RISK ANALYSIS OF LIFE INSURANCE PRODUCTS

by

Christine Zelch

B. S. in Mathematics, The Pennsylvania State University, State College, 2002

B. S. in Statistics, The Pennsylvania State University, State College, 2002

Submitted to the Graduate Faculty of

Arts and Sciences in partial fulfillment

of the requirements for the degree of

Master of Science in Mathematics

University of Pittsburgh

2011

UNIVERSITY OF PITTSBURGH

ARTS AND SCIENCES

This thesis was presented

by

Christine Zelch

It was defended on

April 4, 2011

and approved by

Gregory M. Constantine, PhD, Professor

Satish Iyengar, PhD, Professor and Chair

Thesis Advisor: John M. Chadam, PhD, Professor

RISK ANALYSIS OF LIFE INSURANCE PRODUCTS

Christine Zelch, MS

University of Pittsburgh, 2011

This paper takes a simple life insurance product that pays a benefit upon the death of a person and looks at two separate ways of setting the price, or premium, for the product. The premium is collected only once; on the date that the product is purchased. The first method of pricing uses the actuarial present value of the insurance product to set the single premium. On average, when pricing at the actuarial present value the person selling the product will break even on gains and losses. However, in individual cases there is the risk of large losses that the seller cannot control when pricing at the actuarial present value.

The second method of pricing allows the person setting the premium to have more control over losses by looking at the value at risk. Under the value at risk method, the price charged guarantees that the seller will not lose more than a certain amount of money with a set confidence level. Both pricing methods are analyzed first under a constant rate of return and then later using a yield curve of varying interest rates.

Finally, this paper looks at how changing the rates of return from constant rates to varying rates affect the amount of money that the seller has on hand to pay benefits under both pricing methods. It is determined that the appropriate method to use in pricing the product depends upon the seller's market competition for similar products and the level of risk the seller is willing to undertake.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF EQUATIONS

1.0 INTRODUCTION

The main focus of this paper will be to explore two different options of pricing the Death Benefit Product and the pros and cons of each pricing method under two alternative interest rate structures.

Let the **insurer** be the party who promises to pay future benefits and the **holder** be the party to whom the benefits would be paid. The insurer agrees to pay benefits that are contingent upon the holder's life. The insurer will charge a single premium in exchange for the promise to pay future benefits. Note that once the holder pays the premium, the insurer invests the entire amount in a separate, non-pooled investment account that compounds interest every year at rate r. The investment account is kept self-contained with the goal of assessing gains and losses and analyzing various pricing methods.

Assume that individual and group mortality can be approximated by using one of the standard actuarial life tables, which are based on the mortality experience of a population over a fairly short period of time.

2.0 DESCRIPTION OF PRODUCT

The Death Benefit Product pays a single, lump-sum payment at the end of the year of the holder's death to the holder's beneficiary. In order to determine the amount that the insurer should charge for this product, we must first take a closer look at the actual benefit paid.

Define the **death benefit**, denoted by $DB_t(Y)$, to be the lump-sum payment due t periods (where t is always an integer) from the purchase date ($t = 0$) to the beneficiary of the holder who purchased the product at age Y. By saying that the death benefit is payable at the end of the year of the holder's death, we mean that for a holder who dies after time $t = T - 1$ but on or before time $t = T$, $DB_T(Y)$ is paid at time T:

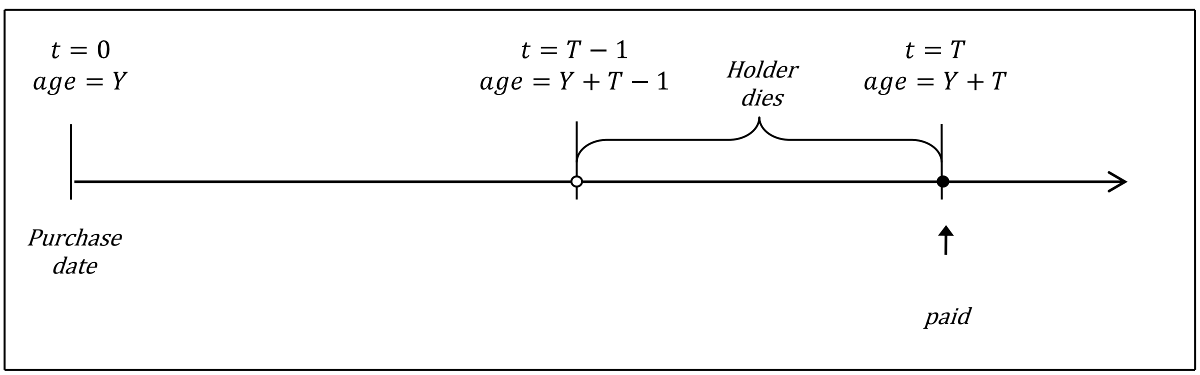

Figure 2. 1

In this illustration, we call the year in which the policy was purchased as "*year 1*" and the year in which the holder died as "*year T*". This connotation for identifying years will be used throughout the rest of the paper. In determining the single premium that will be charged for this product, all possible years of death will be considered.

3.0 PRICING METHODS UNDER A CONSTANT RATE OF RETURN

In this paper, we will analyze the pros and cons of two different pricing methods for determining the premium to charge for the death benefit. Let r be the rate of return for the current financial atmosphere. To begin, assume that r is also the interest applicable to all future years. In other words, we are assuming a constant rate of return of r in all valuations.

First, define the **premium**, denoted by $P_0(Y)$, to be the value of the single payment charged by the insurer to the holder age Y at time 0, the purchase date. The notation $P_t(Y)$ is used to represent the value of the invested premium, $P_0(Y)$, at time $t > 0$.

In order to develop the different pricing methods, we must first introduce some standard terms that are commonly used in actuarial practices.

As identified in *The Theory of Interest* (Kellison, 1991), the ***present value factor***, denoted by v (p. 10), that will be used to represent the time value of money under the concept of compound interest for our example with $i = r$ is defined as:

(a) $$v = (1 + r)^{-1}$$

Using the notation employed in *Actuarial Mathematics* (Bowers, Gerber, Hickman, Jones, & Nesbitt, 1997), let the symbol (x) denote a ***life-age-x*** and the symbol $T(x)$ to be the ***future lifetime of*** $(\boldsymbol{x})$ (p. 52). Bowers et al. identify the following terms which are used to make probability statements about $T(x)$ $(1997, \text{p. } 53)$:

(b) $${}_tq_x = \Pr[T\ x\ \leq t] \quad t \geq 0$$

(c) $${}_tp_x = 1 - {}_tq_x = \Pr\ T(x\ > t]\ \ t \geq 0$$

In other words, ${}_tq_x$ is the probability that (x) will die within the next t years, while ${}_tp_x$ is the probability that (x) will survive to age $(x + t)$.

As mentioned in the model development, we are assuming that the mortality of the holder can be approximated by a standard actuarial life table. There are many different types of actuarial life tables available, some of which have been developed to represent specific population traits (e.g. working class or health status). In order to price the product as accurately as possible, a life table should be chosen that represents the factors affecting the mortality of the individual holder.

Note that when pricing for a group of holders, a life table should be chosen that reflects the key factors affecting the mortality of the entire group. Keep in mind that for a very diverse population, there may be enough off-setting characteristics within the group (e.g. equal numbers of healthy versus non-healthy people) that make generic tables, or ones without specific traits, the best choice.

Once the appropriate life table is chosen, it can be used to provide the specific values for a number of different mortality factors. The basic design of a life table is constructed with a series of ${}_1q_x$, commonly written q_x (Bowers, Gerber, Hickman, Jones, & Nesbitt, 1997, p. 53)�, with increasing lives aged (x). Given the q_x's for all available (x) from the life table, we can easily obtain ${}_1p_x$, or p_x, by using identity (c) above with $t = 1$.

In order to calculate the p's and the q's when $t > 1$, we use the following variation of formula 3.4.2 derived on page 67 of Bowers et al. (1997) by starting at time x instead of time 0 and letting $x = t$:

$$\text{(d)} \qquad {}_tp_x = \;{}^{t-1}_{y=x}\, p_y$$

Lastly, we use the symbol ω to denote the age beyond which no human is expected to live, i.e. Pr $T(x > \omega] = 0$. We call ω the ***limiting age*** (Bowers, Gerber, Hickman, Jones, & Nesbitt, 1997, p. 63).

3.1 ACTUARIAL EQUIVALENCE (CONSTANT RATE OF RETURN)

The first method of pricing the Death Benefit Product sets the premium equal to the actuarial present value of the death benefit. We will call this single payment amount the actuarial present value (APV) premium and this method the APV method. In order to find the actuarial present value of the death benefit at the time of purchase, $t = 0$, we need to take into consideration three elements: (1) the amount paid if the holder dies in year t, (2) probability that the holder will die in year t, (3) the sum the present values of (1) $\times$ (2) for all possible value of $t > 0$.

Finding (1) is straightforward; the amount paid if the holder dies in year t is given by $DB_t\ Y$ in Equation2.1.

In order to find a formula for (2), we must apply a combination of actuarial notation and probability theory. For a holder initially age Y at $t = 0$, the probability that the holder dies in any year $t > 0$, thus triggering the payment of the death benefit at the end of year $Y + t$ is:

[probability holder age Y lives t years] $\times$ [probability holder age $Y + t$ dies before age $Y + t + 1$]

According to Bowers et al., this can be expressed using actuarial notation in the following formula for ***curtate-future-lifetime***, denoted by $K\ x$ with $x = Y$ (1997, p. 54):

$$\Pr\ K\ Y\ = t\ = {}_tp_Y\, q_{Y+t} \qquad where\ t = 0,1,2,\ldots$$

Recall that the death benefit is payable only when $t > 0$. Thus the probability-weighted value of the death benefit for a holder that lives t years and dies before time $t + 1$, or (1) × (2), is:

Equation 3. 1

$$DB_{t+1}\ Y\ \times\ {}_tp_Y\ q_{Y+t} \qquad where\ t = 1,2,3, \ldots$$

This would be the actuarial value of the death benefit when it is paid at a single point in time; however we are interested in pricing at the point in time when the product is sold to the holder. In order to find the value at $t = 0$, use the present value factor for $(t + 1)$-periods, v^{t+1}, to discount this amount back to the purchase date:

Equation 3. 2

$$DB_{t+1}\ Y\ \times\ {}_tp_Y\ q_{Y+t} \times v^{t+1} \qquad where\ t = 1,2,3, \ldots$$

The last step is to sum up the probability-weighted present values of the death benefit for all possible values of t. Since the premium is being priced at time 0 and it is possible for the holder who is currently age Y to live to age ω, the sum range is $0 \leq t \leq \omega - Y$. Thus the actuarial present value of the death benefit, and consequently the APV premium, is given by:

Equation 3. 3

$$P_0(Y) = \sum_{t=1}^{\omega-Y} DB_{t+1}(Y) \times\ {}_tp_Y\ q_{Y+t} \times v^{t+1}$$

Let's look at how this would work in practice:

Example 3.1: An insurer is offering the Death Benefit Product where the amount paid when the holder dies is a flat $1,000. Assume that a holder age 40 purchases the product from the insurer when the current constant rate of return is 4%. The insurer determines that the appropriate life table to use in approximating mortality is the Male RP-2000 Rates for a Non-Annuitant (The Society of Actuaries, 2000), which has a limiting age of 120. Suppose the insurer wishes to find the premium to charge under the APV method with a constant rate of return. We have the following values for this example:

$$DB_t\ Y\ = 1{,}000\ for\ ll\ t,\ \ Y = 40,\ r = 4\%,\ \omega = 120$$

Table 3. 1 (The Society of Actuaries, 2000)

t	(x)	q_x
0	40	0.00108
1	41	0.00114
…	…	…
79	119	0.40000
80	120	1.00000

Using the equations identified earlier for p_x, ${}_tp_x$, v^t and $DB_t\ Y$ we can obtain Table 3.2 from the given information (see Appendix A for a complete table of values):

Table 3. 2

t	(x)	p_{40+t}	${}_tp_{40}$	q_{40+t}	v^{t+1}	$DB_{t+1}\ 40$
0	40	0.99892	1.00000	0.00108	0.96154	1,000.00
1	41	0.99886	0.99892	0.00114	0.92456	1,000.00
2	42	0.99879	0.99778	0.00122	0.88900	1,000.00
…	…	…	…	…	…	…
9	49	0.99801	0.98722	0.00200	0.67556	1,000.00
10	50	0.99786	0.98525	0.00214	0.64958	1,000.00
11	51	0.99771	0.98315	0.00229	0.62460	1,000.00
12	52	0.99755	0.98090	0.00245	0.60057	1,000.00
…	…	…	…	…	…	…
80	120	0.00000	0.00000	1.00000	0.04172	1,000.00

We now have all the components needed to calculate the APV premium. Using Equation 3.3 on the Table 3.2 entries we get that:

$$P_0(40) = \sum_{t=1}^{80} DB_{t+1}\ 40\ \times\ {}_tp_{40}\, q_{40+t} \times v^{t+1} = 205.68$$

Notice that the APV premium charged of $205.68 is considerably less than the death benefit of $1,000 that is promised to be paid in any year of the agreement. The reason for this is twofold: (1) the holder has a very low probability of dying in the earliest years of the agreement and (2) upon receiving $P_0(40)$ from the holder the insurer invests the total amount at the constant rate of return 4%. The invested APV premium is reserved for paying the death benefit when it comes due. For every year that the holder lives, the amount available to pay the death benefit increases with interest.

At first it may not seem that the APV premium will ever accrue enough interest to cover the full death benefit, but Table 3.3 below shows that after 39 years the invested APV premium, $P_{t+1}(40)$, begins to exceed the death benefit payable, $DB_{t+1}(40)$ (see Appendix A for a complete table of values):

Table 3. 3

t	(x)	p_{40+t}	${}_tp_{40}$	q_{40+t}	${}_tp_{40}\ q_{40+t}$	v^{t+1}	$DB_{t+1}\ 40$	$P_{t+1}(40)$
0	40	0.99892	1.00000	0.00108	0.00108	0.96154	1,000.00	213.90
1	41	0.99886	0.99892	0.00114	0.00114	0.92456	1,000.00	222.46
2	42	0.99879	0.99778	0.00122	0.00121	0.88900	1,000.00	231.36
…	…	…	…	…	…	…	…	…
39	79	0.94553	0.71922	0.05447	0.03918	0.20829	1,000.00	987.46
40	80	0.93563	0.68004	0.06437	0.04377	0.20028	1,000.00	1,026.96
41	81	0.92796	0.63627	0.07204	0.04584	0.19257	1,000.00	1,068.03
…	…	…	…	…	…	…	…	…
80	120	0.00000	0.00000	1.00000	0.00000	0.04172	1,000.00	4,930.43

The probability of death, q_x, being low at first and increasing with time means that even though the invested APV premium is not enough to cover the death benefit in the early years, this isn't a cause for too much concern because the probability of having to pay early on is relatively small, less that 1%. It's more likely that the holder will die in the later years and by that time the invested APV premium will be sufficient to cover the death benefit, and likely even leave some profit for the insurer. As we can see in Table 3.3, there is a 68.004% chance that the holder will live to age 80, which is when the insurer first begins to earn money. There is a 6.437% chance of the holder age 80 dying before reaching age 81, thus the overall probability of having to pay the death benefit to an 80 year old who is currently age 40 is 4.377%.

We can obtain a loss distribution at the purchase date for the APV premium in Example 3.1 by plotting the present value of gain/loss in each year against the respective probability. Note that the probability of a gain or loss in any year is just the probability of dying in that year, q_x.

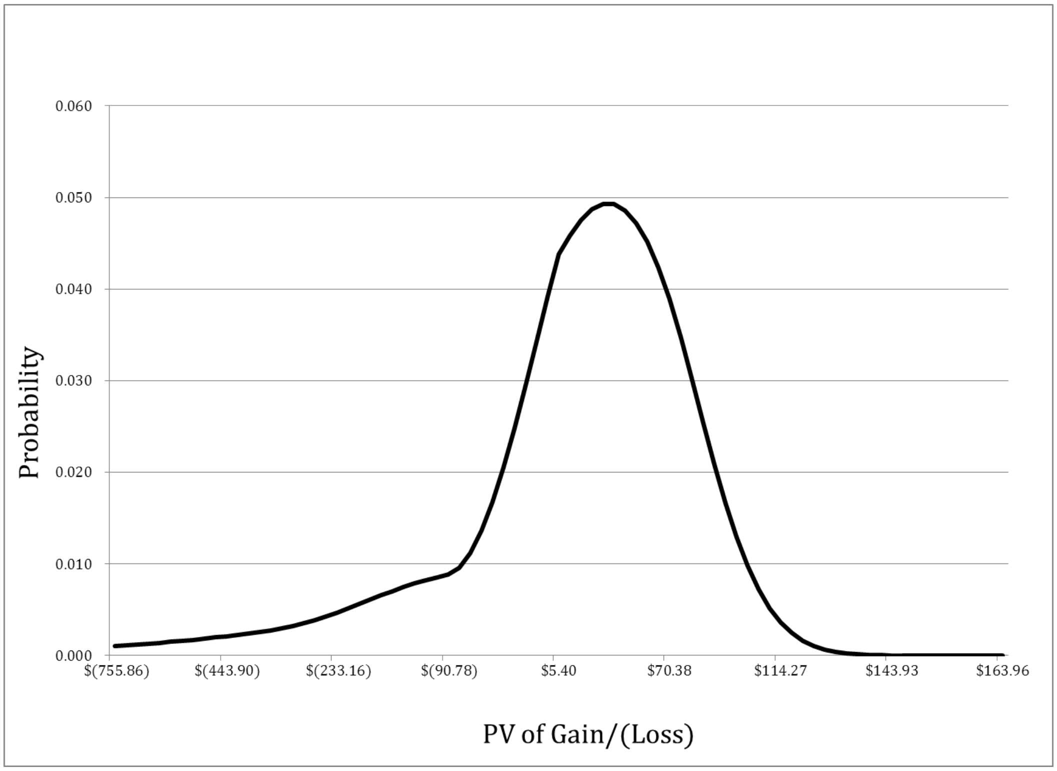

Figure 3. 1: Actuarial present value method loss distribution

A quick glance at Figure 3.1 shows us that the mean of the loss distribution is very close to $0. This analysis illustrates an interesting aspect of setting the premium equal to the actuarial present value of the death benefit. In the APV method, $P_0(Y)$ not only represents the actuarial present value of $DB_t\ Y$, but it also represents the expected, or average, amount of money that the insurer will pay to the holder (Bowers, Gerber, Hickman, Jones, & Nesbitt, 1997, p. 110). Thus on average the insurer will break-even under this pricing policy.

Suppose however that the insurer would like to have more control over their profits than the APV method allows. This could be achieved by limiting the amount of loss they are willing to incur with a certain probability, thus bringing us to the second pricing method.

3.2 VALUE AT RISK (CONSTANT RATE OF RETURN)

The second method of pricing the Death Benefit Product sets the premium equal to an amount that guarantees the insurer won't lose more than a preset maximum loss amount with certain confidence. We will call this single payment amount the value-at-risk (VaR) premium and this method the VaR method. In order to analyze the second pricing policy, we must first take a closer look at what is meant by the term "value-at-risk".

According to McNeil, Frey, & Embrects (Quantitative Risk Management: Concepts Techniques and Tools, 2005), given some confidence level $\alpha \in (0,1)$, the ***VaR*** of a portfolio at the confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is not larger than $(1-\alpha)$. Thus the VaR can be written (McNeil, Frey, & Embrechts, 2005):

$$VaR = \inf\{l \in \boldsymbol{R}: P(L > l) \leq 1-\alpha\}$$

Given that the goal of this method is to allow the insurer to have more control over their profits, they will pre-determine both the maximum loss amount l and confidence level α prior to determining the VaR premium.
In the Death Benefit Product, if a holder who is age Y on the purchase date lives for t years and dies before time $t+1$, we can write the loss at time $t+1$ as the difference between the invested premium and the death benefit payable:

Equation 3. 4

$$L = P_{t+1}(Y) - DB_{t+1}(Y) \qquad where\ t = 1,2,3,\ldots$$

Thus $P(L > l)$ becomes:

Equation 3. 5

$$P(P_{t+1}(Y) - DB_{t+1}(Y) > l) \qquad where\ t = 1,2,3,\ldots$$

Since we are looking to solve for the premium amount, we re-write this as:

Equation 3. 6

$$P(P_{t+1}(Y) > DB_{t+1}(Y) + l) \qquad where\ t = 1,2,3,\ldots$$

In Equation 3.6, it becomes clear that we are cushioning the invested premium amount to be greater than the death benefit by an amount equal to the maximum loss the insurer is willing to incur. In order to determine the value of Equation 3.6, note the following two observations (1) l is a fixed amount and has no associated probability and (2) the $DB_{t+1}(Y)$ is only payable if the holder lives to year t but dies before year $t+1$. Thus, the probability at time $t+1$ that the invested premium is bigger than the sum of the death benefit and the loss cushion (l) is just the probability of the holder dying between times t and $t+1$:

Equation 3. 7

$$P(P_{t+1}(Y) > DB_{t+1}(Y) + l) = {}_tp_Y\, q_{Y+t} \qquad where\ t = 1,2,3,\ldots$$

While Equation 3.7 is true for all value of $t = 1,2,3,\ldots$, the definition of VaR specifies that we are looking for the greatest lower bound of l such that the probability in Equation 3.7 is less than or equal to $1-\alpha$. In other words, we must find the greatest point in time, $T^*$where:

Equation 3. 8

$$\sum_{t=0}^{T^*} {}_{t}p_Y \, q_{Y+t} \leq 1-\alpha$$

In order to find T^*, we can set up an *indicator random variable* (Ross, 2002, p. 25)in our tables to be equal to 1 when Equation 3.8 is satisfied and then find the maximum corresponding point in time:

Equation 3. 9

$$\begin{cases} 1 & when \ \sum_{t=0}^{T} {}_{t}p_Y \, q_{Y+t} \leq 1-\alpha \\ 0 & otherwise \end{cases} \qquad T = 0,1,2,\ldots$$

Equation 3. 10

$$T^* = \max_{T}\{I_T \times T\}$$

Now that we have solved for T^*, we are ready to define the VaR premium. At time T^*, we want:

Equation 3. 11

$$P_{T^*} \; Y \; > DB_{T^*} \; Y \; + l$$

Assuming that the insurer sets the maximum loss to be the exact amount he is willing to lose, it suffices to set:

Equation 3. 12

$$P_{T^*} \; Y \; = DB_{T^*} \; Y \; + l$$

Equation 3.12 is the value that the invested VaR premium needs to be at time T^* in order to not incur a loss more than l with probability α. The value of the VaR premium at the purchase date is then just the value of the invested VaR premium at time T^*, discounted back for T^* years:

Equation 3. 13

$$P_0 \; Y \; = (DB_{T^*} \; Y \; + l) \times v^{T^*}$$

Once the insurer has selected α and l, we can say with α% confidence the maximum loss won't be more than \$ l when a VaR premium of $P_0 \; Y$ is charged to the holder age Y on the purchase date.

Next is an example of the VaR premium in practice:

Example 3.2: Assume the same scenario in Example 3.1 except that instead of charging the APV premium, the insurer wants to price the product so that they do not incur a loss of more than \$100 with confidence level 95%. Thus, we need to find the premium to charge under the VaR method with the following inputs:

$$DB_t \; Y \; = 1{,}000 \; for \; all \; t, \; Y = 40, \; r = 4\%, \; \omega = 120, \; l = -100, \; \alpha = 95\%$$

Table 3. 4 (The Society of Actuaries, 2000)

t	(x)	q_x
0	40	0.00108
1	41	0.00114
…	…	…
79	119	0.40000
80	120	1.00000

We are able to compute the value of the death benefit that is payable at any time t by simply inserting the necessary inputs into Equation 2.1. Since r and l are given, the most complicated component to find is the greatest time T^* such that the following equation is true:

$$\sum_{t=0}^{T^*} {}_tp_{40}\, q_{40+t} \leq 1 - .95$$

To find T^*, we construct a table of all possible probabilities ${}_tp_{40}\, q_{40+t}$ for $t = 0,1,2,3,\ldots$ and then find the cumulative probabilities. Once we have these amounts, the indicator function I_T will return a value of 1 for all cumulative probabilities that are less than or equal to .05 (see Appendix B for a complete table of values):

Table 3. 5

t/T	(x)	${}_tp_{40}$	q_{40+t}	${}_tp_{40}\, q_{40+t}$	$\sum_{t=0}^{T} {}_tp_{40}\, q_{40+t}$	I_T
0	40	1.00000	0.00108	0.00108	0.00108	1
1	41	0.99892	0.00114	0.00114	0.00222	1
…	…	…	…	…	…	…
19	59	0.95967	0.00441	0.00424	0.04456	1
20	60	0.95544	0.00488	0.00466	0.04922	1
21	61	0.95078	0.00538	0.00512	0.05434	0
…	…	…	…	…	…	…
80	120	0.00000	1.00000	0.00000	1.00000	0

Using the information in Table 3.5, we can find T^* by Equation 3.10:

$$T^* = \max_T\{I_T \times T\} = \max \ 1\times 0 \ , \ 1\times 2 \ , \ 1\times 3 \ , \dots , \ 1\times 19 \ , \ 1\times 20 \ , \ 0\times 21 \ , \ 0\times 22 \ , \dots , \ 0\times 80$$

$$T^* = \max \ 0,1,2,3,\dots,19,20,0,0,\dots,0 \ = 20$$

We have successfully solved for T^* and found its value to be 20. It is also known that $DB_{20} \ 40$ is $1,000. The last step is to put the calculated values for T^* and $DB_{20} \ 40$ into Equation 3.13 and solve for the VaR premium on the purchase date ($t = 0$):

$$P_0 \ 40 \ = \ DB_{20} \ 40 \ - 100 \ \times v^{20} = 394.95$$

Thus, with 95% confidence the maximum loss won't be more than $100 when the VaR premium of $394.95 is charged.

To verify that the VaR premium is working properly, let's take a look at Table 3.6 which includes the gain/loss incurred in every year along with the corresponding cumulative probabilities used in determining whether or not the $1 - \alpha$ limit is met (see Appendix B for a complete table of values):

Table 3. 6

t/T	$(\boldsymbol{x})$	$\sum_{t=0}^{T} {}_t p_{40}\, q_{40+t}$	**Confidence Level**	$\boldsymbol{DB_{t+1} \ 40}$	$\boldsymbol{P_{t+1}(40)}$	**Gain/(Loss)**
0	40	0.00108	99.89%	1,000.00	410.75	(589.25)
1	41	0.00222	99.78%	1,000.00	427.18	(572.82)
2	42	0.00343	99.66%	1,000.00	444.27	(555.73)
…	…	…		…	…	…
19	59	0.04456	95.54%	1,000.00	865.38	(134.62)
20	60	0.04922	95.08%	1,000.00	900.00	(100.00)
21	61	0.05434	94.57%	1,000.00	936.00	(64.00)
…	…	…		…	…	…
80	120	1.00000	0.00%	1,000.00	9,467.66	8,467.66

Table 3.6 shows that if the holder survives 20 years and then dies in the following year, the death benefit payable at the end of the 20^{th} year is $1,000. Furthermore, the invested VaR premium at the time the death benefit is payable is $900, which implies that there would be a loss to the insurer of $100. The cumulative probability of the insurer incurring a loss amount of $100 is 4.92%. Once the holder survives to at least age 60, the insurer will never

incur a loss greater than $100. Thus with confidence level 95.08%, the maximum loss that the insurer will sustain is $100.

We can obtain a loss distribution at the purchase date for the VaR premium in Example 3.2 by plotting the present value of gain/loss in each year against the respective probability. Note that as before, the probability of a gain or loss in any year is just the probability of dying in that year, q_x.

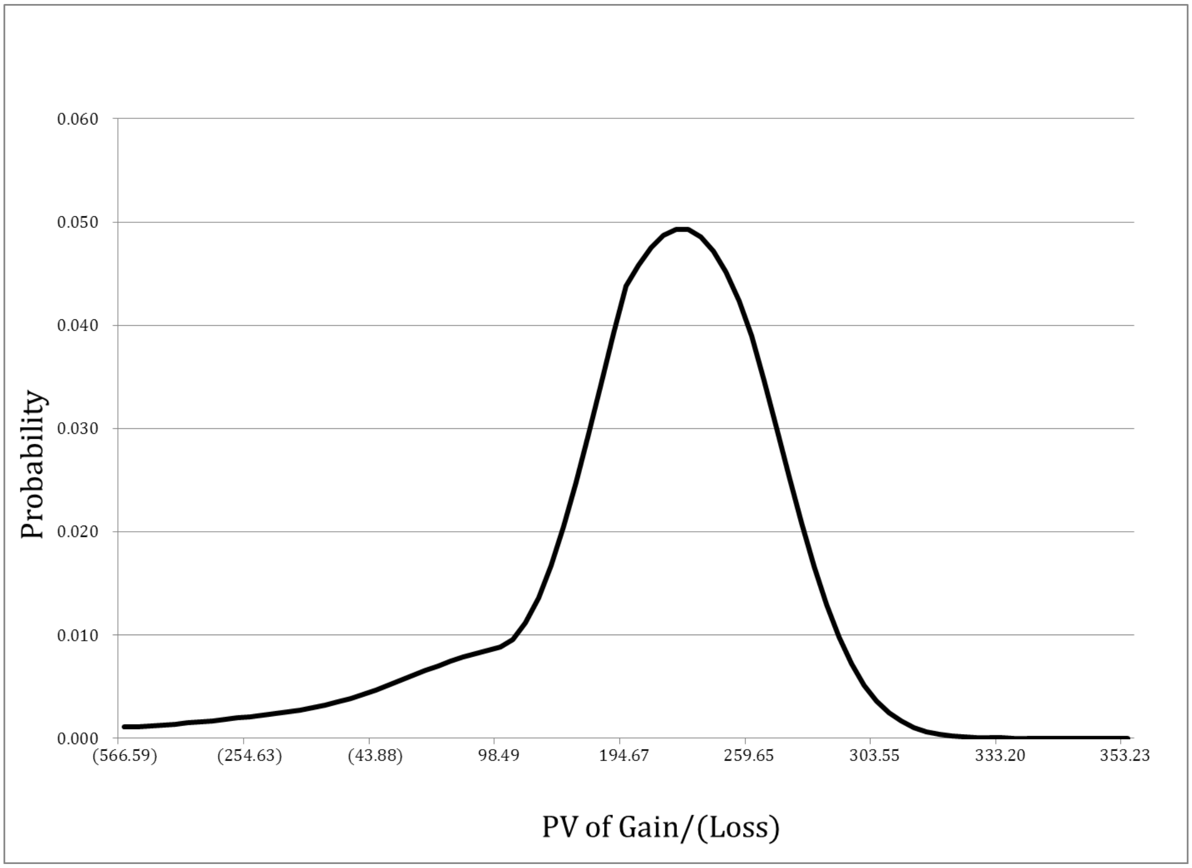

Figure 3. 2: Value at risk method loss distribution

A quick glance at Figure 3.2 shows that the mean of the Value at Risk loss distribution is much greater than $0, which was the mean in the Actuarial Equivalence loss distribution. This means that the insurer is much more likely to make a profit using the VaR premium policy as opposed to the APV premium policy.

Furthermore, pricing under the VaR method gives the insurer greater control over those profits by allowing them to choose l and α. For any given holder, the product could have a wide variety of VaR premiums depending on how much loss the insurer wants to prevent. However, these advantages do not come without cost.

There is a significant difference in the premiums charged under the APV and VaR methods. In Example 3.1 we found the APV premium to be \$205.68. This is considerably less than the VaR premium for the same product, which was \$394.95. In choosing to price under the VaR method, the insurer is more likely to make a profit but it comes at a disadvantage to the holder who now must pay for the insurer's financial security.

Nevertheless, the ability of the insurer to choose l and α also means the VaR method can be tailored to lower the premium. If the insurer thinks potential buyers will not purchase the Death Benefit Product at the elevated cost, then they can either lower the maximum loss they are willing to incur, decrease the confidence level, or do a combination of both. The flexibility of the VaR premium and the power it gives the insurer to control the maximum loss amount makes this method in the end more desirable than the APV premium method for pricing the Death Benefit Product.

4.0 PRICING METHODS UNDER COX-INGERSOLL-ROSS INTEREST RATE MODEL

Up until now we have made the simplification that all premiums were invested at the constant rate of return r. Consequently, r was also used to find the present value of the death benefit payments. While analyzing the results under a constant rate of return has meaningful conclusions, it is not realistic to assume that insurers will invest premiums at the constant rate of return for the lifetime of the product. In this chapter, we look at how using an interest rate process to model future rates of return will affect the insurer's decision on what pricing method to choose.

We use the Cox-Ingersoll-Ross (CIR) (Cox, Ingersoll, & Ross, 1985) model for generating the interest rate process because of the desirable feature that it does not allow rates to become negative. According to *Options, Futures, & Other Derivatives* (Hull, 2000, p. 570) where dz is a standard Weiner process and the parameters a, b and σ are constants, the discrete CIR model for the interest rate process $r(t)$ can be written as:

Equation 4. 1

$$dr(t) = a\left(b - r(t)\right)dt + \sigma\sqrt{r(t)}\,dz$$

The parameter a is the pull-back factor that helps prevent the interest rate from becoming negative, b is the long-term equilibrium of the mean reverting spot rate process and σ is the spot interest rate volatility. Equation 4.1 is used to model the change in the interest rate from an initial rate of r_0 at time 0 due to the randomness of the Weiner process. Note that r_0 here is equal to the constant rate of return r defined in Chapter 1 and used throughout Chapter 3.

The value of a zero-coupon bond at time t that matures for a value of 1 at time T is denoted as $V(t,T)$ and can be derived from the CIR process as (Hull, 2000, p. 570):

$$V(t,T) = A(t,T)\,e^{-B(t,T)r(t)}$$

Where $A(t,T)$, $B(t,T)$ and γ are (Hull, 2000, p. 570):

$$A(t,T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{2ab/\sigma^2}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)+2\gamma}$$

Equation 4. 2

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

Since the insurer prices premiums at time 0 and all values are discounted back to this point in time, we will only need the following shortened versions of these formulas:

Equation 4. 3

$$V_0(t) = A_0(t)\, e^{-B_0(t)\, r(t)}$$

Equation 4. 4

$$A_0(t) = \left[\frac{2\gamma e^{(a+\gamma)t/2}}{(\gamma + a)(e^{\gamma t} - 1) + 2\gamma}\right]^{2ab/\sigma^2}$$

Equation 4. 5

$$B_0(t) = \frac{2(e^{\gamma t} - 1)}{(\gamma + a)(e^{\gamma t} - 1) + 2\gamma}$$

Using equations 4.2 - 4.5 with the proper parameters, we can find a table of $V_0(t)$ values for all future years t that the holder is expected to live. The value of each $V_0(t)$ is simply the price today that a person would pay for a value of 1 at time t. Thus, using the present value factor described in Chapter 3 we can find a single corresponding spot interest rate $s_0(t)$ for each t that would yield the same result:

Equation 4. 6

$$V_0(t) = (1 + s_0(t))^{-t}$$

Solving for $s_0(t)$ gives:

Equation 4. 7

$$s_0(t) = V_0(t)^{t} - 1$$

The resulting values of $s_0(t)$ for each t compose a yield curve of rates that are defined by the CIR model for spot interest rates. In practice, each $s_0(t)$ is the annual effective yield rate used to discount a value at time t to time 0. Consequently each year t will have its own identifying value of $s_0(t)$ that is used only for discounting payments that take place at time t.

Also, because of the randomness present in the derivation of the CIR process, it is expected that a different yield curve will result with each trial of the model. Thus when analyzing results under the CIR process, it helps to look at multiple derivations of the yield curve.

Now let's revisit the pricing methods derived in Chapter 3, but use the yield curve defined from the CIR model instead of a constant rate of return.

4.1 ACTUARIAL EQUIVALENCE (COX-INGERSOLL-ROSS INTEREST RATE MODEL)

In Equation 3.3, we found $P_0(Y) = \sum_{t=1}^{\omega-Y} DB_{t+1}(Y) \times {}_tp_Y\, q_{Y+t} \times v^{t+1}$ to be the value of the APV premium for the Death Benefit Product under a constant rate of return. The only term in this equation affected by using the yield curve instead of a constant interest rate is the present value factor v. In Equation 4.6 we showed that $V_0(t)$ is the present value factor for the yield curve spot rate $s_0(t)$ at time t. Thus the pricing formula for the APV premium under the yield curve can be written as:

Equation 4. 8

$$P_0(Y)_{yield\ curve} = \sum_{t=1}^{\omega-Y} DB_{t+1}(Y) \times {}_tp_Y\ q_{Y+t} \times V_0(t+1)$$

Recall that $V_0(t+1)$ is used instead of $V_0(t)$ to model the fact that death benefits are paid at the end of the year of death for a person who survives to time t.

Since this switch from a constant rate of return to the yield curve doesn't affect any of the other factors, we are ready to take a look at how this works in practice. In order to analyze the impact of a random interest rate on the Death Benefit Product when the APV premium is charged, we will take a look at the hypothetical reserves as time progresses through the lifetime of the holder.

For the end of year 1 the hypothetical reserve, denoted HR_1, is the amount available to pay benefits less the expected benefit payments:

Equation 4. 9

$$HR_1 = P_0(Y)_{yield\ curve} \times e^{r(0)} - DB_1(Y) \times q_Y$$

Note here that $P_0(Y)_{yield\ curve} \times e^{r(0)}$ is the APV premium collected at time 0 increased for one year of interest at the applicable CIR interest rate; this is the amount available to pay benefits. The term $DB_1(Y) \times q_Y$ expresses the benefit payable times the portion of the person expected to die; this is the amount of expected benefit payments.

It may seem odd to think of a portion of a person dying, but generally speaking an insurer will have sold the Death Benefit Product to a large population of holders. Thus, at the end of each year we would expect a portion of the population of holders to die. This concept of a portion of a population dying is being applied to a single person, who can be thought of as representing an entire population.

Getting back to the hypothetical reserve calculations, the amount left over after paying the expected benefits in year 1, or HR_1 is the amount available at the beginning of year 2 for paying the expected benefits in following year. Thus we can find the hypothetical reserves at the end of year 2, HR_2, as follows:

Equation 4. 10

$$HR_2 = HR_1 \times e^{r(1)} - DB_2(Y) \times q_{Y+1}$$

As a result, we have the following recursive formula for the hypothetical reserves in year t, HR_t:

Equation 4. 11

$$HR_t = HR_{t-1} \times e^{r(t-1)} - DB_t(Y) \times q_{Y+(t-1)}$$

Each reserve calculation is found using the applicable CIR interest rate for that year, thus the distribution of the hypothetical reserves over time and many simulations will provide a clear picture of how varying interest rates impacts the insurer's position in the Death Benefit Product when charging the APV premium. Since the APV premium method is being compared to the VaR premium, we ultimately want to compare the hypothetical reserves under the two alternatives.

The VaR premium does not depend as heavily on present value factors as the APV premium, thus we must take a close look at how the yield curve changes the calculation of the VaR premium.

4.2 VALUE AT RISK (COX-INGERSOLL-ROSS INTEREST RATE MODEL)

The first time a present value factor appears in the pricing of VaR premium is in Equation 3.13 which states $P_0(Y) = (DB_{T^*}(Y) + l) \times v^{T^*}$ and only the present value factor associated with time T^* is used. The determination of T^* is dependent solely upon the survival probabilities and thus remains unchanged in the yield curve calculation of the VaR premium.

The present value factor associated with time T^* in the yield curve is Equation 4.6 evaluated at time T^*:

Equation 4. 12

$$V_0(T^*) = (1 + s_0(T^*))^{-T^*}$$

Thus the pricing formula for the VaR premium under the yield curve can be written as:

Equation 4. 13

$$P_0(Y)_{yield\ curve} = (DB_{T^*}(Y) + l) \times V_0(T^*)$$

The calculation of the VaR premium is significantly different than the APV premium. However, the process for finding the hypothetical reserves under the VaR method is exactly the same as finding the hypothetical reserves under the APV method if we assume the same mortality table and yield curve since both premiums are collected at time 0.

In order to be able to compare the hypothetical reserves under the two pricing methods, we will use the following example.

Example 4.1: Assume the same situation as presented in Example 3.2 but suppose now the insurer wishes to find the premium to charge under the VaR method using the CIR process for modeling interest rates. The yield curve will be generated with parameter values $a = 6.46\%$, $b = 4.02\%$, and $\sigma = 6.51\%$ (Chen & Scott, 2003, p. 160) and $r_0 = 4\%$.

Next, 1,000 simulations of the CIR process are used to find 1,000 different yield curves and subsequently 1,000 different values of each present value factor $V_0(t)$ for $t = 0,1, \ldots 80$. For each CIR trial, the following values were calculated and recorded: the APV premium according to Equation 4.8, the VaR premium according to Equation 4.13, and the hypothetical reserves each year, HR_t for $t = 0,1, \ldots 80$. Below are the distributions of the hypothetical reserves for years 1 through 5 under both pricing methods for 1,000 trials and some basic statistics:

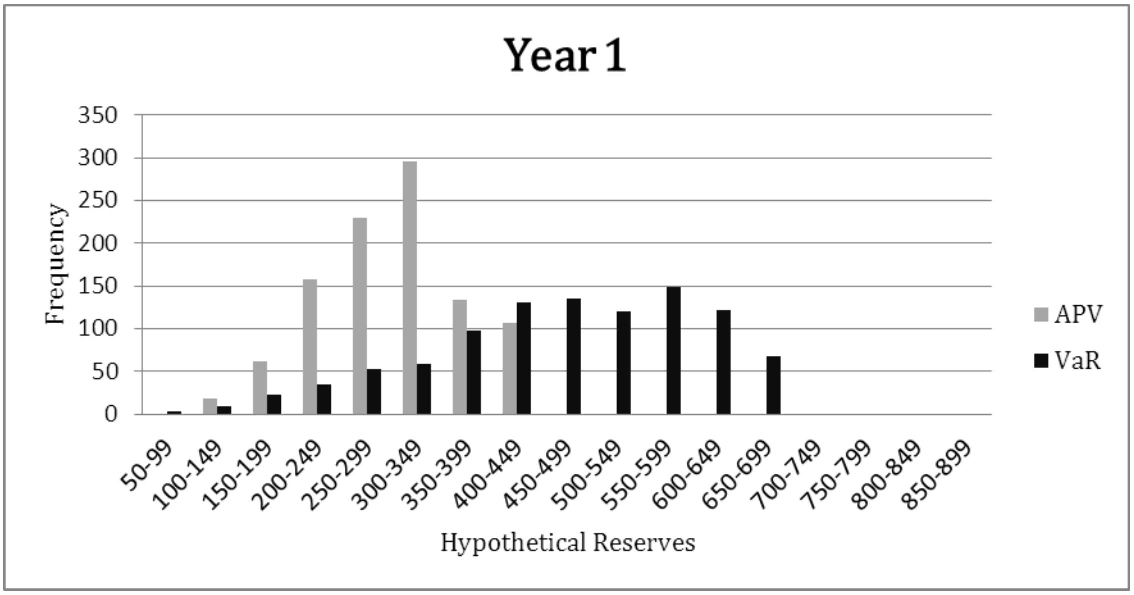

Figure 4. 1: Hypothetical reserves under both methods after 1 year

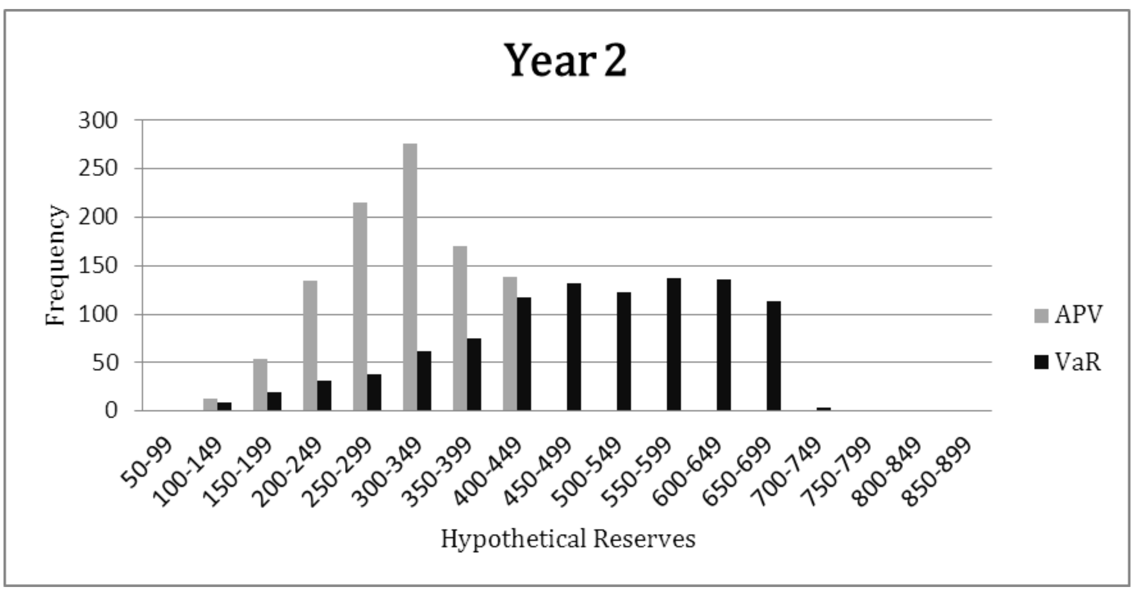

Figure 4. 2: Hypothetical reserves under both methods after 2 years

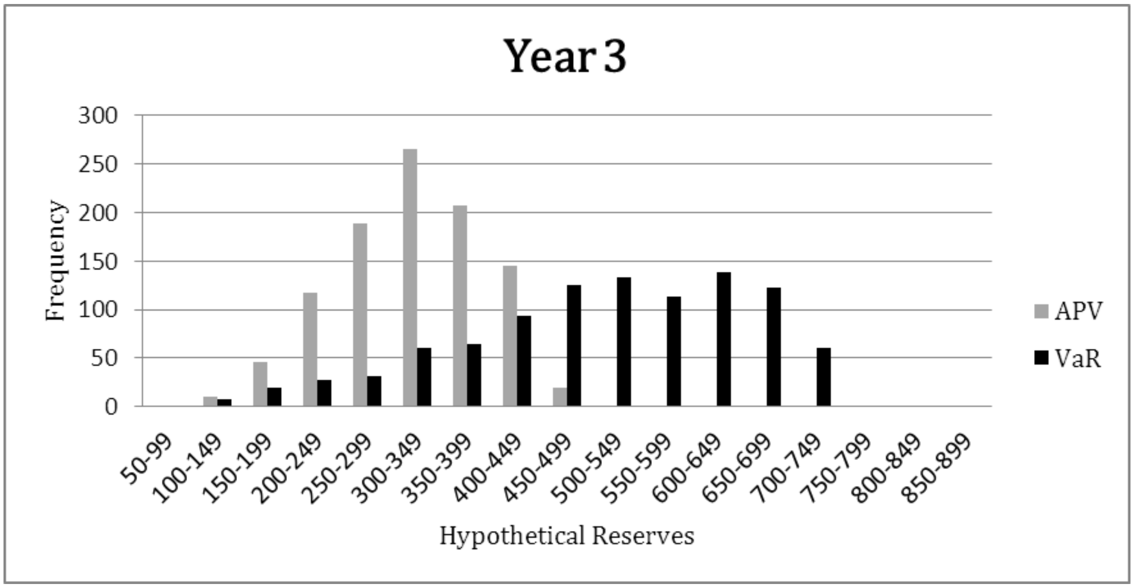

Figure 4. 3: Hypothetical reserves under both methods after 3 years

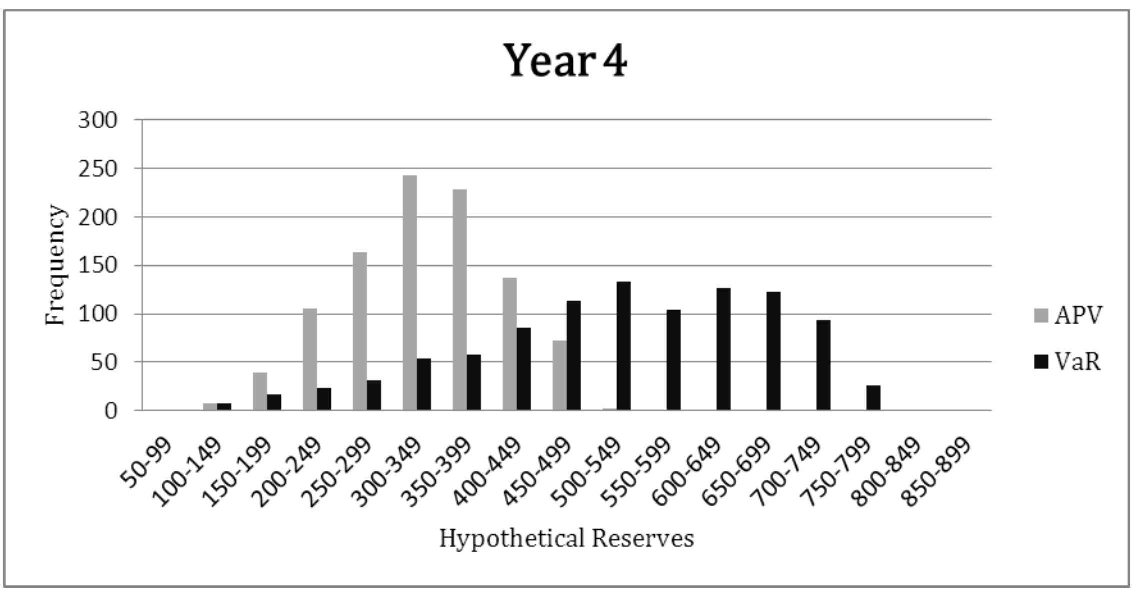

Figure 4. 4: Hypothetical reserves under both methods after 4 years

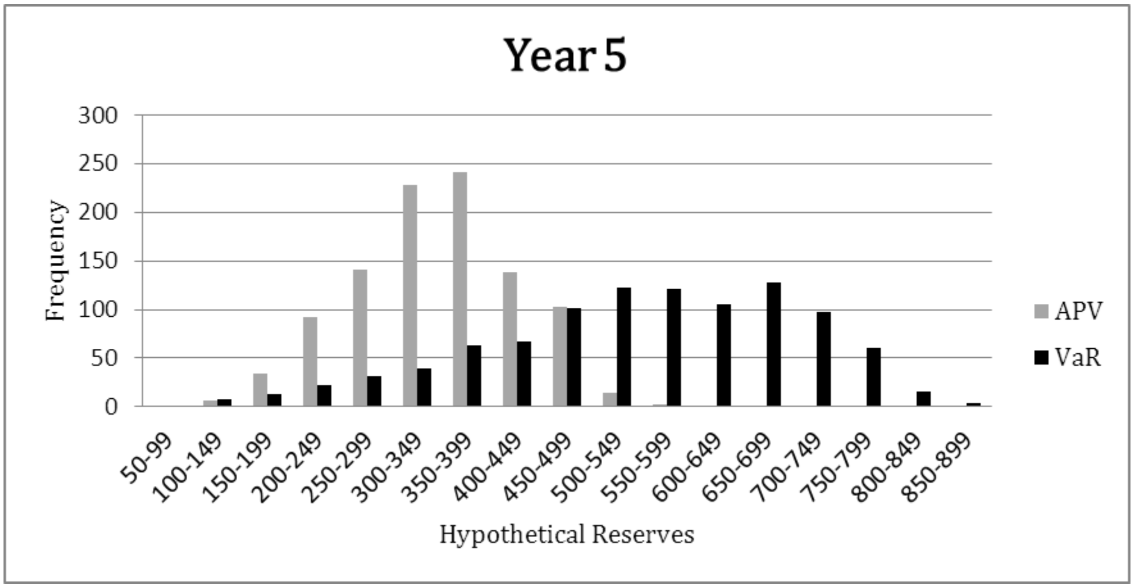

Figure 4. 5: Hypothetical reserves under both methods after 5 years

Table 4. 1

RESERVE STATISTICS	**Year 1**	**Year 2**	**Year 3**	**Year 4**	**Year 5**
APV Average	300.18	311.31	322.81	334.61	346.96
APV Standard Deviation	68.69	71.31	74.34	77.65	81.55
VaR Average	472.66	490.80	509.59	528.86	549.00
VaR Standard Deviation	130.69	135.74	141.27	147.21	153.80

While the distribution of the reserves under the APV method do not vary much in the first five years, the distribution of the reserves under the VaR method slowly begins to shift towards larger reserve values. However, the APV reserve amounts are much more concentrated than the VaR reserve amounts leading towards the conclusion that even though it possible to get higher reserve values under the VaR method, it's at the cost of a broader variance.

After 10 years the distributions have both shifted towards increased reserves amounts, but the shape of the graphs still look similar to those in the first five years:

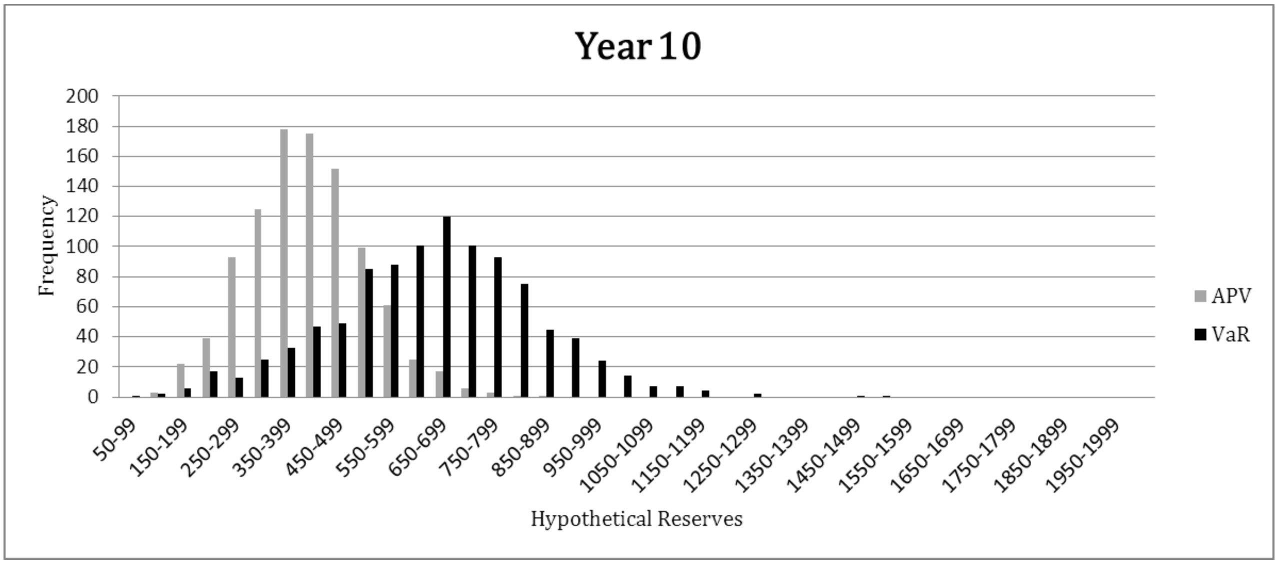

Figure 4. 6: Hypothetical reserves under both methods after 10 years

After 20 years we see a greater variance in the APV method reserves than before, but the VaR method reserves are still more dispersed:

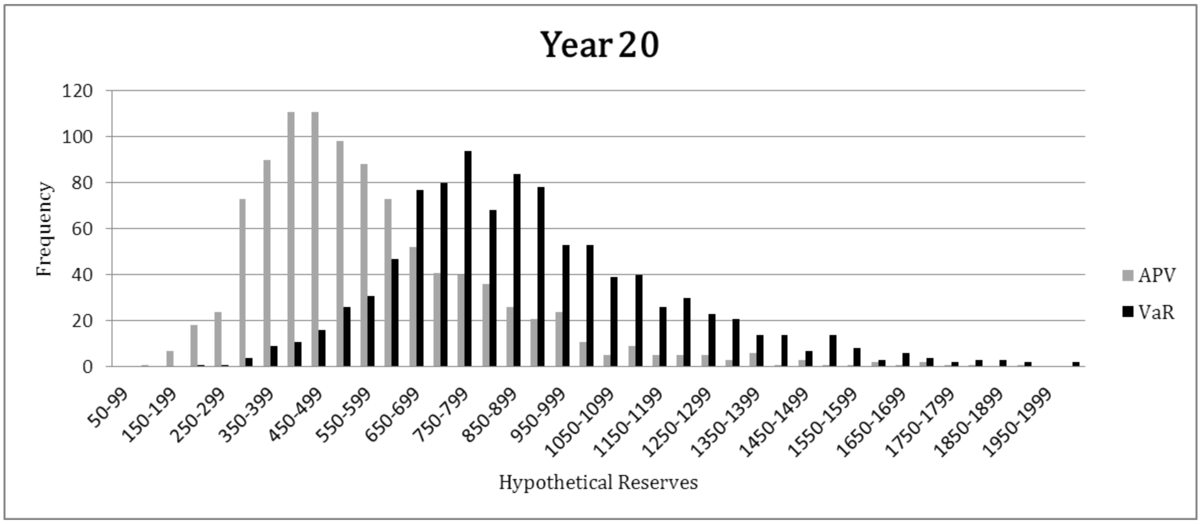

Figure 4. 7: Hypothetical reserves under both methods after 20 years

Table 4. 2

RESERVE STATISTICS	Year 10	Year 20
APV Average	414.65	594.98
APV Standard Deviation	112.38	270.51
VaR Average	657.57	950.88
VaR Standard Deviation	195.84	321.59

Finally, after 40 years there is much more variety in the reserve values under both methods. However, it still appears than an insurer on average will have higher reserve values under the VaR method than the APV method. In fact, all of the lower reserve values in Figure 4.8 are occupied by the APV method:

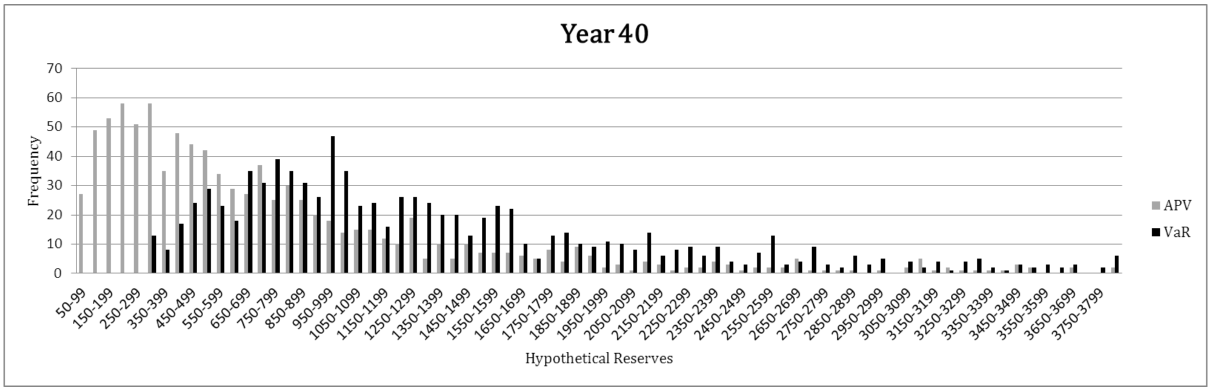

Figure 4. 8: Hypothetical reserves under both methods after 40 years

Table 4. 3

RESERVE STATISTICS	Year 40
APV Average	1,043.89
APV Standard Deviation	1,453.88
VaR Average	1,763.02
VaR Standard Deviation	1,680.41

The APV premium calculation involves the entire yield curve, thus variations due to the CIR process are essentially smoothed out over time. The VaR premium is dependent upon only one spot rate in the yield curve and that one rate has a large impact on the VaR premium. As a result, there is less variation in the 1,000 APV premiums generated from the simulated CIR processes than there were in the 1,000 VaR premiums. Since the starting point for the hypothetical reserves is the premium, we would expect to see less variation in the APV hypothetical reserves over time as compared to the VaR hypothetical reserves. Also, because the VaR premiums tend to be significantly higher than the APV premiums, as shown in Chapter 3, it would also make sense to see higher VaR hypothetical reserves amounts.

However, as time goes farther and farther into the future, these characteristics get diluted until inevitably the positive hypothetical reserves under both methods disappear and become negative. Since insurers would hopefully have new holders entering the population over time, the hypothetical reserve values in the early years are much more important to a solvent insurer than the hypothetical reserves far into the future. The addition of new money to the pool of funds available for paying benefits is crucial for the long term wealth of the insurer.

5.0 CONCLUSIONS

In conclusion, the APV method of pricing the Death Benefit Product under a constant interest rate provides a low cost to the holders and may help the insurer to be more competitive in the market place. Also, when we take into consideration interest rate fluctuations, the APV provides a rather predictable range of hypothetical reserves in the years immediately following the purchase date. The APV method would be desirable to insurers who want to price competitively and have little variance in their future reserves.

The VaR method of pricing the Death Benefit Product comes at a higher cost to the holders under a constant interest rate, but it provides the insurer with a way of controlling their gains and losses. Taking into consideration random interest rates will yield wide-ranging hypothetical reserves, but with average reserves values higher than the APV method during the early years. The VaR method would likely be used by insurers who have a Death Benefit Product that is in high demand (to justify holders willing to pay the increased price) and who are willing to have more variance in their hypothetical reserves. If an insurer anticipates a lot of new holders in the future years, then they may be more tolerable of the varied reserves.

Finally, both methods have their advantages and disadvantages and the method that would benefit a particular insurer depends upon the current premium market, the population of potential and/or existing holders, and their individual financial goals.

APPENDIX A

Table A: ACTUARIAL PRESENT VALUE

t	(x)	p_{40+t}	${}_tp_{40}$	q_{40+t}	${}_tp_{40}\ q_{40+t}$	v^{t+1}	$DB_{t+1}\ 40$	$P_{t+1}(40)$
0	40	0.99892	1.00000	0.00108	0.00108	0.96154	1,000.00	213.90
1	41	0.99886	0.99892	0.00114	0.00114	0.92456	1,000.00	222.46
2	42	0.99879	0.99778	0.00122	0.00121	0.88900	1,000.00	231.36
3	43	0.99870	0.99657	0.00130	0.00129	0.85480	1,000.00	240.61
4	44	0.99860	0.99527	0.00140	0.00139	0.82193	1,000.00	250.24
5	45	0.99849	0.99388	0.00151	0.00150	0.79031	1,000.00	260.25
6	46	0.99838	0.99238	0.00162	0.00160	0.75992	1,000.00	270.66
7	47	0.99827	0.99078	0.00173	0.00172	0.73069	1,000.00	281.48
8	48	0.99814	0.98906	0.00186	0.00184	0.70259	1,000.00	292.74
9	49	0.99801	0.98722	0.00200	0.00197	0.67556	1,000.00	304.45
10	50	0.99786	0.98525	0.00214	0.00211	0.64958	1,000.00	316.63
11	51	0.99771	0.98315	0.00229	0.00225	0.62460	1,000.00	329.29
12	52	0.99755	0.98090	0.00245	0.00240	0.60057	1,000.00	342.47
13	53	0.99738	0.97850	0.00262	0.00256	0.57748	1,000.00	356.17
14	54	0.99719	0.97593	0.00281	0.00274	0.55526	1,000.00	370.41
15	55	0.99697	0.97319	0.00303	0.00295	0.53391	1,000.00	385.23
16	56	0.99669	0.97024	0.00331	0.00321	0.51337	1,000.00	400.64
17	57	0.99637	0.96703	0.00363	0.00351	0.49363	1,000.00	416.66
18	58	0.99600	0.96352	0.00400	0.00385	0.47464	1,000.00	433.33
19	59	0.99559	0.95967	0.00441	0.00424	0.45639	1,000.00	450.66
20	60	0.99512	0.95544	0.00488	0.00466	0.43883	1,000.00	468.69
21	61	0.99462	0.95078	0.00538	0.00512	0.42196	1,000.00	487.44
22	62	0.99408	0.94566	0.00592	0.00560	0.40573	1,000.00	506.93
23	63	0.99353	0.94006	0.00647	0.00608	0.39012	1,000.00	527.21
24	64	0.99297	0.93398	0.00703	0.00656	0.37512	1,000.00	548.30
25	65	0.99243	0.92741	0.00757	0.00702	0.36069	1,000.00	570.23
26	66	0.99190	0.92039	0.00810	0.00745	0.34682	1,000.00	593.04
27	67	0.99140	0.91294	0.00860	0.00785	0.33348	1,000.00	616.76
28	68	0.99093	0.90509	0.00907	0.00821	0.32065	1,000.00	641.43
29	69	0.99049	0.89688	0.00951	0.00853	0.30832	1,000.00	667.09
30	70	0.99008	0.88835	0.00992	0.00881	0.29646	1,000.00	693.77
31	71	0.98909	0.87954	0.01091	0.00960	0.28506	1,000.00	721.52
32	72	0.98711	0.86994	0.01289	0.01122	0.27409	1,000.00	750.39
33	73	0.98414	0.85872	0.01586	0.01362	0.26355	1,000.00	780.40
34	74	0.98018	0.84510	0.01982	0.01675	0.25342	1,000.00	811.62
35	75	0.97523	0.82835	0.02477	0.02052	0.24367	1,000.00	844.08

Table A: ACTUARIAL PRESENT VALUE (continued)

t	(x)	p_{40+t}	${}_tp_{40}$	q_{40+t}	${}_tp_{40}\ q_{40+t}$	v^{t+1}	$DB_{t+1}\ 40$	$P_{t+1}(40)$
36	76	0.96929	0.80783	0.03071	0.02481	0.23430	1,000.00	877.85
37	77	0.96236	0.78302	0.03764	0.02947	0.22529	1,000.00	912.96
38	78	0.95444	0.75355	0.04556	0.03433	0.21662	1,000.00	949.48
39	79	0.94553	0.71922	0.05447	0.03918	0.20829	1,000.00	987.46
40	80	0.93563	0.68004	0.06437	0.04377	0.20028	1,000.00	1,026.96
41	81	0.92796	0.63627	0.07204	0.04584	0.19257	1,000.00	1,068.03
42	82	0.91951	0.59043	0.08049	0.04752	0.18517	1,000.00	1,110.75
43	83	0.91028	0.54291	0.08972	0.04871	0.17805	1,000.00	1,155.18
44	84	0.90022	0.49420	0.09978	0.04931	0.17120	1,000.00	1,201.39
45	85	0.88924	0.44489	0.11076	0.04927	0.16461	1,000.00	1,249.45
46	86	0.87720	0.39562	0.12280	0.04858	0.15828	1,000.00	1,299.43
47	87	0.86396	0.34704	0.13604	0.04721	0.15219	1,000.00	1,351.40
48	88	0.84941	0.29982	0.15059	0.04515	0.14634	1,000.00	1,405.46
49	89	0.83358	0.25467	0.16642	0.04238	0.14071	1,000.00	1,461.68
50	90	0.81659	0.21229	0.18341	0.03894	0.13530	1,000.00	1,520.14
51	91	0.80023	0.17336	0.19977	0.03463	0.13010	1,000.00	1,580.95
52	92	0.78340	0.13872	0.21661	0.03005	0.12509	1,000.00	1,644.19
53	93	0.76634	0.10868	0.23366	0.02539	0.12028	1,000.00	1,709.96
54	94	0.74931	0.08328	0.25069	0.02088	0.11566	1,000.00	1,778.35
55	95	0.73251	0.06240	0.26749	0.01669	0.11121	1,000.00	1,849.49
56	96	0.71610	0.04571	0.28391	0.01298	0.10693	1,000.00	1,923.47
57	97	0.70015	0.03273	0.29985	0.00982	0.10282	1,000.00	2,000.41
58	98	0.68470	0.02292	0.31530	0.00723	0.09886	1,000.00	2,080.42
59	99	0.66979	0.01569	0.33021	0.00518	0.09506	1,000.00	2,163.64
60	100	0.65544	0.01051	0.34456	0.00362	0.09140	1,000.00	2,250.18
61	101	0.64137	0.00689	0.35863	0.00247	0.08789	1,000.00	2,340.19
62	102	0.62832	0.00442	0.37169	0.00164	0.08451	1,000.00	2,433.80
63	103	0.61696	0.00278	0.38304	0.00106	0.08126	1,000.00	2,531.15
64	104	0.60800	0.00171	0.39200	0.00067	0.07813	1,000.00	2,632.40
65	105	0.60211	0.00104	0.39789	0.00041	0.07513	1,000.00	2,737.69
66	106	0.60000	0.00063	0.40000	0.00025	0.07224	1,000.00	2,847.20
67	107	0.60000	0.00038	0.40000	0.00015	0.06946	1,000.00	2,961.09
68	108	0.60000	0.00023	0.40000	0.00009	0.06679	1,000.00	3,079.53
69	109	0.60000	0.00014	0.40000	0.00005	0.06422	1,000.00	3,202.71
70	110	0.60000	0.00008	0.40000	0.00003	0.06175	1,000.00	3,330.82
71	111	0.60000	0.00005	0.40000	0.00002	0.05937	1,000.00	3,464.06
72	112	0.60000	0.00003	0.40000	0.00001	0.05709	1,000.00	3,602.62
73	113	0.60000	0.00002	0.40000	0.00001	0.05490	1,000.00	3,746.72
74	114	0.60000	0.00001	0.40000	0.00000	0.05278	1,000.00	3,896.59
75	115	0.60000	0.00001	0.40000	0.00000	0.05075	1,000.00	4,052.46
76	116	0.60000	0.00000	0.40000	0.00000	0.04880	1,000.00	4,214.55
77	117	0.60000	0.00000	0.40000	0.00000	0.04692	1,000.00	4,383.14
78	118	0.60000	0.00000	0.40000	0.00000	0.04512	1,000.00	4,558.46
79	119	0.60000	0.00000	0.40000	0.00000	0.04338	1,000.00	4,740.80
80	120	0.00000	0.00000	1.00000	0.00000	0.04172	1,000.00	4,930.43

APPENDIX B

Table B: VALUE AT RISK

t	(x)	${}_tp_{40}$	q_{40+t}	${}_tp_{40}\ q_{40+t}$	$^{T}_{t=0}\ {}_tp_{40}q_{40+t}$	Confidence Level	$DB_{t+1}\ 40$	$P_{t+1}(40)$	Gain / (Loss)	I_T
0	40	1.00000	0.00108	0.00108	0.00108	99.89%	1,000.00	410.75	(589.25)	1
1	41	0.99892	0.00114	0.00114	0.00222	99.78%	1,000.00	427.18	(572.82)	1
2	42	0.99778	0.00122	0.00121	0.00343	99.66%	1,000.00	444.27	(555.73)	1
3	43	0.99657	0.00130	0.00129	0.00473	99.53%	1,000.00	462.04	(537.96)	1
4	44	0.99527	0.00140	0.00139	0.00612	99.39%	1,000.00	480.52	(519.48)	1
5	45	0.99388	0.00151	0.00150	0.00762	99.24%	1,000.00	499.74	(500.26)	1
6	46	0.99238	0.00162	0.00160	0.00922	99.08%	1,000.00	519.73	(480.27)	1
7	47	0.99078	0.00173	0.00172	0.01094	98.91%	1,000.00	540.52	(459.48)	1
8	48	0.98906	0.00186	0.00184	0.01278	98.72%	1,000.00	562.14	(437.86)	1
9	49	0.98722	0.00200	0.00197	0.01475	98.53%	1,000.00	584.62	(415.38)	1
10	50	0.98525	0.00214	0.00211	0.01685	98.31%	1,000.00	608.01	(391.99)	1
11	51	0.98315	0.00229	0.00225	0.01910	98.09%	1,000.00	632.33	(367.67)	1
12	52	0.98090	0.00245	0.00240	0.02150	97.85%	1,000.00	657.62	(342.38)	1
13	53	0.97850	0.00262	0.00256	0.02407	97.59%	1,000.00	683.93	(316.07)	1
14	54	0.97593	0.00281	0.00274	0.02681	97.32%	1,000.00	711.28	(288.72)	1
15	55	0.97319	0.00303	0.00295	0.02976	97.02%	1,000.00	739.73	(260.27)	1
16	56	0.97024	0.00331	0.00321	0.03297	96.70%	1,000.00	769.32	(230.68)	1
17	57	0.96703	0.00363	0.00351	0.03648	96.35%	1,000.00	800.10	(199.90)	1
18	58	0.96352	0.00400	0.00385	0.04033	95.97%	1,000.00	832.10	(167.90)	1
19	59	0.95967	0.00441	0.00424	0.04456	95.54%	1,000.00	865.38	(134.62)	1
20	60	0.95544	0.00488	0.00466	0.04922	95.08%	1,000.00	900.00	(100.00)	1
21	61	0.95078	0.00538	0.00512	0.05434	94.57%	1,000.00	936.00	(64.00)	0
22	62	0.94566	0.00592	0.00560	0.05994	94.01%	1,000.00	973.44	(26.56)	0
23	63	0.94006	0.00647	0.00608	0.06602	93.40%	1,000.00	1,012.38	12.38	0
24	64	0.93398	0.00703	0.00656	0.07259	92.74%	1,000.00	1,052.87	52.87	0
25	65	0.92741	0.00757	0.00702	0.07961	92.04%	1,000.00	1,094.99	94.99	0
26	66	0.92039	0.00810	0.00745	0.08706	91.29%	1,000.00	1,138.79	138.79	0
27	67	0.91294	0.00860	0.00785	0.09491	90.51%	1,000.00	1,184.34	184.34	0
28	68	0.90509	0.00907	0.00821	0.10312	89.69%	1,000.00	1,231.71	231.71	0
29	69	0.89688	0.00951	0.00853	0.11165	88.83%	1,000.00	1,280.98	280.98	0
30	70	0.88835	0.00992	0.00881	0.12046	87.95%	1,000.00	1,332.22	332.22	0
31	71	0.87954	0.01091	0.00960	0.13006	86.99%	1,000.00	1,385.51	385.51	0
32	72	0.86994	0.01289	0.01122	0.14128	85.87%	1,000.00	1,440.93	440.93	0
33	73	0.85872	0.01586	0.01362	0.15490	84.51%	1,000.00	1,498.57	498.57	0

Table B: VALUE AT RISK (continued)

t	(x)	${}_tp_{40}$	q_{40+t}	${}_tp_{40}\ q_{40+t}$	${}_{t=0}^{T}\ {}_tp_{40}q_{40+t}$	Confidence Level	$DB_{t+1}\ 40$	$P_{t+1}(40)$	Gain / (Loss)	I_T
34	74	0.84510	0.01982	0.01675	0.17165	82.84%	1,000.00	1,558.51	558.51	0
35	75	0.82835	0.02477	0.02052	0.19217	80.78%	1,000.00	1,620.85	620.85	0
36	76	0.80783	0.03071	0.02481	0.21698	78.30%	1,000.00	1,685.68	685.68	0
37	77	0.78302	0.03764	0.02947	0.24645	75.36%	1,000.00	1,753.11	753.11	0
38	78	0.75355	0.04556	0.03433	0.28078	71.92%	1,000.00	1,823.23	823.23	0
39	79	0.71922	0.05447	0.03918	0.31996	68.00%	1,000.00	1,896.16	896.16	0
40	80	0.68004	0.06437	0.04377	0.36373	63.63%	1,000.00	1,972.01	972.01	0
41	81	0.63627	0.07204	0.04584	0.40957	59.04%	1,000.00	2,050.89	1,050.89	0
42	82	0.59043	0.08049	0.04752	0.45709	54.29%	1,000.00	2,132.93	1,132.93	0
43	83	0.54291	0.08972	0.04871	0.50580	49.42%	1,000.00	2,218.24	1,218.24	0
44	84	0.49420	0.09978	0.04931	0.55511	44.49%	1,000.00	2,306.97	1,306.97	0
45	85	0.44489	0.11076	0.04927	0.60438	39.56%	1,000.00	2,399.25	1,399.25	0
46	86	0.39562	0.12280	0.04858	0.65296	34.70%	1,000.00	2,495.22	1,495.22	0
47	87	0.34704	0.13604	0.04721	0.70018	29.98%	1,000.00	2,595.03	1,595.03	0
48	88	0.29982	0.15059	0.04515	0.74533	25.47%	1,000.00	2,698.83	1,698.83	0
49	89	0.25467	0.16642	0.04238	0.78771	21.23%	1,000.00	2,806.79	1,806.79	0
50	90	0.21229	0.18341	0.03894	0.82664	17.34%	1,000.00	2,919.06	1,919.06	0
51	91	0.17336	0.19977	0.03463	0.86128	13.87%	1,000.00	3,035.82	2,035.82	0
52	92	0.13872	0.21661	0.03005	0.89132	10.87%	1,000.00	3,157.25	2,157.25	0
53	93	0.10868	0.23366	0.02539	0.91672	8.33%	1,000.00	3,283.54	2,283.54	0
54	94	0.08328	0.25069	0.02088	0.93760	6.24%	1,000.00	3,414.88	2,414.88	0
55	95	0.06240	0.26749	0.01669	0.95429	4.57%	1,000.00	3,551.48	2,551.48	0
56	96	0.04571	0.28391	0.01298	0.96727	3.27%	1,000.00	3,693.54	2,693.54	0
57	97	0.03273	0.29985	0.00982	0.97708	2.29%	1,000.00	3,841.28	2,841.28	0
58	98	0.02292	0.31530	0.00723	0.98431	1.57%	1,000.00	3,994.93	2,994.93	0
59	99	0.01569	0.33021	0.00518	0.98949	1.05%	1,000.00	4,154.73	3,154.73	0
60	100	0.01051	0.34456	0.00362	0.99311	0.69%	1,000.00	4,320.92	3,320.92	0
61	101	0.00689	0.35863	0.00247	0.99558	0.44%	1,000.00	4,493.76	3,493.76	0
62	102	0.00442	0.37169	0.00164	0.99722	0.28%	1,000.00	4,673.51	3,673.51	0
63	103	0.00278	0.38304	0.00106	0.99829	0.17%	1,000.00	4,860.45	3,860.45	0
64	104	0.00171	0.39200	0.00067	0.99896	0.10%	1,000.00	5,054.86	4,054.86	0
65	105	0.00104	0.39789	0.00041	0.99937	0.06%	1,000.00	5,257.06	4,257.06	0
66	106	0.00063	0.40000	0.00025	0.99962	0.04%	1,000.00	5,467.34	4,467.34	0
67	107	0.00038	0.40000	0.00015	0.99977	0.02%	1,000.00	5,686.03	4,686.03	0
68	108	0.00023	0.40000	0.00009	0.99986	0.01%	1,000.00	5,913.48	4,913.48	0
69	109	0.00014	0.40000	0.00005	0.99992	0.01%	1,000.00	6,150.01	5,150.01	0
70	110	0.00008	0.40000	0.00003	0.99995	0.00%	1,000.00	6,396.02	5,396.02	0
71	111	0.00005	0.40000	0.00002	0.99997	0.00%	1,000.00	6,651.86	5,651.86	0
72	112	0.00003	0.40000	0.00001	0.99998	0.00%	1,000.00	6,917.93	5,917.93	0
73	113	0.00002	0.40000	0.00001	0.99999	0.00%	1,000.00	7,194.65	6,194.65	0
74	114	0.00001	0.40000	0.00000	0.99999	0.00%	1,000.00	7,482.43	6,482.43	0
75	115	0.00001	0.40000	0.00000	1.00000	0.00%	1,000.00	7,781.73	6,781.73	0
76	116	0.00000	0.40000	0.00000	1.00000	0.00%	1,000.00	8,093.00	7,093.00	0
77	117	0.00000	0.40000	0.00000	1.00000	0.00%	1,000.00	8,416.72	7,416.72	0
78	118	0.00000	0.40000	0.00000	1.00000	0.00%	1,000.00	8,753.39	7,753.39	0
79	119	0.00000	0.40000	0.00000	1.00000	0.00%	1,000.00	9,103.52	8,103.52	0
80	120	0.00000	1.00000	0.00000	1.00000	0.00%	1,000.00	9,467.66	8,467.66	0

BIBLIOGRAPHY

Bowers, J. N., Gerber, H. U., Hickman, J. C., Jones, D. A., & Nesbitt, C. J. (1997). Actuarial Mathematics. Schaumburg: The Society of Actuaries.

Chen, R.-R., & Scott, L. (2003). Multi-Factor Cox-Ingersoll-Ross Models of the Term Structure: Estimates and Tests from a Kalman Filter Model. *Journal of Real Estate Finance and Economics* , 143-172.

Cox, C. J., Ingersoll, J. E., & Ross, S. A. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica* , 385-407.

Hull, J. C. (2000). *Options, Futures, & Other Derivatives.* Upper Saddle River: Prentice Hall.
Kellison, S. G. (1991). *The Theory of Interest.* Boston: Irwin/McGraw-Hill.

McNeil, A. J., Frey, R., & Embrechts, P. (2005). *Quantitative Risk Management: Concepts Techniques and Tools.* Princeton: Princeton University Press.

Ross, S. M. (2002). *Introduction to Probability Models.* San Diego: Academic Press.

The Society of Actuaries. (2000, July 1). *The RP-2000 Mortality Tables.* Retrieved March 28, 2010, from Society of Actuaries: http://www.soa.org/files/pdf/rp00_mortalitytables.pdf