ESSAYS ON DYNAMIC MATCHING MARKETS

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Submitted to the Graduate Faculty of

Arts and Sciences in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2009

UNIVERSITY OF PITTSBURGH

ARTS AND SCIENCES

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The static matching models have been applied to real-life markets such as hospital intern markets, school choice for public schools, kidney exchange for patients, and on-campus housing for college students. However, these markets inherently involve dynamic aspects. This dissertation introduces dynamic frameworks into representative matching models - two-sided matching markets and house allocation problems, and obtained policy implications that cannot be captured by static models.

The first two essays are devoted to two-sided matching models in which two-sided matching interactions occur repeatedly over time, such as the British hospital intern markets. In the first essay, we propose a concept of credible group stability and show that implementing a men-optimal stable matching in each period is credibly group-stable. The result holds for a women-optimal stable matching. Moreover, a sufficient condition for Pareto efficiency is given for finitely repeated markets. In the second essay, we examine another notion of one-shot group stability and prove its existence. Moreover, we investigate to what extent we can achieve coordination across time in the infinite horizon by using the one-shot group stability.

The third essay focuses on the house allocation problem - the problem of assigning indivisible goods, called "houses," to agents without monetary transfers. We introduce an overlapping structure of agents into the problem. This is motivated by the following: In the case of on-campus housing for college students, each year freshmen move in and graduating seniors leave. Each students stays on campus for a few years only. In terms of dynamic mechanism design, we examine two representative static mechanisms of serial dictatorship

(SD) and top trading cycles (TTC), both of which are based on an ordering of agents and give an agent with higher order an opportunity to obtain a better house. We show that for SD mechanisms, the ordering that favors existing tenants is better than the one that favors newcomers in terms of Pareto efficiency. Meanwhile, this result holds for TTC mechanisms under time-invariant preferences in terms of Pareto efficiency and strategy-proofness. We provide another dynamic mechanism that is strategy-proof and Pareto efficient.

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PREFACE

The author gratefully acknowledge the invaluable help and support of:

- M. Utku Ünver for introducing matching theory to me and being a steadfast advisor on all aspects of research and professional life.
- Ted Temzelides for encouraging me to take Prof. Utku's Matching Market Design course and supporting me.
- Andreas Blume for his teaching game theory and microeconomics and his service as a co-chair on the dissertation committee.
- Sourav Bhattacharya, Oliver Board, and Onur Kesten for their service on the dissertation committee
- Masaki Aoyagi for his support since I was a graduate student at Osaka University
- Sachiya and Hitomi Isomura, for their encouragement and financial support
- Chikako and Manaka Kurino, for their love, sacrifice, and enthusiasm for husband and father.
- my parents, Shigemori and Aiko Kurino, and my parents-in-law, Kiyoji and Kazuko Onishi, for their encouragement and financial support.
- God for giving me an opportunity to learn and research at the University of Pittsburgh

1.0 INTRODUCTION

Matching theory is a name referring to research areas that explicitly takes care of indivisibility in economic models. Indivisibility refers to goods such as jobs, dormitory rooms, seats in public schools and so on that are available only in discrete units. Matching (who is matched with whom or what goods) is determined through decentralized markets or a centralized clearinghouse. For commodities, the price is usually enough to induce an efficient matching (allocation), but many markets do not clear by price alone. In fact, in some labor markets, employers do not hire people who want to work at a given wage, but rather interview applicants carefully to find the best one. In addition, for some goods such as transplant organs for patients or seats in public schools, it is actually illegal to set prices for such transactions even though there are possible gains from trade. Finally, universities use various procedures to assign dormitory rooms to college students, each of which has different policy implications.

Matching theory has been extensively studied so that we can now apply the results to real-life markets such as American, British, and Japanese entry-level medical labor markets, school choice in New York and Boston, and kidney exchange for patients. However, the theory still needs to be further explored to understand strategic behavior in decentralized matching markets and dynamic interactions. In this dissertation, we focus on the dynamic aspects of the theory.

Two-sided matching models and house markets are the representative models in the matching theory. Gale and Shapley (1962) introduced the former in which agents are divided into two sides and each agent has preferences over those on the other side, and a suitable solution concept of *stability*. They also showed that a stable matching always exists and proved this result through a simple algorithm known as the *deferred acceptance algorithm*. On the other hand, Shapley and Scarf (1974) together with Gale introduced the latter that is an allocation and exchange of indivisible goods, conventionally called *houses*. The *housing market* consists of agents each of whom owns a house. They showed that such a market always has a strict core matching which is reached by a simple algorithm, called *Gale's top trading cycles algorithm*. The link between the two models were later discovered and explored by Balinski and Sönmez (1999), Ergin (2002), and Abdulkadiroğle and Sönmez (2003), among others.

This dissertation introduces a new dynamic framework for each of the two matching models, and provides theoretical explanations for observations seen in real-life markets that cannot be captured by a static framework. Previous studies exclusively focus on static models.

In developing the dynamic frameworks, we closely look at real-life markets. In the first two essays, presented in chapter 2 and 3, we build dynamic two-sided matching markets whose motivation is from the British entry-level medical labor markets. These markets involve graduating medical students and teaching hospitals. Students seek residency positions for both medical and surgical programs of hospitals - they have one for the first six months and another for the next six months. Teaching hospitals fill positions in both of these periods. The matching interaction repeats twice, and thus involves dynamics. However, this market has been modeled as a "static" models (Roth, 1991). A special clearinghouse based on the Gale and Shapley's deferred acceptance algorithm has been successfully used for the last forty years although it may involve dynamic instability. In chapter 3, we provide theoretical support for the use of the Gale and Shapley's algorithm by developing a new solution concept of *credible group stability*. In the second essay, presented in chapter 3, we investigate another solution concept of *one-shot group stability* which is introduced by Cobae, Temzelides, and Wright (2003) in the framework of random matching models of money.

In the third essay, presented in chapter 4, we develop a house allocation problem with a special dynamic structure of overlapping agents. This is motivated by on-campus housing assignment for college students. Each year freshmen move in and graduating seniors leave. Each student stays on campus for a few years only. In general, students are *overlapping*. A housing office, or a mechanism designer, needs to find a mechanism to assign houses to agents. Many universities in the United States use a variant of *random serial dictatorship* mechanism to allocate dormitory rooms. This mechanism randomly orders the agents and then run the *serial dictatorship (SD)* mechanism in which the first agent in the order is assigned her top choice, and the next agent is assigned her top choice among the remaining rooms, and so on. In real-life markets, the ordering is not entirely random, but rather depends on *seniority*. That is, existing tenants are favored over newcomers. This is used in Northwestern University, the University of Michigan, and the University of Pittsburgh, among others. The exiting literature takes the orderings as given and cannot justify the seniority-based mechanisms. We show that seniority-based serial dictatorship mechanism performs well in terms of Pareto efficiency. In addition, we investigate another static mechanism of *top trading cycles mechanism (TTC)* (Abdulkadiroğle and Sönmez, 1999) which is based on the Gale's top trading cycles algorithm and the serial dictatorship mechanism. This mechanism maintains the same properties of Pareto efficiency and strategy-proofness as the SD mechanism has, yet restores the individual rationality that the SD mechanism lacks in the sense that existing tenants are guaranteed to obtain a house that is at least as good as her occupied house. Under some mild preference restrictions, we show that the seniority-based TTC mechanism performs well in terms of Pareto efficiency and strategy-proofness. Moreover, we propose a new dynamic mechanism which is Pareto efficient and strategy-proof.

2.0 CREDIBILITY, EFFICIENCY, AND STABILITY: A THEORY OF DYNAMIC MATCHING MARKETS

2.1 INTRODUCTION

Much of economic life involves two-sided matching that often spans a long horizon. Examples include most teacher-student interactions such as music lessons, business relationships between firms, and hospital-intern markets.

For example, consider music lessons organized by an institution such as City Music Center of Duquesne University in Pittsburgh, PA.[1] The Center's teachers have preferences over students they would like to teach, and students have preferences over teachers. Moreover, to better play a musical instrument, students have to spend many years taking lessons, and thus they need to be involved in long-term relationships. Hence, this is a dynamic two-sided matching market.

For another example, consider British entry-level medical labor markets. These markets involve graduating medical students and teaching hospitals. Students seek residency positions for both medical and surgical programs of hospitals—they have one for the first six months and another for the next six months. Teaching hospitals fill positions in both of these periods. In each period, the market is a many-to-one matching interaction, since students accept at most one hospital and hospitals accept many students. Moreover, since this interaction repeats twice, it is a dynamic many-to-one matching market with two periods. However, it has been modeled as a "static" matching market (See Roth (1991)).

Until now, although static relationships have been extensively studied in matching mar-

[1] Tuition does not play a decisive role in matching, because the tuition is not differentiated by teachers or students.

kets (cf. Roth and Sotomayor (1990) and Roth (2002)), there has been almost no attempt to analyze dynamic relationships.[2] We introduce a new framework to analyze two-sided dynamic interactions: Time is discrete with either finite or infinite horizon. There are two finite disjoint sets of agents. Each agent is supposed either to be matched with those in the opposite set or to be unmatched in each period. There are no frictions: agents do not have to commit themselves to their prior partners and can freely change partners at any period. Each agent has a time-separable utility function over those in the opposite set and being unmatched in each period. The preferences may *vary* across periods.

In a related paper, Damiano and Lam (2005) consider the finite horizon model where the preferences are constant across time with a discount factor; that is, finitely repeated matching markets. While this is a useful benchmark, it can be unrealistic for some applications. For example, in the example of the music lessons discussed above, as students' skills improve, they prefer teachers with different skills. Violin teachers may not value students who did not learn the piano in the past. That is, their current preferences may depend on the past matchings. Moreover, Damiano and Lam (2005) assume that agents choose an outcome path, or a sequence of matchings but not a contingent plan based on realized matchings. This is restrictive, because agents can change prior partners at any time. In this paper, we consider a contingent plan called a "dynamic matching." The problem in dynamic matching markets is to analyze what kinds of matchings might arise in each period under a dynamic matching.

In static settings, it has been shown that a property known as "stability" is central to determining whether static matchings will be sustainable in real-life applications (cf. Roth (1984, 1991, 2002)). Stability (Gale and Shapley, 1962) requires that (1) no individual would rather stay unmatched than continue with her current partner, and (2) no pair of individuals such as a teacher and a student or a hospital and an intern, would prefer each other to their current partners. Two stable matchings have attracted much attention in real-life applications as well as in theoretical work: "hospital-optimal" and "intern-optimal stable" matchings in the case of hospital-intern markets, where the former (the latter) is the best

[2]See recent exceptions: Damiano and Lam (2005) and Kurino (2009a) for two-sided matching markets, and Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2008), Kurino (2009b) and Ünver (2007) for house allocation problems.

stable matching for hospitals (interns) which is at the same time the worst stable matching for interns (hospitals). For example, several regional markets in the aforementioned British markets use "hospital or intern optimal (statically) stable" mechanisms in their centralized matching process, although the markets are dynamic. As Roth (p430, 1991) noted, this static stable mechanism may produce a "higher-order" instability regarding dynamic aspects. In fact, as we will show in Examples 2 and 3, such matchings need not create "dynamically stable" or even "Pareto efficient" outcomes. However, these centralized clearinghouses have been successfully used for the last forty years in Britain. This creates a puzzle: Why is implementing a hospital-optimal or intern-optimal (statically) stable matching so robust in the British markets? This paper provides a theoretical explanation for the robustness.

In this paper, we are concerned with one-to-one matching markets, conventionally called marriage markets (Gale and Shapley, 1962). In a marriage market there are, so called, "men" and "women," each of whom can be matched with at most one partner of the opposite sex. Although we do not deal with many-to-one matching markets such as hospital-intern and teacher-student markets, conceptual tools and insights developed in this paper can be applied to such markets. The aforementioned British markets have been modeled as many-to-two matching markets (Roth, 1991). The hospital or intern-optimal stable matchings correspond to "men or women-optimal stable" matchings in marriage markets.

Traditionally, the cooperative solution concept known as the "core" has been used in analyzing such markets. We begin by pointing out that coalitional deviations considered in the definition of the core are restrictive in dynamic matching markets. Taking into account more general deviations, we propose a definition of (dynamic) group stability that is stronger than the core. An outcome path, or a sequence of matchings, is in the core if no deviating coalition, by choosing another outcome path only among themselves, can make each agent strictly better off. In other words, after the deviation in the first period, agents in a deviating coalition must be matched with each other from the beginning to the end, and are not allowed to be matched with agents outside the coalition. This notion of a deviation is restrictive.

We propose another concept that allows for more general deviations than those permitted in the core, since in the dynamic relationships we explore, we assume that agents are free to sequentially form new partnerships whenever they want. We define "(dynamic) group stability" by requiring a dynamic matching to be immune against group deviations that do not force agents to be matched within the group during all periods.[3] However, a group stable dynamic matching may not always exist (cf. Examples 2 and 3). This means that a dynamic matching consisting only of men-optimal stable matchings may not be group stable in a dynamic setting. We then introduce a new dynamic stability concept called "credible group stability," and show that such a dynamic matching is justified. That is, we show in Proposition 4 that the dynamic matching that assigns a men-optimal stable matching in each period is "credibly group-stable." Similarly, the result holds for women-optimal stable matchings. The hospital-optimal (or intern-optimal) stable mechanism in the aforementioned British markets turns out to be credibly group-stable if we translate it to marriage markets.

Closely looking at possible group deviations from a dynamic matching, we notice that some of them may not be *defensible* in a certain way. Even if a group benefits by reorganizing its match, some members may have an incentive to deviate further by matching with the other agents inside or "outside" the group. In this case, we say that such group deviations are not "defensible." A "credibly-group stable" dynamic matching is immune against any defensible group deviations, and individually rational (i.e. no agent would rather stay unmatched than her current mate).

Our results on credible group stability have significant policy implications. Since a men-optimal (women-optimal) stable matching is favorable to men (women) but not to women (men), we can think of two compromises: 1) choose men-optimal and women-optimal stable matchings alternately, 2) choose a median stable matching in each period that is neither men-optimal nor women-optimal stable. However, both of compromises may not be credibly group-stable (cf. Example 5). Moreover, static many-to-many markets can be alternative to dynamic markets under restricted preference domains. Konishi and Ünver (2006) show that in a many-to-many market, the set of pairwise stable matchings is equivalent to the set of "credibly group-stable" matchings (their notion of credibility is different from ours) under reasonable preference domains. That is, a stable matching other than hospital-optimal (or student-optimal) ones is supported by their credible group-stability but may not be supported

[3]The word "group" is used as a synonym of coalition that is a collection of agents. The use depends on which solution concept is used. Coalition is used for the characteristic function approach such as the core, while group is for the non-characteristic function approach such as group stability.

by our notion of credibility (cf. Example 5).

The second question we explore is on Pareto efficiency. This question does not arise in a static stable mechanism, since a stable matching is always Pareto efficient in a static market. However, this is not true even for finitely repeated markets (cf. Example 2). Hence, we look at finitely repeated markets and examine whether a credibly group-stable dynamic matching that involves a men-optimal (or women-optimal) stable matching in each period is Pareto efficient. We then introduce a condition, called the "regularity condition," and show in Theorem 3 and Corollary 1 that under this condition, such dynamic matchings are also Pareto efficient.

2.1.1 Related literature

In a closely related paper, Damiano and Lam (2005) consider finitely repeated marriage models. They propose variants of "core"-like solution concepts by taking into account dynamic commitment and credible deviations. Their model studies exclusively "repeated" markets in which preferences are "time independent." Our model explores "dynamic" relationships that may have changes of preferences as in several real-life markets. That is, the dynamic markets are "time dependent." In this sense, our model incorporates theirs. In the framework of random matching models of money (Kiyotaki and Wright, 1989), Corbae, Temzelides and Wright (2003) consider endogenous matching by using a solution concept that is immune to one-shot pairwise deviations. The companion paper (Kurino, 2009a) examines this solution concept in our framework. In addition, Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2008), Kurino (2009b) and Ünver (2007) study another dynamic matching model of house allocation. Roth and Vande Vate (1990) study a static market to see how, starting from an arbitrary matching, decentralized dynamic process reaches stable matchings.

British medical markets have been modeled as a static many-to-two matching market in that medical students look for two positions and hospitals fill many positions (Roth, 1991). This suggests that static many-to-many markets can be used for a dynamic market. However, this modeling involves strong preference restrictions. For many-to-many matching markets, see Sotomayor (1999), Echenique and Oviedo (2006) and Konishi and Ünver (2006).

In a static setting, the matching literature uses group stability instead of the core as a solution concept because the deviation considered in the definition of the core is not realistic. In other words, the non-characteristic function approach is used to define group stability. For example, see Roth and Sotomayor (1990) for many-to-one matching markets and the papers listed in the previous paragraph for many-to-many matching markets. This approach is also used in network games (Jackson and Wolinsky, 1996).

The credibility problem for deviating coalitions has been studied in both static and dynamic settings. In a static setting, the various bargaining sets have been proposed for games in coalitional form since Aumann and Maschler (1964). The idea is to consider an *objection* to an outcome by a coalition, and a *justified* objection in which some member of the coalition can not form a *counterobjection* consisting of members insider or outside the coalition. An outcome in the *bargaining set* has no justified objections. Zhou (1994) introduces the new *bargaining set*. Klijn and Massó (2003) apply Zhou's definition to the marriage model. Moreover, they introduce *weak stability* and investigate the relation with the bargaining set. These two concepts allow members of a deviating coalition to deviate further by matching with agents inside or outside the coalition. We follow the same approach. In fact, weak stability coincides with credible pairwise stability in a static setting that is a special case of our credible group stability in dynamic settings. On the other hand, Konishi and Ünver (2006) require a deviating coalition to have no further pairwise deviation within the coalition in their definition of credible group stability in many-to-many matching problems. Turning to other approaches in a static setting, Bernheim et al. (1987) propose the concept of *coalition-proof Nash equilibrium* for normal form games. Ray (1989) defines the cooperative analogue of this approach called *modified core*. These concepts require a deviating coalition to have no further deviations within the coalitions, where the further deviations satisfy the same requirement. In the same spirit, Bernheim et al. (1987) define *perfect coalition-proofness* for extensive form games. Damiano and Lam (2005) define the cooperate analogue of *self-sustaining stability* for finitely repeated matching markets.

2.2 THE MODEL

2.2.1 Preliminaries: static marriage markets

We define a **static (marriage) market** as a triple $(M, W, \{u_i\}_{i \in I})$. By a static market, we always mean a static marriage market. The set $I := M \cup W$ of agents is divided into two finite disjoint subsets M and W. M is the set of men and W is the set of women. Note that $|M| \neq |W|$ in general. Generic agents are denoted by $i \in I$, while generic men and women are denoted by m and w, respectively. Man m's utility function is $u_m : W \cup \{m\} \to \mathbb{R}$, and woman w's utility function is $u_w : M \cup \{w\} \to \mathbb{R}$. Woman w is **acceptable** to man m if $u_m(w) \geq u_m(m)$, and similarly for m. An agent is said to have **strict preferences** if he or she is not indifferent between any two choices. *We assume throughout the chapter that all agents have strict preferences.* In this market, each agent is either matched with another agent of the opposite sex or is unmatched. An outcome is a **matching** defined by a bijection $\mu : M \cup W \to M \cup W$ such that for each $i \in I, (\mu \circ \mu)(i) = i$, and if $\mu(m) \neq m$ then $\mu(m) \in W$, and if $\mu(w) \neq w$ then $\mu(w) \in M$. Fixing M and W, let $\mathcal{M}$ be the set of all matchings. If $\mu(i) = i$, agent i is said to be **unmatched**, and denote this pair by (i, i). If $\mu(m) = w$, equivalently $\mu(w) = m$, then w is said to be **matched** with m, and denote this pair by (m, w). For notational simplicity, we often use $u_i(\mu)$ instead of $u_i(\mu(i))$. A matching μ is **individually rational** if each agent is acceptable to his or her partner, i.e., $u_i(\mu) \geq u_i(i)$ for each agent i in I. Given a matching μ, a pair (m, w) **blocks** μ if they are not matched with each other in μ but prefer each other to their matched partners in μ, i.e. $u_m(w) > u_m(\mu)$ and $u_w(m) > u_w(\mu)$.

Definition 1 (**Gale and Shapley (1962)**)**.** A matching μ is called **(statically) stable** if it is individually rational, and is not blocked by any pair (m, w) in $M \times W$.

The adverb "statically" is omitted if there is no confusion. Moreover, Gale and Shapley (1962) prove the existence of stable matchings:

Theorem 1 (**Existence: Gale and Shapley (1962)**)**.** *A stable matching exists for each static market. In particular, when all agents have strict preferences, there always exist a men-optimal stable matching (that every man likes at least as well as any other stable matching)* *and a women-optimal stable matching.*

2.2.2 Dynamic marriage markets

We consider a **dynamic (marriage) market** in which one-to-one matching interactions occur repeatedly over time. By a dynamic market, we always mean a dynamic marriage market. Time is discrete with either finite or infinite horizon. We denote the horizon by T. $T < \infty$ stands for a finite horizon, while $T = \infty$ stands for infinite horizon. In this market, there are fixed sets of M and W, where M and W are disjoint and finite. In general, $|M| \neq |W|$. Each agent is supposed either to be matched with at most one agent of the opposite sex or to be unmatched at each period $t = 0, \cdots, T$. There are no frictions: agents do not have to commit themselves to their prior partners and can freely change partners at any period. Each agent has a time-separable utility function over those of the opposite sex and being unmatched. Man m's utility function at period t is given by $u_m^t : W \cup \{m\} \to \mathbb{R}$, while woman w's utility function is $u_w^t : M \cup \{w\} \to \mathbb{R}$. *We assume throughout the chapter that all agents have strict preferences in each period.* An **outcome path** is a sequence of matchings in $\mathcal{M}$, denoted by $\boldsymbol{\mu} := \{\mu^t\}_{t=0}^T$. Given an outcome path $\boldsymbol{\mu} = \{\mu^t\}_{t=0}^T$, agent i's utility function is given by

$$U_i(\boldsymbol{\mu}) \;:=\; \sum_{t=0}^{T} u_i^t(\mu^t),$$

where for notational simplicity we use $u_i^t(\mu^t)$ instead of $u_i^t(\mu^t(i))$. We assume that for an infinite horizon case, $U_i(\boldsymbol{\mu})$ is well-defined for any outcome path $\boldsymbol{\mu}$. Each agent knows his or her utility functions as well as those of the other agents. The above structure is common knowledge. Thus, a dynamic market is a triple $(M, W, \{u_i^t\}_{i \in I, t=0,\cdots,T})$. Looking at period t, $(M, W, \{u_i^t\}_{i \in I})$ is a static market, called a **period** t **(marriage) market**. If we do not need to specify the period, we call it a **constituent (marriage) market**. A dynamic market is called a **repeated (marriage) market** if for each agent $i \in I$ there is a discount factor $\delta_i \in (0, 1]$ and a utility function u_i such that $u_i^t = \delta_i^t u_i$ for each period $t = 0, \cdots, T$.

2.3 DYNAMIC GROUP STABILITY

2.3.1 Core and dynamic group stability

In this dynamic market, the problem is which matchings might arise in each period. In other words, which outcome paths[4] will result from interaction among agents? The core[5] gives an answer:

Definition 2. *1.* An outcome path $\boldsymbol{\mu} = \{\mu^t\}_{t=0}^T$ is in the **core** if no coalition blocks it, i.e. there is no coalition A and outcome path $\hat{\boldsymbol{\mu}} = \{\hat{\mu}^t\}_{t=0}^T$ such that

(a) $\hat{\mu}^t(i) \in A$, for each $t = 0, 1, \cdots, T$ and for each i in A, and

(b) $U_i(\hat{\boldsymbol{\mu}}) > U_i(\boldsymbol{\mu})$, for each i in A.

2. It is **individually rational** if for each i in I, $U_i(\boldsymbol{\mu}) \geq \sum_{t=0}^T u_i(i)$.

We will point out that the core is unrealistic by examining deviations that the core concept considers, and then consider a newly defined deviation to define group stability.[6]

Let's examine the core more closely. Condition (a) in Definition 2 requires that after a coalition deviates from $\boldsymbol{\mu}$, all agents in the coalition must be matched only among themselves "from the beginning to the end." On the other hand, condition (b) says that each agent in A is strictly better off in $\hat{\boldsymbol{\mu}}$ than in $\boldsymbol{\mu}$.

Condition (a) is clearly restrictive. We can think of situations in which agents are matched among themselves for only "several" periods, while still being matched with the old partners at other dates. The following example illustrates this point.

Example 1*.* Consider a two-period dynamic market with $M = \{m_1, m_2\}$ and $W = \{w_1\}$. The constituent markets are illustrated in Figure 1, while the total utilities depending on the outcome paths are shown in Figure 2. In Figure 1, the nodes represent the agents, the lines (or no line) represent matches (or no match). The number attached to a node stands for the utility from the match. In this market, there are two outcome paths in the core: $\boldsymbol{\mu}_1 := (\mu_a, \mu_b)$ and $\boldsymbol{\mu}_2 := (\mu_b, \mu_b)$ the latter of which is indicated by circles in Figure 2.

[4] Damiano and Lam (2005) call an outcome path a matching plan.

[5] In general, the core may be empty in our model. An example is given in APPENDIX A.

[6] This kind of approach has been taken in the matching literature, as we discussed in section 2.1.1.

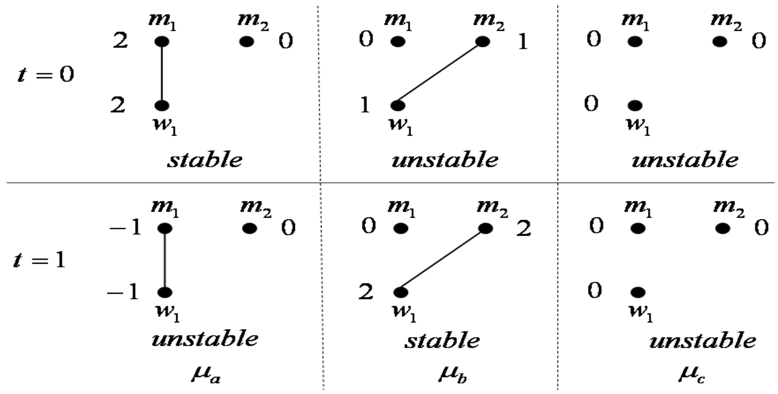

Figure 1: Constituent marriage markets in Example 1

Consider why $\boldsymbol{\mu}_2$ is in the core. We can see that it is individually rational and that no grand coalition blocks it. Consider the coalition $\{m_1, w_1\}$. The outcome paths this coalition can achieve are (μ_a, μ_a), (μ_a, μ_c), and (μ_c, μ_a). Agents (m_1, w_1) obtain $(1, 1)$, $(2, 2)$ and $(-1, -1)$ instead of $(0, 3)$, respectively. However, given that μ_b is chosen at period 1, the pair (m_1, w_1) has an incentive to be matched (i.e. the resulting matching is μ_a) in period 0 and μ_b in period 2. Then, (m_1, w_1) gets $(2, 4)$ instead of $(0, 3)$. Our point is that, instead of requiring that a coalition should be matched only among themselves from the beginning to the end, it may be more appropriate to think that deviators are matched among themselves in only several periods, while still being matched with the old partners in other periods if this results in a superior outcome. We consider these kinds of deviations in the definition of a new solution concept of dynamic group stability. □

Once we allow this kind of deviation, agents become concerned with a contingent plan based on histories of matchings instead of an outcome path. The contingent plan is called a *dynamic matching*.[7] Now we are away from a characteristic function approach,[8] so we use a "group" instead of a coalition for the name of a collection of agents. Our goal is to define dynamic group stability which is "stable" against "group deviations" described above. We

[7]Corbae, Temzelides and Wright (2003) also consider this kind of contingent plan.

[8]Non-characteristic function approaches have been widely used in the many-to-one, many-to-many matching problems and network games, as we discussed in section 2.1.1.

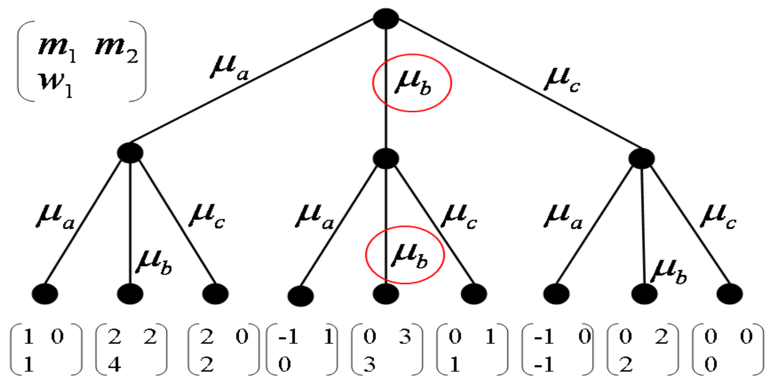

Figure 2: Total utilities in Example 1

need to introduce some new notions:

A **history** at period t, $t \geq 1$, is $h^t := (\mu^0, \mu^1, \cdots, \mu^{t-1}) \in \mathcal{M}^t$, and $h^0 := \emptyset$ is the history at the start of the market. Let $\mathcal{H}^t$ be the set of all histories at period t, i.e. $\mathcal{H}^t = \mathcal{M}^t$. The set of all histories is $\mathcal{H} := \cup_{t=0}^{T} \mathcal{H}^t$.

Definition 3. A **dynamic matching** is a function $\phi : \mathcal{H} \to \mathcal{M}$. Moreover, it is called **history-independent** if in each period, a matching specified by the dynamic matching is independent of histories, i.e., for each $t = 0, 1, \cdots, T$ and for each $h^t, \tilde{h}^t$ in $\mathcal{H}^t$, $\phi(h^t) = \phi(\tilde{h}^t)$.

Note that history independence means that matching in each period is a function of the calender time alone, and that matchings need not be constant across periods. A dynamic matching ϕ induces a unique outcome path $\boldsymbol{\mu}(\phi) := \{\mu^t(\phi)\}_{t=0}^{T}$ recursively as follows: $\mu^0(\phi) := \phi(\emptyset)$, for $t \geq 1$, $\mu^t(\phi) := \phi(\mu^0(\phi), \cdots, \mu^{t-1}(\phi))$. Given ϕ, each agent i's utility function is obtained as $U_i(\phi) := U_i(\boldsymbol{\mu}(\phi))$.

We are interested in whether a given dynamic matching is "stable" (in some sense) against group deviations. To this end, when some group deviates at some history from a given dynamic matching, we must specify how the outsiders respond to the group deviation. This is because the payoffs that agents within the deviating group obtain depend on the outsiders' behavior through the change in histories. In this regard, we make a simple assumption that the outsiders who were matched with agents in the group before the deviation become

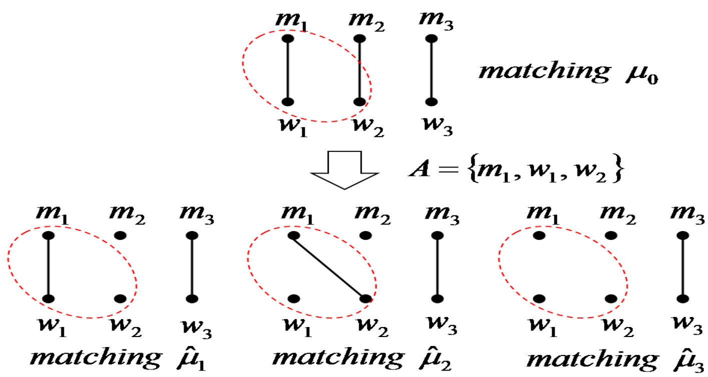

Figure 3: Possible static group deviations by $\{m_1, w_1, w_2\}$

unmatched, and the other outsiders are matched with the same partners as before. With this in mind, we begin to describe how a matching changes in response to a group deviation.

Definition 4. [9] Given a matching μ, a **(static) group deviation** from μ is a pair $(A, \hat{\mu})$ consisting of a group A and a matching $\hat{\mu}$ such that

(a) for each i in A, $\hat{\mu}(i) \in A$,

(b) for each i, j in $I \setminus A$, if $\mu(i) = j$, then $\hat{\mu}(i) = j$, and

(c) for each i in A and for each j in $I \setminus A$, if $\mu(i) = j$, then $\hat{\mu}(j) = j$.

The adjective "static" is omitted when there is no confusion. Condition (a) requires that deviating agents in A should be matched with each other. Condition (b) requires that agents outside the group A should be matched according to μ, while condition (c) requires that any agent who was a partner of an agent outside A should be unmatched under $\hat{\mu}$. Consider the example illustrated in Figure 3, where all of group deviations from μ_0 by the group $A := \{m_1, w_1, w_2\}$ are illustrated. Consider a matching $\hat{\mu}_1$. Condition (a) is satisfied, since m_1 is matched with w_1 and w_2 is unmatched; condition (b) is satisfied, since m_3 and w_3 remain matched; condition (c) is satisfied, since m_2 who was matched with w_2 in A becomes

[9] A special case of pair deviations (the group consisting of one man and one woman) coincides with the one considered by Roth and Vande Vate (1990) where a new matching $\hat{\mu}$ is obtained from μ by *satisfying* the blocking pair. The basic idea is also the same as Corbae, Temzelides and Wright (2003). In addition, this notion is different from *enforcement* used to define a bargaining set in Klijn and Massó (2003).

unmatched.

Definition 5. Given a dynamic matching ϕ, a **(dynamic) group deviation** from ϕ is a pair $(A, \hat{\phi})$ consisting of a group A and a dynamic matching $\hat{\phi}$ such that there is a subset $\mathcal{H}'$ of $\mathcal{H}$,
(a) for each h in $\mathcal{H}'$, a pair $(A, \hat{\phi}(h))$ is a static group deviation from a matching $\phi(h)$, and
(b) for each h in $\mathcal{H} \setminus \mathcal{H}'$, $\hat{\phi}(h) = \phi(h)$.
Moreover, it is called **history-independent** if in each period, a matching inside the group A is history-independent, i.e., for each $t = 0, \cdots, T$, for each h^t and $\tilde{h}^t$ in $\mathcal{H}^t$, if h^t is in $\mathcal{H}'$, then $\tilde{h}^t$ is in $\mathcal{H}'$ and $\hat{\phi}(h^t)|_A = \hat{\phi}(\tilde{h}^t)|_A$.

In the dynamic group deviation $(A, \hat{\phi})$ from ϕ, at histories h in $\mathcal{H}'$ agents in A reorganize their match within A and the others remain matched at $\phi(h)$. In the remaining histories all agents are matched at $\phi(h)$ and possibly matched with agents outside A, which makes our dynamic group deviation different from deviations permitted in the core. In addition, if a dynamic group deviation is history-independent, the matching consisting only of agents in A is a function of calender time alone and need not be constant across periods. However, matchings of the agents outside A can be different across histories in a given period, so $\hat{\phi}$ need not be a history-independent dynamic matching. If the original dynamic matching ϕ is history-independent and a dynamic group deviation $(A, \hat{\phi})$ from ϕ is history-independent, then $\hat{\phi}$ is history-independent by the definition of static group deviation. The adjective "dynamic" is omitted when there is no confusion. For an example, consider a dynamic matching ϕ specifying μ_0 at each history in the repeated market of the constituent market depicted in Figure 3. One possible group deviation $\hat{\phi}$ by $\{m_1, w_1, w_2\}$

$$\begin{aligned} \hat{\phi}(h) &= \mu_1 \quad \text{if } h = \emptyset, \\ &= \mu_2 \quad \text{if } h = \mu_3, \\ &= \mu_0 \quad \text{otherwise.} \end{aligned}$$

For convenience, the group deviation is called **pairwise** if it consists either of an individual or of a pair of one man and one woman.

A group A is said to **block** the dynamic matching ϕ (via $\hat{\phi}$) if $(A, \hat{\phi})$ is a dynamic group deviation from ϕ and $U_i(\hat{\phi}) > U_i(\phi)$ for each i in A. Now we are ready to introduce our concept:[10]

Definition 6.

1. A dynamic matching ϕ is **(dynamically) group-stable** if no group blocks it; i.e., if there is no group deviation $(A, \hat{\phi})$ from ϕ such that $U_A(\hat{\phi}) > U_A(\phi)$.
2. A dynamic matching ϕ is **individually rational** if its outcome path is individually rational.
3. In the special case of a static market (i.e. $T = 0$), a matching μ is called **(statically) group-stable** if it is dynamically group-stable.
4. Moreover, if we consider only pairwise deviations, it is called **(dynamically) pairwise-stable**.

Note that a dynamic market with horizon $T = 0$ is a static market.

Lemma 1. *For a static market, the following are equivalent:*
(a) A matching is stable.
(b) It is in the core.
(c) It is statically group-stable.

For the equivalence of (a) and (b), see Theorem 3.3 in Roth and Sotomayor (1990). To show the equivalence of (b) and (c), observe that in both concepts only a deviating group matters but not the outsiders in a static setting.

Proposition 1. *If a dynamic matching is group stable, then its outcome path is in the core. The converse is not always true.*

The proof of the first part is in APPENDIX A. For the latter part, see Examples 2 and 3 in the next subsection. In addition, we may not have a group stable dynamic matching, as shown in the next subsection. However, if we restrict our attention to *repeated* markets, Proposition 2 can guarantee the existence of *pairwise* stable dynamic matching.

[10]The term "group stability" used in many-to-one or many-to-many matching problems is different from ours, although we adopt the same approach of non-characteristic function. See section 2.1.1.

Proposition 2 (**Existence of a pairwise stable dynamic matching in "repeated" markets**). *There exists a pairwise stable dynamic matching for each finitely or infinitely repeated market.*

Picking a stable matching in the constituent market, consider a dynamic matching assigning this stable matching everywhere. Individuals and a pair of a man and a woman cannot block this dynamic matching, since it assigns a stable matching everywhere and the constituent market is repeated. Thus, the dynamic matching is pairwise stable.

We make three remarks. First, we do not need strict preferences for this proposition to hold. Second, if a matching is not pairwise but group stable, there may be no group stable dynamic matching (cf. Example 2 in the next subsection). Finally, if we have a "dynamic" market, there may be no pairwise stable dynamic matching (cf. Example 3 in the next subsection.).

Before considering some examples, it is useful to characterize dynamic group stability. First, consider a dynamic market with finite horizon $(M, W, \{u_i^t\}_{i\in I, t=0,\cdots,T})$. At history $h^t \in \mathcal{H}$, the **sub-dynamic (marriage) market** is a dynamic market $(M, W, \{u_i^\tau\}_{i\in I,\tau=t,\cdots,T})$. Given a dynamic matching ϕ for the original market, define a **continuation dynamic matching** to be a function $\phi|_{h^t} : \mathcal{M}^{T-t+1} \to \mathcal{M}$ given by $\phi|_{h^t}(h^\tau) = \phi(h^t h^\tau)$ for each $h^\tau \in \mathcal{M}^{T-t+1}$.

Turning to the infinite horizon case ($T = \infty$), at history $h^t \in \mathcal{H}$, the **sub-dynamic (marriage) market** is $(M, W, \{u^\tau_{i\in I,\tau=t,\cdots,\infty}\})$. Given a dynamic matching ϕ for the original market, define a **continuation dynamic matching** to be a function $\phi|_{h^t} : \mathcal{H} \to \mathcal{M}$ given by $\phi|_{h^t} = \phi(h^t h^\tau)$ for each $h^\tau \in \mathcal{H}$. Now we are ready to state:

Lemma 2 (**Partial characterization of group stable dynamic matchings**). *Consider a dynamic market with finite or infinite horizon. If there is a group stable dynamic matching ϕ, then for each history h on the outcome path, the continuation dynamic matching $\phi|_h$ is group stable in the sub-dynamic market starting at h.*

The proof is straightforward and so we omit it.

2.3.2 Examples

Example 1 *(Continued).* The outcome path $\boldsymbol{\mu}_2 := (\mu_b, \mu_b)$ was in the core, and is supported by the following group stable dynamic matching:

$$\begin{aligned} \phi(h) &= \mu_a \quad \text{if } h = \mu_a, \\ &= \mu_b \quad \text{otherwise.} \end{aligned}$$

However, the dynamic matching specifying μ_b at each history cannot be group stable, as we discussed before. Thus, we need to consider history-dependent contingent plans. □

Example 2. (The core is nonempty but there is no group stable dynamic matching)

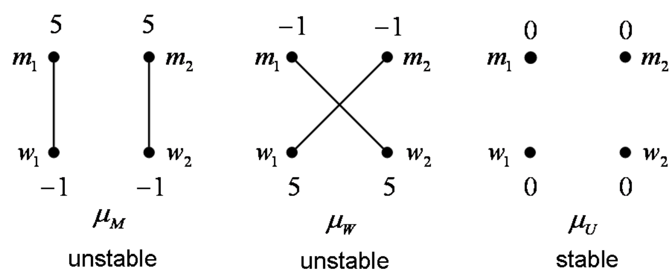

Figure 4: The constituent market in Example 2

Consider a twice repeated market with no discounting whose constituent market[11] is depicted in Figure 4. Here $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. There are seven possible matchings, but only three of them are depicted. Note that the matching μ_M is man-preferred but unstable, μ_W is woman-preferred but unstable, and μ_U is uniquely stable.

First, it can be verified that outcome paths (μ_M, μ_W) and (μ_W, μ_M) are in the core. Next, we show that there is no group stable dynamic matchng. Suppose for a contradiction that there is a group stable dynamic matching ϕ. Let $\{\mu^0, \mu^1\}$ be its outcome path. It follows from Lemma 2 that $\mu^1 = \mu_U$, since μ_U is a unique stable matching in the constituent market. There are two cases to consider. Suppose $\mu^0 \neq \mu_U$. Then, there exists at least one agent i who obtains the payoff of -1 in period 0. In total, his or her payoff is -1 under ϕ. All

[11]This example is from Damiano and Lam (2005).

agents can be unmatched in both periods, which provides a return of 0. Thus, agent i blocks ϕ. This contradicts that ϕ is group stable. On the other hand, suppose $\mu^0 = \mu_U$. Since $\phi(\mu_U) = \mu_U$, each agent gets the payoff of 0. However, the group $I \equiv \{m_1, m_2, w_1, w_2\}$ can make a deviation ϕ' such that $\phi'(\emptyset) = \mu_M$ and $\phi'(\mu_M) = \mu_W$. Then, each agent's payoff is 4. So, the group I blocks ϕ via ϕ'. In any case, some group blocks ϕ. This is a contradiction. □

Example 3. (The core is nonempty but there is no pairwise stable dynamic matching)

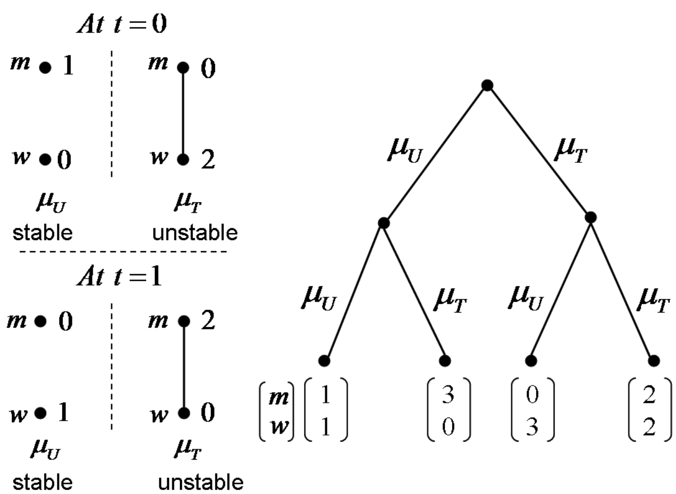

Figure 5: Constituent markets and total utilities in Example 3

Consider a two-period dynamic market[12] depicted in Figure 5. Here, there are man m and woman w. Unlike the previous example, preferences vary across periods. In each period, the matching μ_U (unmatched) is stable, while the matching μ_T (together) is not stable. It can be verified that the outcome path (μ_T, μ_T) is in the core. Similarly to the previous example, we can show by contradiction that there is no pairwise stable dynamic matching.

□

[12]This is adapted from an example in footnote 5 in Corbae, Temzelides and Wright (2003).

2.4 CREDIBLE GROUP STABILITY

2.4.1 Definition

The question on the robustness of clearinghouses in the British medical markets which we raised in section 2.1 can now be restated: What kind of stability concept supports a history-independent dynamic matching assigning a men-optimal stable matching in each period? We saw in the previous section that dynamic group stability does not always work. Remember that we consider all group deviations in the definition of dynamic group stability. Some of them may not be *defensible* in the sense that some members of the deviating group have an incentive to reorganize their match inside or outside the group which makes all of the agents strictly better off. We develop the concept of *defensibility*, and then that of *credible group stability* as immunity against defensible group deviations.

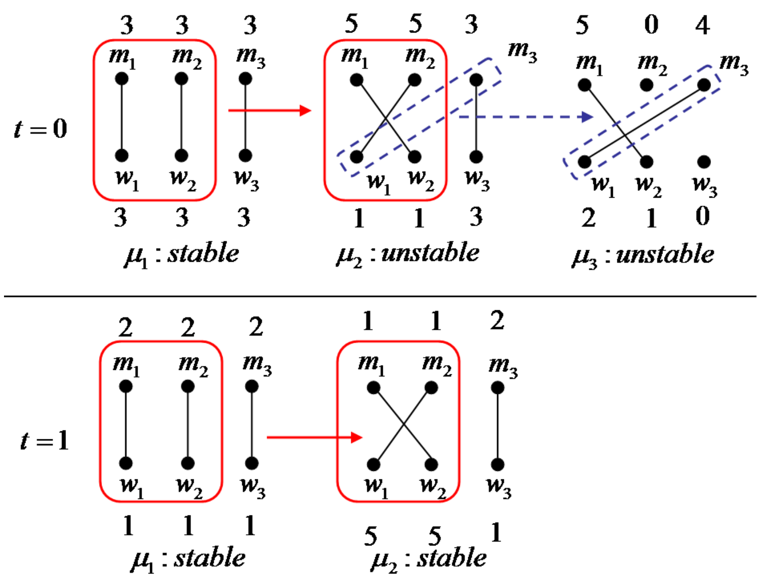

Figure 6: The preferences in the constituent market in Example 4

Example 4. Consider a two-period dynamic market with $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Constituent markets are illustrated in Figure 6, where the utilities of being unmatched for all agents are 0 in both markets. The payoffs depending on outcome paths

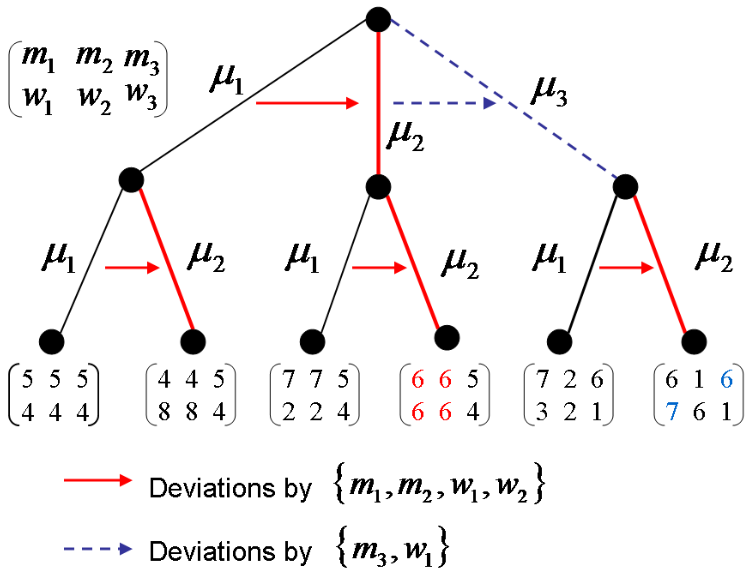

Figure 7: The total utilities in Example 4: The thick arrows stand for a deviation by $\{m_1, m_2, w_1, w_2\}$, and the dotted arrow indicates further deviation by $\{m_3, w_1\}$

are calculated in Figure 7. Note that the figures do not include all possible matchings. In period 0 market there is a unique stable matching μ_1,[13] while in period 1 market μ_1 and μ_2 are men-optimal and women-optimal stable matchings, respectively.

In this market, the dynamic matching ϕ specifying the men-optimal stable matching μ_1 in both periods is not group stable, because the group $A := \{m_1, m_2, w_1, w_2\}$ blocks it via the history-independent dynamic matching $\hat{\phi}$ which specifies μ_2 everywhere. This is illustrated in Figures 6 and 7 by thick circles and arrows.

Consider the possibility of further deviations for the group deviation $(A, \hat{\phi})$. As we can see, no matter how the group A reorganizes its match inside the group, no agent in A can be better off. Note that if we restrict the market to the group A, $\mu_2|_A$ is a men-optimal stable matching in period 0 and women-optimal stable matchings in period 2, although it is not even stable in the original market for period 0. The coordination of men-optimal and women-optimal stable matchings in the restricted markets makes all agents in A better off

[13] We implicitly assume that $u^0_{m_2}(w_3) < u^0_{m_2}(w_2)$ so that we have a unique stable matching.

and has no further deviation within the group. In this sense, the group deviation $(A, \hat{\phi})$ is *credible*, and so the dynamic matching ϕ is not immune to such credible group deviations.

However, closely looking at the group deviation, we notice that woman w_1 in A can be better off with m_3 in period 0 who is "outside" the deviating group A, keeping the matching at period 1 fixed (This is illustrated in Figures 6 and 7 by dotted circles and arrows). That is, the group $\{m_3, w_1\}$ blocks $\hat{\phi}$ via the history-independent dynamic matching $\bar{\phi}$ which specifies μ_3 and μ_2 in periods 0 and 1, respectively. We say that a group deviation is *defensible* if some members of the group have no further profitable deviation by matching with agents inside or outside the group. Although it is *credible* in the sense of the previous paragraph, the group deviation $(A, \hat{\phi})$ is not defensible. We consider only defensible deviations in the solution concept which we define next. □

With this example in mind, we formalize the concept.

Definition 7. Given a dynamic matching ϕ, a group deviation $(A, \hat{\phi})$ from ϕ is **defensible** if

(a) it is history-independent, and

(b) there is no group deviation $(B, \bar{\phi})$ from $\hat{\phi}$ with $A \cap B \neq \emptyset$ such that $U_i(\bar{\phi}) > U_i(\hat{\phi})$ for each i in B.[14]

Any members in a defensible group deviation cannot reorganize their match inside or outside the group via a history-independent group deviation which makes all agents strictly better off. It may seem strong to require a defensible group deviation to be history-independent, but this condition would be acceptable if we consider complexity of contingent plans. Using this defensibility, we introduce the notion of credible group stability:

Definition 8. *1.* A dynamic matching ϕ is **credibly group-stable** if it is individually rational, and there is no defensible group deviation $(A, \hat{\phi})$ such that $U_i(\hat{\phi}) > U_i(\phi)$ for each i in A.

[14] Even if we require the group deviation $(B, \bar{\phi})$ to be history-independent, all of our results are not affected. In this case, since the set of the modified defensible group deviations is larger than that of the original one, the set of credibly group-stable dynamic matchings that use the modified defensibility is smaller than that of the original one.

2. If A is pairwise in the above definition, the the credible group stability is called **credible pairwise stability**.

In other words, a credibly group-stable dynamic matching is individually rational and immune against profitable and defensible group deviations. In a static market, our credible *pairwise* stability coincides with *weak stability*[15] introduced by Klijn and Massó (2003). The idea of our credible group stability is similar to the bargaining set.[16]

Lemma 3. *In a static market,*
(a) a stable matching is credibly group-stable,
(b) a credibly group-stable matching is not always stable, and
(c) a credibly pairwise-stable matching is not always credibly group-stable.

The first statement is obvious, since a stable matching is group stable by Lemma 1. For the rest, examples are given in APPENDIX A. Hence, credible group stability is strictly stronger than credible pairwise stability, and strictly weaker than stability.

The following proposition is the key in proving the existence of credible group stability for dynamic markets. The proof is in APPENDIX A.

Proposition 3. *In a static market, for each stable matching μ, if a group deviation $(A, \hat{\mu})$ from μ is defensible, then $\hat{\mu}$ is stable.*

2.4.2 Existence

Theorem 2 (**Existence**)**.** *For every dynamic market with either finite or infinite horizon, there exists a credibly group-stable dynamic matching.*

Consider any dynamic market with either finite or infinite horizon. From Theorem 1, there exist a men-optimal stable matching and a women-optimal stable matching in each period market. Then, we have either a history-independent dynamic matching assigning a

[15] See Definition 27 and Proposition 7 in the Appendix for the definition and the proof, respectively.

[16] The definition of our group deviation is different from that of enforcement which is used to define the Zhou's bargaining set as formalized by Klijn and Massó (2003) for a marriage model, and thus there is no obvious relationship between our credible group stability and the bargaining set. However, Klijn and Massó (2003) that the set of weakly stable and weakly efficient matchings coincides with the bargaining set. Hence, the set of credible pairwise stable and weakly efficient matchings coincides with the bargaining set.

men-optimal stable matching in each period or the one assigning a women-optimal stable matching in each period. Theorem 2 follows by showing that both are credibly group-stable:

Proposition 4. *In a dynamic market with finite or infinite horizon, a history-independent dynamic matching assigning a men-optimal stable matching in each period is credibly group-stable. Similarly, the result holds for a women-optimal stable matching.*

Proof. Pick a men-optimal stable matching μ_M^t in each period t market. Let ϕ be a history-independent dynamic matching with $\phi(h^t) = \mu_M^t$ for each h^t in $\mathcal{H}$. We show that ϕ is credibly group-stable. First, since each μ_M^t is individually rational in the corresponding period market, ϕ is individually rational. Next, fix a defensible group deviation $(A, \hat{\phi})$ from ϕ. Denote the outcome path of $\hat{\phi}$ by $(\hat{\mu}^0, \hat{\mu}^1, \cdots, \hat{\mu}^T)$. We need to show that $U_i(\phi) \geq U_i(\hat{\phi})$ for some i in A.

Note that from the definition of dynamic group deviation, for each $t = 0, \cdots, T$,

$$\text{either } \hat{\mu}^t = \mu_M^t \text{ or } (A, \hat{\mu}^t) \text{ is a static group deviation from } \mu_M^t. \tag{2.1}$$

There are two cases to consider: First, consider the case where some man m is in A.

<u>Step 1</u>: Show that for each $t = 0, \cdots, T$, either $\hat{\mu}^t = \mu_M^t$, or $\hat{\mu}^t$ is stable in period t market. Suppose for a contradiction that for some period t, $\hat{\mu}^t \neq \mu_M^t$ and $\hat{\mu}^t$ is not stable. Then, it follows from (2.1) that $(A, \hat{\mu}^t)$ is a static group deviation from μ_M^t. By Proposition 3, $(A, \hat{\mu}^t)$ is not defensible. Thus, there exists a static group deviation $(B, \bar{\mu}^t)$ from $\hat{\mu}^t$ with $A \cap B \neq \emptyset$ such that $u_i^t(\bar{\mu}^t) > u_i^t(\hat{\mu}^t)$ for each i in B. Consider the following history-independent dynamic matching:

$$\begin{aligned} \bar{\phi}(h^\tau) &= \hat{\mu}^\tau \quad \text{if } \tau \neq t, \\ &= \bar{\mu}^t \quad \text{if } \tau = t. \end{aligned}$$

Since dynamic matching ϕ is history-independent and dynamic group deviation $(A, \hat{\phi})$ is also history-independent, the dynamic matching $\hat{\phi}$ is history-independent. This implies that

$(B, \bar{\phi})$ is a dynamic group deviation from $\hat{\phi}$. Then, the outcome path of $\hat{\phi}$ is $(\hat{\mu}^0, \cdots, \hat{\mu}^T)$, while the outcome path of $\bar{\phi}$ is $(\hat{\mu}^0, \cdots, \hat{\mu}^{t-1}, \bar{\mu}^t, \hat{\mu}^{t+1}, \cdots, \hat{\mu}^T)$. Thus,

$$U_i(\bar{\phi}) = \sum_{\tau \neq t} u_i^\tau(\hat{\mu}^\tau) + u_i^\tau(\bar{\mu}^t) > \sum_{\tau=0}^{T} u_i^\tau(\hat{\mu}^\tau) = U_i(\hat{\phi}) \text{ for each } i \text{ in } B.$$

This contradicts the assumption that the group deviation $(A, \hat{\phi})$ from ϕ is defensible. This completes the proof of Step 1.

Step 2: Show $U_m(\phi) \geq U_m(\hat{\phi})$. Since μ_M^t is a men-optimal stable matching in period t market, it follows from Step 1 that

$$\text{either } u_m^t(\mu_M^t) = u_m^t(\hat{\mu}^t) \text{ or } u_m^t(\mu_M^t) \geq u_m^t(\hat{\mu}^t).$$

This implies

$$U_m(\phi) \equiv \sum_{t=0}^{T} u_m^t(\mu_M^t) \geq \sum_{t=0}^{T} u_m^t(\hat{\mu}^t) \equiv U_m(\hat{\phi}).$$

This completes the proof of Step 2.

Next, consider the case where A consists only of women. Fix $w \in A$ and period t. Then, if $(A, \hat{\mu}^t)$ is a static group deviation from μ_M^t, all women in A are unmatched. Thus, from (2.1), either $\hat{\mu}^t = \mu_M^t$ or w is unmatched at $\hat{\mu}^t$. Since μ_M^t is individually rational in the period t market, either $u_w^t(\mu_M^t) = u_w^t(\hat{\mu}^t)$ or $u_w^t(\mu_M^t) \geq u_w^t(w) \equiv u_w^t(\hat{\mu}^t)$. Thus, $U_w(\phi) \geq U_w(\hat{\phi})$.

Therefore, we proved that for each defensible group deviation $(A, \hat{\phi})$ from ϕ, $U_i(\phi) \geq U_i(\hat{\phi})$ for some i in A. Hence, ϕ is credibly group-stable. □

2.4.3 Policy implications

Because a men-optimal (women-optimal) stable matching is favorable to men (women) but not to women (men), we can think of two compromises in market design. The first is a mechanism that always selects men-optimal and women-optimal stable matchings alternately. The second is a mechanism that always selects a median stable matching in each period which is neither men-optimal stable nor women-optimal stable. The question is: Is such a dynamic matching always credibly group-stable? The following example indicates that it is not.

Example 5. Consider a two-period dynamic market whose constituent markets are depicted in Figure 8, where utility values of being unmatched are 0. μ_M and μ_W indicate men-optimal and women-optimal stable matchings, respectively. In addition, μ_S denotes another stable matching in Figure 8.

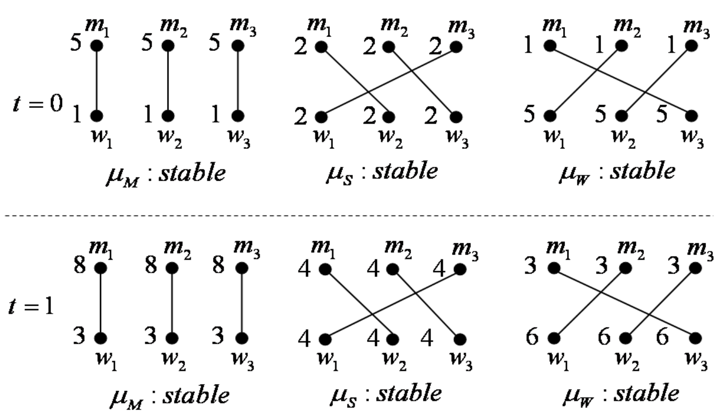

Figure 8: Constituent markets in Example 5

Case 1: A history-independent dynamic matching ϕ consisting only of μ_S is not credibly group-stable.

Consider a history-independent group deviation $(A, \hat{\phi})$ from ϕ where $A = M \cup W$, and $\hat{\phi}$ assigns μ_W and μ_M to period 0 and 1, respectively. All agents in A are better off in $\hat{\phi}$ than in ϕ. We show that $(A, \hat{\phi})$ is defensible. Suppose for a contradiction that there is a group deviation $(B, \bar{\phi})$ from $\hat{\phi}$ with $B \cap A = B \neq \emptyset$ such that $U_i(\bar{\phi}) > U_i(\hat{\phi})$ for each i in B. Note that each man obtains the payoff of 9 and each woman obtains that of 8 at $\hat{\phi}$, and no agent in B is not unmatched at $\bar{\phi}$ in each period. So, B includes matched pairs at $\bar{\phi}$. We have three cases to consider: First, if m_1 is in B, since he has the payoff more than 9 at $\bar{\phi}$, he is either matched with w_1 in both periods, or matched with w_2 and w_1 in periods 0 and 1, respectively. In the former case, w_1 is in B but gets the payoff of 4 at $\bar{\phi}$ that is less than 8. A contradiction. In the latter case, w_2 is in B, and cannot get the payoff more than 8 at $\bar{\phi}$. A contradiction. Similarly, we can obtain a contradiction for the other two cases where m_2 is in B or m_3 is in B. Thus, $(A, \bar{\phi})$ is defensible. Thus, ϕ is not credibly group-stable.

Case 2: A history-independent dynamic matching ϕ consisting of μ_M in the first period and μ_W in the second period is not credibly group-stable.

Consider the same group deviation $(A, \hat{\phi})$ from ϕ as Case 1. All agents in A are better off in $\hat{\phi}$ than ϕ. Since $(A, \hat{\phi})$ is defensible as we verify in Case 1, ϕ is not credibly group-stable.

□

2.5 PARETO EFFICIENCY IN FINITELY REPEATED MARKETS

In a static market, since any stable matching is in the core from Lemma 1, it is weakly Pareto efficient. Thus, the question of welfare does not arise in a static stable mechanism. However, as we saw in Example 2, a history-independent dynamic matching assigning a unique statically stable matching in each period is credibly group-stable, but not Pareto efficient even in a finitely repeated market. In this section, we investigate Pareto efficiency in finitely repeated markets. Whether an outcome path consisting of stable matchings is Pareto efficient depends on preferences of agents in constituent markets. To examine Pareto efficiency, we introduce a condition, called *regularity*, for a static market.

2.5.1 Regularity condition for static markets

To introduce the regularity condition, we define a **restricted market** $(\tilde{M}, \tilde{W}, \tilde{u})$, denoted by $(\tilde{M} \cup \tilde{W})$, of a static market (M, W, u) to be a static market such that $\tilde{M} \subset M$, $\tilde{W} \subset W$, $\tilde{u}_m$ is a restriction of u_m to $\tilde{W} \cup \{m\}$ for each $m \in \tilde{M}$, and $\tilde{u}_w$ is a restriction of u_w to

$\tilde{M} \cup \{w\}$ for each $w \in \tilde{W}$. Moreover, throughout this section, a **pair** (i, j) means that either i belong to the opposite sex of j or $i = j$.

Definition 9. Given a matching μ with the number N of pairs formed in μ, a static market has **regularity** for μ if there is a sequence $\{(i_k, \mu(i_k))\}_{k=1}^{N}$ of pairs formed in μ (called a **regular sequence** for μ) such that
(a) for $k = 1$, i_1's most preferred mate is $\mu(i_1)$ in a restricted market $M \cup W$,
(b) for $k \geq 2$, i_k's most preferred mate is $\mu(i_k)$ in a restricted market $(M \cup W) \setminus \{i_l, \mu(i_l)\}_{l=1}^{k-1}$.

In a regular sequence $\{(i_k, \mu(i_k))\}_{k=1}^{N}$ for μ, agent i_1's partner at μ is $\mu(i_1)$ who is the best partner to i_1 among all agents. Removing this pair $(i_1, \mu(i_1))$ from the market, agent i_2's partner at μ is $\mu(i_2)$ who is the best partner to i_2 among all agents except the pair $(i_1, \mu(i_1))$. Removing the pairs $(i_1, \mu(i_1))$ and $(i_2, \mu(i_2))$ from the market, we repeat the same procedure until no agent is left.

As an example, consider the Example 2. The constituent market has regularity for μ_M, μ_W but not for μ_S. As a regular sequence for μ_M, take $i_1 = m_1$ and $i_2 = m_2$.

Lemma 4. *(1) If a static market has regularity for a matching μ, then μ may not be stable. (2) If a matching is stable in a static market, then the market may not have regularity for it.*

In the constituent market of Example 2, the matching μ_M satisfies regularity, but is not stable. On the other hand, the matching μ_U is stable but does not satisfy regularity. In a special class of markets with acceptability and $|M| = |W|$, the regularity condition is clearly equivalent to a sufficient condition for a unique stable matching identified by Eeckhout (2000). Thus, in this class, if a static market has regularity for a matching μ, then μ is uniquely stable.

2.5.2 Finitely repeated markets

An outcome path $\boldsymbol{\mu}$ is **Pareto efficient** if there is no other outcome path $\boldsymbol{\mu}'$ such that $U_i(\boldsymbol{\mu}') \geq U_i(\boldsymbol{\mu})$ for each i in I with strict inequality for some i in I.

Theorem 3 (**Pareto efficiency**). *In a finitely repeated market, if a matching μ^* satisfies* *regularity in the constituent market, then an outcome path consisting of the matching μ^* is Pareto efficient.*

Proof. Let $(M, W, \{u_i\}_{i \in I})$ be a constituent market. Let the outcome path $\boldsymbol{\mu}^* := (\mu^*, \mu^*, \cdots, \mu^*)$. Take any outcome path $\boldsymbol{\mu} := (\mu^t)_{t=0}^T$ that is different from $\boldsymbol{\mu}^*$. We show that there exists an agent $i \in I$ such that $U_i(\boldsymbol{\mu}^*) > U_i(\boldsymbol{\mu})$.

Take a regular sequence $\{i_k, \mu^*(i_k)\}_{k=1}^N$ of pairs for μ^*. Take $\mathcal{M}(i) := \{\mu \in \mathcal{M} | (i, \mu^*(i)) \notin \mu\}$. We choose a particular agent i_K among $\{i_k\}_{k=1}^N$ in the following way:

Step 1: If there exists $t = 0, \cdots, T$ such that $\mu^t \in \mathcal{M}(i_1)$, then set $i_K = i_1$. Otherwise, go to the next step.

Step k: If there exists $t = 0, \cdots, T$ such that $\mu^t \in \mathcal{M}(i_k)$, then set $i_K = i_k$. Otherwise, go to the next step.

This procedure stops after at most N steps. In addition, we can choose such an agent i_K. Otherwise, we would have a contradiction that $\boldsymbol{\mu}^* = \boldsymbol{\mu}$.

To show that $U_{i_K}(\boldsymbol{\mu}^*) > U_{i_K}(\boldsymbol{\mu})$, it is sufficient to show that for each $\mu^t \in \mathcal{M}(i_K)$, $u_{i_K}(\mu^*) > u_{i_K}(\mu^t)$. Note that for each $\mu^t \in \mathcal{M} \setminus \mathcal{M}(i_K)$, $u_{i_K}(\mu^*) = u_{i_K}(\mu^t)$.

Fix $\mu^t \in \mathcal{M}(i_K)$, i.e., $(i_K, \mu^*(i_K)) \notin \mu^t$. Because of the procedure of finding i_K, agents in $\{i_k, \mu^*(i_k)\}_{k=1}^{K-1}$ are matched with each other, and thus agent i_K is not matched with any mate in $\{i_k, \mu^*(i_k)\}_{k=1}^{K-1}$. By regularity and strict preferences, agent i_K's most preferred mate in the restricted market $(M \cup W) \setminus \{i_k, \mu^*(i_k)\}_{k=1}^{K-1}$ is $\mu^*(i_K)$, and thus $u_{i_K}(\mu^*) > u_{i_K}(\mu^t)$.

□

Corollary 1. *In a finitely repeated market, if a stable matching μ^* satisfies regularity, then an outcome path consisting the matching μ^* is Pareto efficient.*

Any outcome path consisting of the men-optimal (or women-optimal) stable matching of the constituent market can be supported via credible group stability by Theorem 2. However, in the Example 2, such an outcome path consisting only of μ_U is not Pareto efficient, and the regularity condition is not satisfied. Together with the following example, a partial converse[17] of the Corollary 1 may hold:

[17] Conjecture of a partial converse: If a static market does not have regularity for a stable matching μ, there is a period T such that in the T times repeated market, an outcome path consisting of μ is not Pareto efficient.

Example 6. (An example for the partial converse of Corollary 1.) Consider a three-times repeated market with $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the following preferences[18] with no discounting:

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_3 (6)	w_4 (6)	w_1 (6)	w_3 (6)	m_2 (6)	m_1 (6)	m_2 (6)	m_3 (6)
w_1 (2)	w_2 (2)	w_3 (2)	w_4 (2)	m_1 (2)	m_2 (2)	m_3 (2)	m_4 (2)
w_2 (1)	w_3 (1)	w_2 (1)	w_2 (1)	m_3 (1)	m_3 (1)	m_4 (1)	m_1 (1)
w_4 (0)	w_1 (0)	w_4 (0)	w_1 (0)	m_4 (0)	m_4 (0)	m_1 (0)	m_2 (0)

The numbers in parentheses indicate utilities. Each agent is acceptable to all those of the opposite sex. Note that there is a unique stable matching $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$ and the constituent market does not have regularity for μ^*. Consider three matchings $\mu^0 = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$, $\mu^1 = \{(m_1, w_3), (m_2, w_4), (m_3, w_1), (m_4, w_4)\}$ and $\mu^2 = \{(m_1, w_2), (m_2, w_3), (m_3, w_1), (m_4, w_4)\}$. Then, total utilities are calculated as follows:

Total utilities	m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
$U_i(\mu^*, \mu^*, \mu^*)$	6	6	6	6	6	6	6	6
$U_i(\mu^0, \mu^1, \mu^2)$	8	7	12	10	8	14	7	8

Thus, the outcome path consisting only of μ^* is not Pareto efficient.

□

2.6 CONCLUSION

Some real-life dynamic matching markets use a mechanism that finds a men-optimal or a women-optimal stable matching. Our result shows that this approach does not create instability in a dynamic setting. Therefore, this approach is justified.

[18]This example for ordinal preferences is from Example 2 in Eeckhout(2000), which Ahmet Alkan suggests. We attach utility values so that the claim holds.

3.0 A NOTE ON ONE-SHOT GROUP STABILITY IN DYNAMIC MATCHING MARKETS

3.1 INTRODUCTION

In the previous chapter, we introduced a new dynamic framework of dynamic matching markets with either a finite horizon or an infinite horizon in order to analyze two-sided matching interactions that occur repeatedly over time. Moreover, we introduce two solution concepts of *dynamic group stability* and *credible group stability*. However, both concepts assume that agents can perfectly coordinate a contingent plan depending on the realized matchings, called a *dynamic matching*. While this is appropriate for a short horizon, it can be demanding for a long horizon.

In this chapter, we study another solution concept, called *one-shot group stability*, for the same model as chapter 2. This concept is first introduced by Corbae, Temzelides and Wright (2003) in the context of random matching model of money (Kiyotaki and Wright, 1989). In this concept, at each history, a group of agents take the future matching as given and consider a possibility of profitable deviation on this history. A one-shot group-stable dynamic matching is immune against this kind of one-shot group deviations.

Although this concept is myopic, it would be appropriate to analyze the situation where agents are not sure when interactions end. For an example, consider music lessons organized by an institution such as City Music Center of Duquesne University in Pittsburgh, PA.[1] The Center's teachers have preferences over students they would like to teach, and students have preferences over teachers. To better play a musical instrument, students have to spend

[1] See http://www.cmcpgh.org. Tuition does not play a decisive role in matching, because the tuition is not differentiated by teachers or students.

many years taking lessons, and thus they need to be involved in long-term relationships. Hence, this is a dynamic two-sided matching market. On the other hand, students may have preferences over future matchings, but may not be sure when they quit. Myopic solution concepts capture this kind of dynamic matching.

Moreover, in the context of mechanism design, a clearinghouse may not be able to set up a mechanism in which she asks all agents about their preferences on matchings over the entire period. Instead, she sets up a dynamic mechanism where at each period she determines the current matchings based on both the agents' report on their current preferences and the past matchings. When the number of agents is large, it would be harder for participants to coordinate the future matchings. This is one of the cases where one-shot group stability is appropriate. In addition, it is important to examine to what extent we can achieve a coordination across time with one-shot group stability.

This chapter first proves the existence of one-shot group stability, and the next we examine to what extent we can achieve a coordination across time under infinite horizon by using the one-shot group stability.

3.1.1 Related literature

In chapter 2, we observe that there may not always exist a group stable dynamic matching. Thus, we provide the notion of credible group stability[2] and show its existence by showing that implementing a men-optimal stable matching in each period is credibly group-stable. This existence result needs the assumption that the underlying period preference is *strict*, since this assumption guarantees the existence of the men-optimal stable matching. However, we do *not* assume the strict preference in this paper. Under this general condition, we prove the existence of one-shot group stability.

The most closely related paper is Corbae, Temzelides and Wright (2003). They propose a solution concept corresponding to one-shot pairwise stability in this paper. They focus on applications of directed matching to monetary theory and do not deal with the characterization and existence problems. Variants of core in repeated matching markets are extensively

[2]A credible group-stable dynamic matching is individually rational and immune to any defensible group deviations with an appropriate definition of defensibility.

studied by Damiano and Lam (2005).

3.2 THE MODEL

3.2.1 Preliminaries: static marriage markets

We define a **static (marriage) market** as a triple $(M, W, \{u_i\}_{i\in I})$. By a static market, we always mean a static marriage market. The set $I := M \cup W$ of agents is divided into two finite disjoint subsets M and W. M is the set of men and W is the set of women. Note that $|M| \neq |W|$ in general. Generic agents are denoted by $i \in I$, while generic men and women are denoted by m and w, respectively. Man m's utility function is $u_m : W \cup \{m\} \to \mathbb{R}$, and woman w's utility function is $u_w : M \cup \{w\} \to \mathbb{R}$. Woman w is **acceptable** to man m if $u_m(w) \geq u_m(m)$, and similarly for m. An agent is said to have **strict preferences** if he or she is not indifferent between any two choices. Unlike chapter 2, agents may *not* have a strict preference. In this market, each agent is either matched with another agent of the opposite sex or unmatched. An outcome is a **matching** defined by a bijection $\mu : M \cup W \to M \cup W$ such that for each $i \in I, (\mu \circ \mu)(i) = i$, and if $\mu(m) \neq m$ then $\mu(m) \in W$, and if $\mu(w) \neq w$ then $\mu(w) \in M$. Fixing M and W, let $\mathcal{M}$ be the set of all matchings. If $\mu(i) = i$, agent i is said to be **unmatched**, and denote this pair by (i, i). If $\mu(m) = w$, equivalently $\mu(w) = m$, then w is said to be **matched** with m, and denote this pair by (m, w). For notational simplicity, we often use $u_i(\mu)$ instead of $u_i(\mu(i))$. A matching μ is **individually rational** if each agent is acceptable to his or her partner, i.e., $u_i(\mu) \geq u_i(i)$ for each agent i in I. Given a matching μ, a pair (m, w) **blocks** μ if they are not matched with each other in μ but prefer each other to their matched partners in μ, i.e. $u_m(w) > u_m(\mu)$ and $u_w(m) > u_w(\mu)$.

Definition 10 (**Gale and Shapley (1962)**). A matching μ is called **(statically) stable** if it is individually rational, and is not blocked by any pair (m, w) in $M \times W$.

The adverb "statically" is omitted if there is no confusion. Moreover, Gale and Shapley (1962) prove the existence of stable matchings:

Theorem 4 (**Existence: Gale and Shapley (1962)**). *A stable matching exists for each static market.*

See Roth and Sotomayor (1990) for a comprehensive account on static markets.

3.2.2 Dynamic marriage markets

We consider a **dynamic (marriage) market** in which one-to-one matching interactions occur repeatedly over time. By a dynamic market, we always mean a dynamic marriage market. Time is discrete with either a finite horizon or an infinite horizon. We denote the horizon by T. $T < \infty$ stands for a finite horizon, while $T = \infty$ for an infinite horizon. In this market, there are fixed sets of M and W, where M and W are disjoint and finite. In general, $|M| \neq |W|$. Each agent is supposed either to be matched with at most one agent of the opposite sex or to be unmatched at each period $t = 0, \cdots, T$. There are no frictions: agents do not have to commit themselves to their prior partners and can freely change partners at any period. Each agent has a time-separable utility function over those of the opposite sex and being unmatched. Man m's utility function at period t is given by $u_m^t : W \cup \{m\} \to \mathbb{R}$, while woman w's utility function is $u_w^t : M \cup \{w\} \to \mathbb{R}$. Unlike chapter 2, we do *not* assume that all agents have strict preferences in each period. An **outcome path** is a sequence of matchings in $\mathcal{M}$, denoted by $\boldsymbol{\mu} := \{\mu^t\}_{t=0}^T$. Given an outcome path $\boldsymbol{\mu} = \{\mu^t\}_{t=0}^T$, agent i's utility function is given by

$$U_i(\boldsymbol{\mu}) \;\; := \;\; \sum_{t=0}^{T} u_i^t(\mu^t),$$

where for notational simplicity we use $u_i^t(\mu^t)$ instead of $u_i^t(\mu^t(i))$. We assume that for an infinite horizon case, $U_i(\boldsymbol{\mu})$ is well-defined for any outcome path $\boldsymbol{\mu}$. Each agent knows his or her utility functions as well as those of the other agents. The above structure is common knowledge. Thus, a dynamic market is a triple $\Gamma^T := (M, W, \{u_i^t\}_{i \in I, t \geq 0})$. Looking at period t, $(M, W, \{u_i^t\}_{i \in I})$ is a static market, called a **period t (marriage) market**. If we do not need to specify the period, we call it a **constituent (marriage) market**. A dynamic market is called a **repeated (marriage) market** if there is a common discount factor $\delta \in (0, 1)$

such that for each agent $i \in I$ and there is a utility function u_i such that $u_i^t = (1-\delta)\delta^t u_i$. That is,

$$U_i(\boldsymbol{\mu}) = (1-\delta)\sum_{t=0}^{T} \delta^t u_i(\mu^t)$$

3.2.3 Dynamic group stability

Here we summarize basic definitions and results from chapter 2.

A **dynamic matching** is a contingent plan based on histories of realized matchings. Formally, a **history** at period t, $t \geq 1$, is $h^t := (\mu^0, \mu^1, \cdots, \mu^{t-1}) \in \mathcal{M}^t$, and $h^0 := \emptyset$ is the history at the start of the market. Let $\mathcal{H}^t$ be the set of all histories at period t, i.e. $\mathcal{H}^t = \mathcal{M}^t$. The set of all histories is $\mathcal{H} := \cup_{t=0}^{T} \mathcal{H}^t$. Then, a dynamic matching is defined to be a function $\phi : \mathcal{H} \to \mathcal{M}$. Moreover, it is **history-independent** if in each period, a matching specified by the dynamic matching is independent of histories, i.e., for each $t = 0, 1, \cdots, T$ and for each $h^t, \tilde{h}^t$ in $\mathcal{H}^t$, $\phi(h^t) = \phi(\tilde{h}^t)$. Note that history independence means that matching in each period is a function of the calender time alone, and that matchings need not be constant across periods.

A dynamic matching ϕ induces a unique outcome path $\boldsymbol{\mu}(\phi) := \{\mu^t(\phi)\}_{t=0}^{\infty}$ recursively as follows: $\mu^0(\phi) := \phi(\emptyset)$, for $t \geq 1$, $\mu^t(\phi) := \phi(\mu^0(\phi), \cdots, \mu^{t-1}(\phi))$. Given ϕ, each agent i's utility function is obtained as $U_i(\phi) := U_i(\boldsymbol{\mu}(\phi))$.

Definition 11. Given a matching μ, a **(static) group deviation** from μ is a pair $(A, \hat{\mu})$ consisting of a group A and a matching $\hat{\mu}$ such that

(a) for each i in A, $\hat{\mu}(i) \in A$,

(b) for each i, j in $I \setminus A$, if $\mu(i) = j$, then $\hat{\mu}(i) = j$, and

(c) for each i in A and for each j in $I \setminus A$, if $\mu(i) = j$, then $\hat{\mu}(j) = j$.

In a static market, the adjective "static" is omitted when there is no confusion. Condition (a) requires that deviating agents in A should be matched with each other. Condition (b) requires that agents outside the group A should be matched according to μ, while condition (c) requires that any agent who was a partner of an agent outside A should be unmatched under $\hat{\mu}$.

Definition 12. Given a dynamic matching ϕ, a **(dynamic) group deviation** from ϕ is a pair $(A, \hat{\phi})$ consisting of a group A and a dynamic matching $\hat{\phi}$ such that there is a subset $\mathcal{H}'$ of $\mathcal{H}$,
(a) for each h in $\mathcal{H}'$, a pair $(A, \hat{\phi}(h))$ is a static group deviation from a matching $\phi(h)$, and
(b) for each h in $\mathcal{H} \setminus \mathcal{H}'$, $\hat{\phi}(h) = \phi(h)$.

In the dynamic group deviation $(A, \hat{\phi})$ from ϕ, at histories h in $\mathcal{H}'$ agents in A reorganize their match within A and the others remain matched at $\phi(h)$. In the remaining histories all agents are matched at $\phi(h)$ and possibly matched with agents outside A. The adjective "dynamic" is omitted when there is no confusion.

A group A is said to **block** the dynamic matching ϕ (via $\hat{\phi}$) if $(A, \hat{\phi})$ is a dynamic group deviation from ϕ and $U_i(\hat{\phi}) > U_i(\phi)$ for each i in A.

Definition 13.

1. A dynamic matching ϕ is **(dynamically) group-stable** if no group blocks it; i.e., if there is no group deviation $(A, \hat{\phi})$ from ϕ such that $U_i(\hat{\phi}) > U_i(\phi)$ for each i in A.
2. In the special case of a static market (i.e. $T = 0$), a matching μ is called **(statically) group-stable** if it is dynamically group-stable.

Lemma 5. *For a static market, the following are equivalent:*
(a) A matching is stable.
(b) It is in the core.
(c) It is statically group-stable.

3.3 ONE-SHOT GROUP STABILITY

3.3.1 Definition

We first formalize one-shot group deviation and use it to define one-shot group stability.

Definition 14. Given a dynamic matching ϕ, a **one-shot group deviation** from ϕ is a pair (A, ϕ') of a group A and a dynamic matching ϕ' such that there exists a history $\tilde{h} \in \mathcal{H}$

with $\phi'(\tilde{h}) \neq \phi(\tilde{h})$ and $\phi'(h) = \phi(h)$ for each $h \neq \tilde{h}$ in $\mathcal{H}$.

Consider a one-shot group-deviation from a given dynamic matching ϕ where at only one deviating history h^t agents consider a group deviation in the period t market. At this history, agents evaluate their current partner in the future matchings as well as the current one. At the history h^t, agents face a dynamic market starting from h^t, which we call a **sub-dynamic (marriage) market** $\Gamma(h^t) := (M, W, \{u_i^\tau\}_{i \in I, \tau = t, \cdots, T})$. To evaluate the given dynamic matching ϕ, agents consider a dynamic matching starting from h^t, called a **continuation dynamic matching** $\phi|_{h^t}$. Formally, for a finite horizon case, it is a function $\phi|_{h^t} : \mathcal{M}^{T-t+1} \to \mathcal{M}$ given by $\phi|_{h^t}(h^\tau) = \phi(h^t h^\tau)$ for each h^τ in $\mathcal{M}^{T-t+1}$. For an infinite horizon case, it is a function $\phi|_{h^t} : \mathcal{H} \to \mathcal{M}$ given by $\phi|_{h^t}(h^\tau) = \phi(h^t h^\tau)$ for each $h^\tau \in \mathcal{H}$.

Given a dynamic matching ϕ, a group A is said to **one-shot block** ϕ (via ϕ') if (A, ϕ') is a one-shot group-deviation from ϕ such that at the history $\tilde{h}^t$ with $\phi'(\tilde{h}^t) \neq \phi(\tilde{h}^t)$, $U_i(\phi'|_{\tilde{h}^t}) > U_i(\phi|_{\tilde{h}^t})$ for each i in A.

Definition 15. A dynamic matching is **one-shot group-stable** if no group one-shot blocks it, i.e. there is no one-shot group deviation (A, ϕ') from ϕ such that at history $\tilde{h}^t$ with $\phi'(\tilde{h}^t) \neq \phi(\tilde{h}^t)$, $U_i(\phi'|_{\tilde{h}^t}) > U_i(\phi|_{\tilde{h}^t})$ for each i in A.

This one-shot group stability is similar in spirit to the "equilibrium" considered in Corbae, Temzelides and Wright (2003), although they do not prove the existence. In the world of one-shot group-stability, agents and groups are myopic at all histories in the sense that at each history they take the future matchings as given and think about the current matching whose payoff depends not only on the current one but also the future matchings resulting from the current choice.

3.3.2 Characterization by networked (marriage) markets

We provide a tractable method of checking one-shot group stability by using the following notion.

Definition 16. A **networked (marriage) market**[3] is a triple $(M, W, \{v_i\}_{i \in I})$, where

[3] This kind of model was first considered by Sasaki and Toda (1996). They called it a *matching problem with externalities*.

$v_i : \mathcal{M} \to \mathbb{R}$ for each $i \in I$.

The difference from a static marriage market is that the domain of the payoff function v_i is the set $\mathcal{M}$ of all matchings. In other words, an agent's preference over those of the opposite sex depends not only on his or her partner but also on the partners of the others.

Next, we define a solution concept, called *stability**. In general, when a pair of a man and a woman consider divorcing their current partners and to be matched with each other, they need to form expectations on how the other agents, including their former partners, will behave. We use the group deviation as introduced in Definition 11. Given a group deviation (A, μ') from μ, the expectation of group A after deviation is expressed by μ'. A group A is said to **block** μ (via μ') if there is a group deviation (A, μ') from μ such that $v_i(\mu') > v_i(\mu)$ for each i in A.

Definition 17. In a networked market, a matching μ is **(statically) group-stable*** if no group blocks it, i.e. there is no group deviation (A, μ') from μ such that $v_i(\mu') > v_i(\mu)$ for each i in A.

This solution concept is the same as *strong stability* in network games as defined by Dutta and Mutuswami (1997). Note that if in a networked market $\forall i \in I, \forall j \in I, \forall \mu \in \mathcal{M}$ with $\mu(i) = j$, $v_i(\mu)$ is constant, the networked market boils down to a static market.

Lemma 6. *If a networked market is a static market, both group stability** *and group stability coincide.*

The proof follows directly from the definitions. The intuition is that the only difference in the solution concepts is on how a deviating group thinks about the outsiders' behavior, but in a marriage market it does not matter to the deviating group. Because of this lemma, we do not distinguish between group stability* and group stability.

Fix a dynamic matching ϕ and a history h^t. We define the **induced networked (marriage) market** $\tilde{\Gamma}(h^t, \phi) := (M, W, \{v_i\}_{i \in I})$ such that

$$v_i(\mu) := u_i^t(\mu) + U_i(\phi|_{h^t,\mu})$$

for each $i \in I$. Now we can state a useful lemma. The proof is in APPENDIX B.

Lemma 7. *A dynamic matching ϕ is one-shot group-stable if and only if for each history $h \in \mathcal{H}$, the matching $\phi(h)$ is statically group-stable in its induced networked market $\tilde{\Gamma}(h, \phi)$.*

3.3.3 Existence and characterization

We demonstrate the existence of one-shot group-stable dynamic matchings for every dynamic market. To describe a class of one-shot group-stable dynamic matchings, a history-independent dynamic matching is useful in the sense that each induced networked market is simplified:

Lemma 8. *For each history-independent dynamic matching ϕ, for each $t \geq 0$, and for each $h^t \in \mathcal{H}^t$, the induced networked market $\tilde{\Gamma}(h^t, \phi)$ is a static marriage market equivalent to the period t market.*

The proof is in APPENDIX B.

Corollary 2. *Let ϕ be a history-independent dynamic matching. Then, ϕ is one-shot group-stable if and only if for each period, it specifies a statically stable matching of the corresponding period market.*

See APPENDIX B for the proof. Thus, we can fully characterize a history-independent one-shot group-stable dynamic matching in terms of constituent markets.

Theorem 5 (Existence). *A one-shot group-stable dynamic matching always exists for every dynamic market.*

Proof. Consider a history-independent dynamic matching which assigns a statically group-stable matching in the corresponding market for each period. Such a dynamic matching exists by Theorem 4. It follows from Corollary 2 that this is one-shot group-stable. □

Note that we do not need strict preferences for this existence theorem. The same technique can be applied to the other dynamic matching markets such as many-to-one, many-to-many, and roommates matching markets as long as a static market has a stable matching.

Corollary 3. *In a dynamic market under finite horizon, if each of its constituent markets has a unique stable matching, then there is a unique one-shot group-stable dynamic matching.*

A question that we can ask from Corollary 3 is that, when the constituent market involves more than one stable matching, do we have an unstable matching on the outcome path of a one-shot group-stable dynamic matching? The next proposition says Yes. The proof is in APPENDIX B.

Proposition 5. *In a finitely repeated market, a one-shot group-stable dynamic matching may have a statically unstable matching of a constituent market on the outcome path.*

3.4 COORDINATION

Looking at Example 1, we can see that under finite horizon, a one-shot group-stable dynamic matching may not be group stable, i.e. a coordination failure across time. This section investigates to what extent a one-shot group-stable dynamic matching can achieve such a coordination.

3.4.1 Repeated marriage markets

Consider an infinitely repeated market of a static market $(M, W, \{u_i\}_{i \in I})$. The set of static market payoffs generated by matchings in $\mathcal{M}$ is

$$\mathcal{F} := \{u(\mu) \in \mathbb{R}^{|I|} | \mu \in \mathcal{M}\}.$$

The set of **feasible payoffs**,

$$\mathcal{F}^{\dagger} := \text{co}\mathcal{F},$$

is the convex hull of $\mathcal{F}$. A payoff vector $w = (w_1, \cdots, w_{|I|})$ is **individually rational** if $w_i > u_i(i)$ for each $i \in I$. In addition, a payoff vector $w = (w_1, \cdots, w_{|I|})$ is **group-rational** if there exists a stable matching μ such that for each $i \in I$, $w_i > u_i(\mu)$. Obviously, any group-rational payoff is individually rational. Let

$$\mathcal{F}^*(\mu) := \{w \in \mathcal{F}^{\dagger} | w_i > u_i(\mu), \forall i \in I\},$$

for a matching $\mu \in \mathcal{M}$. The set of feasible and group-rational payoffs is given by

$$\mathcal{F}^* := \cup\{\mathcal{F}^*(\mu) |\, \mu \text{ is stable}\}.$$

Lemma 9. *Suppose that $\mathcal{F}^*(\mu)$ is nonempty for some stable matching μ. Then,*
(1) there is no matching μ' such that $u(\mu') \in \mathcal{F}^*(\mu)$, and
(2) $|I| \geq 3$.

The proof is in Appendix B.

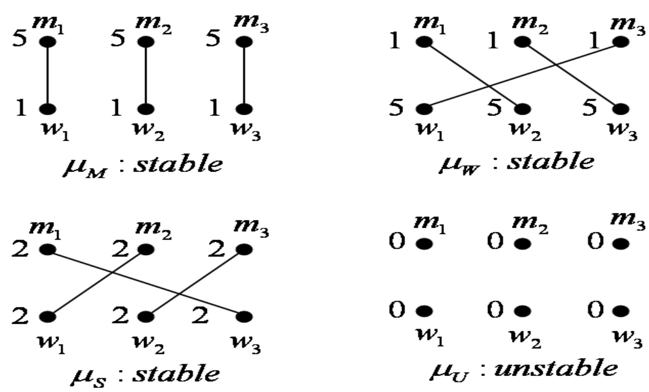

Figure 9: Constituent markets in Example 6

Example 6. Consider the twice repeated market with no discounting whose constituent market is depicted in Figure 9. Here the set of group-rational payoffs is strictly smaller than that of individually rational ones, as shown in Figure 10. The question is whether or not we can sustain a group-rational payoff as a result of one-shot group-stable dynamic matching. The answer is affirmative, and is developed from now on in the general setting. To this end, we need the following lemma. □

Lemma 10 (Lemma 2 in Fudenberg and Maskin (1991) or Lemma 3.7.2 in Mailath and Samuelson (2006))**.** *For any $\epsilon > 0$ there exists $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ and every $w \in \mathcal{F}^\dagger$ there is a sequence of matchings whose discounted average payoffs are w, and whose continuation payoffs at each time t are within ϵ of w.*

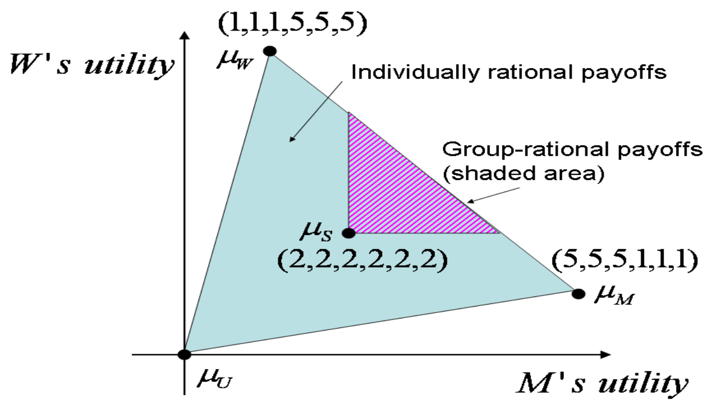

Figure 10: Individually and rational payoffs in Example 6

This lemma is intended for infinitely repeated games under perfect monitoring. Since it does not involve an equilibrium concept, we can get Lemma 10 by replacing the action profile by the matching.

The following theorem informally says that any group-rational and feasible payoff of a static market can arise as a result of one-shot group-stability of the infinitely repeated market if agents are sufficiently patient. The importance of this theorem comes from the fact that one-shot group-stability is seen as myopic decision-making (See the paragraph after Definition 15) but still can achieve any group-rational payoff under infinite horizon.

Theorem 6. *Consider an infinitely repeated market. Then, for each payoff w in $\mathcal{F}^*$, there exists $\underline{\delta} < 1$ such that for each $\delta \in (\underline{\delta}, 1)$, there exists a one-shot group-stable dynamic matching with payoff w.*

Proof. Let $w \in \mathcal{F}^*$. Then, there exists a stable matching $\hat{\mu}$ in the constituent market such that for each $i \in I, w_i > u_i(\hat{\mu})$. Take $\epsilon_i := w_i - u_i(\hat{\mu}) > 0$ for each i, and $\epsilon := \min\{\epsilon_i | i \in I\} > 0$. It follows from Lemma 10 that there exists $\underline{\delta}' < 1$ such that for each $\delta \in (\underline{\delta}', 1)$, there exists $\boldsymbol{\mu} = \{\mu^t\}_{t=0}^{\infty}$ such that $U(\boldsymbol{\mu}) = w$ and whose continuation payoffs at each period t are within ϵ of w.

Consider the following trigger type of dynamic matching ϕ:

$$\begin{aligned}\phi(h^t) &= \mu^t \quad \text{if } h^t = (\mu^0, \cdots, \mu^{t-1}),\\ &= \hat{\mu} \quad \text{otherwise.}\end{aligned}$$

We show that this ϕ is one-shot group-stable. From Lemma 7, it suffices to show that for each history $h^t \in \mathcal{H}$, the matching $\phi(h^t)$ is group stable in the induced networked market $\tilde{\Gamma}(h^t, \phi)$. Take any $h^t \in \mathcal{H}$. There are two cases to consider. First, consider the case where h^t is off the outcome path. Then the continuation dynamic matching $\phi|_{h^t}$ assigns $\hat{\mu}$ to each history, and thus is history-independent. It follows from Lemma 8 that the induced networked market is equivalent to the constituent market. Therefore, since $\hat{\mu}$ is a stable matching in the constituent market, the matching $\phi(h^t) = \hat{\mu}$ is group stable in the induced networked market.

Next, consider the case where the history h^t is on the outcome path, i.e., $h^t = (\mu^0, \mu^1, \cdots, \mu^{t-1})$. Let $K := \max\{u_i(\mu) | i \in I, \mu \in \mathcal{M}\}$. First, show that

$$\exists \underline{\delta} > \underline{\delta}', \forall \delta \in (\underline{\delta}, 1), \forall i \in I, \quad U_i(\phi|_{h^t}) \geq (1-\delta)K + \delta u_i(\hat{\mu}). \tag{3.1}$$

Take any $i \in I$. Since $w_i - u_i(\hat{\mu}) \equiv \epsilon_i \geq \epsilon > 0$ and $|U_i(\phi|_{h^t}) - w_i| < \epsilon$,

$$U_i(\phi|_{h^t}) - u_i(\hat{\mu}) = U_i(\phi|_{h^t}) - w_i + w_i - u_i(\hat{\mu}) > -\epsilon + \epsilon = 0. \tag{3.2}$$

Also,

$$K - u_i(\hat{\mu}) \geq 0. \tag{3.3}$$

It follows from (3.2), (3.3) and the Archimedean property that there exists $\underline{\delta}_i$ such that for each $\delta \in (\underline{\delta}_i, 1)$,

$$U_i(\phi|_{h^t}) - u_i(\hat{\mu}) > (1-\delta)(K - u_i(\hat{\mu})). \tag{3.4}$$

Setting $\underline{\delta} := \max\{\underline{\delta}_1, \cdots, \underline{\delta}_{|I|}, \underline{\delta}'\}$ and arranging (3.4) leads to the desired equation (3.1).

Next, show that μ^t is group stable in the induced networked market. Let (A, μ') be a group deviation from μ^t. Then, equation (3.1) implies that for each $\delta \in (\underline{\delta}, 1)$ and each $i \in I$,

$$v_i(\phi(h^t)) \equiv U_i(\phi|_{h^t}) \geq (1-\delta)u_i(\mu') + \delta u_i(\hat{\mu}) = (1-\delta)u_i(\mu') + \delta U_i(\phi|_{h^t,\mu'}) \equiv v_i(\mu').$$

Thus, the group A does not block μ^t, and so μ^t is group stable. □

3.4.2 Dynamic marriage markets

Now we turn to a dynamic market. Because preferences can vary across periods, a direct application of notions developed in repeated markets is not possible. However, we make the similar notations and definitions. The set of **feasible payoffs** is

$$\mathcal{F}^{\dagger} := \{U(\boldsymbol{\mu}) \in \mathbb{R}^{|I|} | \boldsymbol{\mu} \text{ is an outcome path}\}.$$

Definition 18. A payoff vector $w \in \mathcal{F}^{\dagger}$ is **group-rational** if there exist outcome paths $\boldsymbol{\mu} := \{\mu^t\}_{t=0}^{\infty}$ and $\hat{\boldsymbol{\mu}} := \{\hat{\mu}^t\}_{t=0}^{\infty}$ with $w = U(\boldsymbol{\mu})$ such that
(a) for each $t \geq 0$, $\hat{\mu}^t$ is stable in the period t market,
(b) for each $t \geq 0$ and for each $i \in I$, $\sum_{\tau=t}^{\infty} u_i^{\tau}(\mu^{\tau}) \geq \sum_{\tau=t}^{\infty} u_i^{\tau}(\hat{\mu}^{\tau})$.
Also, the outcome path $\boldsymbol{\mu}$ is called a **group-rational outcome path**.

Theorem 7. *Consider a dynamic market under infinite horizon. Suppose that a payoff w is group-rational. Then, there exists a one-shot group-stable dynamic matching with payoff w.*

Proof. Suppose that a payoff w is group-rational. Take outcome paths $\boldsymbol{\mu}$ with $w = U(\boldsymbol{\mu})$ and $\hat{\boldsymbol{\mu}}$ that satisfy conditions (a) and (b) in Definition 18. Consider the following trigger type of dynamic matching ϕ:

$$\begin{aligned} \phi(h^t) &= \mu^t \quad \text{if } h^t = (\mu^0, \cdots, \mu^{t-1}) \\ &= \hat{\mu}^t \quad \text{otherwise.} \end{aligned}$$

We show that ϕ is one-shot group-stable. From Lemma 7, it suffices to show that for each history h^t, the matching $\phi(h^t)$ is group stable in the induced networked market $\tilde{\Gamma}(h^t, \phi)$. Fix $h^t \in \mathcal{H}$. There are two cases to consider. First, consider the case where h^t is off the outcome path. Then the continuation dynamic matching $\phi|_{h^t}$ assigns $\hat{\mu}^{t+\tau}$ for each history h^{τ}, and thus is history-independent. It follows from Lemma 8 that the induced networked market is equivalent to the period t market. Thus, since $\hat{\mu}^t$ is group stable in the period t market by the condition (a) in Definition 18 and Lemma 5, it follows from Lemma 6 that the matching $\phi(h^t) = \hat{\mu}^t$ is group stable in the induced networked market.

Next, consider the case where the history h^t on the outcome path, i.e., $h^t = (\mu^0, \mu^1, \cdots, \mu^{t-1})$. We show that μ^t is group stable in the induced networked market. Let (A, μ') be a group-deviation from μ^t. First, there exists i in A such that $u_i^t(\hat{\mu}^t) \geq u_i^t(\mu')$. Otherwise, the group A blocks $\hat{\mu}^t$ via μ' in the period t market, which contradicts that $\hat{\mu}^t$ is group stable in the period t market. Together with this claim, the condition (b) in Definition 18 implies that

$$\begin{aligned} v_i(\phi(h^t)) \equiv u_i^t(\mu^t) + U_i(\phi|_{h^t,\mu^t}) &= \sum_{\tau=t}^{\infty} u_i^\tau(\mu^\tau) \\ &\geq \sum_{\tau=t}^{\infty} u_i^\tau(\hat{\mu}^\tau) = u_i^t(\hat{\mu}^t) + \sum_{\tau=t+1}^{\infty} u_i^\tau(\hat{\mu}^\tau) \\ &\geq u_i^t(\mu') + U_i(\phi|_{h^t,\mu'}) \equiv v_i(\mu'). \end{aligned}$$

To obtain the last inequality, we use $u_i^t(\hat{\mu}^t) \geq u_i^t(\mu')$ and $U_i(\phi|_{h^t,\mu'}) = \sum_{\tau=t}^{\infty} u_i^\tau(\hat{\mu}^\tau)$. Therefore, the group A cannot block μ^t via μ' in the induced networked market, and thus μ^t is group stable in the market. □

3.5 CONCLUSION

We have proven the existence of a one-shot group stable dynamic matching, and provided a tractable method of using induced networked markets. In addition, we have shown how it can achieve coordination in an infinite horizon case.

4.0 HOUSE ALLOCATION WITH OVERLAPPING AGENTS: A DYNAMIC MECHANISM DESIGN APPROACH

4.1 INTRODUCTION

The static allocation problem[1] of assigning indivisible goods, called "houses," to agents without monetary transfers has been extensively studied and applied to real-life markets such as on-campus housing for college students (cf. Abdulkadiroğlu Sönmez, 1999; Chen and Sönmez 2002; Guillen and Kesten 2008), kidney exchanges for patients (Roth, Sönmez, and Ünver 2004), and school choice for U.S. public schools (cf. Abdulkadiroğlu and Sönmez, 2003). Until now, there has been little attempt to analyze dynamic house allocations problems.[2]

Considering dynamic aspects enables us to explain aspects of the allocation problem that cannot be captured by static models. For example, in the case of on-campus housing for college students, each year freshmen apply to move in and graduating seniors leave. Each student stays on campus for a few years only. A student is a "newcomer" in the beginning and then becomes an "existing tenant." In general, students are *overlapping*. In this structure, it is not always *dynamically* Pareto efficient to have a static Pareto efficient allocation in each period.

To illustrate this point, suppose in the first period $t = 1$, there is one agent a^0, called an *initial existing tenant*,[3] who came before the market starts and lives only in this period. Moreover, in each period $t \geq 1$, one agent a^t comes to live in a house in periods t and $t + 1$.

[1]See Sönmez and Ünver (2008) for a recent survey.

[2]See recent exceptions: Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2008), and Ünver (2009).

[3]Throughout this paper, the terminology "existing tenants" indicates the agents who came to the market in the previous period. It does not always mean that they have property rights for houses, unlike the ones used by Abdulkadiroğlu and Sönzme (1999).

In each period t, there is an *existing tenant* a^{t-1} who came in the previous period, and a *newcomer* a^t. There are two durable houses h_1 and h_2 available. Each agent prefers h_1 to h_2, and (h_2, h_1) to (h_1, h_2).[4] Consider the allocation:

	$t=1$	$t=2$	$t=3$	$t=4$	$\cdots$
a^0	h_2				
a^1	h_1	h_2			
a^2		h_1	h_2		
a^3			h_1	h_2	
$\vdots$				$\vdots$	$\vdots$

In each period, an existing tenant is assigned h_2 and a newcomer is assigned h_1. This allocation is Pareto efficient for each period's *static* market. However, consider an infinite exchange between an exiting tenant and a newcomer in each period where an existing tenant exchanges her house h_2 for the newcomer's house h_1. As a result, the initial existing tenant is assigned h_1, and each of the other agents is assigned (h_2, h_1). This new allocation is preferred to the original by every agent. Thus, the original allocation is not dynamically Pareto efficient.

Many universities in the United States use a variant of the *random serial dictatorship mechanism* to allocate dormitory rooms.[5] This mechanism randomly orders the agents and then applies the serial dictatorship (SD) mechanism: the first agent is assigned her top choice, and the next agent is assigned her top choice among the remaining rooms, and so on. This ordering is not entirely random, but rather depends on *seniority*. That is, existing tenants are favored over newcomers.

In the previous example, consider *period orderings* which order a newcomer a^t as the first and an existing tenant a^{t-1} as the second in each period. Running an SD mechanism in each period, we obtain the same allocation as indicated in the previous table. As we saw, this outcome is not dynamically Pareto efficient. On the other hand, consider other period

[4] For example, (h_2, h_1) is a consumption path where an agent consumes house h_2 in the first period, and h_1 in the next. Note that this preference violates the discounted utility model. However, considering a critique of the discount utility model as reviewed by Frederick, Lowenstein, and O'Donoghue (2002), we allow for any strict preference relation on $\{h_1, h_2\} \times \{h_1, h_2\}$ in this paper. See footnote 15 for a further discussion.

[5] We will list some of real-life examples later in this section.

orderings that order an existing tenant first, and a newcomer next. The allocation by the SD mechanism with orderings that favor existing tenants over newcomers Pareto dominates the original, and is dynamically Pareto efficient. That is, the ordering based on seniority performs well in terms of Pareto efficiency.

The subject of this paper is to present a new dynamic framework for a house allocation problem by considering overlapping agents,[6] and to analyze the impact of orderings on Pareto efficiency and strategy-proofness.[7] To our knowledge, the existing literature takes orderings as given, and we are the first to examine the importance of seniority-based mechanisms.

Our model extends the standard overlapping generations (OLG) models[8] to a house allocation problem. Time is discrete and lasts forever. There are a finite number of durable houses that are collectively owned by some institution, say a housing office. In each period, a finite number of newcomers arrive and then stay for a finite number of periods, T, while the oldest agents leave the market. Each agent consumes one house in each period. Each agent has a time-separable preference over houses that spans T periods, consisting of T period preferences. Her given preference does not vary across time. That is, her type is drawn as a preference when she arrives, but her type does not change over time. However, we do allow period preferences to vary across periods. Only initial existing tenants who arrived before the market starts may have endowments. In this environment, a housing office needs to find a mechanism to allocate houses to agents. Unlike in a static model, the office is not able to elicit the preferences of agents who will arrive in the future. In each period, the office assigns houses to agents present in the market, and may assign property rights for the future as well as the current assignment. Thus, the office takes into account the previous assignments in order to determine a current assignment. Hence, the office faces a *dynamic mechanism design* problem.

We study two dynamic mechanisms. The first is a *spot mechanism* where in each period a housing office asks agents present in that period about the current period preference, and not the preference over all time periods. In particular, we look at spot mechanisms with

[6] Block and Cantala (2008) independently consider a similar model to ours. One of the difference is that in their model only *one* agent arrives in each period. See the Related literature section for a further discussion.

[7] By strategy-proofness, all agents find it best to report true preferences.

[8] See Samuelson (1958), or Ljungqvist and Sargent (2004).

or without property rights transfer: In a spot mechanism *with* property rights transfer, the houses occupied by the oldest agent become vacant in the next period, but those occupied by the other agents become their endowment. On the other hand, a spot mechanism *without* property rights transfer has no such transfer. Another dynamic mechanism is a *futures mechanism* in which each new agent is asked to reveal her preference over all time periods when she arrives.

At any point of time, our spot mechanism without property rights transfer resembles a *house allocation problem* (Hylland and Zeckhauser, 1979). A random serial dictatorship (RSD) (static) mechanism has been widely studied (Abdulkadiroğlu and Sönmez, 1998). Some colleges such as Davidson College, Lafayyett College, and St. Olaf College use a seniority-based RSD spot mechanism on the condition that all students are forced to participate in the mechanism every year. As we saw in the previous example, in an SD spot mechanism (period orderings are given each period), period orderings that favor existing tenants induce a Pareto efficient allocation (Theorem 9). On the other hand, period orderings that favor newcomers do not always induce a Pareto efficient allocation (Theorem 10).

Although it is simple, Pareto efficient, and strategy-proof (Svensson, 1994), the RSD mechanism is rarely used. Rather, many universities use a modified version of this mechanism, called *a RSD mechanism with squatting rights*, where existing tenants either keep their current rooms, or give up them and participate in the RSD mechanism. The main reason[9] is that universities want to keep students on campus, which makes the universities financially less risky. This seniority-based mechanism is used in Northwestern University, University of Michigan, and the University of Pittsburgh, among others. Students in these colleges can choose stay off-campus. Even colleges that require all students to live on campus use this seniority-based mechanism; for example, Godrdon College, Guilford College, Lawrence University. Although it is ex ante individually rational, this mechanism is not Pareto efficient (Abdulkadiroğlu and Sönmez, 1999), and not ex post individually rational (Some students

[9]James Earle, Assistant Vice Chancellor for Business at the University of Pittsburgh, gave me the following official reason: The goal of the Department of Housing is first and foremost, customer satisfaction. By allowing students the opportunity to retain a room they like, we are guaranteeing the satisfaction of these returning customers. Furthermore, if these students were forced out of their room, they could not only become a dissatisfied customer, if they then get a room they don't like, but they could also decide to live off campus and become someone else's customer. Why risk the loss of revenue, when you have the potential to have a satisfied customer simply by allowing them to retain their room?

who participate in the RSD mechanism may get a worse room than their previously owned one.).

The RSD mechanism with squatting rights motivates us to introduce a *spot mechanism with property rights transfer* where, in each period, the houses occupied by the oldest agents become vacant in the next period, but those occupied by the other agents are inherited as property rights or endowments in the next period. At any one point of time, this spot mechanism resembles a *static house allocation problem with existing tenants* (Abdulkadiroğlu and Sönmez, 1999) in which there are newcomers (agents with no endowments) and existing tenants (agents with endowments). In a static context, since the SD with squatting rights is not Pareto efficient, Abdulkadiroğlu and Sönmez (1999) propose a mechanism based on the *top trading cycles (TTC) mechanism* (Shapley and Scarf, 1974), referred to as *AS-TTC mechanism*. This mechanism restores Pareto efficiency that the RSD mechanism with squatting rights lacks, while satisfying individual rationality and strategy-proofness.

We introduce a notion of *acceptability* in which each agent is made weakly better off as time goes on. This corresponds to the situation, in a spot mechanism with property rights transfer, where the static mechanism in each period is individually rational. Thus, it can be seen as a counterpart of individual rationality in a static problem. As we mentioned, in order to keep students on campus, many universities give property rights to students. In this sense, the acceptability is desirable for dynamic mechanisms. However, we prove an Impossibility Theorem where there is no dynamic mechanism that is Pareto efficient and acceptable (Theorem 8).

Any SD spot mechanism is not acceptable, since all houses that an existing tenant weakly prefers to their previously occupied one can be obtained by agents with higher order. However, since an AS-TTC static mechanism is individually rational, we consider a *TTC spot mechanism* in which an AS-TTC static mechanism is run each period in a spot mechanism with property rights transfer. Since an AS-TTC mechanism is individually rational, a TTC spot mechanism is acceptable. However, by the Impossibility Theorem, this spot mechanism is not Pareto efficient.[10] We restrict the preference domain to time-invariant preferences

[10]Note that in the general preference domain a TTC spot mechanism is not dynamically but statically Pareto efficient. An SD mechanism with squatting rights is not even statically Pareto efficient.

where each agent has preference consisting of identical period preferences. We emphasize that this is not just a repetition of an AS-TTC static mechanism but has two distinct features. First, we have entry and exit of agents with different preferences in each period. Second, endowments or property rights are endogenous. Under time-invariant preferences, we show that period orderings that favor existing tenants perform better than the ones that favor newcomers in terms of Pareto efficiency and strategy-proofness (Theorems 12, 13, 14 and 15).

Finally, we propose a *serial dictatorship (SD) futures mechanism* which is based on orderings of agents. We show that it is strategy-proof and Pareto efficient.

4.1.1 Related literature

There is an extensive literature on static house allocation problems. See Sönmez and Ünver (2008) for a recent and comprehensive survey.

A dynamic house allocation problem can be classified depending on how and when agents arrive and exit. But with the deterministic arrival and exit of agents, Bloch and Cantala (2008) independently consider a model similar to ours. There are several differences that distinguish our work from theirs. First, in their model only *one* agent enters and exits the market in each period, while our model allows for any finite number of agents to enter and exit. Second, in their model the type of an entering agent is drawn as a period preference but does not vary as time goes on, while in our model we allow period preferences to vary across periods. Third, their preference domain is more restricted than ours. They consider two cases: 1) all agents have identical preferences, and 2) agents have heterogenous preferences but the surpluses from matchings are supermodular. They analyze a Markovian assignment mechanism with property rights transfer that is acceptable. Their seniority-rule corresponds to our *constant SD spot mechanism favoring existing tenants* (to be defined in chapter 4) in a specific environment as described above. However, they do not look at how static mechanisms such as a serial dictatorship mechanism or a TTC mechanism behave in a dynamic setting. They characterize the independent convergent rule when agents are homogenous, but do not consider incentive issues.

Ünver(2009) studies a dynamic mechanism design with an application to kidney exchange for patients (Roth, Sönmez, and Ünver, 2004) in which agents arrive stochastically. However, our dynamic model cannot be applied to kidney exchange for two reasons. First, a patient with live donors (i.e. an agent with an endowment) arrives in each period, while in our model only initial existing tenants may have endowments. Second, kidney patients immediately leave the market once their exchange is done, but our model does not allow for this.

Additionally, Abdulkadiroğlu and Loertscher (2007) consider in a dynamic problem without the arrival and exit of agents and with two periods in which each agent's type is drawn in each period. They also introduce a *dynamic mechanism* that depends on the first period allocation, and examine efficiency and optimal dynamic mechanisms. Similarly, the case of multiple-type (static) housing markets, where multiple types of indivisible goods are traded and endowments are given, can be seen as a dynamic house allocation problem with finite horizon the length of which is the same as the number of types. Konishi, Quint and Wako (2001) obtain a negative result in which there is no mechanism that is Pareto efficient, individually rational, and strategy-proof.

Although there are almost no papers on the ordering of agents in a mechanism, Sönmez and Ünver (2005) show that a stochastic AS-TTC mechanism favoring newcomers is equivalent to the *core-based mechanism* in a *static* house allocation with exiting tenants.

Finally, there is a growing literature on dynamic mechanism design with monetary transfers. For example, see Athey and Segal (2007), and Gershkov and Moldovanu (2009).

4.2 THE MODEL

4.2.1 A dynamic problem

Time is discrete, starts at $t = 1$ and lasts forever. There is a finite set, $\hat{H}$, of indivisible goods, called **houses**, which are collectively owned by some institution (say a housing office.). The houses are perfectly durable in that they can be used in each period. The number of available houses is fixed throughout time.

Agents live in houses for T periods, where $T \geq 2$ is finite.[11] An agent who came before the model starts is called an **initial existing tenant**.[12] In particular, an initial existing tenant who came at period $\tau \leq 0$ is called a **newcomer in period** τ, and lives in one house in each period from period 1 to $\tau + T - 1$. For example, an oldest agent in period 1 is a newcomer in period $2 - T$, and lives in a house only in period 1. In each period $t \geq 1$, newcomers arrive to live in a house in every period from period t to $t + T - 1$. Each such agent is called a **newcomer in period** t. The number of newcomers in each period $t \geq 2 - T$ is finite, and is denoted by n. Let $N(t) := \{a_1^t, a_2^t, \cdots, a_n^t\}$ be the set of newcomers in period $t \geq 2 - T$.[13] Table 4.2.1 shows the demographic structure of our model. Note that there are both an infinite number of periods and an infinite number of agents in this model. This "double infinity" is the major source of the theoretical peculiarities of the OLG model (Shell, 1971).

In each period $t \geq 1$, agents in the market are newcomers in periods $t - T + 1, t + T + 1, \cdots, t - 1, t$. Agents who came before period t are also called **existing tenants in period** t. That is, they are newcomers in periods $t - T + 1, \cdots, t - 1$. Let $E(t)$ be the set of all existing tenants in period t. Thus, $E(t) \equiv \cup\{N(\tau) : t - T + 1 \leq \tau \leq t - 1\}$ and $E(1)$ is the set of all initial existing tenants. Moreover, let $A(t) := N(t) \cup E(t)$ be the set of all agents present in period $t \geq 1$. Note $|A(t)| = nT$. We assume that the number of houses is equal to the number of agents present in each period; that is, $|\hat{H}| = |A(t)| = nT$. Throughout this paper, we fix the sets $\hat{H}$ and $A(t)$ for each $t \geq 1$.

For notational simplicity, we introduce a **virtual house**, h_0, which can be assigned to any number of agents. Later we will need to keep track of property rights or endowments assigned in each period. The virtual house will be used to assign no endowment to agents. Let $H := \hat{H} \cup \{h_0\}$. To distinguish houses in $\hat{H}$ from the virtual house, we call a house in $\hat{H}$ a **real house**.

A **period** t **matching**, $\mu(t)$, is an assignment of houses to agents in $A(t)$ such that each agent is assigned one (real or virtual) house and only the virtual house h_0 can be

[11]If $T = 1$, then our model has a different static model in each period, so there is no dynamic issue. Thus, we exclude $T = 1$.

[12]See footnote 3.

[13]A variable indexed by (t) is defined only in period t.

assigned to more than one agent in period $t \geq 1$. For each a in $A(t)$, we refer to $\mu_a(t)$ as the **period t assignment** of agent a under $\mu(t)$. Let $\mathcal{M}(t)$ be the set of all period t matchings. A **matching plan** is a collection of period t matchings from period 1 on, denoted by $\mu := \{\mu(t)\}_{t=1}^{\infty}$. For each a in $A(t)$, we refer to $\mu_a := (\mu_a(t), \mu_a(t+1), \cdots, \mu_a(t+T-1))$ as the **assignment** of agent a under μ. Let $\mathcal{M}$ be the set of all matching plans.

Each initial existing tenant, a, in $N(t)$ has a strict preference relation, R_a, on the product H^{T+t-1}. In other words, R_a is a linear order over H^{T+t-1}.[14] Given assignments μ_a and $\hat{\mu}_a$, $\mu_a \, R_a \, \hat{\mu}_a$ means that agent a weakly prefers μ_a to $\hat{\mu}_a$, and $\mu_a \, P_a \, \hat{\mu}_a$ means that agent a strictly prefers μ_a to $\hat{\mu}_a$ under R_a. On the other hand, a newcomer in period $t \geq 1$ has a strict preference relation, R_a, on the product H^T. In addition, we assume that each agent has a time-separable preference defined as follows:[15]

Definition 19. A preference, R_a, of newcomer a in period $t \geq 1$ is **time-separable** if for each $\tau = t, \cdots, T+t-1$, there exist strict preferences $R_a(\tau)$ on H such that for any two assignments μ_a^1 and μ_a^2 on H^T,

$$\text{if } \forall \tau = t, \cdots, t+T-1, \mu_a^1(\tau) \, R_a(\tau) \, \mu_a^2(\tau) \text{ and } \exists \hat{\tau}, \mu_a^1(\hat{\tau}) \, P_a(\hat{\tau}) \, \mu_a^2(\hat{\tau}), \text{ then } \mu_a^1 \, P_a \, \mu_a^2.$$

The above definition is similarly defined for initial existing tenants. Moreover, $R_a(\tau)$ is called a **period τ preference**. A preference is called **time-invariant** if all period preferences are identical.

The time-separability condition means that preferences between houses in the same period do not depend on the assignment of houses in the other periods. Moreover, we assume that the virtual house is the worst choice for any period preference of each agent.

We write $\mu_a \, R_a \, \mu'_a$ as $\mu \, R_a \, \mu'$ when no confusion arises. Let $\mathcal{R}_a$ be the set of all preference relations of agent a, and $\mathcal{R} := \prod\{\mathcal{R}_a : a \in A\}$ be the set of all preference profiles. Let $\mathcal{R}_a(\tau)$

[14] A linear order is a complete, reflexive, transitive, and antisymmetric binary relation.

[15] As we discussed in the Introduction, our assumption of time-separable strict preference violates the discounted utility (DU) model in two ways. Even if her preference is time-invariant, an agent may prefer improving path of houses over declining paths, which violates the DU model. If not so, a period preference in some period may be affected by houses experienced in prior or future periods, which violates the independence assumption of the DU model. We do not go into details of experimental results on the validity of these assumptions. See Frederick, Lowenstein, and O'Donoghue (2002), especially section 4.2.4 and 4.2.5, for further discussions.

be the set of all period τ preferences of agent a, and $\mathcal{R}(\tau) := \prod\{\mathcal{R}_a(\tau) : a \in A(\tau)\}$ be the set of all period preference profiles for agents present in period τ.

An **endowment profile** is expressed by a matching plan $e := \{e_a\}_{a \in A} \in \mathcal{M}$. An endowment of each agent except the initial existing tenants consists only of the virtual house. We consider two cases: 1) each initial existing tenant has an endowment consisting only of the virtual house, 2) each initial existing tenant has the right to live in *one* real house only in period 1; that is, $e_a = (h, h_0, \cdots, h_0)$ for some real house h and for the virtual house h_0. A **house allocation problem with overlapping agents** or simply, a **dynamic problem** is expressed by (A, H, R, e). The first case (second case) in the above is called a **dynamic problem without endowments (with endowments)**. The problem with endowments is considered for a specific mechanism.[16] Unless stated explicitly, a dynamic problem is either with or without endowments. Throughout this paper, we fix A, H.

As we discussed in the Introduction, instead of the pure RSD mechanism, many universities introduce squatting rights in that existing tenants have the right to extend their lease in order to make on-campus housing more attractive. This motivates the following:

Definition 20. In a dynamic problem without endowments, a matching plan $\{\mu(t)\}_{t=1}^{\infty}$ is **acceptable** if each agent is better off as time goes on. That is:

1. $\forall t$ if $2 - T \leq t \leq 0$, $\forall a \in N(t)$, $\forall \tau = t+1, \cdots, t+T-1$, $\mu(\tau) R_a(\tau) \mu(\tau - 1)$, and

2. $\forall t \geq 1$, $\forall a \in N(t)$, $\forall \tau = t+1, \cdots, t+T-1$, $\mu(\tau) R_a(\tau) \mu(\tau - 1)$.

For a dynamic problem with endowments, it is **acceptable** if the above conditions hold and each initial existing tenant is assigned a house that is at least as good as her endowment in period 1, i.e., $\forall a \in E(1), \mu(1)\, R_a(1)\, e(1)$.

The first condition is for initial existing tenants, and the second is for the other agents.

A matching plan is **Pareto efficient (PE)** if there is no other matching plan that makes all agents weakly better off and at least one agent strictly better off.

[16] The specific mechanism is a spot mechanism with property rights transfer.

4.3 DYNAMIC MECHANISMS

At any given time, the housing office is not able to ask newcomers who will arrive in the future about their preference. In order to reflect preferences, the office cannot determine the houses from the beginning to the future at once. Instead, it determines the assignment in each period. This feature brings about new aspects for mechanism design problems. First, message spaces can take many forms even if we focus on *direct* mechanisms. For example, in each period, the office can ask an agent about her corresponding period preference or her entire preference. Second, in each period, the office can assign not only the houses for the current period but also houses for the future. Finally, in any given period, some of the houses are already assigned, and thus the office has to take into account this past assignment in order to decide on the current assignment.

Generally, for a dynamic problem with or without endowments, a **dynamic mechanism** is a function $\Pi : \mathcal{R} \rightarrow \mathcal{M}$ that determines a matching plan for each preference profile. A dynamic mechanism is **acceptable** if it always selects an acceptable matching plan. Moreover, it is **Pareto efficient** if it always selects a Pareto efficient matching plan.

We restrict attention to two dynamic mechanisms. The first is a **spot mechanism** where, in each period, the office asks each agent present in the period to reveal her corresponding period preference. We also consider a **futures mechanism**. In the first period the office asks all agents present in this period (i.e. initial existing tenants and newcomers in period 1) about their preference. In subsequent periods, the office asks newcomers about their entire preference.

4.3.1 Spot mechanisms

We formally define a spot mechanism by introducing the concepts of a *static problem* and a *static mechanism.*

4.3.1.1 Static mechanisms Fix a dynamic problem with or without endowments, (A, H, R, e). Consider a period $t \geq 1$. A **period t static problem** is defined as $(D(t), U(t), H, R(t), e(t))$.[17] An agent a in $D(t)$ is called an **endowed agent** and occupies a real house, $e_a(t)$, while an agent in $U(t)$ is called an **unendowed agent** and does not have the right to live in any real house. A real house is called **vacant** if it is not occupied by any endowed agent.

In such models, if $D(t) = \emptyset$ and $U(t) = A(t)$, a static problem is a house allocation problem (Hylland and Zeckhauser, 1979). If $D(t) = A(t)$ and $U(t) = \emptyset$, it is a housing market (Shapley and Scarf, 1974). Finally, if $D(t) \neq \emptyset$, $U(t) \neq \emptyset$, and $D(t) \cup U(t) = A(t)$, it is a house allocation problem with existing tenants (Abdulkadiroğle and Sömez, 1999).

Except for a house allocation problem where there is no endowed agent, a matching is **individually rational** if no endowed agent strictly prefers her endowment to her assignment. A matching is **Pareto efficient** if there is no other matching that makes all agents weakly better off and at least one agent strictly better off.

A **period t static mechanism** determines a period t matching for each of both a period preference profile and an endowment profile. That is, it is a function $\gamma^t : \mathcal{R}(t) \times \mathcal{M}(t) \to \mathcal{M}(t)$. It is denoted by $\gamma^t(R(t), e(t))$ for each period preference profile, $R(t)$, and each endowment profile, $e(t)$. A period t static mechanism is **individually rational** (**Pareto efficient**) if it always selects an individually rational (Pareto efficient) period t matching. In addition, it is **strategy-proof** if truth-telling is a dominant strategy in its associated preference revelation game.

4.3.1.2 Spot mechanisms without property rights transfer In this section, we consider a spot mechanism without property rights transfer. To make the mechanism consistent with the problem, we look only at a dynamic problem without endowments.

In this mechanism, $D(t) = 0$ and $U(t) = A(t)$ for each $t \geq 1$, we always have a static house allocation problem in each period. This motivates the following definition:

Definition 21. Given a sequence of static mechanisms $\{\gamma^t\}_{t=1}^{\infty}$, a **spot mechanism without property rights transfer**, $\Pi : \mathcal{R} \to \mathcal{M}$ with $R \mapsto \Pi(R) := (\Pi(R; 1), \Pi(R; 2), \cdots) \in$

[17] Recall that a variable indexed by (t) is defined only in period t.

$\mathcal{M}$, is obtained through static mechanisms as follows: for each period $t \geq 1$,

$$\Pi(R;t) := \gamma^t(R_{A(t)}(t), e(t)),$$

where all $e(t)$'s consist of the virtual house.

4.3.1.3 Spot mechanisms with property rights transfer Unlike in the previous mechanism, we consider the transfer of property rights in either a dynamic problem with or without endowments in a spot mechanism. In each period, the houses occupied by the oldest agents become vacant in the next period, but those occupied by the other agents are inherited as property rights or endowments in the next period.

In a dynamic problem with endowments, we have $\forall t \geq 1$, $D(t) = E(t)$. That is, endowed agents are the existing tenants. On the other hand, in a dynamic problem without endowments, we have $D(1) = \emptyset$ and $U(1) = A(1)$, but $\forall t \geq 2$, $D(t) = E(t)$. In other words, in the first period, there is no endowed agent, but endowed agents are the existing tenants from the second period on. Each agent has a strict period preference, $R(t)$. We let $e(t)$ denote the endowment profile.

Definition 22. Given a sequence of static mechanisms $\{\gamma^t\}_{t=1}^{\infty}$, a **spot mechanism with property rights transfer**, $\Pi : \mathcal{R} \to \mathcal{M}$ with $R \mapsto \Pi(R) := (\Pi(R;1), \Pi(R;2), \cdots) \in \mathcal{M}$, is obtained through static mechanisms as follows: for each preference profile R in $\mathcal{R}$,

1. In period 1,

$$\Pi(R;1) \equiv \mu(1) := \gamma^1\left(R(1), e(1)\right).$$

2. In period $t \geq 2$, set $\hat{e}(t) := \left(\mu_{E(t)}(t-1), e_{N(t)}(t)\right)$,

$$\Pi(R;t) \equiv \mu(t) := \gamma^t\left(R(t), \hat{e}(t)\right).$$

A spot mechanism is thus defined by using a sequence of static mechanisms. The link between the period $t-1$ static mechanism and the period t static mechanism is made possible through the endogenous endowment $\hat{e}(t)$ for $t \geq 2$. This makes it different from just a repetition of a static mechanism. In each period, the current period mechanism depends on the previous mechanism through the assigned endowment. More precisely, in period 1, the office faces a period 1 static market whose endowment corresponds to $e(1)$ from the original dynamic problem. The office asks each agent a present in period 1 about her period 1 preference, $R_a(1)$. Based on the reported period preference profile, $R(1)$, and the endowment, $e(1)$, the office determines a period 1 matching $\Pi(R;1) \equiv \mu(1)$ through the period 1 static mechanism, $\gamma^1(R(1), e(1))$. In the next period, $t=2$, each agent a who is still in the market (i.e., in $E(2)$) has the right to live in the previously assigned house, $\mu_a(1)$. The agents' endowment profile is $\mu_{E(2)}(1)$. Newcomers in period 2 have the virtual endowment. Their endowment profile is $e_{N(2)}(2)$. Thus, we have an endowment profile $\hat{e}(2) := (\mu_{E(2)}(1), e_{N(2)}(2))$ for the period 2 static market. Based on the reported period 2 preference profile, $R(2)$, and the endowment, $\hat{e}(2)$, the office determines a period 2 matching $\Pi(R;2) \equiv \mu(2)$ through the period 2 static mechanism, $\gamma^2(R(2), \hat{e}(2))$. Repeating this process produces the matching plan $\Pi(R) \equiv \{\mu(t)\}_{t=1}^{\infty}$.

4.3.2 Strategy-proofness

Definition 23. A spot or futures mechanism $\Pi : \mathcal{R} \to \mathcal{M}$ is **strategy-proof** if

$$\forall a \text{ in } A, \forall R \text{ in } \mathcal{R}, \forall R'_a \text{ in } \mathcal{R}_a, \quad \Pi(R_a, R_{-a})\, R_a\, \Pi(R'_a, R_{-a}).$$

Given a spot mechanism, agents face an extensive form with simultaneous moves. We are interested in whether they reveal their true period preferences in each period static mechanism. In any given period, revealing the true period preference for an agent does not depend on history, but rather on that period alone. Implicit in the above definition is the restriction of our attention to a class of history-independent strategies. That is, a spot mechanism is strategy-proof if, for each agent, her history-independent strategy of revealing her true period preferences in weakly better than any other history-independent strategy, regardless of the history-independent strategies of the other agents.[18]

On the other hand, a futures mechanism is strategy-proof if truth-telling is a weakly dominant strategy for each agent.

4.3.3 Impossibility result

In this section, we begin with a negative result. We will investigate positive results in later sections. First, we search for a dynamic mechanism that is Pareto efficient and strongly individually rational. The following result rules out the existence of such a mechanism.

Theorem 8. *Consider a dynamic problem with or without endowments. Suppose there are at least three newcomers in each period who live for at least three periods. Then, there is no dynamic mechanism that is Pareto efficient and acceptable.*

Proof. First, we consider a dynamic problem without endowments. Consider the case where there are three newcomers: a_1^t, a_2^t, and a_3^t in each period $t \geq -1$. Agents live for three periods. In each period, there are nine agents a_i^{t-2}, a_i^{t-1} and a_i^t for $i = 1, 2, 3$. There are nine real houses $h_1, \cdots, h_9$. Newcomers in period 1 have preferences satisfying:

a_1^1			a_2^1			a_3^1		
$R_a(1)$	$R_a(2)$	$R_a(3)$	$R_a(1)$	$R_a(2)$	$R_a(3)$	$R_a(1)$	$R_a(2)$	$R_a(3)$
h_1	h_1	h_3	h_3	h_3	h_2	h_2	h_1	h_3
h_3	h_3	h_2	h_1	h_1	h_1	h_3	h_3	h_2
h_2	h_2	h_1	h_2	h_2	h_3	h_1	h_2	h_1

Moreover,

[18] In the definition of strategy-proofness in a *static* mechanism, truth-telling is a weakly dominant strategy for each agent. However, to our knowledge, there is no existing definition of strategy-proofness investigated in our dynamic setting. Instead of requiring truth telling to be a weakly dominant strategy for each agent, we introduce a weaker notion by looking at a class of history-independent strategies in a spot mechanism.

a_1^1	a_2^1	a_3^1
R_a	R_a	R_a
(h, h_2, h_3)	(h, h_2, h_2)	(h, h_1, h_2)
(h, h_1, h_2)	(h, h_3, h_1)	(h, h_1, h_1)
(h, h_1, h_1)		(h, h_2, h_3)
		(h, h_3, h_1)
		(h, h_2, h_2)

where h can be any real house. In the above tables, each column stands for a corresponding preference from best to worst. For example, the period 2 preference $R_a(2)$ of agent a_1^1 is $h_1\, P_a(2)\, h_3\, P_a(2)\, h_2$. In addition, for each other agent in $\{A(1) \cup A(2) \cup A(3)\} \setminus N(1)$, houses h_1, h_2, and h_3 are less preferred to the other houses in each period.

Seeking a contradiction, suppose there exists a Pareto efficient and acceptable matching plan $\mu \equiv \{\mu(t)\}_{t=1}^{\infty}$. First, by Pareto efficiency, we can find possible matching plans $\mu|_{N(1)}$ restricted to $N(1)$. Next, we show that there is no matching plan restricted to $N(1)$ among those that satisfy acceptability.

First, in each period $t = 1, 2, 3$, Pareto efficiency and time-separable preferences imply that newcomers in period 1 are assigned houses among h_1, h_2 and h_3. Next, it follows from Pareto efficiency and time-separable preferences that, for $t = 1, 2, 3$, each period matching $\mu(t)|_{N(1)}$ restricted to $N(1)$ is Pareto efficient in the period t static market $(\{a_1^1, a_2^1, a_3^1\}, \{h_1, h_2, h_3\}, \{R_{a_i^1}(t)\}_{i=1,2,3})$ restricted to $N(1)$. We can find all of such matchings for $\mu(1)|_{N(1)}$, $\mu(2)|_{N(1)}$, and $\mu(3)|_{N(1)}$ as follows:

	$\mu(1)\|_{N(1)}$	$\mu(2)\|_{N(1)}$				$\mu(3)\|_{N(1)}$			
a_1^1	h_1	h_1	h_1	h_2	h_3	h_1	h_2	h_3	h_3
a_2^1	h_3	h_2	h_3	h_3	h_2	h_2	h_1	h_1	h_2
a_3^1	h_2	h_3	h_2	h_1	h_1	h_3	h_3	h_2	h_1

The table above indicates the possible period matchings of each $\mu(t)|_{N(1)}$. For example, $\mu(2)|_{N(1)}$ has four possible matching plans (each column stands for a matching plan).

Now, we consider the possible Pareto efficient matchings satisfy a combination of $\mu(1)|_{N(1)}$, $\mu(2)|_{N(1)}$, and $\mu(3)|_{N(1)}$, as listed in the table above. Using acceptability, we can find three possible matching plans $\mu^1|_{N(1)}$, $\mu^2|_{N(1)}$ and $\mu^3|_{N(1)}$ restricted to $N(1)$:

	$\mu^1\|_{N(1)}$			$\mu^2\|_{N(1)}$			$\mu^3\|_{N(1)}$		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
a_1^1	h_1	h_1	h_3	h_1	h_1	h_1	h_1	h_1	h_2
a_2^1	h_3	h_3	h_1	h_3	h_3	h_2	h_3	h_3	h_1
a_3^1	h_2	h_2	h_2	h_2	h_2	h_3	h_2	h_2	h_3

Now, we show that each of the above is not Pareto efficient. To show that the matching plan $\mu^1|_{N(1)}$ is not Pareto efficient, consider an exchange between agents a_2^1 and a_3^1 in periods 2 and 3. This exchange Pareto dominates $\mu^1|_{N(1)}$, since agent a_2^1 prefers (h_3, h_2, h_2) to (h_3, h_3, h_1) and agent a_3^1 prefers (h_2, h_3, h_1) to (h_2, h_2, h_2).

Next, to show that the matching plan $\mu^2|_{N(1)}$ is not Pareto efficient, consider an exchange between agents a_1^1 and a_3^1 in periods 2 and 3. This exchange Pareto dominates $\mu^2|_{N(1)}$, since agent a_1^1 prefers (h_1, h_2, h_3) to (h_1, h_1, h_1) and agent a_3^1 prefers (h_2, h_1, h_1) to (h_2, h_2, h_3).

Finally, to show that the matching plan $\mu^3|_{N(1)}$ is not Pareto efficient, consider an exchange between agents a_1^1 and a_3^1 in periods 2 and 3. This exchange Pareto dominates $\mu^3|_{N(1)}$, since agent a_1^1 prefers (h_1, h_2, h_3) to (h_1, h_1, h_2) and agent a_3^1 prefers (h_2, h_1, h_2) to (h_2, h_2, h_3). Therefore, we have a contradiction.

For the other case, we can include the previous case to obtain the result. The detailed procedure is in the Appendix.

□

4.4 SERIAL DICTATORSHIP (SD) SPOT MECHANISMS

In this section, we consider a spot mechanism without property rights transfer.

4.4.1 Definition

Since all of the mechanisms examined in this paper are based on an ordering of agents, here we introduce various types of orderings. Given a set $B \subset A$ of agents, an **ordering** in B is a linear order, denoted by f_B. We often denote it as the ordered list:

$$f_B := (b_1, b_2, \cdots, b_m) \text{ if and only if } b_1 \, f_B \, b_2 \, f_B \cdots f_B \, b_m.$$

We say that b_1 is the **first agent in** B, b_2 is the **second agent in** B and so on. In addition, **agent** a **has higher order than agent** b if $a \, f_B \, b$. Specifically, we look at two kinds of orderings. The first is a **period** t **ordering**, $f_{A(t)}$, which is an ordering of $A(t)$, the set of all agents present in period $t \geq 1$. The second is **cohort orderings** $f_{E(1)}$ and $f_{N(t)}$ for $t \geq 1$, where $f_{E(1)}$ is an ordering of $E(1)$ which is the set of all initial existing tenants, while $f_{N(t)}$ is an ordering of $N(t)$ which is the set of all newcomers in period $t \geq 1$.

A **serial dictatorship (SD) spot mechanism** is a spot mechanism without property rights transfer in which each period static mechanism is a **serial dictatorship (SD) static mechanism**. An SD period t static mechanism is based on a period t ordering, $f_{A(t)}$, and is defined as follows. Take any period t ordering, $f_{A(t)}$. Fix a preference profile, $R(t)$, and an endowment profile, $e(t)$. The first agent gets her top choice, the second agent gets her top choice among houses excluding the one assigned to the first agent. The kth agent gets her top choice among houses excluding those assigned to all agents with higher order than her.

It is known that an SD static mechanism is strategy-proof and Pareto efficient (Svensson, 1994). Note that an SD static mechanism is independent of endowments, that is, $\gamma^t(R(t), e(t)) = \gamma^t(R(t), \hat{e}(t))$, $\forall R(t) \in \mathcal{R}(t)$, $\forall e(t), \hat{e}(t) \in \mathcal{M}(t)$. As a result, an existing tenant is not guaranteed to obtain a house that is at least as good as her occupied house in the previous period. Hence, an SD spot mechanism is not acceptable.

4.4.2 Strategy-proofness

We know that an SD *static* mechanism is strategy-proof and Pareto efficient. The question is whether these properties hold for an SD spot mechanism.

Proposition 6. *In a dynamic problem without endowments, an SD spot mechanism is strategy-proof.*

Proof. An SD spot mechanism has no transfer of property rights and consists of SD static mechanisms, and thus each SD period mechanism is independent of the past assignments. Hence, we have the desired result. □

4.4.3 Pareto efficiency: some positive results

In a simple example in the Introduction, we demonstrated Pareto efficiency of an allocation induced by an SD spot mechanism favoring existing tenants. In this subsection, we study under what kind of period orderings the induced SD spot mechanism can achieve Pareto efficiency. To this end, we introduce the following:

Definition 24. *1.* A **period t ordering $f_{A(t)}$ favors existing tenants** if, in period t, each existing tenant has higher order than any newcomer in $f_{A(t)}$. Moreover, it **favors newcomers** if, in period t, each newcomer has higher order than any existing tenant in $f_{A(t)}$.

2. **A sequence of period orderings favors existing tenants (newcomers)** if, in each period t, a period t ordering favors existing tenants (newcomers).

An SD spot mechanism induced by a sequence of period orderings favoring existing tenants (newcomers) is called a **SD spot mechanism favoring existing tenants (newcomers)**.

Definition 25. A sequence of period orderings is **constant** if the relative ranking of agents is the same across periods. That is, if an agent, a, has higher order than another agent, a', in some period, then a has higher order than a' in any other period when they are in the market.

An SD spot mechanism induced by a constant sequence of period orderings is called a **constant SD spot mechanism**. Now, we can state one of the main positive results.

Theorem 9. *In a dynamic problem without endowments, a constant SD spot mechanism favoring existing tenants is Pareto efficient.*

Before proving the theorem, we explore the relation between the period orderings and the cohort orderings for a given constant sequence of periods orderings favoring existing tenants.

The following example illustrates this.

Example. Consider a situation where there are two newcomers, a_1^t and a_2^t, in each period $t \geq -1$. Agents live for three periods. Then, $E(1) = \{a_1^{-1}, a_2^{-1}, a_1^0, a_2^0\}$, $N(t) = \{a_1^t, a_2^t\}$, and $A(t) = N(t-2) \cup N(t-1) \cup N(t)$. Take a sequence $\{f_{A(t)}\}_{t=1}^{\infty}$ of period orderings such that

$$\begin{aligned} f_{A(1)} &= \left(a_1^0,\, a_1^{-1},\, a_2^0,\, a_2^{-1},\, a_1^1,\, a_2^1\right), \\ f_{A(2)} &= \left(a_1^0,\, a_2^0,\, a_1^1,\, a_2^1,\, a_1^2,\, a_2^2\right), \\ f_{A(3)} &= \left(a_1^1,\, a_2^1,\, a_1^2,\, a_2^2,\, a_1^3,\, a_2^3\right). \end{aligned}$$

Notice that this sequence is constant and favors existing tenants. We can take the following cohort orderings:

$$\begin{aligned} g_{E(1)} &= (a_1^0,\, a_1^{-1},\, a_2^0,\, a_2^{-1}), \\ g_{E(1)}|_{A(2)} &= (a_1^0,\, a_2^0), \\ g_{N(t)} &= (a_1^t,\, a_2^t), \quad \text{for each } t = 1, 2, 3. \end{aligned}$$

Notice that

$$\begin{aligned} f_{A(1)} &= (g_{E(1)}, g_{N(1)}), \\ f_{A(2)} &= (g_{E(1)}|_{A(2)}, g_{N(1)}, g_{N(2)}), \\ f_{A(3)} &= (g_{N(1)}, g_{N(2)}, g_{N(3)}). \end{aligned}$$

The corresponding cohort orderings are denoted by using f instead of g. □

In summary, we have the following lemma (the proof is straightforward).

Lemma 11. *Given a constant sequence $\{f_{A(t)}\}_{t=1}^{\infty}$ of period ordering favoring existing tenants, there are corresponding cohort orderings $f_{E(1)}$ and $\{f_{N(t)}\}_{t\geq 1}$ such that*

$$\begin{aligned} f_{A(t)} &= (f_{E(1)}|_{A(t)}, f_{N(1)}, \cdots, f_{N(t)}), \quad \forall t = 1, \cdots, T-1, \text{ and} \\ f_{A(t)} &= (f_{N(t-T+1)}, \cdots, f_{N(t)}), \quad \forall t \geq T. \end{aligned}$$

Now we are ready to prove Theorem 9.

Proof of Theorem 9. Let $\{f_{A(t)}\}_{t=1}^{\infty}$ be given. From Lemma 11, let a sequence $(f_{E(1)}, \{f_{N(t)}\}_{t=1}^{\infty})$ give the corresponding cohort orderings. Let $\mu = \{\mu(t)\}_{t=1}^{\infty}$ be a matching plan generated by a constant SD spot mechanism for some arbitrary preference profile R. To find a contradiction, suppose some matching plan ν Pareto dominates μ. Then,

$$\forall a \in A, \nu R_a \, \mu \quad \text{and } \exists b \in A, \nu P_b \, \mu.$$

Since $A = E(1) \cup (\cup_{t=1}^{\infty} N(t))$, either $b \in E(1)$ or $b \in N(t)$ for some $t \geq 1$. We consider two cases:

Case 1: $b \in E(1)$.

Take an agent $c \in E(1)$ who has the highest order among agents in $\{b \in E(1) : \nu \, P_b \, \mu\}$ with respect to $f_{E(1)}$. Then, since preferences are strict, it follows that

$$\forall a \in E(1) \text{ who has higher order than } c \text{ in } f_{E(1)}, \nu_a = \mu_a. \tag{4.1}$$

Let $\tau \leq 0$ such that $c \in N(\tau)$. Now it is sufficient to show $\forall t = 1, \cdots, T-1+\tau$, $\mu(t) \, R_c(t) \, \nu(t)$. It then follows from time-separable preferences that $\mu R_c \, \nu$, which is a contradiction. For each period t, for an SD static mechanism, given Lemma 11 and (4.1), we have that there is no room for agent c to be strictly better off than $\mu_c(t)$. Hence, $\mu(t) \, R_c(t) \, \nu(t)$.

Case 2: $b \notin E(1)$ and $b \in N(t)$, for some $t \geq 1$.

Take the smallest $\tau \geq 1$ such that $\exists b \in N(\tau)$ with $\nu \, P_b \, \mu$. Choose an agent $c \in N(\tau)$ who has the highest order among agents in $\{b \in N(\tau) : \nu \, P_b \, \mu\}$ with respect to $f_{N(\tau)}$. Then, it follows from strict preferences that

$$\begin{aligned} &\forall a \in E(1) \cup \left(\cup_{t=1}^{\tau-1} N(t)\right), \, \nu_a = \mu_a, \text{ and} \\ &\forall a \in N(\tau) \text{ who has higher order than } c \text{ in } f_{N(\tau)}, \, \nu_a = \mu_a. \end{aligned} \tag{4.2}$$

Now, it is sufficient to show that $\forall t = \tau, \cdots, \tau + T - 1$, $\mu(t) \, R_c(t) \, \nu(t)$. It then follows from time-separable preferences that $\mu R_c \nu$, which is a contradiction. For each period t, for an SD static mechanism, given Lemma 11 and (4.2), we have that there is no room for agent c to be strictly better off than $\mu_c(t)$. Hence, $\mu(t) \, R_c(t) \, \nu(t)$.

□

4.4.4 When is an SD spot mechanism undesirable?

As we saw in the Introduction, Pareto efficiency depends on the ordering structure.

Theorem 10. *In a dynamic problem without endowments, an SD spot mechanism favoring newcomers is not Pareto efficient even under time-invariant preferences.*

Here **time-invariant preferences** mean that each agent has a time-invariant preference.

Proof. Suppose agents have time-invariant preferences and live for T periods. Fix a sequence of period orderings that favors newcomers. Pick the first agent in $f_{A(t)}$ among newcomers in period $t \geq 1$. Each agent a has the same time-invariant preference, where a house h_1 is her top choice, such that

$$(k_1, h_1, \mu_a^{t+2}) P_a (h_1, k_2, \mu_a^{t+2}) \quad \forall k_1, k_2 \neq h_1, \tag{4.3}$$

where $\mu_{a^t}^{t+2}$ is any assignment of agent a^t from period $t+2$ to $t+T-1$. Then, an SD mechanism favoring newcomers assigns houses (without parentheses below) to agents a^t, $t \geq 1$, as follows.

	$t=1$	$t=2$	$t=3$	$t=4$	$\cdots$
a^1	h_1	k_1 (h_1)	$\cdots$		
a^2		h_1 (k_1)	k_2 (h_1)	$\cdots$	
a^3			h_1 (h_2)	k_3 (h_1)	$\cdots$
$\vdots$				$\vdots$	$\vdots$

Here $\{k_t\}_{t=1}^{\infty}$ is some sequence in the set $\{h_2, \cdots, h_T\}$. Consider an infinite exchange of houses h_1 and k_{t-1} between the existing tenant a^{t-1} and the newcomer a^t for $t \geq 2$, keeping the assignments of the other agents be the same. This exchange is shown as houses inside the parentheses on the above table. It follows from (4.3) that the resulting allocation Pareto dominates the induced matching plan. □

4.5 TOP TRADING CYCLES (TTC) SPOT MECHANISMS

In this section, we consider a spot mechanism with property rights transfer for a dynamic problem with or without endowments.

4.5.1 Definition

In a spot mechanism with property rights transfer, we have a house allocation problem with existing tenants in each period. As an example, a random serial dictatorship mechanism with squatting rights is widely used in on-campus housing for college students. In a static setting, a deterministic serial dictatorship mechanism with squatting rights is not Pareto efficient. To restore Pareto efficiency while satisfying individual rationality and strategy-proofness, Abdulkadiroğlu and Sönmez (1999) propose a mechanism referred to as **Abdulkadiroğlu and Sönmez's top trading cycles (AS-TTC) static mechanisms**: Fix a period t ordering, $f_{A(t)}$. For any announced preference profile, $R(t)$, and an endowment profile, $e(t)$, the AS-TTC static mechanism selects a matching through the following **AS-TTC algorithm**:

Assign the first agent her top choice, the second agent her top choice among the remaining houses, and so on, until an agent a demands house $h_{a'}$ of an endowed agent. If at that point the endowed agent whose house is demanded is already assigned a house, then do not disturb the procedure. Otherwise modify the remainder of the ordering by inserting the agent in question to the top and continue the procedure. Similarly, insert any endowed agent who is not already served at the top of the line once her house is demanded. If at any point a loop forms, it is formed by exclusively endowed agents and each of them demands the house of the endowed agent next in the loop. (A **loop** is an ordered list of agents, $(a_1, a_2, \cdots, a_k)$, where agent a_1 demands the house of agent a_2, agent a_2 demands the house of agent a_3, $\cdots$, agent a_k demands the house of a_1.) In such cases, remove all agents in the loop by assigning them the houses they demand and proceed.

Theorem 11 (Abdulkadiroğlu and Sönmez, 1999)**.** *For any ordering, $f_{A(t)}$, the induced AS-TTC static mechanism is Pareto efficient, individually rational, and strategy-proof in a static* *problem.*

Note that, from the above procedure, any AS-TTC static mechanism is individually rational. This is because an endowment of an endowed agent will not be assigned to another agent before this endowed agent is assigned a house. If another agent demands the endowment of this endowed agent, she will be promoted to the top of the ordering. While at the top of the ordering, if there is no house available better than her endowment, then existing tenants demand her own house. At this point, a trivial loop consisting of this agent will form, and she will leave and be assigned at worst her own endowment.

A **top trading cycles (TTC) spot mechanism** is a spot mechanism with property rights transfer in which each period static mechanism is an AS-TTC static mechanism, given a sequence of period orderings. Clearly, this TTC spot mechanism is acceptable, since an AS-TTC static mechanism is individually rational in each period.

4.5.2 Strategy-proofness: some positive results

We know from Theorem 11 that an AS-TTC *static* mechanism satisfies individual rationality, strategy-proofness, and Pareto efficiency. Because acceptability, which can be seen as a counterpart of individual rationality, always holds for a TTC spot mechanism, the question is whether these properties hold in our dynamic problem. By Impossibility Theorem 8, a TTC spot mechanism is not Pareto efficient in general. To answer the above question, we restrict the preference domain to time-invariant preferences. *Throughout this section below, we assume that each agent has a time-invariant preference.*[19] We emphasize that this is not just a repetition of an AS-TTC static mechanism for a corresponding static problem. There are two features that differ from a static problem. First, we have both entry and exit of different agents in each period. Second, the endowment is endogenous. One of our main positive results is the following.

Theorem 12. *Consider a dynamic problem with endowments and time-invariant prefer-*

[19] Under this assumption, it is sufficient in a spot mechanism that the housing office asks each agent about her period preference, not in all periods, *once* when she arrives. However, we can allow the office to do so in each period. For the definition of strategy-proofness, we take the latter approach. Since the latter is stronger than the former in the definition, all results are not affected.

ences. Then, a constant TTC spot mechanism favoring existing tenants is strategy-proof among all agents except initial existing tenants. It can be manipulated by initial existing tenants, provided there are at least three newcomers in each period who live for at least three periods.

Before proving the theorem, we review some of the new concepts stated in Theorem 12. We say that a spot mechanism $\Pi : \mathcal{R} \to \mathcal{M}$ is **strategy-proof among all agents except initial existing tenants** if for each agent a who is not an initial existing tenant, $\forall R \in \mathcal{R}$, $R'_a \in \mathcal{R}_a$, $\Pi(R_a, R_{-a})\, R_a\, \Pi(R'_a, R_{-a})$. As in strategy-proofness, we restrict attention to a class of history-independent strategies.

To prove the theorem, we introduce some additional concepts of *effective ordering* introduced by Sönmez and Ünver (2005). For each ordering $f_{A(t)}$, the AS-TTC algorithm assigns houses in one of two possible ways:

1. There is a sub-order $(a^1, \cdots, a^k)$ of agents where a^1 demands the house of a^2, a^2 demands the house of a^3, $\cdots$, agent a^{k-1} demands house of a^k, and a^k demands any available house. We call such a sub-order a **serial-order** (S).
2. There is a sub-order $(a^1, \cdots, a^k)$ of endowed agents where a^1 receives a^k's house, a^k receives a^{k-1}'s house, $\cdots$, a^2 receives a^1's house. Recall that we call such sub-order a **loop-order** (L).

For a given ordering, $f_{A(t)}$, construct the **effective ordering**, e_t, as follows: Run the AS-TTC algorithm and order agents in the order their assignments are finalized. When there is a loop-order, order these agents as in the loop-order.

Note that a matching produced by an AS-TTC algorithm with the effective ordering yields the same outcome produced by an SD static mechanism induced by this effective ordering. Also note that the effective ordering is endogenous, depending on preferences and the exogenous ordering $f_{A(t)}$.

We now examine how effective orderings behave under time-invariant preferences for a constant sequence of period orderings favoring existing tenants. Fix a preference profile R and a constant sequence $\{f_{A(t)}\}_{t=1}^{\infty}$ of period orderings that favors existing tenants. Let $(f_{E(1)}, \{f_{N(t)}\}_{t=1}^{\infty})$ be a sequence of its corresponding cohort orderings. For convenience, we use

$$f_{N(t)} := (a_1^t, a_2^t, \cdots, a_n^t)$$

for each $t \geq 1$. Observe that in period 1,

$$e_1 = \left(\overbrace{\underbrace{X, \cdots, X}_{\text{initial existing tenants}}}^{E(1)}, \overbrace{\underbrace{S}_{a_1^1}, \underbrace{S}_{a_2^1}, \cdots, \underbrace{S}_{a_n^1}}^{N(1)} \right) = \left(\overbrace{X, \cdots, X}^{E(1)}, f_{N(1)} \right),$$

where X is either S or L. Recall that S stands for a serial-order and L for a loop-order. That is, initial existing tenants are before newcomers, since newcomers do not have any endowment and the ordering $f_{A(1)}$ favors existing tenants. Moreover, because each newcomer has no endowment, she will point to an available house and form a serial-order consisting of herself.

Now, consider period 2. First, existing tenants in period 2 (who are in $E(2)$) have higher order than newcomers in the effective ordering e_2. Second, in period 1, initial existing tenants prefer their assignment to those assigned to agents in $N(1)$. Since their assignment becomes an endowment in period 2, it follows from time-invariant preferences that initial existing tenants never point to the houses of agents in $N(1)$ in the algorithm. This implies that initial existing tenants have higher order than agents in $N(1)$ in e_2. Third, since period orderings are constant, agent a_1^1 never points to agent a_i^1 $(i \geq 2)$, but points to her own house or an available house. The same applies for other agents in $N(1)$. In summary,

$$e_2 = \left(\overbrace{\overbrace{\underbrace{X, \cdots, X}_{\text{initial existing tenants}}}^{E(1)\cap A(2)}, \overbrace{\underbrace{X}_{a_1^1}, \cdots, \underbrace{X}_{a_n^1}}^{N(1)}}^{E(2)}, \overbrace{\underbrace{S}_{a_1^2}, \cdots, \underbrace{S}_{a_n^2}}^{N(2)} \right) = \left(\overbrace{X, \cdots, X}^{E(1)\cap A(2)}, f_{N(1)}, f_{N(2)} \right).$$

That is, each existing tenant in $N(1)$ forms either a trivial loop-order consisting of herself, or a serial order in which all agents in the serial-order receive a better house than their assignment received in period 1. On the other hand, each newcomer forms a serial-order consisting of herself in the algorithm, because she does not have any endowment.

Repeating this process, in period $\tau = 2, \cdots, T-1$,

$$\begin{aligned} e_\tau &= \left(\overbrace{\overbrace{\underbrace{X, \cdots, X}_{\text{initial existing tenants}}}^{E(1)\cap A(\tau)}, \overbrace{\underbrace{X}_{a_1^1}, \cdots, \underbrace{X}_{a_n^1}}^{N(1)}, \cdots, \overbrace{\underbrace{X}_{a_1^{\tau-1}}, \cdots, \underbrace{X}_{a_n^{\tau-1}}}^{N(\tau-1)}}^{E(\tau)}, \overbrace{\underbrace{S}_{a_1^\tau}, \cdots, \underbrace{S}_{a_n^\tau}}^{N(\tau)} \right), \\ &= \left(\overbrace{X, \cdots, X}^{E(1)\cap A(\tau)}, f_{N(1)}, \cdots, f_{N(\tau-1)}, f_{N(\tau)} \right). \end{aligned}$$

Similarly, in period $\tau \geq T$,

$$\begin{aligned} e_\tau &= \left(\overbrace{\overbrace{\underbrace{X}_{a_1^{\tau-T+1}}, \cdots, \underbrace{X}_{a_n^{\tau-T+1}}}^{N(\tau-T+1)}, \cdots, \overbrace{\underbrace{X}_{a_1^{\tau-1}}, \cdots, \underbrace{X}_{a_n^{\tau-1}}}^{N(\tau-1)}}^{E(\tau)}, \overbrace{\underbrace{S}_{a_1^\tau}, \cdots, \underbrace{S}_{a_n^\tau}}^{N(\tau)} \right), \\ &= \left(f_{N(\tau-T+1)}, \cdots, f_{N(\tau-1)}, f_{N(\tau)} \right). \end{aligned}$$

Now we are ready to prove Theorem 12.

Proof of Theorem 12. Fix a preference profile, R. Consider any agent, a, who is not an initial existing tenant. Consider any other preference, $\hat{R}_a$. Let $\mu := \{\mu(t)\}_{t=1}^\infty$ and $\hat{\mu} := \{\hat{\mu}(t)\}_{t=1}^\infty$ be matching plans induced by a constant TTC spot mechanism favoring existing tenants for (R_a, R_{-a}) and $(\hat{R}_a, R_{-a})$. In each period t from $(\hat{R}_a, R_{-a})$, when agent a is in the market, the effective ordering of agents who have higher order than agent a is not affected. Thus, agent a can get a house that makes her indifferent or worse than $\mu_a(t)$. That is, $\mu_a(t)\, R_a(t)\, \hat{\mu}_a(t)$. By time-separability of preferences, $\mu_a\, R_a\, \hat{\mu}_a$. This completes the proof of the first part.

For the second part, suppose there are at least three newcomers in each period $t \geq 2-T$. They live for at least three periods, T. Fix a constant sequence $\{f_{A(t)}\}_{t=1}^\infty$ of period orderings that favors existing tenants. In light of the proof of the first part, we focus on initial existing tenants. Pick agents a_i^{2-T} and a_i^{3-T}, $i = 1, 2, 3$, such that

$$\begin{aligned} f_{A(1)}|_{\{a_i^{2-T}, a_i^{3-T}: i=1,2,3\}} &:= (a_1^{2-T}, a_2^{2-T}, a_3^{2-T}, a_1^{3-T}, a_2^{3-T}, a_3^{3-T}), \\ f_{A(2)}|_{\{a_i^{3-T}: i=1,2,3\}} &:= (a_1^{3-T}, a_2^{3-T}, a_3^{3-T}). \end{aligned}$$

Note that a_i^{2-T} lives only in period 1, and a_i^{3-T} lives only in periods 1 and 2, $i = 1, 2, 3$. Period preferences satisfy the table on the left hand side (from best to worst):

a_1^{2-T}	a_2^{2-T}	a_3^{2-T}	a_1^{3-T}	a_2^{3-T}	a_3^{3-T}	a
h_1	h_5	h_3	h_6	h_1	h_1	h
			h_4	h_2	h_2	h_3, h_4, h_5
					h_6	h_1, h_2, h_6

a_2^{3-T}
h_1
h_6

where h is any real house other than houses h_1 to h_6, and a is an agent except a_i^{2-T}, a_i^{3-T}, $i = 1, 2, 3$. The above table means that agent a prefers h to any of h_3, h_4, h_5, and prefers any of h_3, h_4, h_5 to any of h_1, h_2, h_6. Moreover, the a_2^{3-T}'s preference satisfies

$$(h_6, h_1) \, P_{a_2^{3-T}} \, (h_2, h_2),$$

Endowments are indicated with the parentheses in the first column on Table 2. We will see that a_2^{3-T} manipulates the mechanism by reporting the preference described on the right hand side of the above table.

At period 1, whether a_2^{3-T} manipulates or not, any agent a, who is not a_i^{2-T}, a_i^{3-T} $i = 1, 2, 3$, never points to houses h_1 to h_6 in an AS-TTC algorithm. Thus, we concentrate on a restricted static market consisting of agents a_i^{2-T}, a_i^{3-T}, $i = 1, 2, 3$ and houses h_1 to h_6 in the algorithm. The procedures to obtain period 1 matchings are illustrated in Figure 11.

In the next period $t = 2$, a_1^{3-T} owns h_4, and a_2^{3-T}, a_3^{3-T} own h_2, h_6. In an AS-TTC algorithm, whether a_2^{3-T} manipulates or not, any agent who are not $a_1^{3-T}, a_2^{3-T}, a_3^{3-T}$ never points to h_1, h_2, h_6. Thus, the first agent among $a_1^{3-T}, a_2^{3-T}, a_3^{3-T}$ who points a house is a_1^{3-T} by either being pointed by the other agent or not. The procedure after a_1^{3-T} has an opportunity to choose a house is depicted in Figure 12.

The resulting assignments for agents a_i^{2-T}, a_i^{3-T}, $i = 1, 2, 3$ are described in Table 2. Thus, a_2^{3-T} obtains an assignment (h_6, h_1) from lying, while she obtains a worse assignment (h_2, h_2) from truth-telling.

Finally, consider why agent a_2^{3-T} manipulates the mechanism. Given that agent a_1^{3-T} points to h_6 in $t = 2$ and an agent whose assigned house is assigned h_6 in $t = 1$ and becomes an endowment in $t = 2$ can be upgraded in $t = 2$, agent a_2^{3-T} lies so that she can obtain a worse house, h_6, in $t = 1$, but a better house, h_1, at $t = 2$. $\square$

We state two corollaries:

Corollary 4. *Consider a dynamic problem with endowments and time-invariant preferences. Suppose each agent lives for two periods. Then, a constant TTC spot mechanism favoring existing tenants is strategy-proof.*

Proof. Initial existing tenants live for only one period. Since the static mechanism is strategy-proof, truth-telling is a dominant strategy for each initial existing tenant. □

Corollary 5. *Consider a dynamic problem without endowments and with time-invariant preferences. Then, a constant TTC spot mechanism favoring existing tenants is strategy-proof.*

Proof. Note that the induced effective ordering takes the same form for agents, except initial existing tenants as the one in a dynamic problem with endowments. Thus, consider an induced effective ordering $e_\tau|_{E(1)\cap A(\tau)}$ restricted to initial existing tenants for each period $\tau = 1, \cdots, T-1$. Similar arguments to the above lead to $e_\tau|_{E(1)\cap A(\tau)} = f_{E(1)}|_{A(\tau)}$. Using the same logic as Theorem 12, we can conclude that the history-independent strategy of true period preferences is weakly better off than any other history-independent strategy for each initial existing tenant. □

4.5.3 How can a TTC spot mechanism be manipulated by agents who are not initial existing tenants?

Remember that an SD spot mechanism is strategy-proof. This is because, in each period, it ignores an endowment or the past assignment. On the other hand, a TTC spot mechanism guarantees each agent a house that is at least weakly better than the previously assigned house. This opens up the possibility of manipulation in which an agent obtains a worse house than she can obtain in truth-telling, expecting her to be upgraded in an ordering by being pointed out by some other agent in the next period. As we saw, a constant TTC spot mechanism favoring existing tenants effectively excludes such a possibility. However, this is not the case if it favors newcomers.

Theorem 13. *Consider a dynamic problem with time-invariant preferences either with endowments or without endowments. Suppose there are at least two newcomers in each period. Then, a TTC spot mechanism favoring newcomers is not strategy-proof among all agents except initial existing tenants.*

Proof. Suppose there are at least two newcomers in each period $t \geq 2-T$. Agents live for T periods. Pick two newcomers a_1^t and a_2^t in each period. Fix a sequence of period orderings that favors newcomers. Without loss of generality, a_1^t has higher order than a_2^t in each period t. Period preferences $R_{a_i^t}(t)$ $(t = 2-T, 3-T, 2, 3;\ i = 1, 2)$ of each agent a_i^t satisfy the table on the left hand side (from best to worst):

a_1^{2-T}	a_2^{2-T}	a_1^{3-T}	a_2^{3-T}	a_1^2	a_2^2	a_1^3	a_2^3
h_2	h_3	h_1	h_4	h_1	h_1	h_2	h_4
				h_3	h_2		
				h_2			

a_1^2
h_1
h_2
h_3

For the other agents, houses h_1 to h_4 are less preferred to any other house. Moreover, agent a_1^2's preference satisfies

$$(h_2, h_1, \mu^4_{a_1^2}) P_{a_1^2} (h_3, h_3, \mu^4_{a_1^2}),$$

where $\mu^4_{a_1^2}$ is any assignment of agent a_1^2 from period 4 on. Unspecified preferences are assumed to be arbitrary.

Endowments are indicated by the parentheses in the first column on the table below. If $T = 2$, agents a_1^{3-T} and a_2^{3-T} are not initial existing tenants but newcomers in period 1. However, the allocations to be calculated will not be affected, even for the case without endowments, because of preferences.

In each period, we concentrate on a static problem consisting of agents $a_i^{2-T}, a_i^{3-T}, a_i^2, a_i^3$, $i = 1, 2$, since houses h_1 to h_4 are less preferred to any other house for each of the other agents.

We will see that agent a_1^2 manipulates the mechanism by reporting the preference described on the right hand side of the above table.

	$t=1$	$t=2$	$t=3$	$\cdots$
a_1^{2-T} (h_2) a_2^{2-T} (h_3)	h_2 h_3			
a_1^{3-T} (h_1) a_2^{3-T} (h_4)	h_1 h_4	h_1 h_4		
$\vdots$		$\vdots$		
$\boldsymbol{a_1^2}$ a_2^2		$\boldsymbol{h_3}$ h_2	$\boldsymbol{h_3}$ h_1	$\cdots$ $\cdots$
a_1^3 a_2^3			h_2 h_4	$\cdots$ $\cdots$
$\vdots$				$\ddots$

	$t=1$	$t=2$	$t=3$	$\cdots$
a_1^{2-T} (h_2) a_2^{2-T} (h_3)	h_2 h_3			
a_1^{3-T} (h_1) a_2^{3-T} (h_4)	h_1 h_4	h_1 h_4		
$\vdots$		$\vdots$		
$\boldsymbol{a_1^2}$ a_2^2		$\boldsymbol{h_2}$ h_3	$\boldsymbol{h_1}$ h_3	$\cdots$ $\cdots$
a_1^3 a_2^3			h_2 h_4	$\cdots$ $\cdots$
$\vdots$				$\ddots$

The left hand side shows an allocation by the TTC spot mechanism when agent a_1^2 reveals her true preference, while the right hand side shows an allocation by the TTC spot mechanism when a_1^2 lies. Note that, whether a_1^2 has higher order than a_2^2 in the period 3 ordering or not, the above assignments are not affected. The procedures to obtain each allocation for period 3 static markets are illustrated in Figure 13 in the case that a_1^2 has higher order than a_2^2 in the period 3 ordering.

Thus, a_1^2 obtains an assignment $(h_2, h_1, h_1, \cdots, h_1)$ from lying, while she obtains a worse assignment $(h_3, h_3, \mu^4_{a_1^2})$ from truth-telling, where $\mu^4_{a_1^2}$ is some assignment of a_1^2 from period 4 on.

Consider why agent a_1^2 manipulates the mechanism. Given that newcomer a_1^3 points to h_2 in $t=3$, and an agent whose assigned house is assigned h_2 in $t=2$ and becomes an endowment in $t=3$ can be upgraded in $t=3$, agent a_1^2 lies so that she can obtain a worse h_2 in $t=2$, but a better house h_1 in $t=3$.

□

The reason for the failure of strategy-proofness in the previous proof is that, provided that a newcomer has a favorable house, by lying, an existing tenant obtains this house in the previous period and in the next period she gets a better house by being pointed by the newcomer. As we saw in Theorem 12, such an opportunity for all agents except initial existing tenants is excluded by making period orderings favor existing tenants.

To contrast a TTC spot mechanism favoring newcomers with the one favoring existing tenants and an SD spot mechanism, see the last section of the Summary.

4.5.4 Pareto efficiency: some positive results

We now turn our attention to Pareto efficiency. We saw in the previous subsection that strategy-proofness among all agents except initial existing tenants makes a difference between a constant TTC spot mechanism favoring newcomers and the one favoring existing tenants. Similarly, we introduce a weaker notion for Pareto efficiency.

Definition 26. A matching plan ν **Pareto dominates** another matching plan μ **among all agents except initial existing tenants** if

1. $\{\mu_a(t) : a \in A \setminus E(1)\} = \{\nu_a(t) : a \in A \setminus E(1)\}$ for each $t \geq 1$, and
2. $\forall a \in A \setminus E(1)$, $\nu R_a \mu$ and $\exists a \in A \setminus E(1)$, $\nu P_a \mu$.

Moreover, a matching plan is **Pareto efficient among all agents except initial existing tenants** if it is not Pareto dominated by any other matching plan among all agents except initial existing tenants.

As with strategy-proofness, a constant TTC spot mechanism favoring existing tenants is Pareto efficient among all agents except initial existing tenants, but not Pareto efficient.

Theorem 14. *Consider a dynamic problem with endowments and time-invariant preferences. Then, a constant TTC spot mechanism favoring existing tenants is Pareto efficient among all agents except initial existing tenants, but not Pareto efficient, provided there are at least two newcomers in each period who live for at least three periods.*

Proof. For the first part, let $\mu = \{\mu(t)\}_{t=1}^{\infty}$ be a matching plan generated by a constant TTC spot mechanism favoring existing tenants for some arbitrary preference profile, R. Let $\{e_t\}_{t=1}^{\infty}$ be a corresponding sequence of effective orderings. To find a contradiction, suppose some matching plan, ν, Pareto dominates μ among all agents except initial existing tenants. Then,

$$\forall a \in A \setminus E(1), \nu R_a \mu \quad \text{and } \exists a \in A \setminus E(1), \nu P_a \mu.$$

Since $A \setminus E(1) \equiv \cup_{t=1}^{\infty} N(t)$, take the smallest $\tau \geq 1$ such that $\exists a \in N(\tau)$, $\nu P_a \, \mu$. It follows from strict preferences that

$$\forall t \leq \tau - 1, \forall a \in N(t), \nu_a = \mu_a. \tag{4.4}$$

In addition, take an agent $b \in N(\tau)$ who has the highest order among agents in $\{a \in N(\tau) : \nu P_a \, \mu\}$. Then, it follows from strict preferences that

$$\forall a \in N(\tau) \text{ who has a higher order than } b \text{ does}, \nu_a = \mu_a. \tag{4.5}$$

Now, it is sufficient to show that $\forall t = \tau, \cdots, \tau + T - 1$, $\mu(t) R_b(t) \nu(t)$, since this leads to a contradiction, namely, that $\mu R_b \, \nu$ and $\nu P_b \, \mu$. For each $t = \tau, \cdots, \tau + T - 1$, it follows from (4.4) and (4.5) that in the effective ordering e_t, each agent, a, ordered before agent b has $\nu_a(t) = \mu_a(t)$. Thus, the AS-TTC algorithm implies that there is no room for agent b to be strictly better off than $\mu_b(t)$. Hence, $\mu(t) R_b(t) \nu(t)$.

For the second part, suppose there are at least two newcomers in each period who live for at least three periods, T. Fix a constant sequence, $\{f_{A(t)}\}_{t=1}^{\infty}$, of period orderings that favors existing tenants. Pick initial existing tenants a_1^{2-T}, a_2^{2-T}, a_1^{3-T}, and a_2^{3-T} such that

$$f_{A(1)}|_{\{a_1^{2-T}, a_2^{2-T}, a_1^{3-T}, a_2^{3-T}\}} = (a_1^{2-T}, a_2^{2-T}, a_1^{3-T}, a_2^{3-T}), \text{ and } f_{A(2)}|_{\{a_1^{3-T}, a_2^{3-T}\}} = (a_1^{3-T}, a_2^{3-T}).$$

Note that a_i^{2-T} lives only in period 1, and a_i^{3-T} lives only in period 1 and 2, $i = 1, 2$. Each agent $a_i^t \neq a_2^{2-T}$ has an identical preference (from best to worst):

$$P_{a_i^t}(t) : h_1, h_2, h_3.$$

Agent a_2^{2-T}'s top choice is h_4. For the other agents, houses h_1 to h_4 are less preferred to any other house. Moreover,

$$(h_2, h_2) \, P_{a_1^{3-T}} \, (h_3, h_1) \quad \text{and} \quad (h_3, h_1) \, P_{a_2^{3-T}} \, (h_2, h_2).$$

Endowments are indicated in the first column on the table below.

The induced TTC spot mechanism produces the following assignments:

	$t=1$	$t=2$	$t=3$	$\cdots$
a_1^{2-T} (h_1) a_2^{2-T} (h_2)	h_1 h_4			
a_1^{3-T} (h_3) a_2^{3-T} (h_4)	h_3 (h_2) h_2 (h_3)	h_1 (h_2) h_2 (h_1)		
$\vdots$				

Consider another matching plan in which a_1^{3-T} exchanges the first two periods assignments (h_3, h_1) for (h_2, h_2) with a_2^{3-T}. This exchange is described by houses inside the parentheses on the above table. This matching plan Pareto dominates the one induced by the TTC spot mechanism.

□

We state two corollaries:

Corollary 6. *Consider a dynamic problem with endowments and time-invariant preferences. Suppose each agent lives for two periods. Then, a constant TTC spot mechanism favoring existing tenants is Pareto efficient.*

Proof. Pareto efficiency among all agents except initial existing tenants does not consider any matching that involves an exchange between initial existing tenants and the other agents. However, when agents live for two periods, initial existing tenants live for only one period. Since a static AS-TTC spot mechanism is Pareto efficient, any other matching plan involving such a exchange necessarily hurts the initial existing tenants. Note that this logic does not work for the case where agents live for at least three periods. Thus, any matching plan induced by a constant TTC spot mechanism favoring existing tenants is Pareto efficient.

□

Corollary 7. *Consider a dynamic problem without endowments and with time-invariant preferences. Then, a constant TTC spot mechanism favoring existing tenants is Pareto efficient.*

Proof. The same argument applies on the induced effective ordering as the one in Corollary 5. Using the same logic used in Case 2 in the proof of Theorem 9, we obtain the desired result. □

4.5.5 When is a TTC spot mechanism undesirable?

In an example taken up in Theorem 10 that shows Pareto inefficiency in an SD spot mechanism favoring newcomers, we demonstrated that an infinite exchange between existing tenants and newcomers Pareto dominates a matching plan induced by the SD spot mechanism. Looking at this example closely, we might think that acceptability precludes such an infinite exchange. Since a TTC spot mechanism satisfies acceptability, one might conjecture that a TTC spot mechanism favoring newcomers is Pareto efficient. However, this is not the case, as shown in the following theorem:

Theorem 15. *Consider a dynamic problem with time-invariant preferences either with endowments or without endowments. Suppose there are at least two newcomers in each period. Then, a TTC spot mechanism favoring newcomers is not Pareto efficient among all agents except initial existing tenants.*

Proof. Suppose there are at least two newcomers in each period $t \geq 2 - T$. They live for T periods. Pick two newcomers a_1^t and a_2^t in each period. Fix a sequence of period ordering $\{f_{A(t)}\}_{t=1}^{\infty}$ that favors newcomers. Without loss of generality, a_1^t is the first agent in $f_{A(t)}$ in each period. Note that this sequence may not be constant; e.g., a_1^t may not be the first in the subsequent periods. Period preferences satisfy: For each $m \geq 0$,

a_1^{2Tm+2}	a_2^{2Tm+2}	a_1^{2Tm+3}	a_2^{2Tm+3}	$a_1^{T(2m+1)+2}$	$a_2^{T(2m+1)+2}$	$a_1^{T(2m+1)+3}$	$a_2^{T(2m+1)+3}$
h_2	h_2	h_3	h_3	h_1	h_1	h_4	h_4
	h_4		h_2		h_3		h_1
	h_1		h_4		h_2		h_3
	h_3		h_1		h_4		h_2

a_2^{2Tm+2}	a_2^{2Tm+3}	$a_2^{T(2m+1)+2}$	$a_2^{T(2m+1)+3}$
(h_3, μ_a, h_4)	(μ_a, h_1, h_2)	(h_4, μ_a, h_3)	(μ_a, h_2, h_1)
(h_1, μ_a, h_1)	(μ_a, h_4, h_4)	(h_2, μ_a, h_2)	(μ_a, h_3, h_3)

where $\mu_a \in H^{T-2}$ is an arbitrary assignment. Moreover,

a_1^{2-T}	a_2^{2-T}	a_1^{3-T}	a_2^{3-T}
h_1	h_2	h_3	h_4
h	h	h	h

where h is an arbitrary house other than the second row in each column. In the above tables, each column indicates the corresponding preference where an upper house is strictly preferred to the lower one. For any other agent not specified above, houses h_1 to h_4 are less preferred to any other house.

Endowments are indicated by the parentheses in the first column in Table 3. If $T = 2$, agents a_1^{3-T} and a_2^{3-T} are not initial existing tenants but newcomers in period 1. However, the allocations to be calculated will not be affected, even for the case without endowments, because of preferences.

In each period, we concentrate on a static problem consisting of agents a_i^{2-T}, a_i^{3-T}, a_i^{2Tm+2}, a_i^{2Tm+3}, $a_i^{T(2m+1)+2}$, and $a_i^{T(2m+1)+3}$ for $i = 1, 2$ and $m \geq 0$, since houses h_1 to h_4 are less preferred to any other house for each of the other agents.

The induced TTC spot mechanism produces the matching plan μ, where houses without parentheses are the assignments, on Table 3. Note that this matching plan is not affected by whether a sequence of period orderings is constant or not.

Consider an infinite exchange depicted by houses inside the parentheses in Table 3. Clearly, the resulting allocation Pareto dominates the induced matching plan μ among all agents except initial existing tenants. □

See the last section of the Summary to contrast a TTC spot mechanism favoring newcomers with the one favoring existing tenants and an SD spot mechanism.

4.6 SERIAL DICTATORSHIP (SD) FUTURES MECHANISMS

In this section, we consider a dynamic problem without endowments and propose a simple futures mechanism termed **serial dictatorship (SD) futures mechanism**. Fix a sequence $(f_{A(1)}, \{f_{N(t)}\}_{t\geq 2})$ of an ordering of initial agents and an ordering of newcomers in period $t \geq 2$. For any announced preference profile, R, an SD futures mechanism finds a matching plan by using the following algorithm.

Period 1: The first agent in $f_{A(1)}$ gets her top *assignment* (consisting of houses up to the period when she leaves the market) under her reported preference. The kth agent in $f_{A(1)}$ gets her top assignment excluding the houses assigned to all agents before her under her reported preference. This produces the current matching and the future assignment for agents in $A(1)$.

Period t: Given the assignment determined in past periods, each of the existing tenant is assigned a house according to her assignment as previously determined. The first newcomer in $f_{N(t)}$ gets her top assignment excluding the houses assigned to the existing tenants under her reported preference. A kth newcomer in $f_{N(t)}$ gets her top assignment excluding the houses assigned to all existing tenants and all newcomers before this agent under her reported preference. The procedure for newcomers generates the current matching and the future assignment for agents in $A(t)$.

As such, we have the following.

Theorem 16. *In a dynamic problem without endowments, an SD futures mechanism is strategy-proof and Pareto efficient, but not acceptable under the same assumptions as Impossibility Theorem 8.*

Proof. First, we show strategy-proofness. In period 1, the first agent in $f_{A(1)}$ cannot do better by reporting any other preference, since she already receives her top assignment under her reported preference. The kth agent in $f_{A(1)}$ cannot do better than reporting her true preference, since the house distributed until the kth agent is independent of her preference and receives her top assignment among the remaining houses. The argument for any other period is similar.

Next, we show Pareto efficiency. To find a contradiction, suppose for some preference profile, $R \in \mathcal{R}$, a matching plan $\Pi(R)$ given by an SD futures mechanism is not Pareto efficient. For notational simplicity, let $\mu := \Pi(R)$. Then, there exists a matching plan, ν, that Pareto dominates μ in R. Thus, $\forall a \in A$, $\nu_a R_a \mu_a$ and $\exists b \in A$ such that $\nu_b P_b \mu_b$. Since $A \equiv A(1) \cup (\cup_{t \geq 2} N(t))$, agent b is either in $A(1)$ or in $N(t)$ for some $t \geq 2$. Suppose $\forall a \in A(1)$ $\mu_a R_a \nu_a$. Otherwise, the proof is similar to the following and is therefore omitted. Take the smallest $\hat{t} \geq 2$ such that $\exists b \in N(\hat{t})$ with $\nu_b P_b \mu_b$. It follows from strict preferences that $\forall a \in A(1)$, $\nu_a = \mu_a$ and $\forall t$ with $2 \leq t \leq \hat{t} - 1$, $\forall a \in N(t)$, $\nu_a = \mu_a$. Consider an agent $b \in N(\hat{t})$ who has the highest order in $f_{N(\hat{t})}$ such that $\nu_b P_b \mu_b$. Then, it follows from strict preferences that for each agent a who has higher order than b, $\nu_a = \mu_a$. Thus, in the SD futures mechanism, when it is agent b's turn to choose, two assignments, ν_b and μ_b, are still available. Thus, since agent b chooses μ_b in the SD futures mechanism, $\mu_b R_b \nu_b$. This is a contradiction.

Since the SD futures mechanism is proved to be Pareto efficient, it follows from Impossibility Theorem 8 that it is not acceptable.

□

4.7 SUMMARY

We summarize some of our results in the tables below.

Note: AC stands for acceptability, SP stands for strategy-proofness, and PE stands for Pareto efficiency. The mark "✓" in a cell indicates that a corresponding dynamic mechanism in the first column satisfies the corresponding properties in the first row. On the other hand, a blank cell indicates that the dynamic mechanism does not satisfy the property. Moreover, ✓* indicates that the spot mechanism is acceptable for a problem without endowments (See Proposition 8 in the Appendix). ✓** shows that it is SP (PE) for a problem without endowments and SP (PE) among all agents except initial existing tenants for a problem with endowments.

These results verify the desirableness of seniority-based mechanisms.

Table 1: Demographic structures

	1	2	$\cdots$	$T-1$	T	$\cdots$	t	$t+1$	$\cdots$	$t+T-2$	$t+T-1$
a_i^{2-T}	✓										
a_i^{3-T}	✓	✓									
$\vdots$	$\vdots$	$\vdots$	$\ddots$								
a_i^0	✓	✓	$\cdots$	✓							
a_i^1	✓	✓	$\cdots$	✓	✓						
$\vdots$						$\ddots$					
a_i^{t-T+1}							✓				
a_i^{t-T+2}							✓	✓			
$\vdots$							$\vdots$	$\vdots$	$\ddots$		
a_i^{t-1}							✓	✓	$\cdots$	✓	
a_i^t							✓	✓	$\cdots$	✓	✓
$\vdots$											

Note: The mark "✓" indicates when a newcomer a_i^τ in period τ is in the market with the corresponding period in the first row.

Table 2: Assignments under the truthful preference (left) and the manipulated preference (right)

	$t=1$	$t=2$	$\cdots$
a_1^{2-T} (h_1)	h_1		
a_2^{2-T} (h_2)	h_5		
a_3^{2-T} (h_3)	h_3		
a_1^{3-T} (h_4)	h_4	h_6	
$\boldsymbol{a_2^{3-T}}$ (h_5)	$\boldsymbol{h_2}$	$\boldsymbol{h_2}$	
a_3^{3-T} (h_6)	h_6	h_1	
$\vdots$		$\vdots$	

	$t=1$	$t=2$	$\cdots$
a_1^{2-T} (h_1)	h_1		
a_2^{2-T} (h_2)	h_5		
a_3^{2-T} (h_3)	h_3		
a_1^{3-T} (h_4)	h_4	h_6	
$\boldsymbol{a_2^{3-T}}$ (h_5)	$\boldsymbol{h_6}$	$\boldsymbol{h_1}$	
a_3^{3-T} (h_6)	h_2	h_2	
$\vdots$		$\vdots$	

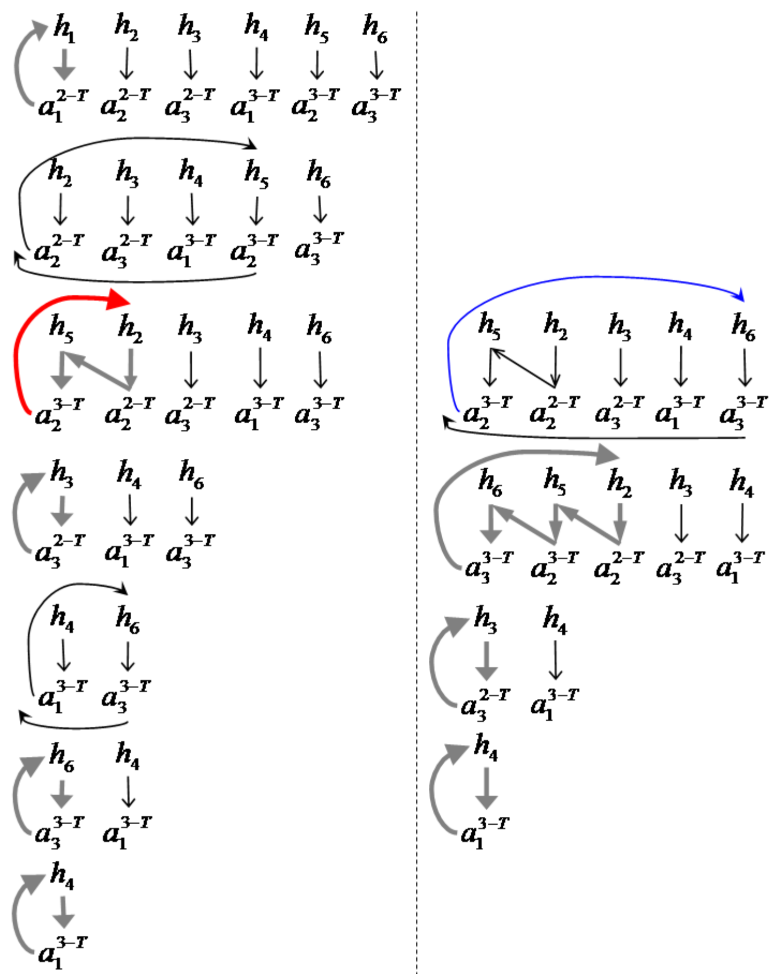

Figure 11: AS-TTC algorithms in period $t = 1$ under the truthful preference (left) and the manipulated preference (right). Thick arrows indicate a cycle in each step.

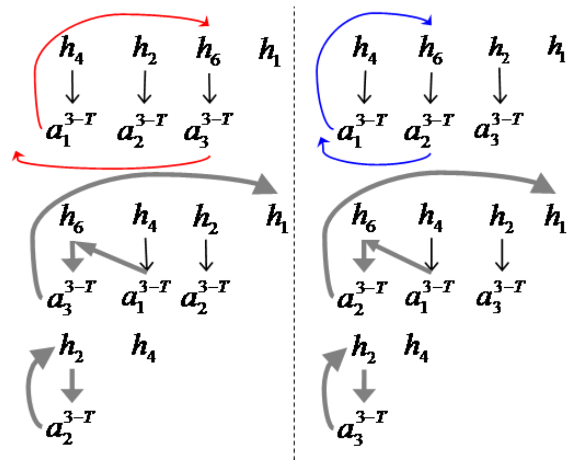

Figure 12: AS-TTC algorithms in period $t = 2$ under the truthful preference (left) and the manipulating preference (right). Thick arrows indicate a cycle in each step.

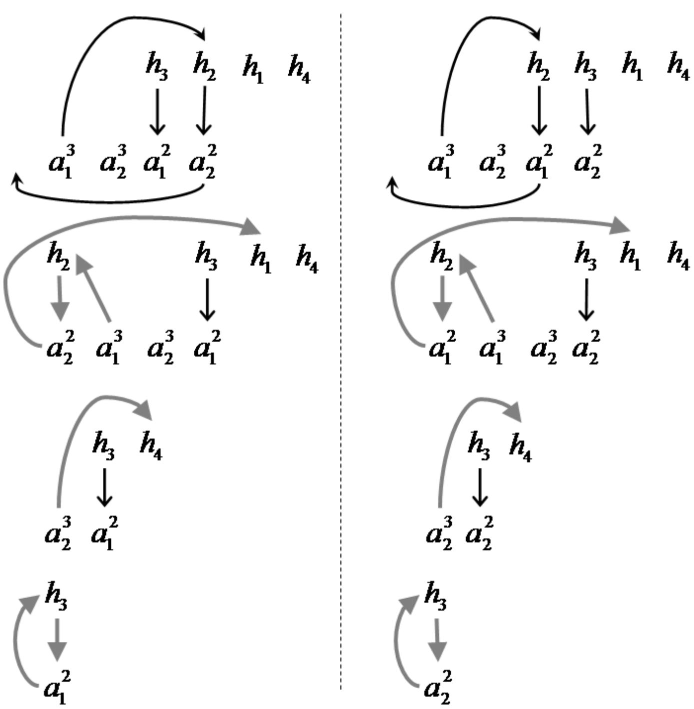

Figure 13: AS-TTC algorithms in period $t = 3$ under the truthful preference (left) and the manipulating preference (right). Thick arrows indicate a cycle in each step.

Table 3: Matching plans in Theorem 15

	1	2	3	$\cdots$	$T+1$	$T+2$	$T+3$	$\cdots$	$2T+1$	$2T+2$	$2T+3$	$\cdots$
$a_1^{2-T}(h_1)$	h_1											
$a_2^{2-T}(h_2)$	h_2											
$a_1^{3-T}(h_3)$	h_3	h_3										
$a_2^{3-T}(h_4)$	h_4	h_4										
$\vdots$		$\vdots$										
a_1^2		h_2	h_2	$\cdots$	h_2							
a_2^2		h_1	h_1	$\cdots$	$h_1(h_4)$							
a_1^3			h_3	$\cdots$	h_3	h_3						
a_2^3			h_4	$\cdots$	$h_4(h_1)$	$h_4(h_2)$						
$\vdots$						$\vdots$						
a_1^{T+2}						h_1	h_1	$\cdots$	h_1			
a_2^{T+2}						$h_2(h_4)$	h_2	$\cdots$	$h_2(h_3)$			
a_1^{T+3}							h_4	$\cdots$	h_4	h_4		
a_2^{T+3}							h_3	$\cdots$	$h_3(h_2)$	$h_3(h_1)$		
$\vdots$										$\vdots$		
a_1^{2T+2}										h_2	h_2	$\cdots$
a_2^{2T+2}										$h_1(h_3)$	h_1	$\cdots$
a_1^{2T+3}											h_3	$\cdots$
a_2^{2T+3}											h_4	$\cdots$
$\vdots$												$\vdots$

Table 4: Properties of dynamic mechanisms under "general" preferences

	AC	SP	PE
General SD spot mechanism		✓	
Constant SD spot mechanism favoring existing tenants		✓	✓
SD spot mechanism favoring newcomers		✓	
TTC spot mechanism	✓		
SD futures mechanism		✓	✓

Table 5: Properties of dynamic mechanisms under "time-invariant" preferences

	AC	SP	PE
General SD spot mechanism		✓	
Constant SD spot mechanism favoring existing tenants	✓*	✓	✓
SD spot mechanism favoring newcomers		✓	
General TTC spot mechanism	✓		
Constant TTC spot mechanism favoring existing tenants	✓	✓**	✓**
TTC spot mechanism favoring newcomers	✓		
SD futures mechanism		✓	✓

APPENDIX A

EXAMPLES AND PROOFS IN CHAPTER 2

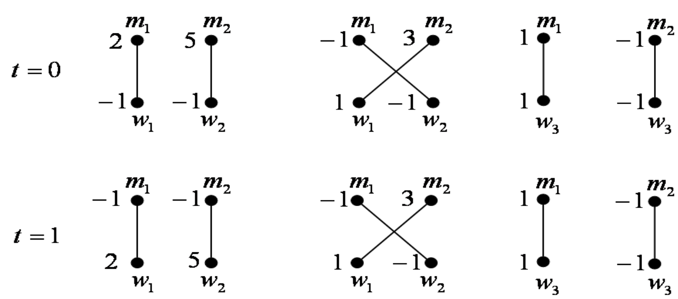

Figure 14: The preferences in the constituent market

An example of empty core. Consider a two-period dynamic market with $M = \{m_1, m_2\}$, $W = \{w_1, w_2, w_3\}$. The preferences are depicted in Figure 14. In addition, the utility of being unmatched is 0 to each agent. Note that the Figure 14 just indicates the preferences for all agents, but does not show all matchings. There are eleven possible matchings. Denote μ_{ij} by the matching in which m_i is matched with w_j and the other agents are unmatched. Denote $\mu_{ij,kl}$ by the matching in which m_i (m_k) is matched with w_j (w_l) and the other agent is unmatched. μ_U is the matching where all agents are unmatched. In total, we have $121 = 11 \times 11$ outcome paths. Out of them, we have 15 individual rational outcome paths: (μ_{11}, μ_{11}), (μ_{11}, μ_{21}), (μ_{13}, μ_{11}), (μ_{13}, μ_{13}), (μ_{13}, μ_{21}), (μ_{13}, μ_U), (μ_{21}, μ_{13}), (μ_{21}, μ_U), (μ_U, μ_{13}), (μ_U, μ_{21}), (μ_U, μ_U); (μ_{21}, μ_{21}), (μ_{21}, μ_{22}), (μ_{22}, μ_{22}),$(\mu_{11,22}, \mu_{11,22})$. The first eleven outcome paths are blocked by the pair (m_2, w_2) via (μ_{22}, μ_{22}), and the last four are blocked by (m_1, w_3) via (μ_{13}, μ_{13}). $\square$

Proof of Proposition 1. Let a dynamic matching ϕ be group stable. Suppose for a contradiction that its outcome path $\boldsymbol{\mu}(\phi) = \{\mu^t(\phi)\}_{t=0}^T$ is not in the core. Then, there exist a group A and an outcome path $\hat{\boldsymbol{\mu}} := \{\hat{\mu}^t\}_{t=0}^T$ such that $U_i(\hat{\boldsymbol{\mu}}) > U_i(\boldsymbol{\mu}(\phi))$ for each i in A. Then, for each t take a matching $\tilde{\mu}^t$ such that $(A, \tilde{\mu}^t)$ is a static group deviation from $\mu^t(\phi)$ and $\tilde{\mu}^t(i) = \hat{\mu}^t(i)$ for each i in A. Consider the dynamic group deviation $(A, \tilde{\phi})$:

$$\begin{aligned}\tilde{\phi}(h) &= \tilde{\mu}^t \quad \text{if } h = (\hat{\mu}^0, \cdots, \hat{\mu}^{t-1})),\\ &= \phi(h) \quad \text{otherwise.}\end{aligned}$$

Then, $U_i(\tilde{\phi}) > U_i(\phi)$ for each i in A. A contradiction. $\square$

Proof of equivalence between credible pairwise stability and weak stability.

Proposition 7. *In a static market, a matching is credibly pairwise-stable if and only if it is weakly stable.*

Definition 27 (Klijn and Massó, 2003). Consider a static market.

1. A blocking pair (m, w) for μ is **weak** if there is a woman $w' \in W$ such that $u_m(w') > u_m(w)$ and (m, w') is a blocking pair for μ, or a man $m' \in M$ such that $u_w(m') > u_w(m)$ and (m', w) is a blocking pair for μ. Here, (m, w) is a blocking pair for μ if the pair blocks μ.
2. A matching μ is **weakly stable** if it is individually rational and all blocking pairs are weak.

To prove the equivalence, we show that if a matching μ is individually rational,

all blocking pairs for μ are weak

$\Leftrightarrow$ there is no pairwise deviation $(A, \hat{\mu})$ from μ, $u_i(\hat{\mu}) > u_i(\mu)$ for each i in A

$\Leftrightarrow$ for each pairwise deviation $(A, \hat{\mu})$ from μ, if $u_i(\hat{\mu}) > u_i(\mu)$ for each i in A, then $(A, \hat{\mu})$ is not defensible.

The equivalence of the second and the third statements is a logical consequence. We show the equivalence of the first and the third statements. Suppose that μ is individually rational.

First, we show the direction ($\Rightarrow$). Suppose that all blocking pairs for μ is weak. Let $(A, \hat{\mu})$ be a pairwise deviation from μ such that $u_i(\hat{\mu}) > u_i(\mu)$ for each i in A. Without loss of generality, take $m \in A$. Then, since m is in A and μ is individually rational, $u_m(\hat{\mu}) > u_m(\mu) \geq u_m(m)$. This implies that m is matched with some woman in A at $\hat{\mu}$. Denote this woman by w. Then, the pair (m, w) blocks μ. We show that a pairwise deviation $(A, \hat{\mu}) = (\{m, w\}, \hat{\mu})$ is not defensible. Since all blocking pairs are weak by our hypothesis, without loss of generality,

$$\exists w' \in W, u_m(w') > u_m(w), \text{and} \tag{A.1}$$

$$(m, w') \text{ is a blocking pair for } \mu. \tag{A.2}$$

By the definition of pairwise deviation, either $w' \in A$, w' is unmatched at $\hat{\mu}$, or w is matched with $\mu(w)$ at $\hat{\mu}$. If w' were in A, $w' \neq w$ by (A.4), which would contradict that $A = \{m, w\}$. If w' were unmatched at $\hat{\mu}$, then $(m, w') \in \mu$ from the definition of pairwise deviation, and thus would contradict (A.2). Thus, w is matched with $\mu(w)$ at $\hat{\mu}$. Now, we consider a pairwise deviation $(\{m, w'\}, \bar{\mu})$ with $(m, w') \in \bar{\mu}$. Then, it follows from (A.2) that $u_{w'}(m) \equiv u_{w'}(\bar{\mu}) > u_{w'}(\mu) \equiv u_w(\hat{\mu})$. Moreover, it follows from (A.4) that $u_m(w') \equiv u_m(\bar{\mu}) > u_m(w) \equiv u_m(\hat{\mu})$. Thus, the pairwise deviation $(A, \hat{\mu})$ is not defensible.

Next, we show the other direction ($\Leftarrow$). Suppose that the hypothesis is true. Let (m, w) be a blocking pair of μ. Then, consider the pairwise deviation $(\{m, w\}, \hat{\mu})$ from μ with $(m, w) \in \hat{\mu}$. Then, $u_m(\hat{\mu}) > u_m(\mu)$ and $u_w(\hat{\mu}) > u_w(\mu)$. By our hypothesis, the pairwise deviation is not defensible. Thus, there is a group deviation $(B, \bar{\mu})$ from $\hat{\mu}$ with $\{m, w\} \cap B \neq \emptyset$ such that $u_i(\bar{\mu}) > u_i(\hat{\mu})$ for each i in B. Without loss of generality, take m in $\{m, w\} \cap B$. Then,

$$u_m(\bar{\mu}) > u_m(\hat{\mu}) > u_m(\mu) \geq u_m(m). \tag{A.3}$$

The last inequality follows from individual rationality of μ. The inequalities (A.3) imply that m is matched with some woman at $\bar{\mu}$ who is in B. Denote this woman by w'. We show that the pair (m, w') is a blocking pair for μ. Since $(m, w') \in \bar{\mu}$, the inequalities (A.3) imply that

$u_m(w') > u_m(\mu)$. Now, it is sufficient to show $u_{w'}(m) > u_w(\mu)$. By the definition of pairwise deviation, either $w' = w$, w' is unmatched at $\hat{\mu}$, or w' is matched with $\mu(w')$ at $\hat{\mu}$. However, $w' \neq w$, since we have (A.3), $(m, w) \in \hat{\mu}$ and $(m, w') \in \bar{\mu}$. Moreover, if she were unmatched at $\hat{\mu}$, then it would follow from the definition of pairwise deviation that w' is matched with m at $\hat{\mu}$, that is, we would have (m, w') in μ and $\bar{\mu}$, contradicting the inequalities (A.3). Hence, w' is matched with $\mu(w')$ at $\hat{\mu}$ and thus $u_{w'}(m) \equiv u_{w'}(\bar{\mu}) > u_{w'}(\hat{\mu}) = u_{w'}(\mu)$. The inequality holds because w' is in B.

□

Lemma* 3 *(b). Consider a static market (Knuth, 1976) with $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the following preferences:

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
$\boldsymbol{w_1}$	w_2	w_3	$\boldsymbol{w_4}$	m_4	$\boldsymbol{m_3}$	$\boldsymbol{m_2}$	m_1
w_2	w_1	w_4	w_3	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_1	m_4	m_3
w_4	$\boldsymbol{w_3}$	$\boldsymbol{w_2}$	w_1	$\boldsymbol{m_1}$	m_2	m_3	$\boldsymbol{m_4}$

where each column indicates the preference of an agent in the first row, all mates are acceptable in each column, and an upper mate is strictly preferred to the lower one. Consider the matching $\mu := \{(m_1, w_1), (m_2, w_3), (m_3, w_2), (m_4, w_4)\}$. Each of boldfaced cells in the table indicates his or her partner from this matching. This matching is not stable (for example, a pair (m_2, w_1) blocks it) but individually rational. We show by contradiction that μ is credibly group-stable. Suppose for a contradiction that there exists a defensible group deviation $(A, \hat{\mu})$ such that $u_A(\hat{\mu}) > u_A(\mu)$. Note that A does not contain the agents m_1, m_4, w_2 nor w_3, because m_1, m_4, w_2 and w_3 have the best mate in the matching μ.

First, consider the case where A is a pair. Then, since A blocks μ, A is (m_2, w_1), (m_3, w_1), (m_2, w_4), or (m_3, w_4). If $A = (m_2, w_1)$, then the pair (m_3, w_1) blocks $\hat{\mu}$. If $A = (m_3, w_1)$, then the pair (m_3, w_4) blocks $\hat{\mu}$. If $A = (m_2, w_4)$, then (m_2, w_1) blocks $\hat{\mu}$. Finally, if $A = (m_3, w_4)$, then (m_2, w_4) blocks $\hat{\mu}$. Hence, the deviation $(A, \hat{\mu})$ is not defensible. A contradiction.

If A consists of three agents, it is not defensible since the deviation is similar to pairwise ones. A contradiction. Thus, $A = \{m_2, m_3, w_1, w_4\}$. By the defensibility, the restriction

$\hat{\mu}|_A$ to A is stable in the restricted market consisting of A. This implies that $\hat{\mu}|_A$ is either $\{(m_2, w_1), (m_3, w_4)\}$ or $\{(m_2, w_4), (m_3, w_1)\}$. In both cases, since m_4 is unmatched at $\hat{\mu}$, (m_4, w_1) blocks $\hat{\mu}$. A contradiction. □

Lemma 3 (c) . Consider a static market with $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$, and the following preferences:

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
$\boldsymbol{w_1}$	w_1	w_4	w_1	m_3	$\boldsymbol{m_2}$	$\boldsymbol{m_3}$	m_2
	w_4	w_1	$\boldsymbol{w_4}$	m_4			m_3
	$\boldsymbol{w_2}$	$\boldsymbol{w_3}$		m_2			$\boldsymbol{m_4}$
				$\boldsymbol{m_1}$			

where each column indicates the preference of an agent in the first row, only acceptable mates are listed in each column, and an upper mate is strictly preferred to the lower one. Consider the matching $\mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$. Each of boldfaced cells in the table indicates his or her partner from this matching. This matching is not stable but individually rational. All of blocking pairs are (m_2, w_1), (m_3, w_1), (m_4, w_1), (m_2, w_4), and (m_3, w_4).

First, we show that μ is credibly pairwise-stable. Suppose for a contradiction that there is a defensible pairwise deviation $(A, \hat{\mu})$ from μ such that $u_A(\hat{\mu}) > u_A(\mu)$. Then, since A is a blocking pair, A is (m_2, w_1), (m_3, w_1), (m_4, w_1), (m_2, w_4), or (m_3, w_4). If $A = (m_2, w_1)$, then the pair (m_3, w_1) blocks $\hat{\mu}$. If $A = (m_3, w_1)$, then the pair (m_3, w_4) blocks $\hat{\mu}$. If $A = (m_4, w_1)$, then the pair (m_3, w_1) blocks $\hat{\mu}$. If $A = (m_2, w_4)$, then the pair (m_2, w_1) blocks $\hat{\mu}$. Finally, if $A = (m_3, w_4)$, then the pair (m_2, w_4) blocks $\hat{\mu}$. Thus, we have a contradiction: the pair deviation $(A, \hat{\mu})$ is not defensible. Hence, μ is credibly pairwise-stable.

Next, we show that μ is not credibly group-stable. Consider the group deviation $(A, \hat{\mu})$ from μ where $A = \{m_2, m_3, w_1, w_4\}$, $(m_2, w_4) \in \hat{\mu}$, and $(m_3, w_1) \in \hat{\mu}$. Note that both w_1 and w_4 are matched with the best mate. Thus, the only possibility that an agent in A is strictly better off by further deviation is that either m_2 is matched with w_1 or m_3 is matched with w_4. w_1 is worse off in the former case, while w_4 is worse off in the latter case. Thus, $(A, \hat{\mu})$ is defensible. Moreover, each agent in A is better off in $\hat{\mu}$ than in μ. Hence, μ is not credibly group-stable. $\square$

Proof of Proposition 3. Fix a stable matching μ and a defensible group deviation $(A, \hat{\mu})$ from μ. Let B be the set of all agents outside A who are matched according to μ, and C be the set of all agents outside A whose partner is in A. That is, agents in B (C) satisfy condition (b) (condition (c)) in Definition 11. Note that all agents in C are unmatched at $\hat{\mu}$ and $\hat{\mu}|_B = \mu|_B$.

First, we show that if C is empty, then $\hat{\mu}$ is stable. Let C be empty. Then, $M \cup W = A \cup B$. Suppose that some agent i blocks $\hat{\mu}$. If i is in A, the blocking contradicts the defensibility of $(A, \hat{\mu})$. If i is in B, since $\hat{\mu}|_B = \mu|_B$, i blocks μ. This contradicts the stability of μ. Thus, no agent blocks $\hat{\mu}$. On the other hand, suppose that some pair (m, w) blocks $\hat{\mu}$. If either $m \in A$ or $w \in A$, then the blocking contradicts the defensibility of $(A, \hat{\mu})$. If $m \in B$ and $w \in B$, then since $\hat{\mu}|_B = \mu|_B$, (m, w) blocks μ. This contradicts the stability of μ. Hence, no pair blocks $\hat{\mu}$. Therefore, $\hat{\mu}$ is stable.

Now, to show that $\hat{\mu}$ is stable, it is sufficient to show that C is empty. Suppose for a contradiction that C is not empty. Without loss of generality, take a woman w_0 in $C \cap W$. Using the stability of μ and the defensibility of $(A, \hat{\mu})$, we will recursively construct an infinite sequence $\{(m_k, w_k)\}_{k=1}^{\infty}$ of distinct pairs in $M \times W$ such that for each $k = 1, 2, \cdots$

(a) $(m_k, w_{k-1}) \in \mu$,

(b) $(m_k, w_k) \in \hat{\mu}$,

(c) $m_k, w_k \in A$,

(d) $u_{m_k}(\mu) < u_{m_k}(\hat{\mu})$,

(e) $u_{w_k}(\mu) > u_{w_k}(\hat{\mu})$.

This contradicts the finiteness of M and W.

First, construct m_1 and w_1 that satisfy conditions (a) to (e). By the definition of group deviation, w_0 is matched with some man in A at μ. Denote this man by m_1. Thus, (a) is satisfied. Since w_0 is unmatched at $\hat{\mu}$ and μ is individually rational,

$$u_{w_0}(m_1) \equiv u_{w_0}(\mu) > u_{w_0}(\hat{\mu}) \equiv u_{w_0}(w_0), \tag{A.4}$$

from strict preferences. If m_1 were unmatched at $\hat{\mu}$, $u_{m_1}(w_0) \equiv u_{m_1}(\mu) > u_{m_1}(\hat{\mu}) \equiv u_{m_1}(m_1)$ by strict preferences and the individual rationality of μ. Then, the pair (m_1, w_0) would block

$\hat{\mu}$, violating the defensibility of $(A, \hat{\mu})$ as m_1 is in A. Thus, it follows from the definition of group deviation that m_1 is matched with some woman in A at $\hat{\mu}$. Denote this woman by w_1. Now, $m_1, w_1 \in A$ and $(m_1, w_1) \in \hat{\mu}$ so that (b) and (c) are satisfied. Note $w_0 \neq w_1$. Since $w_0 \neq w_1$, it follows from strict preferences that

$$\text{either} \quad u_{m_1}(w_0) \equiv u_{m_1}(\mu) > u_{m_1}(w_1) \equiv u_{m_1}(\hat{\mu}), \tag{A.5}$$

$$\text{or} \quad u_{m_1}(w_0) \equiv u_{m_1}(\mu) < u_{m_1}(w_1) \equiv u_{m_1}(\hat{\mu}). \tag{A.6}$$

If the inequality (A.5) were true, then with the inequality (A.4), the pair (m_1, w_0) would block $\hat{\mu}$, violating the defensibility of $(A, \hat{\mu})$ as m_1 is in A. Thus, the inequality (A.6) is true so that (d) is satisfied. Now, $\mu(w_1) \neq \hat{\mu}(w_1) \equiv m_1$, otherwise we would have a contradiction that $w_0 = w_1$. Since μ is stable, it follows from the inequality (A.6) and strict preferences that $u_{w_1}(\mu) > u_{w_1}(\hat{\mu})$ so that (e) is satisfied. Now, $\{m_1, w_1, w_0\}$ satisfies the conditions (a) to (e).

Suppose that we are given w_0 and $\{(m_k, w_k)\}_{k=1}^{K-1}$ which satisfy conditions (a) to (e) and all of whom are distinct. We construct m_K and w_K that satisfy the conditions. First, by our hypothesis,

$$u_{w_{K-1}}(\mu) > u_{w_{K-1}}(\hat{\mu}). \tag{A.7}$$

If w_{K-1} were unmatched at μ, then w_{K-1} would block $\hat{\mu}$ from the inequality (A.7), violating the defensibility of $(A, \hat{\mu})$ as w_{K-1} is in A by our hypothesis. Thus, w_{K-1} is matched with some man at μ. Denote this man by m_K so that $(m_K, w_{K-1}) \in \mu$ and thus (a) is satisfied. Since by our hypothesis w_{K-1} is different from $w_1, \cdots, w_{K-2}$ and $(m_k, w_{k-1}) \in \mu$ for each $k = 1, \cdots, K-1$, $(m_K, w_{K-1}) \in \mu$ implies that $m_K \neq m_1, \cdots, m_{K-1}$, and thus $m_1, \cdots, m_K$ are distinct. If m_K were not in A, then m_K would be unmatched at $\hat{\mu}$ from the definition of group deviation. Then, since μ is individually rational, $u_{m_K}(w_{K-1}) \equiv u_{m_K}(\mu) > u_{m_K}(\hat{\mu}) \equiv u_{m_K}(m_K)$ from strict preferences. Thus, with the inequality (A.7), the pair (m_K, w_{K-1}) would block $\hat{\mu}$, violating the defensibility as w_{K-1} is in A by our hypothesis. Thus, m_K is in A. If m_K were unmatched at $\hat{\mu}$, then we would violate the defensibility like before. So, it follows from the definition of group deviation that m_K is matched with some woman in A at $\hat{\mu}$. Denote this woman by w_K so that (m_K, w_K) is in $\hat{\mu}$ and w_K is in A, and now (b) and

(c) are satisfied. Since $m_1, \cdots, m_K$ are distinct and $(m_k, w_k) \in \hat{\mu}$ for each $k = 1, \cdots, K$, we have $w_K \neq w_1, \cdots, w_{K-1}$, and thus $w_1, \cdots, w_K$ are distinct. Now, because $w_{K-1} \neq w_K$, strict preferences imply that

$$\text{either} \quad u_{m_K}(w_{K-1}) \equiv u_{m_K}(\mu) > u_{m_K}(w_K) \equiv u_{m_K}(\hat{\mu}), \tag{A.8}$$

$$\text{or} \quad u_{m_K}(w_{K-1}) \equiv u_{m_K}(\mu) < u_{m_K}(w_K) \equiv u_{m_K}(\hat{\mu}). \tag{A.9}$$

If the inequality (A.8) were true, then with the inequality (A.7), the pair (m_K, w_{K-1}) would block $\hat{\mu}$, violating the defensibility as m_K and w_{K-1} are in A. Thus, the inequality (A.9) holds so that (d) is satisfied. Finally, $\mu(w_K) \neq m_K$ as m_K is matched with $w_{K-1} \neq w_K$ at μ. This implies from the stability of μ and the inequality (A.9) that $u_{w_K}(\mu) > u_{w_K}(\hat{\mu}) \equiv u_{w_K}(m_K)$ so that (e) is satisfied. Now, we have the desired sequence. □

APPENDIX B

PROOFS IN CHAPTER 3

Proof of Lemma 7. We prove the result for the infinite horizon case. The finite horizon case is similar and thus the proof is omitted. Suppose that a dynamic matching ϕ is one-shot group-stable. Fix $h^t \in \mathcal{H}$. Let $(A, \hat{\mu})$ be a group-deviation from $\phi(h^t)$. Consider a one-shot group deviation $(A, \hat{\phi})$ such that $\hat{\phi}(h^t) = \hat{\mu}$ and $\hat{\phi}(h) = \phi(h)$ for each $h \neq h^t$. Since ϕ is one-shot group-stable, there exists i in A such that $U_i(\hat{\phi}|_{h^t}) \leq U_i(\phi|_{h^t})$. This implies

$$u_i^t(\hat{\mu}) + U_i(\hat{\phi}|_{h^t,\hat{\mu}}) \leq u_i^t(\phi(h^t)) + U_i(\phi|_{h^t,\phi(h^t)}). \tag{B.1}$$

Since $\hat{\phi}$ is a one-shot group deviation, $\hat{\phi}|_{h^t,\hat{\mu}} = \phi|_{h^t,\hat{\mu}}$. Thus, (B.1) implies

$$v_i(\hat{\mu}) \equiv u_i^t(\hat{\mu}) + U_i(\phi|_{h^t,\hat{\mu}}) \leq u_i^t(\phi(h^t)) + U_i(\phi|_{h^t,\phi(h^t)}) \equiv v_i(\phi(h^t)).$$

This means that the group A cannot block $\phi(h^t)$ via $\hat{\mu}$ in the induced networked market $\tilde{\Gamma}(h^t, \phi)$, i.e., the matching $\phi(h^t)$ is group stable in the induced networked market.

Conversely, consider a dynamic matching ϕ such that for each history h the matching $\phi(h)$ is group stable in the induced networked market $\tilde{\Gamma}(h, \phi)$. Let $(A, \hat{\phi})$ be a one-shot group deviation from ϕ with $\phi(h^t) \neq \hat{\phi}(h^t)$ at some history h^t. Since $\phi(h^t)$ is group stable in $\tilde{\Gamma}(h^t, \phi)$, for some $i \in A$

$$v_i(\phi(h^t)) \equiv u_i(\phi(h^t)) + U_i(\phi|_{h^t,\phi(h^t)}) \geq u_i(\hat{\phi}(h^t)) + U_i(\phi|_{h^t,\hat{\phi}(h^t)}) \equiv v_i(\hat{\phi}(h^t)). \tag{B.2}$$

Since $\hat{\phi}$ is one-shot group deviation, $\phi|_{h^t,\hat{\phi}(h^t)} = \hat{\phi}|_{h^t,\hat{\phi}(h^t)}$. Thus, (B.2) implies

$$U_i(\phi|_{h^t}) \equiv u_i(\phi(h^t)) + U_i(\phi|_{h^t,\phi(h^t)}) \geq u_i(\hat{\phi}(h^t)) + U_i(\hat{\phi}|_{h^t,\hat{\phi}(h^t)}) \equiv U_i(\hat{\phi}|_{h^t}).$$

Thus, the group A cannot one-shot block ϕ via $\hat{\phi}$, and therefore ϕ is one-shot group-stable.

□

Proof of Lemma 8 . We prove the result for the finite horizon case. The infinite horizon case is similar and thus the proof is omitted. Let $(M, W, \{u_i^t\}_{i\in I})$ be the period t market. For the history h^T, the claim is obvious. Fix $t \leq T-1$ and a history $h^t \in \mathcal{H}$. Since the dynamic matching ϕ is history-independent, for each matching $\mu \in \mathcal{M}$, the continuation dynamic matching $\phi|_{h^t,\mu}$ induces the same outcome path, say $(\hat{\mu}^{t+1}, \hat{\mu}^{t+2}, \cdots, \hat{\mu}^T)$. Then, for each $i \in I$, the continuation payoff is

$$\begin{aligned} v_i(\mu) \equiv U_i(\mu, \phi|_{h^t,\mu}) &= u_i^t(\mu(i)) + U_i(\phi|_{h^t,\mu}) = u_i^t(\mu(i)) + \sum_{\tau=0}^{T-t} u_i^\tau(\hat{\mu}^\tau) \\ &= u_i^t(\mu(i)) + c_i, \quad \text{where } c_i := \sum_{\tau=0}^{T-t} \delta^\tau u_i^\tau(\hat{\mu}^\tau) \end{aligned}$$

Since the second term c_i is independent of μ, the induced networked market $\tilde{\Gamma}(h^t, \phi)$ is a static market $(M, W, \{u_i^t + c_i\}_{i\in I})$. Since static group stability concept does not depend on any positive affine transformation of utility functions, it is equivalent to the period t market $(M, W, \{u_i^t\}_{i\in I})$. □

Proof of Corollary 2 . ϕ is one-shot group-stable if and only if $\forall t \geq 0$ $\forall h^t \in \mathcal{H}^t$, $\phi(h^t)$ is group stable in the induced networked market $\tilde{\Gamma}(h^t, \phi)$ (Lemma 7) if and only if $\forall t \geq 0$ $\forall h^t \in \mathcal{H}^t$, $\phi(h^t)$ is group stable in the period t market (Lemma 8) if and only if $\forall t \geq 0$ $\forall h^t \in \mathcal{H}^t$, $\phi(h^t)$ is stable in the period t market (Lemma 5). □

Proof of Proposition 5 . Consider the following twice repeated market consisting of $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. We take $\delta = 1$ for simplicity. The utilities are given by

	w_1	w_2	m
$u_{m_1}(\cdot)$	1	2	0
$u_{m_2}(\cdot)$	1	0	0

	m_1	m_2	w
$u_{w_1}(\cdot)$	1	0	0
$u_{w_2}(\cdot)$	0	1	0

Then, there are seven possible matchings: $\mu_U = \{m_1, m_2, w_1, w_2\}$, $\mu_{11} = \{m_1w_1, m_2, w_2\}$, $\mu_{12} = \{m_1w_2, m_2, w_1\}$, $\mu_{21} = \{m_2w_1, m_1, w_2\}$, $\mu_{22} = \{m_2w_2,\ m_1, w_1\}$, $\mu_M = \{m_1w_2, m_2w_1\}$ and $\mu_W = \{m_1w_1, m_2w_2\}$. Then, $u(\mu_U) = (u_{m_1}(\mu_U), u_{m_2}(\mu_U), u_{w_1}(\mu_U), u_{w_2}(\mu_U)) = (0, 0, 0, 0)$, $u(\mu_{11}) = (1, 0, 1, 0)$, $u(\mu_{12}) = (2, 0, 0, 0)$, $u(\mu_{21}) = (0, 1, 0, 0)$, $u(\mu_{22}) = (0, 0, 0, 1)$, $u(\mu_M) = (2, 1, 0, 0)$ and $u(\mu_W) = (1, 0, 1, 1)$. It can be verified that μ_S, μ_{21} and μ_{22} are unstable and the others are stable in the constituent market. Consider the following dynamic matching ϕ:

$$\begin{aligned} \phi(\emptyset) &= \mu_{22} \\ \phi(h^1) &= \mu_M \quad \text{if } h^1 = \mu_{22} \\ &= \mu_{11} \quad \text{otherwise.} \end{aligned}$$

We can verify that this dynamic matching is one-shot group-stable. □

Proof of Lemma 9. Suppose that $\mathcal{F}^*(\mu)$ is nonempty for some stable matching μ. To show (1), suppose for a contradiction that there is a matching μ' such that $u(\mu')$ is in $\mathcal{F}^*(\mu)$. Then, μ cannot be statically stable, a contradiction. To show (2), suppose instead that $|M| = |W| = 1$. Then, there are two possible matchings, say μ and μ'. Since $\mathcal{F}^*(\mu) \neq \emptyset$, take $v \in \mathcal{F}^*(\mu)$. Thus, since $w \in \mathcal{F}^\dagger$ and (1), w is a convex combination of two points $u(\mu)$ and $u(\mu')$ outside $\mathcal{F}^*(\mu)$. But any convex combination of $u(\mu)$ and $u(\mu')$ cannot be in $\mathcal{F}^*(\mu)$, a contradiction. □

APPENDIX C

PROOFS IN CHAPTER 4

Proof of Theorem 8. We describe the detailed procedure for the other cases in the proof of Theorem 8. Call the case considered in the proof of the main body the *Case 1-1*.

Case 1-2: $T > 3$ and $n = 3$

Let $R_{a_i^1}$ be the preference considered in Case 1-1 for $i = 1, 2, 3$. Consider the preference $\hat{R}_{a_i^1}$ such that

$$(h^1, h^2, h^3, h^4, \cdots, h^T)\, \hat{R}_{a_i^1}\, (\hat{h}^1, \hat{h}^2, \hat{h}^3, h^4, \cdots, h^T) \Leftrightarrow (h^1, h^2, h^3)\, R_{a_i^1}\, (\hat{h}^1, \hat{h}^2, \hat{h}^3),$$

for each $h^1, \hat{h}^1, h^2, \hat{h}^2, h^3, \hat{h}^3, h^4, \cdots, h^T$ in H. For each other agent, houses h_1, h_2, h_3 are less preferred to the other houses in each period. Then, the similar argument to the Case 1-1 leads to our desired conclusion.

Case 1-3: $T > 3$ and $n > 3$

Fix $T > 3$ and $n > 3$. Agent a_i^1 $(i = 1, 2, 3)$ has a preference $\bar{R}_{a_i^1}$ with $\bar{R}_{a_i^1} = \hat{R}_{a_i^1}$ where $\hat{R}_{a_i^1}$ is the preference used in Case 1-2. Let $a_4^1, \cdots, a_n^1$ be the other agents in $N(1)$. For agents a_i^1, $i = 4, \cdots, n$, houses h_1, h_2, h_3 are less preferred to the other houses in each period. Then, the similar argument to the Case 1-1 leads to our desired conclusion.

Next, consider a dynamic problem *with* endowments.

Case 2-1: $n = 3$ and $T = 3$.

We look at newcomers a_1^4, a_2^4, a_3^4 in period 4. Consider a preference $R_{a_i^4}$ with $R_{a_i^4} = R_{a_i^1}$ for $i = 1, 2, 3$ where a_i^1 and $R_{a_i^1}$ is the agent and her preference used in Case 1-1. For the other agents, houses h_1, h_2, h_3 are less preferred to the other houses in each period. A similar argument to Case 1-2 or 1-3 leads to our desired result.

For the remaining cases, the same idea as in Cases 1-2 and 1-3 leads to our desired conclusion. □

Proposition 8. *Consider a dynamic problem with time-invariant preferences and without endowments. A constant SD spot mechanism favoring existing tenants is acceptable.*

Proof. Suppose that an agent, a, obtains some house h at some period t. Since agents before agent a do not prefer the house h in period t, in the next period, it follows from time-invariant preferences that agents before agent a do not obtain this house, and thus agent a has an option of getting house h or one of the remaining houses. Hence, agent a is weakly better off as time goes on. □

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