frac{G^2m^2\dot{r}\nu}{r^4}\Bigg[\mathbf{v}^6(-\tfrac{69474}{245}+\tfrac{80370}{49}\nu-\tfrac{541266}{245}\nu^2)+\dot{r}^2\mathbf{v}^4(\tfrac{213694}{147}-\tfrac{1151509}{147}\nu+\tfrac{1473538}{147}\nu^2)\\ &\quad+\dot{r}^4\mathbf{v}^2(-\tfrac{14458}{7}+\tfrac{207226}{21}\nu-\tfrac{237472}{21}\nu^2)+\dot{r}^6(\tfrac{6498}{7}-\tfrac{25380}{7}\nu+\tfrac{22257}{7}\nu^2)\Bigg]\Bigg\}\mathbf{x}^i\end{aligned}$$

$$
\begin{aligned}
&+\left\{\frac{G^5m^5\nu}{r^6}\left[\tfrac{169810}{1323}+\tfrac{5596856}{6615}\nu-\tfrac{74216}{6615}\nu^2\right]\right.\\
&+\frac{G^4m^4\nu}{r^5}\left[\mathbf{v}^2(\tfrac{57748}{735}-\tfrac{4564916}{2205}\nu+\tfrac{347552}{2205}\nu^2)+\dot{r}^2(-\tfrac{105092}{147}+\tfrac{5839346}{735}\nu-\tfrac{188532}{245}\nu^2)\right]\\
&+\frac{G^3m^3\nu}{r^4}\left[\mathbf{v}^4(-\tfrac{228836}{2205}+\tfrac{3125284}{2205}\nu-\tfrac{193838}{441}\nu^2)\right.\\
&\qquad\left.+\dot{r}^2\mathbf{v}^2(\tfrac{3621206}{2205}-\tfrac{28750504}{2205}\nu+\tfrac{5766616}{2205}\nu^2)+\dot{r}^4(-\tfrac{275258}{105}+\tfrac{321100}{21}\nu-\tfrac{18286}{7}\nu^2)\right]\\
&+\frac{G^2m^2\nu}{r^3}\left[\mathbf{v}^6(\tfrac{1504}{147}-\tfrac{16178}{147}\nu+\tfrac{206434}{735}\nu^2)+\dot{r}^2\mathbf{v}^4(-\tfrac{19328}{245}+\tfrac{31637}{49}\nu-\tfrac{518338}{245}\nu^2)\right.\\
&\qquad\left.\left.+\dot{r}^4\mathbf{v}^2(\tfrac{2984}{21}-\tfrac{3638}{21}\nu+\tfrac{39176}{21}\nu^2)+\dot{r}^6(-\tfrac{816}{7}-\tfrac{8440}{21}\nu+\tfrac{6619}{21}\nu^2)\right]\right\}\mathbf{v}^i. \qquad (376)
\end{aligned}
$$

C. Current quadrupole

The current quadrupole acceleration contribution is given by

$$
\begin{aligned}
\mathbf{a}_{\rm cq}^{4.5\rm PN} =&\left\{\frac{G^5m^5\dot{r}\nu}{r^7}\left[-\tfrac{21208}{315}+\tfrac{89672}{315}\nu-\tfrac{3872}{63}\nu^2\right]\right.\\
&+\frac{G^4m^4\dot{r}\nu}{r^6}\left[\dot{r}^2(-\tfrac{49052}{63}+\tfrac{120368}{35}\nu-\tfrac{409088}{315}\nu^2)+\mathbf{v}^2(\tfrac{27428}{105}-\tfrac{358184}{315}\nu+\tfrac{116192}{315}\nu^2)\right]\\
&+\frac{G^3m^3\dot{r}\nu}{r^5}\left[\dot{r}^4(-\tfrac{5076}{5}+\tfrac{73172}{15}\nu-\tfrac{9808}{3}\nu^2)\right.\\
&\qquad\left.+\mathbf{v}^4(-\tfrac{2696}{35}+\tfrac{46204}{105}\nu-\tfrac{55408}{105}\nu^2)+\dot{r}^2\mathbf{v}^2(\tfrac{27044}{35}-\tfrac{27200}{7}\nu+\tfrac{111296}{35}\nu^2)\right]\\
&+\frac{G^2m^2\dot{r}\nu}{r^4}\left[\mathbf{v}^6(-\tfrac{3328}{35}+\tfrac{11148}{35}\nu+\tfrac{8656}{35}\nu^2)+\dot{r}^2\mathbf{v}^4(\tfrac{12368}{21}-\tfrac{14516}{7}\nu-\tfrac{23696}{21}\nu^2)\right.\\
&\qquad\left.\left.+\dot{r}^4\mathbf{v}^2(-\tfrac{2816}{3}+\tfrac{10372}{3}\nu+\tfrac{3568}{3}\nu^2)+\dot{r}^6(432-1668\nu-240\nu^2)\right]\right\}\mathbf{x}^i\\
&+\left\{\frac{G^5m^5\nu}{r^6}\left[\tfrac{21208}{315}-\tfrac{89672}{315}\nu+\tfrac{3872}{63}\nu^2\right]\right.\\
&+\frac{G^4m^4\nu}{r^5}\left[\mathbf{v}^2(-\tfrac{35692}{315}+\tfrac{159944}{315}\nu-\tfrac{68704}{315}\nu^2)+\dot{r}^2(\tfrac{198668}{315}-\tfrac{295024}{105}\nu+\tfrac{72320}{63}\nu^2)\right]\\
&+\frac{G^3m^3\nu}{r^4}\left[\mathbf{v}^4(\tfrac{3344}{315}-\tfrac{12092}{105}\nu+\tfrac{18320}{63}\nu^2)\right.\\
&\qquad\left.+\dot{r}^2\mathbf{v}^2(-\tfrac{20588}{45}+\tfrac{37192}{15}\nu-\tfrac{116896}{45}\nu^2)+\dot{r}^4(\tfrac{80476}{105}-\tfrac{79732}{21}\nu+\tfrac{307024}{105}\nu^2)\right]
\end{aligned}
$$

$$+\frac{G^2m^2\nu}{r^3}\Big[\mathbf{v}^6(\tfrac{816}{35}-\tfrac{1956}{35}\nu-\tfrac{5232}{35}\nu^2)+\dot{r}^2\mathbf{v}^4(-\tfrac{9248}{35}+\tfrac{24268}{35}\nu+\tfrac{50896}{35}\nu^2)$$
$$+\dot{r}^4\mathbf{v}^2(\tfrac{10480}{21}-\tfrac{29140}{21}\nu-\tfrac{17040}{7}\nu^2)+\dot{r}^6(-\tfrac{736}{3}+716\nu+\tfrac{3184}{3}\nu^2)\Big]\Big\}\mathbf{v}^i. \quad (377)$$

D. Mass octupole

The mass octupole contribution to the NNLO radiation-reaction acceleration is

$$\mathbf{a}^{4.5\mathrm{PN}}_{\mathrm{mo}}=\Big\{\frac{G^5m^5\dot{r}\nu}{r^7}\Big[\tfrac{26216}{315}-\tfrac{21016}{63}\nu+\tfrac{96}{35}\nu^2\Big]$$
$$+\frac{G^4m^4\dot{r}\nu}{r^6}\Big[\mathbf{v}^2(\tfrac{32804}{105}-\tfrac{439918}{315}\nu+\tfrac{5288}{9}\nu^2)+\dot{r}^2(-\tfrac{27116}{189}+\tfrac{5470}{7}\nu-\tfrac{156904}{189}\nu^2)\Big]$$
$$+\frac{G^3m^3\dot{r}\nu}{r^5}\Big[\mathbf{v}^4(-\tfrac{23466}{35}+\tfrac{301666}{105}\nu-\tfrac{80296}{105}\nu^2)$$
$$+\dot{r}^2\mathbf{v}^2(\tfrac{286688}{105}-\tfrac{1240282}{105}\nu+\tfrac{74824}{21}\nu^2)+\dot{r}^4(-\tfrac{220658}{105}+\tfrac{322724}{35}\nu-\tfrac{9776}{3}\nu^2)\Big]$$
$$+\frac{G^2m^2\dot{r}\nu}{r^4}\Big[\mathbf{v}^6(-\tfrac{8062}{35}+\tfrac{6066}{5}\nu-\tfrac{40856}{35}\nu^2)+\dot{r}^2\mathbf{v}^4(\tfrac{10510}{7}-\tfrac{57978}{7}\nu+\tfrac{63752}{7}\nu^2)$$
$$+\dot{r}^4\mathbf{v}^2(-2550+\tfrac{43226}{3}\nu-\tfrac{50504}{3}\nu^2)+\dot{r}^6(1278-7314\nu+8808\nu^2)\Big]\Big\}\mathbf{x}^i$$
$$+\Big\{\frac{G^5m^5\nu}{r^6}\Big[-\tfrac{19616}{945}+\tfrac{22256}{315}\nu+\tfrac{46784}{945}\nu^2\Big]$$
$$+\frac{G^4m^4\nu}{r^5}\Big[\mathbf{v}^2(-\tfrac{10972}{105}+\tfrac{27758}{63}\nu-\tfrac{4072}{45}\nu^2)+\dot{r}^2(\tfrac{13516}{315}-\tfrac{29458}{105}\nu+\tfrac{27448}{63}\nu^2)\Big]$$
$$+\frac{G^3m^3\nu}{r^4}\Big[\mathbf{v}^4(\tfrac{3256}{35}-\tfrac{35848}{105}\nu-\tfrac{12896}{105}\nu^2)$$
$$+\dot{r}^2\mathbf{v}^2(-\tfrac{94636}{105}+\tfrac{383086}{105}\nu-\tfrac{6056}{35}\nu^2)+\dot{r}^4(\tfrac{89996}{105}-\tfrac{126458}{35}\nu+\tfrac{2216}{3}\nu^2)\Big]$$
$$+\frac{G^2m^2\nu}{r^3}\Big[\mathbf{v}^6(\tfrac{5296}{105}-\tfrac{9112}{35}\nu+\tfrac{24608}{105}\nu^2)+\dot{r}^2\mathbf{v}^4(-\tfrac{23304}{35}+\tfrac{127364}{35}\nu-\tfrac{136592}{35}\nu^2)$$
$$+\dot{r}^4\mathbf{v}^2(1440-\tfrac{24128}{3}\nu+\tfrac{27392}{3}\nu^2)+\dot{r}^6(-824+4636\nu-5360\nu^2)\Big]\Big\}\mathbf{v}^i. \quad (378)$$

E. Current octupole

The current octupole contribution to the acceleration is found to be

$$\begin{aligned}\mathbf{a}_{\rm co}^{4.5\rm PN} =&\Bigg\{\frac{G^5m^5\dot{r}\nu}{r^7}\Big[\tfrac{344}{315}-\tfrac{688}{105}\nu+\tfrac{344}{35}\nu^2\Big]\\
&+\frac{G^4m^4\dot{r}\nu}{r^6}\Big[\mathbf{v}^2(-\tfrac{64}{105}+\tfrac{128}{35}\nu-\tfrac{192}{35}\nu^2)+\dot{r}^2(\tfrac{4232}{315}-\tfrac{8464}{105}\nu+\tfrac{4232}{35}\nu^2)\Big]\\
&+\frac{G^3m^3\dot{r}\nu}{r^5}\Big[\mathbf{v}^4(-\tfrac{710}{21}+\tfrac{1420}{7}\nu-\tfrac{2130}{7}\nu^2)\\
&\quad+\dot{r}^2\mathbf{v}^2(\tfrac{11138}{105}-\tfrac{22276}{35}\nu+\tfrac{33414}{35}\nu^2)+\dot{r}^4(-60+360\nu-540\nu^2)\Big]\\
&+\frac{G^2m^2\dot{r}\nu}{r^4}\Big[\mathbf{v}^6(\tfrac{330}{7}-\tfrac{1980}{7}\nu+\tfrac{2970}{7}\nu^2)+\dot{r}^4\mathbf{v}^2(\tfrac{1438}{3}-2876\nu+4314\nu^2)\\
&\quad+\dot{r}^2\mathbf{v}^4(-\tfrac{6110}{21}+\tfrac{12220}{7}\nu-\tfrac{18330}{7}\nu^2)+\dot{r}^6(-234+1404\nu-2106\nu^2)\Big]\Bigg\}\mathbf{x}^i\\
&+\Bigg\{\frac{G^5m^5\nu}{r^6}\Big[-\tfrac{344}{315}+\tfrac{688}{105}\nu-\tfrac{344}{35}\nu^2\Big]\\
&+\frac{G^4m^4\nu}{r^5}\Big[\mathbf{v}^2(-\tfrac{1184}{315}+\tfrac{2368}{105}\nu-\tfrac{1184}{35}\nu^2)+\dot{r}^2(-\tfrac{136}{15}+\tfrac{272}{5}\nu-\tfrac{408}{5}\nu^2)\Big]\\
&+\frac{G^3m^3\nu}{r^4}\Big[\mathbf{v}^4(\tfrac{86}{5}-\tfrac{516}{5}\nu+\tfrac{774}{5}\nu^2)\\
&\quad+\dot{r}^2\mathbf{v}^2(-\tfrac{1262}{15}+\tfrac{2524}{5}\nu-\tfrac{3786}{5}\nu^2)+\dot{r}^4(\tfrac{164}{3}-328\nu+492\nu^2)\Big]\\
&+\frac{G^2m^2\nu}{r^3}\Big[\mathbf{v}^6(-\tfrac{90}{7}+\tfrac{540}{7}\nu-\tfrac{810}{7}\nu^2)+\dot{r}^2\mathbf{v}^4(150-900\nu+1350\nu^2)\\
&\quad+\dot{r}^4\mathbf{v}^2(-\tfrac{910}{3}+1820\nu-2730\nu^2)+\dot{r}^6(\tfrac{494}{3}-988\nu+1482\nu^2)\Big]\Bigg\}\mathbf{v}^i. \qquad (379)\end{aligned}$$

F. Mass hexadecapole

The mass hexadecapole contribution is given by

$$\begin{aligned}\mathbf{a}_{\rm mh}^{4.5\rm PN} =&\Bigg\{\frac{G^5m^5\dot{r}\nu}{r^7}\Big[-\tfrac{13784}{6615}+\tfrac{27568}{2205}\nu-\tfrac{13784}{735}\nu^2\Big]\\
&+\frac{G^4m^4\dot{r}\nu}{r^6}\Big[\mathbf{v}^2(-\tfrac{61792}{2205}+\tfrac{123584}{735}\nu-\tfrac{61792}{245}\nu^2)+\dot{r}^2(\tfrac{29528}{1323}-\tfrac{59056}{441}\nu+\tfrac{29528}{147}\nu^2)\Big]\end{aligned}$$

$$
\begin{aligned}
&+\frac{G^3m^3\dot{r}\nu}{r^5}\Big[\mathbf{v}^4(\tfrac{10826}{735}-\tfrac{21652}{245}\nu+\tfrac{32478}{245}\nu^2)\\
&\quad+\dot{r}^2\mathbf{v}^2(-\tfrac{21806}{147}+\tfrac{43612}{49}\nu-\tfrac{65418}{49}\nu^2)+\dot{r}^4(\tfrac{2764}{21}-\tfrac{5528}{7}\nu+\tfrac{8292}{7}\nu^2)\Big]\\
&+\frac{G^2m^2\dot{r}\nu}{r^4}\Big[\mathbf{v}^6(\tfrac{19990}{147}-\tfrac{39980}{49}\nu+\tfrac{59970}{49}\nu^2)+\dot{r}^2\mathbf{v}^4(-\tfrac{33450}{49}+\tfrac{200700}{49}\nu-\tfrac{301050}{49}\nu^2)\\
&\quad+\dot{r}^4\mathbf{v}^2(\tfrac{6786}{7}-\tfrac{40716}{7}\nu+\tfrac{61074}{7}\nu^2)+\dot{r}^6(-\tfrac{2970}{7}+\tfrac{17820}{7}\nu-\tfrac{26730}{7}\nu^2)\Big]\Big\}\mathbf{x}^i\\
&+\Big\{\frac{G^5m^5}{r^6}\nu\Big[\tfrac{296}{3969}-\tfrac{592}{1323}\nu+\tfrac{296}{441}\nu^2\Big]\\
&+\frac{G^4m^4\nu}{r^5}\Big[\dot{r}^2(-\tfrac{3560}{441}+\tfrac{7120}{147}\nu-\tfrac{3560}{49}\nu^2)+\mathbf{v}^2(\tfrac{3616}{441}-\tfrac{7232}{147}\nu+\tfrac{3616}{49}\nu^2)\Big]\\
&+\frac{G^3m^3\nu}{r^4}\Big[\mathbf{v}^4(\tfrac{622}{147}-\tfrac{1244}{49}\nu+\tfrac{1866}{49}\nu^2)\\
&\quad+\dot{r}^2\mathbf{v}^2(\tfrac{7790}{147}-\tfrac{15580}{49}\nu+\tfrac{23370}{49}\nu^2)+\dot{r}^4(-\tfrac{1180}{21}+\tfrac{2360}{7}\nu-\tfrac{3540}{7}\nu^2)\Big]\\
&+\frac{G^2m^2\nu}{r^3}\Big[\mathbf{v}^6(-\tfrac{11890}{441}+\tfrac{23780}{147}\nu-\tfrac{11890}{49}\nu^2)+\dot{r}^2\mathbf{v}^4(\tfrac{12330}{49}-\tfrac{73980}{49}\nu+\tfrac{110970}{49}\nu^2)\\
&\quad+\dot{r}^4\mathbf{v}^2(-\tfrac{3090}{7}+\tfrac{18540}{7}\nu-\tfrac{27810}{7}\nu^2)+\dot{r}^6(\tfrac{1530}{7}-\tfrac{9180}{7}\nu+\tfrac{13770}{7}\nu^2)\Big]\Big\}\mathbf{v}^i. \qquad (380)
\end{aligned}
$$

G. Reduced contributions

The reduced contributions consist of acceleration pieces arising from conservative accelerations with radiative acceleration reductions. In addition, we have 3.5PN center-of-mass corrections to the 1PN conservative acceleration. These sum to

$$
\begin{aligned}
\mathbf{a}^{4.5\mathrm{PN}}_{\mathrm{red.}} =&\Big\{\frac{G^5m^5\dot{r}\nu}{r^7}\Big[-\tfrac{9224}{105}-\tfrac{8704}{35}\nu+\tfrac{584}{15}\nu^2\Big]\\
&+\frac{G^4m^4\dot{r}\nu}{r^6}\Big[\mathbf{v}^2(\tfrac{85942}{105}-\tfrac{18502}{21}\nu+\tfrac{33988}{105}\nu^2)+\dot{r}^2(-\tfrac{7178}{5}+\tfrac{25432}{35}\nu-\tfrac{26584}{105}\nu^2)\Big]\\
&+\frac{G^3m^3\dot{r}\nu}{r^5}\Big[\mathbf{v}^4(\tfrac{793}{35}+\tfrac{276712}{105}\nu-\tfrac{11034}{35}\nu^2)\\
&\quad+\dot{r}^2\mathbf{v}^2(-\tfrac{953}{5}-\tfrac{359673}{35}\nu+\tfrac{32028}{35}\nu^2)+\dot{r}^4(84+\tfrac{57459}{7}\nu-\tfrac{4716}{7}\nu^2)\Big]\\
&+\frac{G^2m^2\dot{r}\nu}{r^4}\Big[\mathbf{v}^6(-\tfrac{552}{35}+\tfrac{2199}{35}\nu+\tfrac{306}{35}\nu^2)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \dot{r}^2\mathbf{v}^4(5 + \tfrac{347}{7}\nu - \tfrac{2473}{7}\nu^2) + \dot{r}^4\mathbf{v}^2(14 - 147\nu + 406\nu^2)\Big]\Big\}\mathbf{x}^i \\
&+ \Big\{\frac{G^5m^5\nu}{r^6}\Big[\tfrac{5368}{35} + \tfrac{6764}{105}\nu - \tfrac{488}{105}\nu^2\Big] \\
&+ \frac{G^4m^4\nu}{r^5}\Big[\mathbf{v}^2(-\tfrac{5274}{35} + \tfrac{6246}{35}\nu - \tfrac{6316}{105}\nu^2) + \dot{r}^2(\tfrac{85298}{105} - \tfrac{15956}{35}\nu - \tfrac{512}{15}\nu^2)\Big] \\
&+ \frac{G^3m^3\nu}{r^4}\Big[\mathbf{v}^4(-\tfrac{1867}{35} - \tfrac{9808}{35}\nu + \tfrac{54}{5}\nu^2) \\
&\qquad + \dot{r}^2\mathbf{v}^2(\tfrac{16239}{35} + \tfrac{261923}{105}\nu - \tfrac{2220}{7}\nu^2) + \dot{r}^4(-\tfrac{1606}{5} - \tfrac{98743}{35}\nu + \tfrac{10588}{35}\nu^2)\Big] \\
&+ \frac{G^2m^2\nu}{r^3}\Big[\mathbf{v}^6(\tfrac{468}{35} - \tfrac{3063}{35}\nu + \tfrac{678}{5}\nu^2) + \dot{r}^2\mathbf{v}^4(-\tfrac{1797}{35} + \tfrac{1749}{7}\nu - \tfrac{6129}{35}\nu^2) \\
&\qquad + \dot{r}^4\mathbf{v}^2(10 + \tfrac{1273}{7}\nu - \tfrac{6422}{7}\nu^2) + \dot{r}^6(28 - 294\nu + 812\nu^2)\Big]\Big\}\mathbf{v}^i. \qquad (381)
\end{aligned}
$$

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