GEOMETRIC MOTIVIC INTEGRATION ON ARTIN N-STACKS

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We construct a measure on the Boolean algebra of sets of formal arcs on an Artin stack which are definable in the language of Denef-Pas. The measure takes its values in a ring that is obtained from the Grothendieck ring of Artin stacks over the residue field by a localization followed by a completion. This construction is analogous to the construction of motivic measure on varieties by Denef and Loeser. We also obtain a "change of base" formula which allows us to relate the motivic measure on different stacks.

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PREFACE

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First of all, I would like to thank my advisor, Professor Hales. He introduced me to motivic integration and was extremely patient as I struggled through the early stages to find the right approach to this problem. However, his contribution to this work has been more than just mathematical. I first met Professor Hales when I was an undergraduate student and he has never failed to offer me encouragement. At times, he has chosen to have faith in my abilities when I had none.

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Finally, I wish to thank my family and friends for their support. My parents have always supported me in my mathematical pursuits since I was in high school and I owe them a great deal. I would also like to thank my friend Linda Henderson who supported me through some very difficult times.

I dedicate this work to my first mathematics teacher, Mr. Prakash Mulbagal from Pune, India. It was thanks to his remarkable teaching ability, that I chose mathematics as a career.

1.0 INTRODUCTION

Motivic measure on a given variety is a measure on the set of points of the space of formal arcs on the variety, taking values in the Grothendieck ring of varieties. Thus the measure of a set of arcs is essentially a geometrical object, not a mere number. As a result, motivic measure encodes a great deal of geometrical information about the variety. This technique has found numerous applications in algebraic geometry and representation theory.

In this dissertation, we construct an analogue for Artin stacks. We follow the definition of Artin stacks introduced by Toën and Vezzosi in [TV1] and [TV2]. The measure that we construct takes its values in the Grothendieck ring of Artin stacks which was defined by Toën in [To1]. We also obtain a "change of variables" formula which allows us to relate the motivic measure on different stacks.

1.1 HISTORICAL OVERVIEW OF MOTIVIC INTEGRATION

The idea of motivic integration was first formulated by Kontsevich. Let k be a field and let X be a variety over k. A *formal arc* on X is a morphism

$$\mathrm{Spec}(K[[t]]) \longrightarrow X$$

where K is some field extension of k. In other words, it is a point on X with coordinates in a power series ring. Similarly, for any non-negative integer n, an *n-arc* is a morphism

$$\mathrm{Spec}(K[[t]]/(t^{n+1})) \longrightarrow X.$$

A well-known theorem of Greenberg implies that the space of n-arcs on a variety can be represented by a scheme $L_n(X)$, of finite type over k. Indeed, for $n = 1$, this is simply the tangent bundle over X and for $n > 1$, we obtain the higher analogues - the jet-bundles of X. The truncation homomorphisms $k[[t]]/(t^m) \to k[[t]]/(t^n)$ for $m > n$ induce morphisms $L_m(X) \to L_n(X)$. This defines a projective system in the category of varieties and we define the *space of arcs on* X to be the pro-variety defined by this system and denote it by

$$L(X) := \varprojlim L_n(X).$$

(The morphisms $L_m(X) \to L_n(X)$ for $m > n$ are affine and thus $L(X)$ is actually a scheme over k. However, it is not of finite type over k.)

The idea of motivic measure originates in the observation that if X is smooth, the morphism $L_m(X) \to L_n(X)$ for $m > n$ is an affine linear bundle of rank $(m - n)\dim(X)$. Thus if $C = C_n$ is a subset of $L_n(X)$, its preimages C_m in $L_m(X)$, as m varies over integers $\geq n$, must share a geometrical invariant. We associate this invariant to the projective limit $C \subset Gr(X)$ and call it the "measure" of C. We will review the precise definition in the following chapters.

The requirement that X should be smooth is rather stringent. Indeed, for an arbitrary variety, we can obtain similar results if we "stay away from the singular locus". Since the singular locus is a nowhere dense in X, it seems plausible that this construction can be generalized to singular varieties. This is done in [DL].

It is not really necessary to restrict ourselves to varieties over k. Indeed, if X is a scheme of finite type over $k[[t]]$, the theorem of Greenberg mentioned above implies that the set of n-arcs on X can be represented by a scheme of finite type over k, which we denote by $Gr_n(X)$. As before, the *space of arcs on* X can be defined as the projective limit of the $Gr_n(X)$. It is proved in [Lo] that the construction of motivic measure can be extended to schemes X which are flat, reduced and of finite type over $k[[t]]$.

A further development in the theory is the work of Cluckers and Loeser (see [CL]). They consider subsets of products of the form $k((t))^m \times k^n \times \mathbb{Z}^r$ which can be defined by formulas involving the symbols:

– 0, 1, $+$, $-$, $\times$ for the $k((t))$ and k-variables,

– 0, 1, +, −, ≤, $\equiv_n$ for the $\mathbb{Z}$-variables, and

– the symbol ord for the valuation and $\overline{\text{ac}}$ for the first non-trivial coefficient of elements of $k((t))$ (written as a Laurent series).

To these *definable sets*, they associate a group of "constructible functions" and to every definable map between definable sets they associate a natural "push-forward". This push-forward corresponds to integration along the fibers with respect to the motivic measure. As such, this construction provides an elegant framework for working with parametrized integrals.

Some other contributions to the theory include the work of Kazhdan and Hrushovski, the construction of motivic integration for formal varieties by Julien Sebag ([Se]), applications to representation theory by Julia Gordon and Tom Hales (for example, [Go] and [Ha]), applications to the study of the McKay correspondence ([Re]), etc. There is also a construction of motivic measure on Deligne-Mumford stacks developed by Takehiko Yasuda ([Ya]). However, it is not clear if that construction could be generalized to Artin stacks in any way and the construction presented in this dissertation follows a different path.

1.2 ARTIN STACKS

As mentioned above, the definition of stacks that we use was presented by Toën and Vezzosi. In the homotopical algebraic geometry context (see [To2]) that we work in, a *prestack* is just a presheaf on a Grothendieck site taking values in the model category of simplicial sets. The category of simplicial sets comes with a notion of equivalence that is far more interesting than isomorphism - two simplicial sets are weakly equivalent if there is a morphism between them inducing an isomorphism of the sets of components and of the homotopy groups. Just as the category of sheaves is obtained from the category of presheaves by localizing with respect to the class of "local isomorphisms", the category of stacks is obtained by localizing the category of prestacks with respect to "local equivalences". While this construction may seem different from the classical construction of stacks as categories fibered in groupoids ([LMB]), these two notions are equivalent via the functor which associates to each simplicial set its fundamental groupoid. (We will not review this equivalence. See [TV2], Chapter 2.)

Once the category of stacks has been defined, the notion of geometricity is defined in a manner similar to that of the classical notion of stacks. The main references for this material that were used in this work are [TV1], [TV2] and [Si1].

1.3 AN OVERVIEW OF THE FOLLOWING CHAPTERS

The definition of the category of stacks requires a considerable amount of machinery from the theory of model categories. Hence we spend some time giving a brief overview of this theory. Chapter 2 consists of the main definitions and concepts that we require. A thorough treatment of this material is best left to textbooks and hence this presentation is unlikely to serve as a comprehensive introduction. Hopefully, it will serve to provide enough sufficient understanding to make the remainder of this work accessible. Needless to say, this chapter can be skipped by the expert, or indeed by anyone with a good grasp of the standard techniques of algebraic topology.

Chapter 3 is a review of the theory of n-stacks. Apart from introducing the definitions, we also present the generalizations of some of the standard constructions for varieties. For instance, Section 3.7 defines the notion of the set of points on a stack. This section is a straightforward generalization of ([LMB], Chapter 5). The proofs of these results are no different from the proofs in [LMB]. Hence we prove the basic results and merely state their consequences. The detailed proofs can be found in [LMB] and can be applied with with little or no changes. We also review the definition of the Grothendieck ring of Artin stacks as presented in [To1]. The only difference is that we use the etale topology instead of the fpqc topology.

Chapter 4 generalizes the construction of "arc spaces" to Artin stacks. The definitions in this chapter are fairly natural and most of the results follow immediately from the definitions. The most important result from this Chapter is Lemma 4.3.2 which shows that the homotopy groups of the arc spaces are well-behaved with respect to truncations.

Chapters 5 and 6 constitute the core of this dissertation. The material in these chapters follows closely the treatment of [DL]. Indeed, we generalize the entire machinery to the point where the construction of motivic measure and the verification of its properties becomes a formal exercise. The most important result is Proposition 6.3.4.

In Chapter 7 we define the motivic measure and verify that it is well-defined. The proof of Theorem 7.1.1 is reproduced from [DL] with no changes except for elaborating on some of the steps in the argument. We also obtain the "change of variables" formula for a special class of morphisms.

Most of the results from Chapter 4 onwards pertaining to *stacks* are original to this work, at least to the extent that they were hitherto only formulated for schemes. For some results, references to the papers [DL] and [Lo] are provided (e.g. we provide [DL] as a reference for Lemma 5.4.2). However this usually refers to the analogue of the result for schemes. Some of these results follow rather easily from the corresponding result for schemes. Some others require more work. In most cases, we reproduce in full or at least sketch the arguments from [DL] and [Lo]. I owe a great debt to the authors of these two papers.

Remark concerning notation: The principle notations and conventions used in this dissertation are introduced at the beginning of Section 4.1 and Section 5.2 and remain in effect through the rest of the work.

2.0 MODEL CATEGORIES

Before we examine the precise definition of a stack, a quick look ahead might help explain the motivation behind the numerous definitions that will precede it:

Suppose S is a fixed affine base scheme. Suppose $(\mathrm{Aff}\,/S)$ denotes the category of affine schemes over S with some Grothendieck topology attached to it. Let $\mathbf{Set}^{(\mathrm{Aff}\,/S)^{op}}$ denote the category of presheaves on this category taking values in the category of sets. Yoneda's lemma tells us that the functor

$$X \to h_X := Hom_{(\mathrm{Aff}\,/S)}(-, X)$$

is a full embedding of $(\mathrm{Aff}\,/S)$ into $\mathbf{Set}^{(\mathrm{Aff}\,/S)^{op}}$. In algebraic geometry, we commonly identify an object X with the functor h_X. It is a commonly known fact that if the topology on $(\mathrm{Aff}\,/S)$ is weaker than the fpqc topology, a presheaf of the form h_X is actually a *sheaf.* A sheaf is a presheaf satisfying certain "descent conditions". However, there is another way of looking at sheaves. The topology allows us to define a notion of *local isomorphism* on the category of presheaves. A sheaf is simply an equivalence class of presheaves that are locally isomorphic. In other words, *the category of sheaves is the localization of the category of presheaves with respect to the class of local isomorphisms.*

The definition of a stack is a natural generalization of these ideas. A prestack is a presheaf on the given category taking values in the category of simplicial sets, or equivalently, a simplicial object in the category of presheaves (a simplicial presheaf). A stack is an equivalence class of prestacks that are *locally equivalent.* In other words, the category of stacks is the localization of the category of simplicial presheaves with respect to local equivalences. Simplicial sets have a rich structure that allows us to compute this localization with great efficiency. In particular, we will require the notion of *homotopy fiber product* in order to define the fiber product of stacks.

This section is not meant to be a comprehensive (or precise) introduction to the theory of model categories. The objective of this discussion is merely to provide an intuitive grasp of the machinery. The results are not necessarily stated in the most general form in order to avoid introducing too many definitions.

2.1 AN EXAMPLE: TOPOLOGICAL SPACES

Let **Top** denote the category of topological spaces.

Homotopy equivalence: We say that two maps $f, g : X \to Y$ are *homotopic*, and write $f \sim g$, if in the square

$$\begin{array}{ccc} X \coprod X & \xrightarrow{f \coprod g} & Y \\ \downarrow & & \downarrow \\ X \times [0,1] & \longrightarrow & *, \end{array} \tag{2.1.1}$$

there exists a morphism $X \times [0, 1] \to Y$ making the diagram commutative. Homotopy is an equivalence relation on the set $\mathbf{Top}(X, Y)$ of continuous maps from X into Y. We will denote the set of homotopy classes of maps from X into Y by $[X, Y]$. If we are working within the category $\mathbf{Top}_*$ of pointed topological spaces, we have a corresponding notion of homotopies which leave the basepoint fixed.

We say that a morphism $f : X \to Y$ is a *homotopy equivalence* if there exists a morphism $g : Y \to X$ such that $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$.

Weak equivalence: For any topological space X, let $\pi_0(X)$ denote the set of path components of X. Let S^i denote the i-dimensional sphere with a chosen base point e_i. Let X be a topological space and let $x \in X$ be some point in X. Let $\pi_i(X, x) = [(S^i, e_i), (X, x)]$. This set has a natural group structure (in which the composition map is defined by means of the map $S^i \to S^i \vee S^i$ which collapses the equator). We call it the *i-th homotopy group* of (X, x). A morphism $f : X \to Y$ is called a *weak equivalence* if it induces a bijection $\pi_0(X) \to \pi_0(Y)$

and a group isomorphism $\pi_i(X, x) \to \pi_i(Y, f(x))$.

It is easy to check that *a homotopy equivalence is a weak equivalence.* The converse is not true, however homotopy equivalence does allow us to get a more manageable description of the localization of **Top** with respect to weak equivalences.

Cofibrations: Let $i : A \to X$ be a morphism. Let $X \coprod_{A\times\{0\}}(A \times [0,1])$ denote the space obtained by gluing $A \times [0,1]$ and X via the map

$$A \times \{0\} \cong A \xrightarrow{i} X.$$

A morphism $i : A \to B$ has the *homotopy extension property* or is a *cofibration* if in any diagram of the form

$$\begin{array}{ccc} B\coprod_{A\times\{0\}}(A\times[0,1]) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B\times[0,1] & \longrightarrow & * \end{array} \tag{2.1.2}$$

there exists a morphism $B \times [0,1] \to Y$ making the diagram commutative. In other words, a homotopy from A into Y can be extended to B for any given initial map on B. It can be checked that i is a cofibration if it is a closed inclusion and if the image of A in B is a *neighbourhood deformation retract* (i.e. A has a neighbourhood in B which can be deformed into A). Thus, roughly speaking, "cofibrations are well-behaved inclusions". (See [Ma] for details.)

Fibrations: We say that a morphism $p : X \to Y$ has the *homotopy lifting property* or that it is a *fibration* if for any diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ A\times[0,1] & \longrightarrow & Y. \end{array} \tag{2.1.3}$$

there exists a morphism $A \times [0,1] \to X$ making the diagram commutative. In other words, given a homotopy from A into Y and given any lift of the initial map to X, the entire homotopy can be lifted to X.

It can be proved that if $p : X \to Y$ is a fibration, and if $f : [0, 1] \to Y$ is a path, the fibers of p over $f(0)$ and $f(1)$ are *homotopy equivalent*. Intuitively, a fibration is a morphism $p : X \to Y$ such that $X \to p(X)$ is a "well-behaved surjection". (See [Ma] for details.)

Weak equivalences and homotopy equivalences: Let the localization of **Top** with respect to weak equivalences be denoted by $Ho(\mathbf{Top})$. As mentioned before, a weak equivalence is not necessarily a homotopy equivalence. Also, a map $X \to Y$ in $Ho(\mathbf{Top})$ is not necessarily the image of a map $X \to Y$ in **Top** under localization. All this bad behavior can be eliminated if the spaces in question are "simple enough", or to be precise, if the spaces are constructed by glueing spheres together. The following results are well-known:

– Any object X of **Top** is weakly equivalent to a CW-complex.

– Two CW-complexes are weakly equivalent if and only if they are homotopy equivalent.

– Given two CW-complexes X and Y, the set $Ho(\mathbf{Top})(X, Y)$ is simply the set of homotopy classes of $\mathbf{Top}(X, Y)$.

Let $\mathbf{Top}_{cw}$ be the full subcategory of **Top**, the objects of which are all CW-complexes. A consequence of the two observations above is that $Ho(\mathbf{Top})$ is equivalent to the category $Ho(\mathbf{Top}_{cw})$ which can be described in terms of homotopies.

Thus we see that the extra structure at our disposal when studying **Top** allows us to obtain a much more manageable description of the localization of this category with respect to weak equivalences. The theory of model categories attempts to generalize this machinery.

2.2 MODEL CATEGORIES: BASIC DEFINITIONS

We begin by introducing a convenient language to describe situations such as the one in the diagrams (2.1.1), (2.1.2), (2.1.3).

Definition 2.2.1. *(Lifting-extension) Let $\mathcal{M}$ be a category. Suppose $i : A \to B$ and $p : X \to Y$ be two morphisms in $\mathcal{M}$. We say that (i, p) is a* lifting-extension pair *if for any* *commutative square of the form,*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

there exists a map $h : B \to X$ *such that* $f = h \circ i$ *and* $g = p \circ h$. *We also describe this condition by saying that* i *has the* left-lifting property *with respect to* p *or that* p *has the* right-lifting property *with respect to* i.

Definition 2.2.2. *(Retract) Let* $f : A \to B$ *and* $g : C \to D$ *be two morphisms in* $\mathcal{M}$. *We say that* f *is a retract of* g *if there exists a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & B \end{array}$$

such that $\beta \circ \alpha = 1_A$ *and* $\delta \circ \gamma = 1_B$.

Recall that a category is *complete* (*co-complete*) if it admits all small limits (resp. colimits).

Definition 2.2.3. *(Model category) A model category is a complete and co-complete category* $\mathcal{M}$ *with three given subcategories* $\mathcal{W}$ *(*weak equivalences*),* $\mathcal{C}$ *(*cofibration*) and* $\mathcal{F}$(fibrations) *satisfying the following properties:*

1. *(2-out of-3 axiom) If* f *and* g *are morphisms in* $\mathcal{M}$ *such that* $g \circ f$ *is defined and if any two of* f,g *and* $g \circ f$ *is a weak equivalence, so is the third.*
2. *(Retracts axiom) If* f *and* g *are maps in* $\mathcal{M}$ *such that* f *is a retract of* g *and* g *is a weak equivalence, cofibration or a fibration, then so is* f.
3. *(Lifting axiom) Let* i *and* p *be two morphisms in* $\mathcal{M}$. *Then* (i, p) *is a lifting-extension pair if:*
   a. i *is a cofibration and* p *is a* trivial fibration *(i.e. a fibration which is also a weak equivalence).*
   b. i *is a* trivial cofibration *(i.e. a cofibration which is also a weak equivalence) and* p *is a fibration.*
4. *Any map* f *in* $\mathcal{M}$ *can be functorially factored in the following two ways:*

a. $f = \beta(f) \circ \alpha(f)$ *where* $\alpha(f)$ *is a trivial cofibration and* $\beta(f)$ *is a fibration.*

b. $f = \delta(f) \circ \gamma(f)$ *where* $\gamma(f)$ *is a cofibration and* $\delta(f)$ *is a trivial fibration.*

Duality principle: The axioms above are self-dual. Thus given any results following from these axioms, we can obtain a dual result by reversing all arrows and interchanging fibrations and cofibrations.

2.3 EXAMPLES OF MODEL CATEGORIES

1. Let **Sset** denote the category of simplicial sets. This category is defined as follows:
   Let $\mathbf{\Delta}$ denote the category whose objects are ordered sets of the form

$$[n] := \{0 < 1 < \ldots < n\}$$

   where n varies over all non-negative integers. The morphisms of $\mathbf{\Delta}$ are the order-preserving maps. The category **Sset** is the category of contravariant functors $X : \mathbf{\Delta} \to$ **Set**.
   If X is a simplicial set, we will denote the set $X([n])$ by X_n. The elements of X_n are called the *n-simplices* of X. A *simplex* of X is an n-simplex for some n.

   Face and degeneracy maps: The category $\mathbf{\Delta}$ is generated by the following two sets of maps:

   - For any integers $n \geq 1$ and $0 \leq i \leq n$, $d^{i,n} : [n-1] \to [n]$ is the injective, order-preserving map, the image of which does not include i.
   - For any integers $n \geq 1$ and $0 \leq i \leq n-1$, the map $s^{i,n} : [n] \to [n-1]$ is the surjective map which identifies i and $i+1$.

   We will simply write d^i and s^i for $d^{i,n}$ and $s^{i.n}$. If X is a simplicial set, the maps $d_i := X(d^i)$ are called the *face maps* of X and the maps $s_i := X(s^i)$ are called the *degeneracy maps* of X. A simplex of X is *degenerate* if it is in the image of a degeneracy map.

   Standard simplices: We define the following standard simplices:

– The *standard n-simplex*, denoted by Δ^n is the one defined by the formula $\Delta^n([k]) = \mathbf{\Delta}([k],[n])$. It is easy to check that for any simplicial set X, there is a natural bijection $X_n \cong \mathbf{Sset}(\Delta^n, X)$.

– The *boundary of the standard n-simplex*, denoted by $\partial\Delta^n$ is the one whose non-degenerate simplices are all the non-degenerate k-simplices of Δ^n for all $k \leq n-1$.

– The *r-horn of* Δ^n is the simplicial set Λ^n_k whose non-degenerate simplices are all the non-degenerate simplices of Δ^n except for the $(n-1)$-simplex corresponding to the map $d^r : [n-1] \to [n]$ via the bijection

$$\mathbf{\Delta}([n-1],[n]) \cong \mathbf{Sset}(\Delta^{n-1}, \Delta^n) \cong \Delta^n_{n-1}.$$

The *geometric realization* of a simplicial set is a topological space defined as follows: For every $n \geq 0$, let $|\Delta^n|$ denote the topological space

$$\{(t_0, \ldots, t_n)| \sum_i t_i = 1, t_i \geq 0\}.$$

To every topological space S, we can associate a simplicial set $Sing(S)$ by the formula $Sing(S)_n = \mathbf{Top}(|\Delta^n|, S)$. This gives us a functor $Sing : \mathbf{Sset} \to \mathbf{Top}$. This functor has a left adjoint which we denote by $S \to |S|$. $|S|$ is called the *geometric realization* of S. (It is easy to check that this definition of $|\Delta^n|$ coincides with the earlier definition.)

Now we can describe the model structure on **Sset**:

– A morphism $f : K \to L$ of simplicial sets is a weak equivalence if morphism $|f| : |K| \to |L|$ is a weak equivalence of topological spaces.

– A morphism $f : K \to L$ of simplicial sets is a cofibration if $K_n \to L_n$ is an injection for all n.

– A morphism $f : K \to L$ of simplicial sets is a fibration if it has the right lifting property with respect to all trivial cofibrations. It can be checked that this is equivalent to having the right lifting property with respect to all maps of the form $\Lambda^n_k \to \Delta^n$.

In this model category, all objects are *cofibrant* (see Section 2.4).

2. **Top** is a model category. The model structure is as follows:

– A morphism $f : X \to Y$ is a weak equivalence if it induces a bijection $\pi_0(X) \to \pi_0(Y)$ and group isomorphisms $\pi_i(X, x) \to \pi_i(Y, f(x))$ for all integers $i > 0$.

– A morphism $f : X \to Y$ is a fibration if it has the right lifting property with respect to all morphisms of the form $|\Delta^n| \to |\Delta^n| \times [0, 1]$ for all n. (Note that this is a weaker condition than the one described earlier.)

– A morphism is a cofibration if and only if it has the homotopy extension property, or equivalently if it has the left-lifting property with respect to every map that is a weak equivalence as well as a fibration.

In this model category, all objects are *fibrant* (see Section 2.4).

2.4 BASIC RESULTS ABOUT MODEL CATEGORIES

We review some elementary concepts and some easy consequences of the axioms defining a model category.

Cofibrant and fibrant objects: Since a model category is complete and co-complete, it has an initial object $\emptyset$ and a terminal object $*$. We say that an object X is *cofibrant* if the morphism $\emptyset \to X$ is a cofibration and we say that it is *fibrant* if $X \to *$ is a fibration. If X is an arbitrary object in X, a *cofibrant approximation of* X is a cofibrant object $\widetilde{X}$ and a weak equivalence $\widetilde{X} \to X$. A *fibrant approximation of* X is a fibrant object $\widehat{X}$ and a weak equivalence $X \to \widehat{X}$. By applying the functorial factorizations in condition (4) of the definition of a model category, we see that every object has *functorial* cofibrant and fibrant approximations. We will denote them by $QX \to X$ and $X \to RX$ respectively.

Retract argument:

Lemma 2.4.1. *Let $\mathcal{M}$ be any category and let $g : X \to Y$ be a map in $\mathcal{M}$. The following lemma is easy to prove.*

1. *If g can be factored as $g = p \circ i$ where p has the right lifting property with respect to g, then g is a retract of i.*

2. If g can be factored as $g = p \circ i$ where i has the left lifting property with respect to g, then g is a retract of p.

An important consequence: fibrations, cofibrations, trivial fibrations and trivial cofibrations are characterized by their lifting properties. To be precise, a morphism is a fibration if it has the right lifting property with respect to all trivial cofibrations, etc. It follows that the model structure is uniquely determined if $(\mathcal{W}, \mathcal{C})$ or $(\mathcal{W}, \mathcal{F})$ are given.

Homotopy and weak equivalences:

Definition 2.4.2. *(Homotopy category) The* homotopy category *of a model category $\mathcal{M}$ is the localization of $\mathcal{M}$ with respect to weak equivalences and is denoted by $Ho(\mathcal{M})$.*

As in the case of topological spaces, we would like to obtain a more manageable description of the homotopy category of a model category. The following is a brief sketch notion of how this is achieved. While this discussion does not include any of the technical details, it should give a good intuitive grasp of the role that cofibrant and fibrant objects play in the study of model categories.

Let $\mathcal{M}$ be a fixed model category.

- Let A be an object in $\mathcal{M}$. A *cylinder object* for A is a factorization of the map $A \coprod A \to A$ into a cofibration $A \coprod A \to A'$ followed by a weak equivalence $A' \to A$. The two maps $A \to A \coprod A$ give us two morphisms $i_0, i_1 : A \to A'$. We say that two morphisms $f, g : A \to X$ are *left homotopic* if there exists a cylinder object A' and a morphism $H : A' \to X$ such that $f = Hi_0$ and $g = Hi_1$.
- Let X be an object in $\mathcal{M}$. A *path object* for X is a factorization of the diagonal map $X \to X \times X$ into a weak equivalence $X \to X'$ followed by a fibration $X' \to X \times X$. The two projections $X \times X \to X$ give us two maps $p_0, p_1 : X' \to X$. We say that two morphisms $f, g : A \to X$ are *right homotopic* if there exists a path object X' and a morphism $H : A \to X'$ such that $f = p_0 H$ and $g = p_1 H$.
- If two maps $f, g : A \to X$ are left homotopic as well as right homotopic, we say that they are *homotopic*. This allows us to define a corresponding notion of *homotopy equivalence* of objects. In general, the conditions of left and right homotopy do not imply each other.

Also, left and right homotopy are not equivalence relations in general.

- If A is a cofibrant object and X is a fibrant object, two morphisms $f, g : A \to X$ are left homotopic if and only if they are right homotopic. In this case, we say that f and g are *homotopic*. Also, under this hypothesis, homotopy is an equivalence relation on $\mathcal{M}(A, X)$.

- If A is cofibrant and X is fibrant, $Ho(\mathcal{M})(A, X)$ is the set of homotopy classes of $\mathcal{M}(A, X)$. Thus, if X and Y are arbitrary, a morphism $X \to Y$ in $Ho(\mathcal{M})$ is the image of a map $X' \to Y'$ where X' is a cofibrant approximation of X and Y' is a fibrant approximation of Y.

- Let $\mathcal{M}_{cf}$ denote the category of objects of $\mathcal{M}$ that are cofibrant-fibrant (i.e. cofibrant as well as fibrant). Two objects in $\mathcal{M}_{cf}$ are weakly equivalent if and only if they are homotopy equivalent. The category $Ho(\mathcal{M}_{cf})$ is a equivalent to $Ho(\mathcal{M})$.

Ken Brown's lemma:

Lemma 2.4.3. *Let $\mathcal{M}$ be a model category. If $f : A \to B$ be a weak equivalence between cofibrant objects. Then there is a functorial factorization of f as $f = ji$ where i is a trivial cofibration and j is a trivial fibration that has a right inverse that is a trivial cofibration.*

Proof. Suppose A and B are cofibrant objects. Factor the map $A \coprod B \to B$ as $A \coprod B \to C \to B$ where $A \coprod B \to C$ is a cofibration and $C \to B$ is a trivial fibration. The map $A \to A \coprod B$ is a cofibration. Thus $A \to C$ is a cofibration. It is a trivial cofibration since $A \to B$ and $C \to B$ are weak equivalences. Similarly $B \to A \coprod B \to C$ is a trivial cofibration. □

An important consequence:

Corollary 2.4.4. *Let $\mathcal{M}$ be a model category and let $F : \mathcal{M} \to \mathcal{C}$ be a functor that maps trivial cofibrations between cofibrant objects to isomorphisms. Then F maps weak equivalences between cofibrant objects to isomorphisms.*

2.5 FUNCTORS BETWEEN MODEL CATEGORIES

We wish to find conditions on a functor $F : \mathcal{M} \to \mathcal{N}$ which will ensure the existence of a (left or right) Kan extension $Ho(F) : Ho(\mathcal{M}) \to Ho(\mathcal{N})$. To require F to preserve the entire model structure is too stringent. It is enough for F to respect only half the model structure:

Definition 2.5.1. *(Quillen functors) Let $\mathcal{M}$ and $\mathcal{N}$ be model categories.*

1. *A functor $F : \mathcal{M} \to \mathcal{N}$ is a* left Quillen functor *if it is a left adjoint and preserves cofibrations and trivial cofibrations.*

2. *A functor $G : \mathcal{N} \to \mathcal{M}$ is a* right Quillen functor *if it is a right adjoint and preserves fibrations and trivial fibrations.*

Facts:

– If F is a left Quillen functor, its right adjoint is a right Quillen functor. If G is a right Quillen functor, its left adjoint is a left Quillen functor. If $(F, G) : \mathcal{M} \to \mathcal{N}$ is an adjunction and F is a left Quillen functor, we say that (F, G) is a *Quillen adjunction.*

– A left Quillen functor preserves weak equivalences between cofibrant objects and a right Quillen functor preserves weak equivalences between fibrant objects. (This follows from Corollary 2.4.4.)

– Let $(F, G) : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction. Then we can define a functor $\mathbb{L}F : Ho(\mathcal{M}) \to Ho(\mathcal{N})$ by the formula $\mathbb{L}F(X) = F(QX)$ where QX denotes a functorial cofibrant approximation of X (see the remark on cofibrant and fibrant objects following Definition 2.2.3). We say that $\mathbb{L}F$ is the *total left derived functor of F.* Similarly, we have a functor $\mathbb{R}G : Ho(\mathcal{N}) \to Ho(\mathcal{M})$ defined by the formula $\mathbb{R}G(X) = G(RX)$ where RX is a functorial fibrant approximation of X. We say that $\mathbb{R}G$ is the *total right derived functor of G.*

2.6 COFIBRANTLY GENERATED MODEL CATEGORY

In our description of the model structure on **Sset** in Section 2.3, we stated that in order for a morphism to be a fibration, it suffices that it should satisfy the right-lifting property with respect to morphisms of the form $\Lambda^n_k \to \Delta^n$ where $n \geq 1$ and $0 \leq k \leq n$. In this section, we generalize this property.

Small objects:

Let α be some cardinal. We say that an ordinal λ is α-filtered if it is a limit ordinal and if for any subset A of λ such that $|A| \leq \alpha$, we have $\sup A < \lambda$.

We say that an object A in a category $\mathcal{C}$ is *α-small relative to a subcategory $\mathcal{D}$ of $\mathcal{C}$* if for any α-filtered ordinal λ and a λ-sequence in $\mathcal{D}$

$$A_0 \to A_1 \to \ldots A_\beta \to A_{\beta+1} \to \ldots$$

the natural map

$$\mathrm{colim}_{\beta<\lambda}\, \mathcal{C}(A, A_\beta) = \mathcal{C}(A, \mathrm{colim}_{\beta<\lambda}(A_\beta))$$

is an isomorphism.

We say that an object is small relative to a subcategory $\mathcal{D}$ of $\mathcal{C}$ if it is α-small relative to $\mathcal{D}$ for some cardinal α.

Cell complexes: We have the following analogue of CW-complexes.

Definition 2.6.1. *Let $\mathcal{C}$ be a co-complete category. Let I be a set of maps in $\mathcal{C}$.*

1. *A* relative I-cell complex *is a morphism in $\mathcal{C}$ which is the transfinite composition of pushouts of elements of I. (See [Hir] for a precise definition of transfinite composition. Roughly speaking, this notion allows us to compose an infinite sequence of morphisms.)*
2. *An object is an* I-cell complex *if the morphism from the initial object of $\mathcal{C}$ into it is a relative I-cell complex.*

Definition 2.6.2. *(I-cofibrations) Let $\mathcal{C}$ be a category and I be a set of morphisms in $\mathcal{C}$.*

1. *A morphism is* I-injective *if it has the right-lifting property with respect to all elements of I.*

2. *A morphism is an I-cofibration if it has the left-lifting property with respect to all I-injectives.*

The following lemma is easy:

Lemma 2.6.3. *A retract of a relative I-cell complex is an I-cofibration.*

Cofibrantly generated categories:

Proposition 2.6.4. *(Small object argument) Let $\mathcal{C}$ be a co-complete category and let I be a set of maps in $\mathcal{C}$. Suppose that the domains of the elements of I are small relative to the category of relative I-cell complexes. Then there exists a functorial factorization of every map of $\mathcal{C}$ into a relative I-cell complex followed by an I-injective map*

(See [Hir], Chapter 10 for proof.)

Definition 2.6.5. *(Cofibrantly generated model category) A cofibrantly generated model category is a model category $\mathcal{M}$ such that:*

1. *There exists a set I of cofibrations such that the domains of I are small relative to the category of relative I-cell complexes and such that the category of cofibrations coincides with the category of I-cofibrations.*
2. *There exists a set J of trivial cofibrations such that the domains of J are small relative to the category of relative J-cell complexes and such that the category of trivial cofibrations coincides with the category of J-cofibrations.*

Proposition 2.6.6. *Let $\mathcal{M}$ be a complete and co-complete category with a subcategory $\mathcal{W}$ which satisfies the 2-out of-3 property and is closed under retracts. Let I and J be sets of morphisms in $\mathcal{M}$ such that:*

1. *The domains of I (resp. J) are small relative to the category of relative I-cell complexes (resp. J-cell complexes).*
2. *Every J-cofibration is both an I-cofibration and an element of $\mathcal{W}$.*
3. *Every I-injective is both a J-injective and an element of $\mathcal{W}$.*
4. *One of the following two conditions are true:*
    a. *A map that is both a I-cofibration and an element of $\mathcal{W}$ is a J-cofibration.*

b. A map that is both a J-injective and an element of $\mathcal{W}$ is an I-injective.

Then there is a cofibrantly generated model structure on $\mathcal{M}$ in which $\mathcal{W}$ is the class of weak equivalences, I is a set of generating cofibrations and J is a set of generating trivial cofibrations.

Cellular model category: Let I be a set of morphisms in $\mathcal{M}$. Let $X \to Y$ be a relative I-cell complex. A *relative subcomplex* of $X \to Y$ is a factorization $X \to Z \to Y$ such that $X \to Z$ and $Z \to Y$ are relative I-cell complexes.

Suppose $X \to Y$ is a relative I-cell complex. A *presentation* of $X \to Y$ is a λ-sequence (for some ordinal λ) of pushouts of elements of I, the transfinite composition of which is the morphism $X \to Y$. A *presented relative I-cell complex* is a relative I-cell complex with a given presentation. The *size* of a presented relative I-cell complex given by a λ-sequence of pushouts of I is the cardinal $|\lambda|$.

For any cardinal α, an object W is said to be *α-compact relative to I* if for every presented relative I-cell complex $X \to Y$, every morphism $W \to Y$ factors through a subcomplex of size at most α. An object is said to be *compact relative to I* if it is α-compact relative to I for some cardinal α.

Definition 2.6.7. *(Cellular model category) A* cellular model category *is a cofibrantly generated model category $\mathcal{M}$ for which there exists a set I of generating cofibrations and a set J of trivial cofibrations such that:*

1. *Both the domains and codomains of I are compact relative to I.*
2. *The domains of J are small relative to I.*
3. *If $f : A \to B$ is a cofibration, it is the equalizer of the natural inclusions $B \rightrightarrows B \coprod_A B$.*

A category $\mathcal{C}$ is *locally presentable* if there exists a set of objects S which are α-small for some cardinal α and such that any object in $\mathcal{C}$ is an α-filtered colimit of objects in S.

Definition 2.6.8. *A* combinatorial model category *is a cofibrantly generated model category $\mathcal{M}$ whose underlying category is locally presentable.*

We look at some examples, but omit the proofs.

Examples:

1. In the category **Sset**, consider the sets

$$I := \{\partial\Delta^n \to \Delta^n | n \geq 0\}$$

and

$$J := \{\Lambda^n_k \to \Delta^n | n \geq k \geq 0\}.$$

Then I and J are the sets of generating cofibrations and generating cofibrations for the model structure on **Sset**. **Sset** is a cellular as well as combinatorial model category.

2. Let $\mathcal{C}$ be a small category. Let $\mathcal{M}$ denote a cofibrantly generated model category in which I and J are sets of generating cofibrations and generating trivial cofibrations respectively. Consider the category $\mathcal{M}^{\mathcal{C}}$ of functors from $\mathcal{C}$ into $\mathcal{M}$. This is a model category, the model structure on which is described as follows:

   – A morphism $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence if for any object c of $\mathcal{C}$, the morphism $f(c) : X(c) \to Y(c)$ is a weak equivalence.

   – A morphism $f : X \to Y$ in $\mathcal{M}$ is a fibration if for any object c of $\mathcal{C}$, the morphism $f(c) : X(c) \to Y(c)$ is a fibration.

   This model structure is called the *projective model structure* on $\mathcal{M}^{\mathcal{C}}$.

   Let P denote the category which has a unique object $*$ and only one morphism $Id_* : * \to *$. Let $i_c : P \to \mathcal{C}$ be the functor which takes $*$ to C. This defines a functor $(i_c)_* : \mathbf{Sset}^{\mathcal{C}} \to \mathbf{Sset}^P \cong \mathbf{Sset}$ which is given by the formula $(i_c)_*(X) = X(c)$. Clearly, this is a right Quillen functor. Let $(i_c)_!$ denote its left adjoint.

   For any set K of morphisms of $\mathcal{M}$, let $(i_K)_!$ denote the set of maps of $\mathcal{M}^{\mathcal{C}}$ of the form $(i_c)_!(X) \to (i_c)_!(Y)$ where $X \to Y$ is in K.

   The sets $(i_I)_!$ and $(i_J)_!$ are sets of generating cofibrations and generating trivial cofibrations respectively for the model structure on $\mathcal{M}^{\mathcal{C}}$.

   If $\mathcal{M}$ is cellular or combinatorial, so is $\mathcal{M}^{\mathcal{C}}$.

Remark: If we fix a universe $\mathbb{U}$, and require all the cardinals in this section to be $\mathbb{U}$-small and all the sets to be $\mathbb{U}$-small, we obtain notions such as $\mathbb{U}$-cofibrantly generated model structure $\mathbb{U}$-compactness, $\mathbb{U}$-cellular model categories and $\mathbb{U}$-combinatorial model categories.

2.7 HOMOTOPY PULLBACKS AND PUSHOUTS

If in the diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longleftarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & C' & \longleftarrow & B' \end{array}$$

all the vertical maps are weak equivalences, the map $A \times_C B \to A' \times_{C'} B'$ is a *not* necessarily a weak equivalence. We wish to define the *homotopy fiber product* in such a way that it is "homotopy invariant". We saw in the case of topological spaces that the fibers of a fibration over a path are all homotopy equivalent. This motivates the following definition of homotopy pullback. We will use the notation in Definition 2.2.3 and the remarks regarding cofibrant and fibrant objects in Section 2.4.

Definition 2.7.1. *(Homotopy pullbacks) Let $A \xrightarrow{f} C \xleftarrow{g} B$ be two morphisms in a model category $\mathcal{M}$. Let $\rho : C \to RC$ denote the functorial fibrant approximation of C. Let*

$$A \overset{\alpha(\rho\circ f)}{\longrightarrow} A' \overset{\beta(\rho\circ f)}{\longrightarrow} RC$$

and

$$B \overset{\alpha(\rho\circ g)}{\longrightarrow} B' \overset{\beta(\rho\circ g)}{\longrightarrow} RC$$

denote factorizations of the maps $A \to RC$ and $B \to RC$ respectively into a trivial cofibration followed by a fibration.

The square

$$\begin{array}{ccc} W & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

is a homotopy pullback square *if the morphism $W \to A' \times_{RC} B'$ is a weak equivalence. We say that W is the* homotopy pullback *of the diagram $A \to C \leftarrow B$. We will denote the homotopy pullback (which is only determined up to weak equivalence) by $A \times^h_C B$.*

In the above definition, it is not necessary to use functorial factorizations. Indeed, if we use any fibrant approximation $C \to \widehat{C}$ of C and any factorizations of the maps $A \to \widehat{C}$ and $B \to \widehat{C}$ into a weak equivalence followed by a fibration, the resultant fiber products are weakly equivalent.

Definition 2.7.2. *Let $\mathcal{M}$ be a model category. $\mathcal{M}$ is* left proper *if every pushout of a weak equivalence along a cofibration is a weak equivalence. $\mathcal{M}$ is* right proper *if every pullback of a weak equivalence along a fibration is a weak equivalence. $\mathcal{M}$ is* proper *if it is both left proper and right proper.*

Remark. It can be proved that in any model category, the pullback of a weak equivalence between fibrant objects is a weak equivalence (Reedy's theorem, see [Hir]). This should provide some perspective for the above definition.

Example: Top and **Sset** are proper. If $\mathcal{M}$ is left proper, right proper or proper, so is $\mathcal{M}^{\mathcal{C}}$ for any small category $\mathcal{C}$.

Fact: (See [Hir], Chapter 13.) If the category $\mathcal{M}$ right proper, the homotopy pullback of a diagram

$$A \xrightarrow{f} C \xleftarrow{g} B$$

is weakly equivalent to the pullback of the diagrams

$$A \xrightarrow{f} C \overset{\beta(g)}{\longleftarrow} B'$$

and

$$A' \overset{\beta(f)}{\to} C \overset{g}{\leftarrow} B.$$

Consequently, homotopy pullbacks in right proper categories can be calculated with greater ease. Indeed, in [Hir], homotopy pullbacks are only defined for right proper model categories.

Example: (Loop spaces) Suppose X is a topological space. Let $x, y : |\Delta^0| \to X$ be points of X. Let $P_x X$ denote the space of paths in X beginning at x. $|\Delta^0| \to P_x X \to X$ is a factorization of $|\Delta^0| \to X$ into a weak equivalence followed by a fibration. Thus,

$$|\Delta^0| \times^h_{x,X,y} |\Delta^0| \cong P_x X \times_{X,y} |\Delta^0|$$

which is the *space of paths from x into y* which we denote by $\Omega_{x,y} X$. If $x = y$, we denote this object by $\Omega_x X$ and call it the *loop space of X at x.*

Remark: The construction in the example above is not really complete unless we specify how the set of paths in X beginning at x is a topological space. The topology on $P_x X$ is the compact open topology. One would naturally expect a similar construction to be possible in the model category of simplicial sets if we could define the notion of a "function space" for simplicial sets. Fortunately, the model category of simplicial sets comes with a natural structure which makes this possible.

2.8 SIMPLICIAL MODEL CATEGORY

Definition 2.8.1. *(Simplicial category) A simplicial category is a category $\mathcal{C}$ with a given* mapping space functor

$$Map_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \boldsymbol{Sset}$$

with the following properties:

1. *For any objects X and Y in $\mathcal{C}$, $Map_{\mathcal{C}}(X,Y)_0 = \mathcal{C}(X,Y)$.*
2. *The functor $Map_{\mathcal{C}}(X,-) : \mathcal{C} \to \boldsymbol{Sset}$ has a left adjoint*

$$X \otimes (-) : \boldsymbol{Sset} \to \mathcal{C}$$

which is associative in the sense that there is an isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L$$

natural in $X \in \mathcal{C}$ and $K, L \in \boldsymbol{Sset}$.

3. *For any object Y in $\mathcal{C}$, the functor $Map_{\mathcal{C}}(-, Y)$ has a left adjoint which we denote by*

$$Y^{(-)} : \textbf{\textit{Sset}} \to \mathcal{C}^{op}.$$

An easy consequence of the above axioms is that if X and Y are two objects of a simplicial category $\mathcal{C}$,

$$Map_{\mathcal{C}}(X, Y)_n = \mathcal{C}(X \otimes \Delta^n, Y).$$

Definition 2.8.2. *(Simplicial model category) A* simplicial model category *is a model category $\mathcal{M}$ which is also a simplicial category with the following property:*

If $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration, then the morphism

$$Map(B, X) \to Map(A, X) \times_{Map(A,Y)} Map(B, Y)$$

is a fibration which is trivial if either one of i or p is trivial.

Remark. This is a stronger version of the lifting axiom in the definition of a model category. Indeed, (i, p) is a lifting extension pair if and only if the morphism

$$\mathcal{M}(B, X) \to \mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y)$$

is surjective. Since a trivial fibration of simplicial sets is surjective on simplices, the above condition strengthens the lifting-extension axiom.

Examples:

1. **Sset** is a simplicial model category. The simplicial set $Map(X, Y)$ is defined by the formula $Map(X, Y)_n := Map(X \times \Delta^n, Y)$. For this simplicial structure, $X \otimes Y = X \times Y$ and $X^Y = Map(Y, X)$.
2. Let $\mathbf{Top}_{CGHaus}$ be the model category of of compactly generated, Hausdorff topological spaces. The simplicial model structure on this category is defined by

   $$X \otimes K := X \times_{Ke} |K|$$

   where $(-) \times_{Ke} (-)$ denotes the Kelley product, which is the product in the category $\mathbf{Top}_{CGHaus}$.

3. Let $\mathcal{C}$ be a small category and let $\mathcal{M}$ be a cofibrantly generated simplicial model category. Then the projective model structure on $\mathcal{M}^{\mathcal{C}}$ is simplicial. For any object F of $\mathcal{M}^{\mathcal{C}}$ and any simplicial set K, we define $F \otimes K$ and F^K by the formulas

$$(F \otimes K)(c) := F(c) \otimes K$$

and

$$F^K(c) := F(c)^K$$

for all objects c of C.

All the examples listed above for a simplicial category are simplicial model categories.

Remark: If X is a fibrant simplicial set, and $x : \Delta^0 \to X$ is a 0-simplex, the *space of paths in X beginning at x* is the simplicial set $X^{\Delta^1} \times_{X,x} \Delta_0$. As we remarked before in the case of topological spaces, the homotopy pullback

$$\Omega_x X := \Delta^0 \times^h_{x,X,x} \Delta^0$$

is called the *loop space of X at x*.

One of the advantages of working with simplicial model categories is that the additional structure allows us to define cylinder objects and path objects in a canonical way. This gives us a somewhat stronger notion of homotopy. This is straightforward generalization of the definition of homotopy in **Top**. The simplicial set Δ^1 is clearly an obvious candidate to play the role of the interval $[0, 1]$. However, since we need homotopy to be an equivalence relation, we need to come up with a more general definition.

Definition 2.8.3. *(Generalized interval) A* generalized interval *J is a simplicial set that is a union of finitely many copies of Δ^1 with their vertices (i.e. 0-simplices) identified so that its geometric realization of J is homeomorphic to $[0, 1] \in$ **Top**. If J is a generalized interval, we let i_0 and i_1 denote the inclusions of Δ^0 into J at the two end vertices of J.*

Definition 2.8.4. *Let $f, g : K \to L$ be a morphism in a simplicial category $\mathcal{M}$. We say that f and g are* simplicially homotopic *if there exists an interval J and a morphism $h : K \otimes J \to L$ such that $f = h \circ (Id_K \otimes i_0)$ and $g = h \circ (Id_K \otimes i_1)$.*

Facts:

- If A is a cofibrant object in $\mathcal{M}$ and $K \to L$ is a cofibration in **Sset**, the morphism $A \otimes K \to A \otimes L$ is a cofibration. In particular, $A \otimes K$ is cofibrant.
- Dually, if X is a fibrant object in $\mathcal{M}$ and $K \to L$ is a fibration in **Sset**, the morphism $X^L \to X^K$ is a fibration. In particular, if K is fibrant, X^K is fibrant.
- Let J be an interval. If A is a cofibrant object, $A \otimes J$ is a cylinder object for A. If X is a fibrant object, X^J is a path object for X.
- If A is a cofibrant object and X is a fibrant object, two morphisms $f, g : A \to X$ are homotopic if and only if they are simplicially homotopic.
- Let A and B be two cofibrant objects. A morphism $f : A \to B$ is a weak equivalence if and only if for any fibrant object X, the morphism $Map(B, X) \to Map(A, X)$ is a weak equivalence.

2.9 HOMOTOPY LIMITS AND COLIMITS

Let $\mathcal{C}$ be a small category. We would like to define homotopy invariant versions of the limit and colimit functors for $\mathcal{C}$-diagrams in a simplicial model category $\mathcal{M}$.

First we need the following definitions:

- The classifying space of $\mathcal{C}$ is a simplicial set $B\mathcal{C}$ such that:
  1. An n-simplex of $B\mathcal{C}$ is a diagram σ of the form

$$c_0 \to c_1 \to \ldots \to c_n.$$

  2. The face maps are defined by

$$d^i(\sigma) = \begin{cases} c_1 \to c_2 \to \ldots \to c_n & \text{if } i = 0, \\ c_0 \to \ldots \to c_{i-1} \to c_{i+1} \to \ldots c_n & \text{if } 0 < i < n, \\ c_0 \to c_1 \to \ldots \to c_{n-1} & \text{if } i = n. \end{cases}$$

3. The degeneracy maps are defined by

$$s^i(\sigma) = c_0 \to c_1 \ldots \to c_{i-1} \to c_1 \xrightarrow{1_{c_i}} c_i \to c_{i+1} \to \ldots \to c_n.$$

– If c is any object of $\mathcal{C}$, let $(\mathcal{C} \downarrow c)$ denote the comma category of objects of $\mathcal{C}$ over c and let $(c \downarrow \mathcal{C})$ denote the comma category of objects of $\mathcal{C}$ below c.

Now suppose X is a $\mathcal{C}$-diagram in $\mathcal{M}$ and K is a $\mathcal{C}^{op}$-diagram in **Sset**. We define the object $X \otimes_{\mathcal{C}} K$ as the coequalizer of the diagram

$$\coprod_{(\sigma : c \to c') \in \mathcal{C}} X_c \otimes K_{c'} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{c \in Ob(\mathcal{C})} X_c \otimes K_c$$

where ϕ is defined on the summand corresponding to the map $\sigma : c \to c'$ as the composition of the map

$$\sigma_* \otimes 1_{K_{c'}} : X_c \otimes K_{c'} \to X_{c'} \otimes K_{c'}$$

with the natural injection into the coproduct. The map ψ is defined on the summand corresponding to $\sigma : c \to c'$ as the composition of the map

$$1_{X_c} \otimes \sigma^* : X_c \otimes K_{c'} \to X_c \otimes K_c$$

with the natural injection into the coproduct.

Dually, if X is a $\mathcal{C}$-diagram in $\mathcal{M}$ and K is a $\mathcal{C}$-diagram in **Sset**, we can define the object $\hom^{\mathcal{C}}(K, X)$ as the equalizer of the dual of the above diagram (where the dual of $X_c \otimes K_{c'}$ is the object $X_c^{K_{c'}}$).

Examples: If P is the constant point diagram, $X \otimes_{\mathcal{C}} P = \operatorname{colim}(X)$ and $\hom^{\mathcal{C}}(P, X) = \lim(X)$.

Definition 2.9.1. *Let $\mathcal{M}$ be a simplicial model category and $\mathcal{C}$ be a small category.*

1. *If X is a $\mathcal{C}$-diagram, then the* homotopy colimit *of X, denoted by* $\operatorname{hocolim}(X)$ *is the object $X \otimes_{\mathcal{C}} B(- \downarrow \mathcal{C})^{op}$.*
2. *If X is a $\mathcal{C}$-diagram, then the* homotopy limit *of X, denoted by* $\operatorname{holim}(X)$ *is the object* $\hom^{\mathcal{C}}(B(\mathcal{C} \downarrow -), X)$.

Fact: If X and Y are $\mathcal{C}$-diagrams in $\mathcal{M}$ and $f : X \to Y$ is an objectwise weak equivalence of cofibrant objects. Then the induced map $\text{hocolim}(X) \to \text{hocolim}(Y)$ is a weak equivalence.

Remark. If $X \to Z \leftarrow Y$ is a diagram of *fibrant* objects, the homotopy pullback $X \times^h_Z Y$ is weakly equivalent to the homotopy limit of this diagram. (See Prop. 19.5.3 in [Hir] which proves this statement under the assumption that the model category is right proper. However, this assumption seems unnecessary if one uses Prop. 15.10.10 of [Hir].)

2.10 SOME RESULTS AND CONSTRUCTIONS IN THE CATEGORY OF SIMPLICIAL SETS

In this section, we review some basic results and constructions in **Sset**.

Long exact sequence of homotopy groups:

In algebraic topology, we encounter the long exact sequence of homotopy groups corresponding to a fibration $f : X \to Y$. This long exact sequence gives us a relationship between the homotopy groups of the X, Y and a fiber F of f at some point of X. A similar argument (which we will omit) provides us with the following:

Fact: Let $X \to Y$ be a morphism in **Sset**. Let $v : \Delta^0 \to Y$ be a 0-simplex. Let $u : \Delta^0 \to X$ be a lifting of v. Let $w : \Delta^0 \to F := X \times^h_{Y,y} \Delta_0$ be induced by u and v. Then we have a long exact sequence

$$\ldots \longrightarrow \pi_n(F, w) \longrightarrow \pi_n(X, u) \longrightarrow \pi_n(Y, v) \longrightarrow \ldots$$
$$\ldots \pi_1(Y, v) \longrightarrow \pi_0(F) \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y).$$

(The last three terms in this sequence are not groups, but the sequence is exact as a sequence of pointed sets.)

Eilenberg-MacLane spaces: Let G be a group and let $n \geq 1$ be an integer. If $n \neq 1$, we require G to be an abelian group. Then $K(G, n)$ is a fibrant simplicial set such that

$$\pi_i(K(G, n), *) = \begin{cases} 0 & \text{for } i \neq n \\ G & \text{for } i = n. \end{cases}$$

(Here $*$ denotes some fixed base-point. Since we assume that $\pi_0(K(G,n))$ is a singleton set, the choice of the base-point does not matter much.)

Fact: Let G and n be as above. Then the Eilenberg-Maclane space $K(G,n)$ exists and is unique up to homotopy equivalence. For example, if $K(G,1)$ is simply the classifying space $B(G)$ where we interpret G as a category with a single object.

Coskeleton:

Let $\mathbf{\Delta}_n$ denote the full subcategory of $\mathbf{\Delta}$ the objects of which are [0],[1] $\ldots [n]$. Let $i_n : \mathbf{\Delta}_n \to \mathbf{\Delta}$ be the inclusion functor.

There is a natural pullback functor $(i_n)_* : \mathbf{Set}^{\mathbf{\Delta}} \to \mathbf{Set}^{\mathbf{\Delta}_n}$. It can be proved that $(i_n)_*$ has a right adjoint $(i_n)^*$ and a left adjoint $(i_n)_!$. We define the *n-coskeleton functor*

$$cosk_n : \mathbf{Sset} \to \mathbf{Sset}$$

by the formula $cosk_n(X) := (i_n)^*(i_n)_*(X)$. Clearly, the functor $sk_n := (i_n)_!(i_n)_*$ (called the *n-skeleton functor*) is the right adjoint of $cosk_n$. It is easy to verify that for any simplicial set X,

$$(sk_n(X))_m = \begin{cases} X_m & \text{for } m \leq n, \\ X_n & \text{for } m > n. \end{cases}$$

Along with the adjointness of sk_n and $cosk_n$, this implies the following:

Fact:

$$\pi_i(cosk_n(X), *) = \begin{cases} \pi_i(X, *) & \text{for } i < n \\ 0 & \text{for } i \geq n. \end{cases}$$

We will say that a simplicial set X is *n-truncated* if $X = cosk_{n+1}(X)$.

2.11 LOCALIZATIONS OF MODEL CATEGORIES

Note: The definitions and results in this definition can be formulated and proved for general model categories. However, since the model categories that we are concerned with are simplicial, we will not state the results in the most general form.

Let $\mathcal{M}$ be a model category and S be a class of morphisms in $\mathcal{M}$. The localization of $\mathcal{M}$ is a functor $F : \mathcal{M} \to \mathcal{N}$ such that all the morphisms in S are mapped to weak equivalences in $\mathcal{N}$ (of course, we require F to satisfy a universal property). Depending on whether the functor F is a left Quillen functor or a right Quillen functor we obtain two notions of localization. We will focus on *left localizations*.

Definition 2.11.1. *(Left localization) Let $\mathcal{M}$ be a model category and let S be a class of morphisms of $\mathcal{M}$. A* left localization *of $\mathcal{M}$ with respect to S is a left Quillen functor $j : \mathcal{M} \to L_S\mathcal{M}$ such that:*

1. *the total derived functor $\mathbb{L}j : Ho(\mathcal{M}) \to Ho(L_S\mathcal{M})$ takes the images in $Ho(\mathcal{M})$ of elements of S into isomorphisms in $Ho(L_S\mathcal{M})$, and*
2. *if $\mathcal{N}$ is a model category and $\phi : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor such that $\mathbb{L}\phi : Ho(\mathcal{M}) \to Ho(\mathcal{N})$ takes the images in $Ho(\mathcal{M})$ of the elements of S into isomorphisms in $Ho(\mathcal{N})$, then there is a unique left Quillen functor $\delta : L_S\mathcal{M} \to \mathcal{N}$ such that $\delta j = \phi$.*

We will restrict ourselves to looking at the few basic facts that we will need. Additional details may be found in ([Hir], Chapter 3).

Left Bousfield localization:

Let $\mathcal{M}$ be a *simplicial model category.*

An object W is *S-local* if W is fibrant and for every element $f : A \to B$ of S, the induced map $Map(B, W) \to Map(A, W)$ is a weak equivalence of simplicial sets. A morphism $g : X \to Y$ in $\mathcal{M}$ is a S-local equivalence if for any S-local object W , the induced map $Map(Y, W) \to Map(X, W)$ is a weak equivalence. (Remark: Clearly these definitions could be generalized to the case of a general model category if we could define a "mapping space functor" for a general model category. See [Hir], Chapter 17.)

The *left Bousfield localization of $\mathcal{M}$ with respect to S* (if it exists) is a model category structure $L_S\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that:

1. The class of weak equivalences of $L_S\mathcal{M}$ equals the class of S-local equivalences of $\mathcal{M}$.
2. The class of cofibrations of $L_S\mathcal{M}$ equals the class of cofibrations of $\mathcal{M}$.
3. The class of fibrations of $L_S\mathcal{M}$ is the class of maps having the right lifting property with respect to maps that are both cofibrations and S-local equivalences.

A left Bousfield localization, if it exists, is a left localization ([Hir], Theorem 3.3.19).

Theorem 2.11.2. *([Hir], Theorem 4.1.1) Let $\mathcal{M}$ be a model category and let S be a set of maps in $\mathcal{M}$. Then the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ with respect to S exists if either of the following is true:*

1. *$\mathcal{M}$ is a left proper cellular model category.*
2. *$\mathcal{M}$ is a combinatorial model category.*

In particular, if $\mathcal{C}$ is a small category and S is a set of morphisms in $\mathbf{Sset}^{\mathcal{C}}$ (with the projective model structure), the left Bousfield localization of $\mathbf{Sset}^{\mathcal{C}}$ with respect to S exists.

If the left Bousfield localization exists, it has the following properties:

- If X and Y are S-local objects, then $f : X \to Y$ is a fibration in $L_S\mathcal{M}$ if and only if it is a fibration in $\mathcal{M}$ ([Hir], Proposition 3.3.16).
- If $\mathcal{M}$ is left proper, an object is S-local if and only if it is fibrant in the $L_S\mathcal{M}$ model structure ([Hir],Proposition 3.4.1).
- Let X be a cofibrant object of $\mathcal{M}$ and let Y be an S-local object. Then two maps from X to Y are homotopic in $\mathcal{M}$ if and only if they are homotopic in $L_S\mathcal{M}$ ([Hir], Lemma 3.5.1).
- $\mathbb{R}(Id) : Ho(L_S\mathcal{M}) \to Ho(\mathcal{M})$ is fully faithful. (This is an easy consequence of the above statements.)

3.0 INTRODUCTION TO N-STACKS

3.1 MODEL CATEGORY OF STACKS: DEFINITION AND EXISTENCE

In this section, we review the construction and basic properties of stacks as presented in [TV1] and [TV2]. Some of the definitions we present here are very simplified versions of the ones in [TV2] since we will only work with the homotopical algebraic context of [TV2], Chapter 2.1.

Let $\mathcal{T}$ be a small category with a Grothendieck topology τ. Let $Pr(\mathcal{T})$ denote the category of presheaves on T. For any object T of $\mathcal{T}$, let h_T denote the presheaf $\mathcal{T}(-, T)$.

The category $Sh(T)$ is the localization of $Pr(T)$ with respect to the class of local isomorphisms. It can also be described as the localization of $Pr(T)$ with respect to morphisms of the form $\mathrm{colim}(T_*) \to h_T$ where T_* is a simplicial object in $Pr(T)$ such that:

1. T_0 is of the form $\coprod_i h_{U_i}$ such that $\coprod_i h_{U_i} \to T$ is a local epimorphism of presheaves.
2. $T_n = \underbrace{T_0 \times_T \ldots \times_T T_0}_{n+1 \text{ times}}$.

Definition 3.1.1. *(Prestacks) Let $\mathcal{T}^\wedge$ denote the category of $\boldsymbol{Sset}^{\mathcal{T}^{op}}$ of simplicial presheaves on $\mathcal{T}$ with the projective model structure. We call this the model category of* prestacks *on $\mathcal{T}$.*

Recall that $\mathcal{T}^\wedge$ is a simplicial model category. The simplicial structure is defined by the formula

$$(F \otimes K)(T) := F(T) \otimes K$$

where F is an object of $\mathcal{T}^\wedge$, T is an object of $\mathcal{T}$ and K is an object of **Sset**.

Any set S defines a simplicial set $\underline{S}$ such that $(\underline{S})_n = S$ for all n. Thus we have a functor

$Pr(\mathcal{T}) \to \mathcal{T}^\wedge$ which is easily seen to be fully faithful. In particular, if T is any object of $\mathcal{T}$, we may use $\underline{h}_T$ to denote the prestack

$$U \to \underline{\mathcal{T}(U,T)}.$$

This defines a functor $\mathcal{T} \to \mathcal{T}^\wedge$.

We have the following simplicial version of the Yoneda lemma.

Lemma 3.1.2. *Let $\mathcal{T}$ be a small category, let T be an object of $\mathcal{T}$ and let F be an object of $\mathcal{T}^\wedge$. Then there is an isomorphism*

$$F(T) \cong Map_{(\mathcal{T}^\wedge)}(\underline{h}_T, F).$$

which is natural in T and F.

If there is no possibility of confusion, we will not distinguish between an object T and the prestack $\underline{h}_T$.

The category of stacks on $\mathcal{T}$ will be defined to be the localization of $\mathcal{T}^\wedge$ with respect to local *weak equivalences* which we define as follows:

Homotopy group sheaves and local equivalences:

Let F be a prestack on $\mathcal{T}$.

– Let $\pi_0^{pr}(F)$ be the presheaf defined by the formula $\pi_0^{pr}(F)(T) = \pi_0(F(T))$. Let $\pi_0(F)$ denote the associated sheaf.

– Let U be an object of $\mathcal{T}$. Let $i > 0$ be an integer. Let $\underline{h}_U$ denote the prestack represented by U. Let $u : \underline{h}_U \to F$ denote a morphism of prestacks (u is a 0-simplex of $F(U)$ by the simplicial version of Yoneda's lemma). Let $\pi_i^{pr}(F, u)$ denote the presheaf on (Aff $/U$) defined by

$$\pi_i^{pr}(F, u)(V) = \pi_i(F(V), u|_V).$$

Let $\pi_i(F, u)$ denote the associated sheaf. We call this the *i-th homotopy group sheaf of F at u*. Equivalently,

$$\pi_i(F, u) = \pi_0((F^{\Delta^n}|_U) \times_{(F^{\partial\Delta^n}|_U)} *)$$

where $*$ is the point of $F|_U$ defined by u.

– Let $f : F \to G$ be a morphism of prestacks. We say that f is a *local equivalence* if:

1. The induced morphism $\pi_0(F) \to \pi_0(G)$ is an isomorphism of sheaves.
2. For any object U of $\mathcal{T}$, any morphism $u : \underline{h}_U \to F$ and any integer $i > 0$, the induced morphism $\pi_i(F, u) \to \pi_i(G, f(u))$ is an isomorphism of sheaves.

Model category of stacks:

Theorem 3.1.3. *([DHI], Theorem 6.2) There exists a model structure on the category $\mathcal{T}^\wedge$ for which the weak equivalences are the local equivalences and the cofibrations are the cofibrations for the projective model structure. We denote this model structure by $\mathcal{T}^{\sim,\tau}$ and call it the* local projective model structure*.*

The following outline is from ([DHI]).

Proof. (Sketch)

– ([DHI], Section 4.1) For any object F of $\mathcal{T}^\wedge$, let F_n denote the presheaf of $T \to F(T)_n$. Then a we have the following generalization of the notion of a cover:
Let T be an object in $\mathcal{T}$. A *hypercover of* T is a morphism $U \to T$ in $\mathcal{T}^\wedge$ such that each U_n is a coproduct of representable presheaves on $\mathcal{T}$ and the morphism

$$U_n \to (U^{\partial\Delta^n})_0 \times_{((\underline{h}_T)^{\partial\Delta^n})_0} (\underline{h}_T)_n$$

is a local epimorphism of presheaves for all $n \geq 0$. This is similar to the case of presheaves (see the remarks at the beginning of this section) except that in that situation we require this map to be an *isomorphism*. Suppose $U_n = \coprod_{V \in K_n} h_V$. (Note: h_V is the presheaf represented by V, not the prestack.) The *size* of a hypercover $U \to X$ is the cardinality of the set $\coprod_n K_n$. (This definition is from [DHI].)

– We say that a set S of hypercovers is *dense* if every hypercover $U \to X$ can be refined by one in S.
Let λ be a regular cardinal sufficiently large compared to the set of morphisms in $\mathcal{T}$ and let S_λ denote the set of all hypercovers of size less than λ. Then any hypercover $U \to X$ can be refined by one in S ([DHI], Proposition 6.6).

– The left Bousfield localization of $\mathcal{T}^\wedge$ with respect to the set S_λ (which exists by Theorem 2.11.2) is the same as the left Bousfield localization of $\mathcal{T}^\wedge$ with respect to the class of local equivalences ([DHI], Theorem 6.2).

□

3.2 PROPERTIES OF THE MODEL CATEGORY OF STACKS

We state, without proofs, some of the basic properties of the local projective model structure.

1. The model category $\mathcal{T}^\wedge$ is cofibrantly generated and proper ([TV1], Theorem 3.4.1).
2. An object $F \in \mathcal{T}^{\sim,\tau}$ is fibrant if and only if it is objectwise fibrant and if for any object $X \in Ob(\mathcal{T})$ and any hypercover $U \to X$, the induced morphism

$$F(X) \to Map(\text{hocolim}(U), F)$$

   is a weak equivalence in **Sset**, i.e. F satisfies the *hyperdescent condition* with respect to the hypercover $U \to X$. (See [DHI], Corollary 7.1.)
3. Let $(\mathcal{T}, \tau)$ and $(\mathcal{T}', \tau')$ be two small Grothendieck sites. Let $f : \mathcal{T} \to \mathcal{T}'$ be a functor. We say that f is *continuous* if the pullback functor $f^* : (\mathcal{T}')^{\sim,\tau'} \to \mathcal{T}^{\sim,\tau}$ is a right Quillen functor. Let $f_!$ denote the left adjoint of f^*. The following result is an easy consequence of (2).

   Proposition 3.2.1. *([DHI], Proposition 8.2) With the above notation, suppose there exists a dense set S of hypercovers of $\mathcal{T}$ such that $f_!$ takes elements of S to hypercovers in $\mathcal{T}'$. Then the adjunction $(f_!, f^*)$ is a Quillen adjunction for the local projective model structure.*

   Indeed, this adjunction is easily seen to be a Quillen adjunction for the projective model structure. Let H and H' denote the classes of hypercovers in $\mathcal{T}$ and $\mathcal{T}'$. Then, due to (2), f^* takes fibrant objects (in the local projective model structure) to fibrant objects if $f_!(H) \subset H'$. By adjointness, if f^* preserves local objects, $f_!$ preserves local equivalences. Thus $f_!$ is a left Quillen functor for the local projective model structure.

4. The following is the model theoretic analogue of a groupoid. Roughly speaking, we require arrows to be invertible upto homotopy.

Definition 3.2.2. *A* Segal groupoid *in a model category $\mathcal{M}$ is a simplicial object $X : \mathbf{\Delta}^{op} \to \mathcal{M}$ such that:*

- *For any $n > 0$, the natural morphism*

$$X_n \to \underbrace{X_1 \times^h_{X_0} \times X_1 \times^h_{X_0} \ldots \times^h_{X_0} X_1}_{n \text{ times}}$$

induced by the n morphisms $s_i : [1] \to [n]$ defined as $s_i(0) = i$, $s_i(1) = i + 1$, is a weak equivalence.

- *The morphism*

$$d_0 \times d_1 : X_2 \to X_1 \times^h_{d_0, X_0, d_0} X_1$$

is a weak equivalence.

Definition 3.2.3. *We say that* Segal equivalence relations are homotopy effective in $\mathcal{M}$ *if for any Segal groupoid X_* in $\mathcal{M}$ with homotopy colimit*

$$|X_*| := \operatorname{hocolim}_{n \in \mathbf{\Delta}} X_n,$$

and any $n > 0$, the natural morphism

$$X_n \to \underbrace{X_0 \times^h_{|X_*|} X_0 \times^h_{|X_*|} \times^h_{|X_*|} \ldots \times^h_{|X_*|} X_0}_{n+1 \text{ times}}$$

induced by the $n + 1$ morphisms $[0] \to [n]$ is a weak equivalence in $\mathcal{M}$.

Segal equivalence relations are homotopy effective in **Sset**. This implies the following:

Theorem 3.2.4. *([TV1], Theorem 4.9.2) Let $\mathcal{T}$ be a small Grothendieck site. Segal equivalence relations are homotopy effective in $\mathcal{T}^{\sim,\tau}$.*

3.3 TRUNCATION FUNCTORS

We say that a stack F is *n-truncated* or that it is an *n-stack* if for any object T of $\mathcal{T}$, any $t : \underline{h}_T \to F$ and any integer $i > n$, the sheaf $\pi_i(F, t)$ is trivial for $i > n$.

We saw in Section 2.10 that there exists an endofunctor on **Sset** which associates an n-truncated object to a simplicial set. We would like to construct an analogous functor for stacks.

Definition 3.3.1. *Let $\mathcal{T}$ be a small Grothendieck site. Let $n \geq 0$ be an integer. Let $f : F \to G$ be a morphism in $\mathcal{T}^\wedge$. We say that f is a* local n-equivalence *if the following two conditions are satisfied:*

1. *The morphism $\pi_0(F) \to \pi_0(G)$ is an isomorphism of sheaves.*
2. *For any object T in $\mathcal{T}$, and $t : \underline{h}_T \to F$ and any i such that $n \geq i > 0$, the morphism $\pi_i(F, t) \to \pi_i(G, f(t))$ is an isomorphism of sheaves.*

Theorem 3.3.2. *([TV1], Theorem 3.7.3) There exists a model structure on $\mathcal{T}^\wedge$ called the* n-truncated local projective model structure *for which the weak equivalences are the $\pi_{\leq n}$-equivalences and the cofibrations are the cofibrations for the projective model structure. This model structure is cofibrantly generated and proper. We will denote this model structure by $\mathcal{T}^{\sim,\tau}_{\leq n}$.*

The model structure on $\mathcal{T}^{\sim,\tau}_{\leq n}$ is simply the left Bousfield localization of $\mathcal{T}^{\sim,\tau}$ with respect to morphisms of the form $\partial\Delta^n \otimes \underline{h}_T \to \Delta^n \otimes \underline{h}_T$ for all $i > n$ and all objects T of $\mathcal{T}$. Thus we have a Quillen adjunction

$$(Id, Id) : \mathcal{T}^{\sim,\tau} \to \mathcal{T}^{\sim,\tau}_{\leq n}$$

which induces an adjunction between the homotopy categories

$$(\mathbb{L}(Id), \mathbb{R}(Id)) : Ho(\mathcal{T}^{\sim,\tau}) \to Ho(\mathcal{T}^{\sim,\tau}_{\leq n}).$$

Then the *n-truncation endofunctor* is the functor

$$t_{\leq n} = \mathbb{R}(Id) \circ \mathbb{L}(Id) : Ho(\mathcal{T}^{\sim,\tau}) \to Ho(\mathcal{T}^{\sim,\tau}).$$

The adjunction also gives us a natural transformation $Id_{Ho(\mathcal{T}^{\sim,\tau})} \to t_{\leq n}$ such that $F \to t_{\leq n}(F)$ is a local n-equivalence for every F.

We saw in Section 2.11 that the functor

$$\mathbb{R}(Id) : Ho(\mathcal{T}^{\sim,\tau}_{\leq n}) \to Ho(\mathcal{T}^{\sim,\tau})$$

is fully faithful. The essential image of this functor is called the *subcategory of n-truncated stacks* or *n-stacks*.

Proposition 3.3.3. *([TV1], Proposition 3.7.8) The functor $Pr(\mathcal{T}) \to \mathcal{T}^{\wedge}$ induces an equivalence of the category of sheaves with the category of 0-stacks*

$$Shv(\mathcal{T}) \to \mathcal{T}^{\sim,\tau}.$$

A quasi-inverse of this functor is obtained by restricting the functor π_0 to the category of 0-stacks.

3.4 LONG EXACT SEQUENCE OF HOMOTOPY GROUP SHEAVES

Let $F \to G$ be a fibration in $\mathcal{T}^{\sim,\tau}$. Let T be any object of $\mathcal{T}$. Let $g : \underline{h}_T \to G$ be any morphism and let $f : \underline{h}_T \to F$ be a lifting of t. Let $h : \underline{h}_T \to F \times^h_G T$ be induced by f and g. Using the long exact sequence in Section 2.10, we obtain an exact sequence of homotopy group sheaves

$$\begin{aligned} \ldots \longrightarrow \pi_n(F \times^h_G T, h) \longrightarrow \pi_n(F, f) \longrightarrow \pi_n(G, g) \longrightarrow \ldots \\ \ldots \pi_1(G, g) \longrightarrow \pi_0(F \times^h_G T) \longrightarrow \pi_0(F) \longrightarrow \pi_0(G). \end{aligned}$$

We now define an analogue of the loop spaces we defined for **Sset** and **Top**.

Definition 3.4.1. *(Loop stack) Let F be a stack over $\mathcal{T}$. Let T be any object of $\mathcal{T}$. Let $t_1, t_2 : \underline{h}_T \to F$ be any two morphisms. Then the* stack of paths from t_1 and t_2 *is a stack over the site $\mathcal{T}/T$ defined by the formula*

$$\Omega_{t_1,t_2}F := Id_T \times^h_{t_1,F|_T,t_2} Id_T.$$

If $t_1 = t_2 = t$, we denote $\Omega_{t_1,t_2}F$ by $\Omega_t F$.

The sheaf $\pi_0(\Omega_{t_1,t_2}F)$ is called the sheaf of homotopy classes of paths from t_1 to t_2 *and is denoted by $\pi_1(F, t_1, t_2)$.*

The n-th iterated loop stack of F at t is defined inductively by the formula

$$\Omega^{(n)}_t F := \Omega_t(\Omega^{(n-1)}_t F).$$

We record the following lemma for later use:

Lemma 3.4.2. *Let F be a stack over the site $\mathcal{T}$. Let T be an object of T and let $t_1, t_2 : \underline{h}_T \to F$ be two morphisms.*

1. If $t_1 = t_2 = t$, we have the isomorphism $\pi_n(F, t) = \pi_0(\Omega^{(n)}_t F)$.

2.

$$\Omega_{t_1,t_2}F := ((T \times^h_{t_1,F,t_2} T) \times^h_{(T\times T)} T)|_T.$$

Proof. (1) is a direct consequence of the long exact sequence of homotopy group sheaves. The proof of (2) is simply a category theoretic calculation of fiber products. □

Eilenberg-MacLane stacks:

Let G be a group sheaf. Consider the simplicial presheaf $K(G, n)$ defined by the formula

$$X \to K(G(X), n).$$

This defines an object of $\mathcal{T}^{\sim,\tau}$ which we denote by $K(G, n)$. Then for any affine scheme X over S, $\pi_0(K(G, n)(X)) = \{*\}$.

$$\pi_i(K(G(X), n), *) = \begin{cases} 0 & \text{for } i \neq n, \\ G(X) & \text{for } i = n. \end{cases}$$

It follows immediately from Lemma 3.4.2 that

$$\Omega_* K(G,n) \cong \begin{cases} K(G,n-1) & \text{for } n > 1, \\ G & \text{for } n = 1. \end{cases}$$

3.5 GEOMETRIC STACKS

We now focus on stacks in the context of algebraic geometry.

Let $S = \mathrm{Spec}(R)$ be a fixed affine scheme. Let $(\mathrm{Aff}\,/S)$ denote the category of affine schemes over S which are spectra of rings within some universe $\mathbb{U}$. We will use the etale topology on $(\mathrm{Aff}\,/S)$. If $\mathbb{V}$ is a universe such that $\mathbb{U} \in \mathbb{V}$, the category $(\mathrm{Aff}\,/S)$ is $\mathbb{V}$-small and thus the construction above may be used to define the model category $(\mathrm{Aff}\,/S)^{\sim,et}$ of stacks over the site $(\mathrm{Aff}\,/S)$.

For the sake of convenience we will denote the category $Ho((\mathrm{Aff}\,/S)^{\sim,et})$ by $St(S)$ or $St(R)$ and call it the *category of stacks over* S.

Recall from ([LMB], Chapter 1) that an algebraic space is a sheaf that satisfies a certain "geometric" condition. We quote this definition in order to motivate the definition of geometricity of stacks:

Definition 3.5.1. *(Algebraic spaces) An algebraic space over* S *is a sheaf* F *over* $(\mathrm{Aff}\,/S)$ *such that:*

1. *There exists a scheme* U *and a map of sheaves* $U \to F$ *such that for all schemes* V *and all morphisms* $V \to F$*, the sheaf* $V \times_F U$ *is a scheme and the morphism* $U \times_F V \to V$ *is an etale surjection.*
2. *If* $V \to F \times F$ *is any morphism, the sheaf* $F \times_{(F\times F)} V$ *is a scheme and the morphism*

$$F \times_{(F\times F)} V \to V$$

is a quasi-compact morphism of schemes.

First we define monomorphisms and epimorphisms of stacks:

Definition 3.5.2. *Let $f : F \to G$ be a morphism of stacks.*

1. *We say that f is a* monomorphism *or that F* is a substack of G *if the morphism*

$$\Delta_{F/G} : F \to F \times^h_G F$$

   is an isomorphism in $St(S)$.

2. *f is an* epimorphism *if the morphism $\pi_0(F) \to \pi_0(G)$ is an epimorphism of sheaves.*

Remark. f is a monomorphism if and only if f induces a monomorphism of π_0 sheaves and an isomorphism of π_i sheaves for $i > 0$.

The definition of geometricity for stacks is a natural generalization of the above definition.

Definition 3.5.3. *([TV2], Definition 1.3.3.1) We define the notions of a k-*geometric stack, k-representable morphism *and k-*smoothness *by induction on k for $k \geq -1$ as follows:*

1. *A stack is (-1)-geometric if it is represented by an affine scheme.*
2. *Suppose the notion of k-geometric stack has been defined. Then a morphism $F \to G$ is* k-representable *if for any affine scheme Z and any morphism $Z \to G$, the stack $F \times^h_G Z$ is k-geometric.*
3. *Suppose that the notion of l-smoothness has been defined for all $l < k$ and that the notion of k-representable morphism has been defined. Let $f : F \to G$ be a k-representable morphism. Then f is k-smooth if for any affine scheme Z and any morphism $Z \to G$, there exists X which is a coproduct of affine schemes $X = \coprod_i X_i$ and a $(k-1)$-smooth morphism $X \to F \times^h_G Z$ which is an epimorphism of stacks such that for each i, the morphism $X_i \to X \to Z$ is a smooth morphism of schemes.*
4. *Suppose that the notion of a $(k-1)$-geometric stack has been defined. Then a stack F is k-geometric if it satisfies the following two conditions:*
   a. *There exists a X which is a coproduct of affine schemes $X = \coprod_i X_i$ and an epimorphism $X \to F$ such that each $X_i \to F$ is $(k-1)$-smooth. (We say that $X \to F$ is a* k-atlas *for F.)*
   b. *The morphism $F \to F \times^h F$ is $(k-1)$-representable.*

Suppose $k < m$. It is easy to verify that a k-geometric stack is m-geometric and a k-representable morphism is m-representable. A k-smooth morphism is m-smooth and a k-representable morphism which is known to be m-smooth is also k-smooth. (See [TV2], Section 1.3.3.)

Lemma 3.5.4. *([TV2], Lemma 2.1.1.2) A k-geometric stack is $(k+1)$-truncated.*

Definition 3.5.5. *1. An* Artin n-stack *is an n-stack which is m-geometric for some integer m. An* Artin stack *is a stack which is an Artin n-stack for some integer n.*

A sheaf is an algebraic space if and only if it is the quotient of an etale equivalence relation (see [Knu]). This result can be generalized to Artin stacks as follows:

Definition 3.5.6. *A Segal groupoid object X_* in $St(S)$ is an* n-smooth Segal groupoid *if it satisfies the following two conditions:*

1. *The stacks X_0 and X_1 are disjoint unions of n-geometric stacks.*
2. *The morphism $d_0 : X_1 \to X_0$ is n-smooth. (It is easy to check that all the face morphisms of X_* are n-smooth.)*

Definition 3.5.7. *Let $f : F \to G$ be a morphism in $St(S)$. The* homotopy nerve *of f is the Segal groupoid defined by*

$$X_0 := F$$

and

$$X_n := \underbrace{X_0 \times^h_G X_0 \times^h_G \times^h_G \dots \times^h_G X_0}_{n+1 \text{ times}}$$

with the face and degeneracy maps being defined by the projection maps in the obvious manner.

Proposition 3.5.8. *([TV2], Proposition 1.3.4.2) Let $F \in St(S)$ be a stack and let $n \geq 0$. The following conditions are equivalent:*

1. *The stack F is n-geometric.*
2. *There exists an $(n-1)$-smooth Segal groupoid object X_* in $St(S)$ such that X_0 is a disjoint union of representable stacks and an isomorphism in $St(S)$*

$$F \cong |X_*| = \mathrm{hocolim}_{[n] \in \mathbf{\Delta}} X_n.$$

In fact, if $U \to F$ is an atlas for F, X_* can be chosen to simply be the homotopy nerve of the map $U \to F$.

3. *There exists an $(n-1)$-smooth Segal groupoid object X_* in $St(S)$ and an isomorphism in $St(S)$*

$$F \cong |X_*| := \operatorname{hocolim}_{[n]\in\mathbf{\Delta}} X_n.$$

If these conditions are satisfied we say that F is the quotient stack of the $(n-1)$-Segal groupoid X_*.

Corollary 3.5.9. *([TV2], Corollary 1.3.4.5) Let $f : F \to G$ be an epimorphism of stacks and let $n \geq 0$. If F is n-geometric and f is $(n-1)$-smooth, then G is n-geometric.*

Lemma 3.5.10. *Let $f : F \to G$ be a morphism of stacks. Let X_* denote the homotopy nerve of f. Then the induced map $F \to |X_*|$ is an epimorphism and the map $|X_*| \to G$ is a monomorphism.*

Proof. The proof of this lemma is essentially contained in the proof of ([TV2], Lemma 1.3.4.3). □

Definition 3.5.11. *(Image stacks) Let $X \to Y$ be a morphism of stacks. Let $X \to W \to Y$ be a factorization of $X \to Y$ such that $X \to W$ is an epimorphism and $W \to Y$ is a monomorphism. Then we say that the stack W is the* image *of the morphism $X \to Y$.*

3.6 PROPERTIES OF MORPHISMS OF STACKS

Properties of schemes and morphisms of schemes are extended to stacks in the expected manner. (For example, see [Knu], [LMB].)

Stable properties:

Definition 3.6.1. *Let $\mathbf{Q}$ be a property of morphisms of affine schemes. We say that $\mathbf{Q}$ is* stable with respect to the etale topology *if:*

1. *If a morphism $X \to Y$ of affine schemes has the property $\mathbf{Q}$, Z is any affine scheme and $Z \to Y$ is any morphism, then $X \times_Y Z \to Z$ has the property $\mathbf{Q}$.*

2. *Suppose $X \to Y$ is a morphism of affine schemes. If $\{U_i \to Y\}_i$ is an etale cover of Y, the morphism $X \to Y$ has the property $\mathbf{Q}$ if and only if the morphism $X \times_Y U_i \to U_i$ has the property $\mathbf{Q}$ for each i.*

The following properties are stable with respect to the etale topology (see [LMB], Chapter 3):

Surjective, radiciel, universally bijective, universally open, universally closed, separated, quasi-compact, locally of finite type, locally finitely presented, of finite type, finitely presented, immersion, open immersion, closed immersion, flat, unramified, smooth, etale, etc.

Definition 3.6.2. *Let $X \to Y$ be a morphism of stacks such that for any affine scheme Z and any morphism $Z \to Y$, the stack $X \times^h_Y Z$ is a scheme. Let $\mathbf{Q}$ be a property of morphisms of schemes that is stable with respect to the etale topology. We say that $X \to Y$ has the property $\mathbf{Q}$ if for any scheme Z and any morphism $Z \to Y$, the morphism $X \times^h_Y Z \to Z$ has the property $\mathbf{Q}$.*

Remark. Suppose $X \to Y$ is a morphism of *Artin stacks*. Let $V = \coprod_i V_i \to Y$ be an atlas of Y where each V_i is an affine scheme. Suppose that for each i, the stack $X \times_Y V_i$ is a scheme and the morphism $X \times_Y V_i \to V_i$ is a morphism having a stable property $\mathbf{Q}$. Then it is easy to check that the same is true for *any* morphism $Z \to Y$ from an affine scheme into Y. In other words, the condition in the above definition only needs to be checked for a fixed atlas.

In particular, we obtain notions of open and closed immersions for stacks. So we are able to define.

Definition 3.6.3. *Let $F \to G$ be a monomorphism of stacks. We say that F is a* closed (resp. open, resp. locally closed) substack *of G if the morphism $F \to G$ is a closed immersion (resp. open immersion, resp immersion).*

Lemma 3.6.4. *Let F be an Artin stack. Let G be an open substack of F. Let $U \to F$ be an atlas for F. Let H_0 be the reduced, closed complement of the open substack $U \times^h_F G$ of U. Let H be the image of $H_0 \to F$. Then H is a closed substack of F. For any atlas $V \to F$, $V \times^h_F H$ is the reduced, closed complement of $V \times^h_F G$ in V.*

Definition 3.6.5. *Let F be an Artin stack. Let G be an open substack of F. Let U be an atlas for F and let H_0 be the reduced, closed complement of $U \times^h_F G$ of U. Then the* reduced, closed complement of G *is the closed substack of* F *which is the image of* $H_0 \to F$.

Local properties:

Definition 3.6.6. *Let* $\mathbf{P}$ *be a property of affine schemes. We say that* $\mathbf{P}$ *is* local with respect to the smooth topology *if for any smooth surjection* $X \to Y$ *of affine schemes,* X *has the property* $\mathbf{P}$ *if and only if* Y *has the property* $\mathbf{P}$.

The following properties of schemes are local with respect to the smooth topology (see [LMB], Chapter 4):

Locally noetherian, reduced, normal, locally notherian and Cohen-Macaulay, regular, of given characterstic p, etc.

Definition 3.6.7. *Let* $\mathbf{P}$ *be a property of affine schemes that is local with respect to the smooth topology. We say that an Artin stack* X *has the property* $\mathbf{P}$ *if and only if there exists an atlas* $\coprod_i U_i \to X$ *such that every* U_i *has the property* $\mathbf{P}$.

Definition 3.6.8. *Let* $\mathbf{Q}$ *be a property of morphisms of schemes. We say that* $\mathbf{Q}$ *is* local for the smooth topology *if the following is true:*

Let $f : X \to Y$ *be a morphism of schemes. Let* Z *be a scheme and let* $Z \to Y$ *and* $W \to X \times_Y Z$ *be a smooth surjections. Then* f *has the property* $\mathbf{Q}$ *if and only if* $W \to Z$ *has the property* $\mathbf{Q}$.

The following properties of morphisms of schemes are local with respect to the smooth topology:

Surjective, universally open, locally finitely presented, locally of finite type, flat, smooth, universally generisant, etc.

Definition 3.6.9. *Let* $\mathbf{Q}$ *be a property of morphisms of schemes that is local with respect to the smooth topology. Let* $X \to Y$ *be a morphism of Artin stacks. We say that* $X \to Y$ *has the property* $\mathbf{Q}$ *if for any affine scheme* Z *and any morphism* $Z \to Y$*, there exists an atlas* $\coprod_i U_i \to X \times_Y^h Z$ *such that* $U_i \to Z$ *has the property* $\mathbf{Q}$ *for each* i.

Remark. Applying this definition to *smoothness*, we obtain a notion of smoothness for morphisms of Artin stacks. It is easy to see that a morphism is smooth in this sense if and only if it is k-smooth for some k.

Strongly quasi-compact morphisms:

We defined a notion of quasi-compactness for morphisms $X \to Y$ of stacks such that for any affine scheme Z and any morphism $Z \to Y$ the stack $X \times^h_Y Z$ is a scheme. However by defining *quasi-compact stacks*, we can generalize this notion to arbitrary morphisms of stacks.

We say that a stack X is *quasi-compact* if there exists an epimorphism $X' \to X$ where X' is an affine scheme. We say that a morphism $X \to Y$ is *quasi-compact* if for any affine scheme Z and any morphism $Z \to Y$, the stack $X \times^h_Y Z$ is quasi-compact.

If X is a quasi-compact stack and $U \to X$ is a morphism from an affine scheme U into X, the stack $U \times^h_X U$ is not necessarily quasi-compact. Hence, if we wish to obtain a notion that is invariant under passage to loop stacks, we need to strengthen this definition as follows (see [To1]):

Definition 3.6.10. *We define the notion of* strong quasi-compactness *for k-geometric stacks and k-representable morphisms of stacks for all $k \geq 0$ by induction on k:*

- *A 0-geometric stack is* strongly quasi-compact *if it is an affine scheme.*
- *Suppose that the notion of a strongly quasi-compact k-geometric stack has been defined. Then we say that a k-representable morphism $X \to Y$ of stacks is* strongly quasi-compact *if for any affine scheme Z and any morphism $Z \to Y$, the n-geometric stack $X \times^h_Y Z$ is strongly quasi-compact.*
- *Suppose that the notion of a strongly quasi-compact l-geometric stack has been defined for all $l < k$. Let X be a k-geometric stack. We say that X is* strongly quasi-compact *if:*
  - *(i) The morphism $X \to X \times^h X$ (which is $(k-1)$-representable) is strongly quasi-compact.*
  - *(ii) The stack X is quasi-compact.*

Lemma 3.6.11. *Let X be a strongly quasi-compact stack. Then if U and V are affine schemes and $U \to X$, $V \to X$ are any morphisms, the stack $U \times^h_X V$ is strongly quasi-compact.*

Proof. This follows immediately from the fact that the square

$$\begin{array}{ccc} U \times^h_X V & \longrightarrow & U \times V \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times^h X \end{array}$$

is homotopy cartesian. □

Definition 3.6.12. *We say that a stack X is* strongly finitely presented *if it is locally finitely presented and strongly quasi-compact.*

3.7 POINTS ON ARTIN STACKS

The following results are generalizations of the contents of ([LMB], Chapter 5) where they are proved for 1-stacks. The proofs for most of these results apply verbatim to the case of higher stacks. Indeed, they are formal consequences of ([LMB], Proposition 5.6) which is generalized in Proposition 3.7.4 below. Hence we will merely state the remaining results, referring to [LMB] for proofs.

Definition 3.7.1. *Let X be a stack. The* set of points on X *is defined as follows:*

Consider the set

$$P_X := \coprod_K \pi_0(X(K))$$

as K varies over all fields over S. We say that two elements x and x' in P_X represented by morphisms $x' : \mathrm{Spec}(K') \to X$ and $x'' : \mathrm{Spec}(K'') \to X$ are equivalent if there exists a common extension K of K' and K'' such that the two morphisms

$$\mathrm{Spec}(K) \to \mathrm{Spec}(K') \xrightarrow{x'} X$$

$$\mathrm{Spec}(K) \to \mathrm{Spec}(K'') \xrightarrow{x''} X$$

lie in the same component of the simplicial set $X(K)$. Let $|X|$ denote the set of equivalence classes of P_X modulo this equivalence relation. We say that the $|X|$ is the set of points of X.

Remark. In the above definition, we have not assumed that X is an Artin stack. However all the following topological results in this section only apply to Artin stacks.

Definition 3.7.2. *(Zariski topology) Let X be an Artin stack over S. The* Zariski topology *on $|X|$ is the topology for which the the open subsets are those of the form U where U is an open substack of X.*

Let $f : X \to Y$ be a morphism of Artin stacks. Clearly, the induced morphism $|f| : |X| \to |Y|$ is a continuous map. We will abuse notation and write f instead of $|f|$ if there is no likelihood of confusion.

We will say that a map f is *(Zariski) open* if the induced map $|f|$ is open. Note that the definitions in Section 3.6 do not provide us with a notion of openness of morphisms. However, they do provide us with a notion of a *universally open* map. On the other hand, we can also define a universally open map to be a map $X \to Y$ such that for any morphism $Y' \to Y$, the induced morphism $X \times_Y Y' \to Y'$ is (Zariski) open. However, these notions coincide as can be seen from the following results:

Lemma 3.7.3. *Let $X \to Y$ be a universally open morphism of Artin stacks. Then the image of $X \to Y$ is an open substack of Y.*

Proof. Let W be the image of $X \to Y$. Let $Z \to Y$ be any morphism from an affine scheme Z into Y. Let $V \to X \times^h_Y Z$ be an atlas of the stack $X \times^h_Y Z$. Then $V \to Z$ is a universally open morphism of schemes, the image of which is the stack $W \times^h_Y Z$. Thus $W \times^h_Y Z$ is actually an open subcheme of Z. In other words, the morphism $W \times^h_Y Z \to Z$ is an open immerson. Since $Z \to Y$ was arbitrary, it follows from the definitions that $W \to X$ is an open immersion. $\square$

Proposition 3.7.4. *([LMB], Proposition 5.6) Let $f : X \to Y$ be a morphism of Artin stacks. The following conditions are equivalent:*

(a) f is universally open.

(b) For any morphism $Y' \to Y$ of Artin stacks, the induced morphism

$$X \times^h_Y Y' \to Y'$$

is open.

In particular, flat, locally finitely presented morphisms between Artin stacks are open.

Proof. We prove that (a) implies (b). Since the two properties in (a) and (b) are both invariant under change of base, it follows that it suffices to prove that if f is universally open then $f(|X|)$ is an open subset of $|Y|$. But this is immediate from the above lemma since $f(|X|)$ is easily seen to be the same as the image of $|W|$ in $|Y|$ where $W \to Y$ is the image of f.

For the converse, let $Y' \to Y$ be an atlas of Y and let $X' \to X \times^h_Y Y'$ be an atlas of $X \times^h_Y Y'$. Then it suffices to prove that the morphism $X' \to Y'$ is an open morphism of schemes in the conventional sense (i.e. an open morphism of the underlying topological space of the schemes X' and Y' viewed as ringed topological spaces). The map $|X \times^h_Y Y'| \to |Y|$ is open by the assumption on f. The morphism $|X'| \to |X \times^h_Y Y'|$ is open by part (a). Thus the morphism $|X'| \to |Y'|$ is open. As X' and Y' are schemes, the morphism $X' \to Y'$ is an open morphism of schemes (in the conventional sense). □

Generalizing results about the Zariski topology on schemes:

An important application of Proposition 3.7.4 is when the map being considered is an atlas $p : X' \to X$ of an Artin stack and X' is a scheme. Then the following results are easily proved from known results about schemes:

– The topology on $|X|$ admits a basis consisting of quasi-compact open sets.

– A morphism $X \to Y$ of Artin stacks is quasi-compact if and only if the morphism $|X| \to |Y|$ is quasi-compact (i.e. the inverse image of a quasi-compact open subset of $|Y|$ is quasi-compact). (See [LMB], Corollary 5.6.3.)

– Let $f : X \to Y$ be a quasi-compact map. Then a point y of $|Y|$ is in the closure of $f(|X|)$ if and only if it is the specialization of a point of $f(|X|)$. (See [LMB], Proposition 5.7 or [Har], Chapter II, Lemma 4.5.)

– An open morphism of stacks is generisant (i.e. for any $x \in |X|$, a generization of $f(x)$ is the image of a generization of x in $|X|$). (See [LMB], Corollary 5.7.1)

– Any closed irreducible subset of $|X|$ has a generic point.

We will now focus specially on the notion of *constructible sets of points.*

Constructible subsets:

We recall the definition of a *constructible* subsets of a topological space T.

Definition 3.7.5. *Let T be a topological space.*

1. *A subset S of T is* retrocompact *if the inclusion $S \to T$ is quasi-compact, i.e. the intersection of S with any quasi-compact open subsets of T is quasi-compact.*
2. *We say that a set S of T is* constructible *if it is in the smallest family $\mathfrak{F}$ of subsets of T which is closed under finite intersection and complementation and which contains all the retrocompact open subsets of T.*

The following results about constructible subsets are well-known (see [EGA]).

– An open subset is constructible if and only if it is retrocompact.

– A set is retrocompact if and only if it is the finite union of subsets of T of the form $U \cap (T \backslash V)$ where U and V are retrocompact open subsets of T. If T is *noetherian*, a set is constructible if and only if it is the finite union of locally closed subsets of T.

– (Local characterization) Let $\{U_i\}_{i \in I}$ be a *finite* covering of T by *retrocompact* open subsets. Then a set S is constructible if and only if $S \cap U_i$ is constructible for every i.

– If T is an irreducible noetherian space, a constructible set S is dense in T if and only if it contains a non-empty open subset of T.

Remark. As an immediate consequence of the characterization of quasi-compact morphisms above, we see that if U is an open substack of X, then $|U|$ is retrocompact if and only if $U \to X$ is a quasi-compact morphism of stacks in the sense of Section 3.6. Thus if $f : X \to Y$ is a morphism of Artin stacks and $|U|$ is a retrocompact open subset of $|Y|$, its inverse image in $|X|$ is a retrocompact open subset as well. In particular, we see that given a morphism of Artin stacks, *the preimage of a constructible subset is constuctible.*

Lemma 3.7.6. *Let $f : X \to Y$ be a surjective and open morphism of Artin stacks. If $f^{-1}(Z)$ is constructible (resp. retrocompact and open) in $|X|$, then Z is constructible (resp. retrocompact and open) in $|Y|$.*

Proof. It will suffice to prove the result when $f^{-1}(Z)$ is a retrocompact open subset. (The case when $f^{-1}(Z)$ is constructible is an easy consequence.)

Suppose Z is a retrocompact open subset. Replacing $|Y|$ by a quasi-compact open subset, we see that it will be enough to prove that Z is *quasi-compact*. Since $|X|$ has a basis consisting of quasi-compact open subsets and since f is open, we can find a quasi-compact subset $|X'|$ of $|X|$ such that $f|_{|X'|}$ is surjective. $f^{-1}(Z) \cap |X'|$ is quasi-compact. Thus $|Z|$ is quasi-compact. □

Thus we have the following useful description of constructible subsets on stacks:

Corollary 3.7.7. *Let X be an Artin stack and let $U \to X$ be an atlas. Then a subset Z of $|X|$ constructible if and only if its preimage in $|U|$ is constructible.*

This easily yields the following result:

Proposition 3.7.8. *(Chevalley's theorem, [EGA], I, 7.1.4) Let $f : X \to Y$ be a finitely presented morphism of Artin stacks. Suppose Z is a constructible subset of $|X|$. Then $f(Z)$ is a constructible subset of $|Y|$.*

Pro-constructible and ind-constructible sets:

We recall some results about pro-constructible and ind-constructible sets on schemes from ([EGA], I, §7). We will not present any of the proofs since we require these results in a single instance, namely Lemma 5.2.4, which is proved in [DL]. Also the reference for this material ([EGA]) is easily accessible.

Let T be a topological space. A subset S of T is *locally constructible* if for each $t \in T$, there exists an open neighbourhood U of t in T such that $T \cap U$ is constructible in U. A subset S of T is said to be *pro-constructible (resp. ind-constructible)* if for any $t \in T$, there exists an open neighbourhood U of t in T such that $S \cap U$ is the intersection (resp. union) of a family of locally constructible sets.

Lemma 3.7.9. *([EGA], I, Proposition 7.2.1 (vi)) Let $f : X' \to X$ be a morphism of schemes. Then a subset E of $|X|$ is pro-constructible (resp. ind-constructible) in $|X|$ if and only if $f^{-1}(E)$ is pro-constructible (resp. ind-constructible) in $|X'|$.*

Lemma 3.7.10. *([EGA], I, Corollary 7.2.7) Let X be a quasi-compact scheme. Let F be a pro-constructible subset of $|X|$ and $\{E_\lambda\}_{\lambda \in L}$ be a family of ind-constructible subsets of $|X|$ such that $F \subset \bigcup_{\lambda \in L} E_\lambda$. Then there exists a finite subset J of L such that $F \subset \bigcup_{\lambda \in J} E_\lambda$.*

3.8 DIMENSION

We define the notion of dimension for locally noetherian Artin stacks over S. The following definition is a straightforward generalization of the definition in ([LMB], Chapter 11).

We define the notion of *dimension of a locally noetherian k-geometric stack* and *relative dimension of a k-representable morphism of locally finite type* by induction on k as follows:

– If X is a (-1)-geometric stack (i.e. an affine scheme) that is locally noetherian and x is a point of $|X|$, we define $\dim_x(X)$ to be the Krull dimension of the local ring $\mathcal{O}_{X,x}$, i.e. the length of the maximal chain of prime ideals in $\mathcal{O}_{X,x}$.

– Suppose that the notion of the dimension of a k-geometric stack has been defined. Let $f : X \to Y$ be a k-representable morphism of locally finite type. Let x be a point of X and y be its image. Suppose y is represented by a morphism $\mathrm{Spec}(K) \to Y$. Let X_y denote the fiber of f over y i.e. the stack $X \times^h_{Y,y} \mathrm{Spec}(K)$. X_y is independent of the representative $\mathrm{Spec}(K) \to Y$ of y (see [Hir], Proposition 13.4.7). We define $\dim_y(f) := \dim_x(X_y)$.

– Suppose that the notion of the dimension of l-geometric stack has been defined for all $l < k$. Let X be a locally noetherian k-geometric stack. Let $p : U \to X$ be an atlas of X where U is a locally notherian scheme. Let x be a point of X and let u be a lift of x to U. Let $pr_1 : U \times^h_X U \to U$ be the first projection. Then we define $\dim_x(X) := \dim_u(U) - \dim_{(u,u)}(pr_1)$.

If X is a locally noetherian stack, we define $\dim(X)$ as the supremum of $\dim_x(X)$ as x varies over all elements of $|X|$.

If X is a noetherian stack and C is a constructible subset of $|X|$, we saw that C can be written as a finite union $\coprod_i |F_i|$ where each F_i is a locally closed substack of X. Then we define

$$\dim(C) := \max_i \dim(F_i).$$

It is easy to check that this definition is independent of the choice of $\{F_i\}_i$.

3.9 PRESENTABILITY OF HOMOTOPY GROUP SHEAVES

For the duration of this section, we assume that the base scheme S is *noetherian*.

Let F be a strongly finitely presented Artin stack over (Aff/S). Let $U \to F$ be an atlas of F such that U is an affine scheme of finite type over S. Then $U \times^h_F U$ is also a strongly finitely presented Artin stack over S. Let $R \to U \times^h_F U$ be an atlas for $U \times^h_F U$ such that R is an affine scheme of finite type over S. Then $\pi_0(F)$ is the quotient of the equivalence relation

$$R \rightrightarrows U$$

in the category of sheaves. Thus, it is natural to expect that it should inherit some of the reasonable properties of sheaves that are represented by noetherian schemes. The following results from [Si1] make this precise. We will restrict ourselves to the bare essentials that we require since a detailed treatment of this topic would be involve an unreasonablely long digression.

Vertical maps: A morphism $F \to G$ of sheaves on (Aff/S) is said to be *vertical* if the following is statement is true:

Let $n \geq 1$ be any integer. Suppose Y is a noetherian scheme with n-closed subschemes $Y_i \subset Y$ and retractions $r_i : Y \to Y_i$ which commute pairwise ($r_i r_j = r_j r_i$) and such that for all $j \leq i$, r_i retracts Y_i to $Y_j \cap Y_i$. Suppose given a morphism $Y \to G$ and liftings $\lambda_i : Y_i \to F$ such that $\lambda_i|_{Y_i \cap Y_j} = \lambda_j|_{Y_j \cap Y_i}$. Then for any point $P \in Y$, lying on at least one of the Y_i, there exists an etale neighbourhood $Y' \to Y$ of P and a lifting $\lambda : Y' \to F$ which agrees with the given liftings $\lambda_i|_{Y_i \times_Y Y'}$.

Example: Etale maps, smooth maps, injective maps and surjective morphisms of group sheaves are examples of vertical maps ([Si1], Theorem 2.2).

We define the following properties of sheaves over (Aff/S):

P4: A sheaf F is $P4$ if there exist vertical surjective morphisms $X \to F$ and $Y \to X \times_F X$ such that X and Y are affine schemes of finite type over S.

P5: A sheaf F is $P5$ if it is $P4$ and the morphism $F \to S$ is vertical. A $P5$ *group* sheaf is said to be *presentable*.

Remark. As the reader might expect, the properties $P4$ and $P5$ are part of a sequence - $P1$, $P2$, $P3$, $P3\frac{1}{2}$, $P4$ and $P5$. However, we omit these details to avoid a digression. See [Si1] for details.

Lemma 3.9.1. *([Si1], Lemma 1.6) If $F \to H$ and $G \to H$ are morphisms of $P4$ schemes then the scheme $F \times_H G$ is $P4$.*

Lemma 3.9.2. *Let K be an uncountable field of characteristic zero.*

1. *Let $f : S \to T$ be a morphism of schemes of finite type over* $\mathrm{Spec}(K)$*. Then f is* $\mathrm{Spec}(K)$*-vertical if and only if it is smooth. (Lemma 6.2 of [Si1])*
2. *A group sheaf over* $\mathrm{Spec}(K)$ *is presentable if and only if it is respresented by a group scheme of finite type over K. (Theorem 6.4 of [Si1])*

The above results imply the following lemma which will be useful for us:

Lemma 3.9.3. *Let $F \to G$ be a morphism of $P4$ sheaves such that $|F| \to |G|$ is a surjection. Then $F \to G$ is a surjection.*

Proof. Let $X \to F$, $Y \to G$ be surjections such that X and Y are schemes of finite type over S. Then the sheaf $X \times_G Y$ is $P4$. Let $Z \to X \times_G Y$ be a surjection such that Z is a scheme of finite type over S. As $|X| \to |G|$ is a surjection, so is $|X \times_G Y| \to |Y|$. Thus $|Z| \to |Y|$ is a surjection. As Z and Y are schemes, this implies that $Z \to Y$ is a surjection. Thus $Z \to G$ is a surjection and hence $F \to G$ is a surjection. □

Proposition 3.9.4. *(Proposition 5.1 in [Si2]) Let X be a strongly finitely presented Artin stack. Let $i > 0$ be an integer. Let T be a scheme and let $t : T \to X$ be some morphism. Then $\pi_i(X, t)$ is a presentable group sheaf over T. In particular, if $T =$ $\mathrm{Spec}(K)$ where K is an uncountable field of characteristic zero, then $\pi_i(X, t)$ is a smooth group scheme.*

Remark. 1. The results in [Si1] and [Si2] are formulated over the site of *noetherian* schemes over S. We are working with a much larger site. However, the results apply easily to this situation since if X is a strongly finitely presented Artin stack over S and T is an arbitrary scheme over T, any morphism $T \to X$ factors as $T \to T_0 \to X$ where T_0 is noetherian over S.

2. The assumption that K is *uncountable* is adopted in [Si1] from section 4 onwards. It does not seem to be necessary for the proof of Lemma 3.9.2 (1) (Lemma 6.2 in [Si1]). The proof of Lemma 3.9.2 (2) (Theorem 6.4 in [Si1]) explicitly assumes $k = \mathbb{C}$, which of course, implies the result for any uncountable k of characteristic zero. It is not clear if the result is true or not without this assumption. Regardless of whether this assumption is necessary or not in what follows, we will always adhere to it in order to be consistent with the original source.

3.10 GROTHENDIECK RING OF ARTIN STACKS

In this section we define the ring in which motivic measure takes its values.

Let k be a field of characteristic zero.

First we recall the definition of the Grothendieck ring of varieties over k (see [DL]):

Definition 3.10.1. *Let $K_0(Var/k)$ denote the free abelian group generated by symbols of the form $[V]$ where V is a variety over k subject to the following relations:*

1. If V and V' are isomorphic, then

$$[S] = [S'].$$

2. If V is a variety over k and W is a closed sub-variety, then

$$[V] = [W] + [V \backslash W].$$

We define a product on this group by $[V_1] \cdot [V_2] = [V_1 \times_k V_2]$. This gives $K_0(Var/k)$ the structure of a commutative ring. This is called the Grothendieck ring of varieties over k.

Let $\mathbb{L} = [\mathbb{A}^1_k]$. Let $M_k = K_0(Var/k)[\mathbb{L}^{-1}]$. For any $[V]/\mathbb{L}^m \in M_k$, define

$$\dim([V]/\mathbb{L}^m) = \dim(V) - m.$$

Let $F^m(M_k) := \{v \in M_k | \dim(v) \leq -m\}$. This is a decreasing filtration on M_k. Let $\widehat{M^k}$ denote the completion of M_k with respect to the filtration F^*.

The analogous notion for Artin stacks is defined as follows (see [To1]):

Definition 3.10.2. *Let F_0 be a stack. We say that $F \to F'$ is a* Zariski locally trivial F_0-fibration *if for any affine scheme X and any morphism $X \to F'$ there exists a Zariski cover $\{U_i\}_i$ of X such that each morphism $F \times^h_{F'} U_i \to U_i$ is isomorphic to the projection $F_0 \times^h U \to U_i$.*

Let $St^{fp}(k)$ denote the full subcategory of $St(k)$ whose objects are strongly finitely presented stacks over k.

Definition 3.10.3. *Let $K_0(St^{fp}(k))$ be quotient of the free abelian group generated by the symbols $[F]$ where F varies over all strongly finitely presented stacks over k subject to following three relations:*

1. *If F and F' are isomorphic in $St^{fp}(k)$, we have*

$$[F] = [F'].$$

2. *For any stacks F and G in $St^{fp}(k)$ such that G is a closed substack of F, we have*

$$[F] = [G] + [F \backslash G]$$

3. *Let F_0 be an affine scheme or a stack of the form $K(\mathbb{G}_a, n)$ for some integer $n > 0$. If $F \to F'$ in $St^{fp}(k)$ is a Zariski locally trivial F_0-fibration then we have*

$$[F] = [F' \times F_0].$$

As in the case of varieties, this group can be given the structure of a commutative ring. We call this the ring the Grothendieck ring of Artin stacks over k.

As in the case of varieties, we define a filtration on this ring:

Let $\mathbb{L} = [\mathbb{A}^1_k]$. Let $\mathcal{M}_k = K_0(St^{fp}(k))[\mathbb{L}^{-1}]$. For any $[F]/\mathbb{L}^m \in \mathcal{M}_k$, define

$$\dim([F]/\mathbb{L}^m) = \dim(F) - m.$$

Let $F^m(\mathcal{M}_k) := \{v \in \mathcal{M}_k | \dim(v) \leq -m\}$. This is a decreasing filtration on $\mathcal{M}_k$. Let $\widehat{\mathcal{M}^k}$ denote the completion of $\mathcal{M}_k$ with respect to the filtration F^*.

In Section 3.4 we defined the Eilenberg-MacLane stacks $K(G, n)$ for a group sheaf G. If G is a group scheme of finite type over k, it can be proved that $K(G, n)$ is a strongly finitely presented Artin stack over k. Also, by our computation of the loop stacks of the Eilenberg-MacLane stacks, it is clear that $\mathrm{Spec}(k) \to K(\mathbb{G}_a, n)$ is a $K(\mathbb{G}_a, n-1)$ fibration. It can be proved that this is a Zariski locally trivial fibration (see [To1], Section 3.1). Thus, for $n > 1$,

$$[\mathrm{Spec}(k)] = [K(\mathbb{G}_a, n-1)] \cdot [K(\mathbb{G}_a, n)]$$

and similarly,

$$[\mathrm{Spec}(k)] = \mathbb{L} \cdot [K(\mathbb{G}_a, 1)].$$

Since, $[\mathrm{Spec}(k)]$ is the multiplicative identity in the Grothendieck ring, it follows that

$$[K(\mathbb{G}_a, n)] = \mathbb{L}^{(-1)^m}$$

.

Definition 3.10.4. *Let X be a strongly finitely presented Artin stack over k. Let C be a constructible subset of $|X|$. Let $C = \coprod_i |F_i|$ be a decomposition of C such that F_i is a locally closed substack of X for each i. Then we define*

$$[C] = \sum_i [F_i] \in K_0(St^{fp}(k)).$$

It is easy to check that $[C]$ is independent of the choice of $\{F_i\}_i$.

4.0 GREENBERG FUNCTOR FOR STACKS

From now onwards, k is a field of characterstic zero. This restriction is in effect because we wish to apply Lemma 3.9.2 and Lemma 3.9.4.

4.1 DEFINITION

We adopt the following notation.

Notation:

1. $R_n := k[t]/(t^{n+1})$ for $n \in \mathbb{Z}_{\geq 0}$.
2. $R_\infty = R := k[[t]]$.
3. $\mathbb{D}_n := \operatorname{Spec} R_n$ for $n \in \mathbb{Z}_{n\geq 0}$, $\mathbb{D}_\infty = \mathbb{D} := \operatorname{Spec} R$.

For any scheme T over $\operatorname{Spec}(k)$, let $\sigma_n^T : \operatorname{Aff}/T \to \operatorname{Aff}/(T \times \mathbb{D}_n)$ be the functor $U \to U \times_T (T \times \mathbb{D}_n) = U \times \mathbb{D}_n$. This induces a Quillen adjunction

$$((\sigma_n^T)_!, (\sigma_n^T)_*) : (\operatorname{Aff}/T)^\wedge \to (\operatorname{Aff}/(T \times \mathbb{D}_n))^\wedge. \tag{4.1.1}$$

The following result is well-known:

Proposition 4.1.1. *(Greenberg, [Gr1])*

1. *If $n < \infty$ and if X is a scheme of finite type over $\mathbb{D}_n$, the presheaf $(\sigma_n^{\mathbb{D}_0})_*(X)$ is a scheme of finite type over $\mathbb{D}_0$.*
2. *If $X \to Y$ is a smooth morphism of schemes of finite type over $\mathbb{D}_n$, the morphism $(\sigma_n^{\mathbb{D}_0})_*(X) \to (\sigma_n^{\mathbb{D}_0})_*(Y)$ is also a smooth morphism.*

We will not present a proof of this result. However, we briefly describe the construction as follows:

Suppose X is an affine scheme over R_n. Let $X = \mathrm{Spec}(A)$ where $A \cong R_n[x]/I$ where $x = (x_1, \ldots x_p)$ is a p-tuple of variables and I is an ideal in $R_n[x]$. Choose variables $(a_{ij})_{1\leq i\leq p, 0\leq j\leq n}$ and let $a(t) = (a_i(t))_{1\leq i\leq p}$ be the p-tuple of polynomials defined by $a_i(t) = \sum_{j=0}^{n} a_{ij}t^j$. Suppose f is a polynomial in I. Let $f(a(t)) = \sum_{j=0}^{n} \theta_{f,j}t^j (\mathrm{mod}\ t^{n+1})$ where $\theta_{f,j}$ is a polynomial in $k[(a_{ij})_{i,j}]$. Let I' be the ideal of $k[\{a_{ij}\}_{i,j}]$ generated by the polynomials $\theta_{f,j}$ for $0 \leq j \leq n$ as f varies over all elements of I. Then $(\sigma_n)_*(X)$ is represented by the closed subscheme of $\mathbb{A}_k^{(n+1)p}$ defined by the ideal I'.

We would like to extend this result to strongly finitely presented stacks over $\mathbb{D}_n$. To begin with, we first confirm that the adjunction (4.1.1) is a Quillen adjunction for the local projective model structure.

Proposition 4.1.2. *Let $U \to V$ be a morphism of affine schemes. Consider the functor $\sigma : (\mathrm{Aff}/V) \to (\mathrm{Aff}/U)$ defined by $\sigma(T) = T \times_V U$. Then the induced adjunction*

$$(\sigma_!, \sigma_*) : (\mathrm{Aff}/U)^{\sim,et} \to (\mathrm{Aff}/V)^{\sim,et}$$

(where σ_* is defined by $\sigma_*(F)(U) := F(\sigma(W))$, i.e. it is the "pullback functor") is a Quillen adjunction. In other words, the functor σ is continuous (in the sense of Section 3.2).

Proof. To prove this, it will suffice (see Proposition 3.2.1) to check that $(\sigma_n)_!$ takes hypercovers to hypercovers. This is almost immediate. Indeed, let T be an object of (Aff/V). Consider a hypercover $A \to T$. We need to check that $\sigma_!(A) \to \sigma_!(\underline{h}_T) = \underline{h}_{T\times_V U}$ is a hypercover.

Suppose $A_n = \coprod_{C\in K_n} h_V$. Then

$$(\sigma_!(A))_n = \coprod_{C\in K_n} \sigma_!(h_C) = \coprod_{C\in K_n} h_{(C\times_V U)}$$

which is a coproduct of representable presheaves. Now we need to check that

$$\coprod_{C\in K_n} h_{C\times_V U} \to (\sigma_!(A)^{\partial\Delta^n})_0 \times_{(\sigma_!(\underline{h}_T)^{\partial\Delta^n})_0} \sigma_!(\underline{h}_T)_n$$

is a local epimorphism of presheaves. σ takes covers in the site $(\mathrm{Aff}\,/V)$ to covers in the site $(\mathrm{Aff}\,/U)$. In other words, $\sigma_!$ takes local epimorphisms of representable presheaves into local epimorphisms. Since every presheaf is the colimit of representable presheaves, $\sigma_!$ takes local epimorphisms of presheaves to local epimorphisms. Thus, it will suffice to check that for any prestack F

$$\sigma_!(F^{\partial\Delta^n})_0 = (\sigma_!(F)^{\partial\Delta^n})_0.$$

There is an obvious coequalizer diagram of the form

$$\coprod_i 1 \leq i < j \leq n\Delta^{n-2} \rightrightarrows \coprod_{i=0}^n \Delta^{n-1} \longrightarrow \partial\Delta^n.$$

(This is simply the diagram that expresses $\partial\Delta^n$ as a simplicial object obtained by glueing together n copies of Δ^{n-1} along their faces.)

It follows that for any prestack F over a site $\mathcal{T}$, if T is an object of T, the set

$$(F^{\partial\Delta_n}(T))_0 = \mathbf{Sset}(\partial\Delta^n, F(T))$$

can be computed by an equalizer diagram of the form

$$\mathbf{Sset}(\partial\Delta_n, F(T)) \longrightarrow \prod_{i=0}^n F_{n-1} \rightrightarrows \prod_{0\leq i<j\leq n} F_{n-2}.$$

In other words, it suffices to check that $(\sigma)_!$ commutes with the formation of finite limits for diagrams involving coproducts of representable presheaves. But this is evident since coproducts of representable presheaves are represented by schemes and for any scheme W, $(\sigma)_!(h_W) = h_{(W\times_V U)}$ which commutes with limits for diagrams involving schemes since its restriction to the category of schemes over V is actually the right adjoint to the forgetful functor from the category of U-schemes to the category of V-schemes. $\square$

Definition 4.1.3. *(Greenberg functor) Let T be any scheme over* $\operatorname{Spec}(k)$*. Let X be a stack over $T \times_{\mathbb{D}_0} \mathbb{D}$. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. The* stack of n-arcs of X *(or simply* stack of arcs over X *if $n = \infty$) is defined by the formula*

$$Gr_n^T(X) := \mathbb{R}(\sigma_n^T)_*(X \times_{\mathbb{D}} \mathbb{D}_n).$$

This defines a functor $Gr_n^T : Ho((\mathrm{Aff}\,/(T \times_{\mathbb{D}_0} \mathbb{D}))^{\sim,et}) \to Ho((\mathrm{Aff}\,/T)^{\sim,et})$. For any stack X over $\mathbb{D}$, we refer to the stack $Gr_n(X)$ as the stack of n-arcs on X *(or simply* the stack of arcs on X *if $n = \infty$). A point on $Gr_n(X)$ (i.e. an element of $|Gr_n(X)|$) will be called an* n-arc on X *(or* an arc on X*).*

Remark. We will usually omit the reference to T and simply write Gr_n instead of Gr_n^T. Indeed, we will usually only need $T = \mathbb{D}_0$ except in the situation of Section 4.3.

Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For any stack X over $\mathbb{D}$, and any affine scheme T over $\mathbb{D}_0$, we have, by definition, an isomorphism

$$\Theta_n^T : X(T \times_{\mathbb{D}_0} \mathbb{D}_n) \to Gr_n(X)(T).$$

As we did in the above remark, we will usually drop the reference to T and write Θ_n instead of Θ_n^T. Also, we will refer to the induced map on the sets of components

$$\pi_0(X(T \times_{\mathbb{D}_0} \mathbb{D}_n)) \to \pi_0(Gr_n(X)(T))$$

by Θ_n.

Truncation maps:

For any $n, m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $n \geq m$, the morphism $\mathbb{D}_m \to \mathbb{D}_n$ induces natural morphisms

$$\tau_{m,X}^n : Gr_n(X) \to Gr_m(X).$$

When $n = \infty$, we write $\tau_{m,X}$ instead of $\tau_{m,X}^n$. Also, we will omit the reference to X if it is clear from the context.

For integers $m \geq n \geq 0$, we denote the map $\tau_m(|Gr(X)|) \to \tau_n(|Gr(X)|)$ by ρ_n^m.

Before we generalize Proposition 4.1.1, we need to examine the behavior of covers.

We will also need the following lemma:

Lemma 4.2.1. *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Let U a scheme over $\mathbb{D}_0$. Then any cover $\{V_i \to U \times_{\mathbb{D}_0} \mathbb{D}_n\}_{i \in I}$ of $U \times_{\mathbb{D}_0} \mathbb{D}_n$ over $\mathbb{D}_n$ has a refinement of the form $\{U_i \times_{\mathbb{D}_0} \mathbb{D}_n \to U \times_{\mathbb{D}_0} \mathbb{D}_n\}_{j \in J}$ for some cover $\{U_i \to U\}_{j \in J}$ of U over $\mathbb{D}_0$.*

Proof. For any scheme T, let $Et(T)$ denote the caetgory of etale schemes over T.

First consider the case $n < \infty$. The two morphisms $\mathbb{D}_0 \to \mathbb{D}_n$ and $\mathbb{D}_n \to \mathbb{D}_0$ induce morphisms $U \to U \times_{\mathbb{D}_0} \mathbb{D}_n$ and $U \times_{\mathbb{D}_0} \mathbb{D}_n \to U$. The pullbacks along these morphisms give functors $\alpha : Et(U \times_{\mathbb{D}_0} \mathbb{D}_n) \to Et(U)$ and $\beta : Et(U) \to Et(U \times_{\mathbb{D}_0} \mathbb{D}_n$. The morphism $U \to U \times_{\mathbb{D}_0} \mathbb{D}_n$ is a closed immersion which induces a homeomorphism of underlying topological spaces. Thus the functor α is an equivalence of categories. The composition $\alpha \circ \beta$ is the identity functor on $Et(U)$. Thus, β is also an equivalence of categories. This proves the lemma for $n < \infty$.

Now consider the case $n = \infty$. Let $V \to U \times_{\mathbb{D}_0} \mathbb{D}$ be an etale morphism. Let $V_0 = V \times_{\mathbb{D}} \mathbb{D}_0$. Without any loss of generality, we may assume that V is affine. Suppose $V = \mathrm{Spec}(S)$ and $V_0 = \mathrm{Spec}(S_0)$. By the above argument, for every n, we have a unique isomorphism $V_0 \times_{\mathbb{D}_0} \mathbb{D}_n \to V \times_{\mathbb{D}} \mathbb{D}_n$. Thus we have a consistent system of morphisms $V_0 \times_{\mathbb{D}_0} \mathbb{D}_n \to V$, i.e. a system of ring homomorphisms $S \to S_0[t]/(t^{n+1})$. This induces a ring homomorphism morphism

$$S \to \varprojlim S_0[t]/(t^{n+1}) = S_0[[t]].$$

Thus, we have a morphism $V_0 \times_{\mathbb{D}_0} \mathbb{D} \to V$. (Of course, this need not be an isomorphism.) In order to complete the proof, it will suffice to show that if $\{V_i \to U \times_{\mathbb{D}_0} \mathbb{D}\}_{i \in I}$ is an etale cover, the morphism $\coprod_{i \in I} (V_i)_0 \times_{\mathbb{D}_0} \mathbb{D} \to U \times_{\mathbb{D}_0} \mathbb{D}$ is surjective, which is obvious since $\coprod_{i \in I} (V_i)_0 \to U$ is a covering. ☐

Corollary 4.2.2. *Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.*

1. *$(\sigma_n)_*$ preserves epimorphisms of stacks.*

2. *$(\sigma_n)_*$ commutes with sheafification.*

Proof. A morphism $F \to G$ of prestacks is an epimorphism if and only if it induces a local epimorphism of presheaves $\pi_0^{pr}(F) \to \pi_0^{pr}(G)$. This is an immediate consequence of the preceding lemma. This proves (1).

(2) follows from the definition of the sheafification functor. □

Now we can prove that Gr_n preserves geometricity of strongly finitely presented Artin stacks by the usual inductive process:

Proposition 4.2.3. *Let $n \in \mathbb{Z}_{\geq 0}$.*

1. *$(\sigma_n)_*$ maps strongly finitely presented m-geometric stacks over $\mathbb{D}_n$ to strongly finitely presented m-geometric stacks over $\mathbb{D}_0$.*
2. *$(\sigma_n)_*$ maps m-representable morphisms between strongly finitely presented m-geometric stacks over $\mathbb{D}_n$ to m-representable morphisms.*
3. *$(\sigma_n)_*$ maps m-smooth morphisms between strongly finitely presented m-geometric stacks over $\mathbb{D}_n$ to m-smooth morphisms.*

Proof. All three statements are proved simultaneously by induction on m.

The cases $m = -1$ and $m = 0$ are proved in [Gr1]. We assume that the result is proved for $m = l$ and prove it for $m = l + 1$. Given an $(l + 1)$-geometric stack X over $\mathbb{D}_n$, let $\{U_i \to X\}_{i \in I}$ be an atlas. Then Corollary 4.2.2 and the assumption that $(\sigma_n)_*$ preserves l-representable morphisms and l-smooth morphisms imply that $\{(\sigma_n)_*(U_i) \to (\sigma_n)_*(X)\}_{i \in I}$ is an atlas for $(\sigma_n)_*(X)$ and also that the diagonal of $(\sigma_n)_*(X)$ is l-geometric. Thus $(\sigma_n)_*(X)$ is $(l+1)$-geometric. $(\sigma_n)_*$ is a right Quillen functor and thus preserves homotopy pullbacks for fibrant objects. Thus it follows that $(\sigma_n)_*$ takes $(l + 1)$-representable morphisms to $(l + 1)$-representable morphisms. The fact that $(\sigma_n)_*$ preserves $(l + 1)$-smoothness also follows easily. □

Remark. Note that the above result requires $n < \infty$. However, if X is a scheme, the morphism $Gr_{n+1}(X) \to Gr_n(X)$ is affine for each n. Thus, the projective limit $Gr(X)$ is also a scheme. If X is an affine scheme, so is $Gr_n(X)$ for all $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

4.3 HOMOTOPY GROUP SHEAVES OF ARC SPACES

Let X be a stack over $\mathbb{D}$. Let S be a scheme over $\mathbb{D}_0$ and $\widetilde{s} : S \to X$ be any morphism. For any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, this induces a morphism $s_n : (S \times_{\mathbb{D}_0} \mathbb{D}) \times_{\mathbb{D}} \mathbb{D}_n = S \times_{\mathbb{D}_0} \mathbb{D}_n \to X$. By the definition of the functor $(\sigma_n)_*$, this defines a morphism $s_n : S \to (\sigma_n)_*(X) = Gr_n(X)$.

Lemma 4.3.1. *Let X be a stack over $\mathbb{D}$. Let S be a scheme over $\mathbb{D}_0$ and $\widetilde{s} : S \times_{\mathbb{D}_0} \mathbb{D} \to X$ be any morphism. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let s_n be as defined above. Then we have isomorphisms:*

1. $\pi_0(Gr_n(X)) \cong Gr_n(\pi_0(X))$.
2. $\pi_i(Gr_n(X), s_n) \cong Gr_n(\pi_i(X, \widetilde{s}))$.

Proof. Let F be a prestack over $(\mathrm{Aff}\ /\mathbb{D}_n)$. The presheaves

$$\pi_0^{pr}((\sigma_n)_*(F)) : U \to \pi_0(((\sigma_n)_*(F))(U))$$

and

$$(\sigma_n)_*(\pi_0^{pr}(F)) : U \to \pi_0^{pr}(F(\sigma_n(U)))$$

are equal by definition. Since $(\sigma_n)_*$ commutes with sheafification, this proves (1).

The isomorphism in (2) is a consquence of (1) since the π_i sheaves are simply the π_0 sheaves of loop stacks. □

Lemma 4.3.2. *Let X be a strongly finitely presented Artin stack over $\mathbb{D}$. Let S is an affine scheme over $\mathbb{D}_0$. Let $\widetilde{s} : S \times_{\mathbb{D}_0} \mathbb{D} \to X$ be any morphism. For any $p \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let $s_p : S \to Gr_p(X)$ be as defined above. Then for integers $n \geq m \geq 0$ and any $i > 0$, the morphism*

$$\pi_i(Gr_n(X), s_n) \to \pi_i(Gr_m(X), s_m)$$

is surjective.

Remark. Note that this is *not* true for the π_0 sheaves.

Proof. First we consider the case that $S = \mathrm{Spec}(K)$ for an algebraically closed extension K of k. (See remark at the end of Section 3.9.)

Let $\widetilde{s}_n$ denote the induced map $S \times_{\mathbb{D}_0} \mathbb{D}_n \to S \times_{\mathbb{D}_0} \mathbb{D} \to X$. Then

$$\pi_i(X, \widetilde{s}_n) = \pi_i(X, \widetilde{s}) \times_{(S \times_{\mathbb{D}_0} \mathbb{D})} (S \times_{\mathbb{D}_0} \mathbb{D}_n)$$

and

$$\pi_i(Gr_n(X), s_n) = Gr_n(\pi_i(X, \widetilde{s})) = (\sigma_n)_*(\pi_i(X_n, \widetilde{s}_n))$$

where X_n is the stack on $(\mathrm{Aff}\,/(S \times_{\mathbb{D}_0} \mathbb{D}_n))$ obtained by restricting X.

We know from Proposition 3.9.4 that the group sheaf $\pi_i(X, \widetilde{s})$ is presentable. Thus there exists a $S \times_{\mathbb{D}_0} \mathbb{D}$ scheme Z and a surjective morphism $Z \to \pi_i(X, \widetilde{s})$ which is $(S \times_{\mathbb{D}_0} \mathbb{D})$-vertical. For any $n \in \mathbb{Z}_{\geq 0}$, let $Z_n = Z \times_{\mathbb{D}} \mathbb{D}_n$. Then the morphism $Z_n \to \pi_i(X_n, \widetilde{s}_n)$ is $(S \times_{\mathbb{D}_0} \mathbb{D}_n)$-vertical. Thus $Z_n \to S \times_{\mathbb{D}_0} \mathbb{D}_n$ is a $(S \times_{\mathbb{D}_0} \mathbb{D}_n)$-vertical morphism of schemes over $\mathbb{D}_n$.

However, there exists a retraction $\mathbb{D}_n \to \mathbb{D}_0$. Thus a $(S \times_{\mathbb{D}_0} \mathbb{D})$-vertical morphism of schemes is automatically an S-vertical morphism (viewed as a morphism of S-schemes via the forgetful functor corresponding to the morphism $S \times_{\mathbb{D}_0} \mathbb{D}_n \to S$). The schemes Z_n and $S \times_{\mathbb{D}_0} \mathbb{D}_n$ are of finite type over S and thus, by Lemma 3.9.2, Z_n is a smooth scheme over $S \times_{\mathbb{D}_0} \mathbb{D}_n$ for each n.

Applying Hensel's lemma, to Z_n, we see that for any $m < n$ the morphism

$$Gr_n(Z) = (\sigma_n)_*(Z_n) \longrightarrow (\sigma_m)_*(Z_m) = Gr_m(Z)$$

is surjective. Since the morphisms $Gr_n(Z) \to Gr_n(\pi_i(X, \widetilde{s}))$ and $Gr_m(Z) \to Gr_m(\pi_i(X, \widetilde{s}))$ are surjective, it follows that the morphism $Gr_n(\pi_i(X, \widetilde{s})) \to Gr_m(\pi_i(X, \widetilde{s}))$ is surjective. Thus the result is proved when $S = \mathrm{Spec}(K)$ where K is an algebraically closed field.

If S is a general scheme over $\mathbb{D}_0$, for every point of S represented by $\mathrm{Spec}(K) \to S$, we obtain a morphism

$$\mathrm{Spec}(K) \times_{\mathbb{D}_0} \mathbb{D} \longrightarrow S \times_{\mathbb{D}_0} \mathbb{D} \xrightarrow{\widetilde{s}} X.$$

Applying the above arguments to all such morphisms, we see that for all integers $n > m$, the morphism $|\pi_i(Gr_n(X), s_n)| \to |\pi_i(Gr_m(X), s_m)|$ is a surjection. Applying Lemma 3.9.3, we see that $\pi_i(Gr_n(X), s_n) \to \pi_i(Gr_m, s_m)$ is a surjection of sheaves.

□

Remark. Note that in the above proof, we do not say that Z is smooth. For a general scheme T, a T-vertical morphism $Z \to T$ need not be smooth.

5.0 DEFINABLE SETS OF ARCS

5.1 LANGUAGE OF DENEF-PAS AND QUANTIFIER ELIMINATION

In this section we recall an important quantifier elimination result of Denef-Pas. First we need some definitions:

Angular component: Let K be a valued field. Let R denote its valuation ring, k its residue field and Γ its valuation group. We denote the residue map $R \to k$ by $x \to \overline{x}$. An *angular component map* on K is a multiplicative map $\overline{\mathrm{ac}} : K^\times \to k^\times$ extended by defining $\overline{\mathrm{ac}}(0) = 0$ and satisfying $\overline{\mathrm{ac}}(x) = \overline{x}$ for all x such that $\mathrm{ord}(x) = 0$.

For instance, if $K = k((t))$ and $R = k[[t]]$, we have $\Gamma = \mathbb{Z}$. Then there is a natural angular component map which is defined as follows:

Any element x of K can be written in the form $x = \sum_{i \geq l} a_i t^i$ with $a_i \in k$ and $a_l \neq 0$. Then $\overline{\mathrm{ac}}(x) = a_l$.

Language of Denef-Pas: A *language of Denef-Pas* is a three-sorted language of the form

$$\mathcal{L}_{DP} = (\mathcal{L}_{Val}, \mathcal{L}_{Res}, \mathcal{L}_{Pres}, \mathrm{ord}, \overline{\mathrm{ac}})$$

with three sorts - Val-sort (for valued fields), Res-sort (for residue field) and Ord-sort (for ordered groups). The languages $\mathcal{L}_{Val}$, $\mathcal{L}_{Res}$ and $\mathcal{L}_{Pres}$ used for these sorts are defined as follows:

– $\mathcal{L}_{Pres}$: The *Presburger language* is an expansion of the language of ordered groups.

$$\mathcal{L}_{Pres} = \{+, -, 0, 1, \leq\} \cup \{\equiv_n | n \in \mathbb{N}, n > 1\}$$

where $\equiv_n$ denotes the equivalence relation module n and 1 is a constant symbol.

– $\mathcal{L}_{Res}$ is an expansion of the language $(+,-,\cdot,0,1)$ of rings.

– $\mathcal{L}_{Val}$ is the language of rings.

We consider structures $(\mathbf{k},k,\Gamma)$ of $\mathcal{L}_{DP}$ such that $\mathbf{k}$ is a valued field withvalue group Γ, residue field k, valuation map ord and an angular component map $\overline{\mathrm{ac}}$.

Quantifier elimination: Let $H_{\overline{\mathrm{ac}},0}$ denote the $\mathcal{L}_{DP}$-theory of $\mathcal{L}_{DP}$-structures whose valued field is Henselian and whose residue field is of characterstic zero. Let $(\mathbf{k},k,\Gamma)$ denote a model for $H_{\overline{\mathrm{ac}},0}$. Let S denote a subring of K and let T_S denote the diagram of $\mathbf{k}$ in $\mathcal{L}_{DP}$, i.e. it is the set of atomic $\mathcal{L}_{DP}\cup S$ formulas and negations of atomic $\mathcal{L}_{DP}\cup S$-formulas ϕ such that $\mathbf{k}\models\phi$. Let H_S denote the theory which is the union of $H_{\overline{\mathrm{ac}},0}$ and T_S. We recall the following quantifier-elimination result (see [CL], Theorem 2.1.1 and Corollary 2.1.2)

Theorem 5.1.1. *(Denef-Pas) Every $\mathcal{L}_{DP}\cup S$ formula $\phi(x,\xi,\alpha)$ with variables x in the Val-sort, ξ in the Res-sort and α in the Ord-sort is H_S equivalent to a finite disjunction of formulas of the form*

$$\psi(\overline{\mathrm{ac}}\,f_1(x),\overline{\mathrm{ac}}\,f_2(x),\ldots,\overline{\mathrm{ac}}\,f_k(x),\psi)\wedge\vartheta(\mathrm{ord}\,f_1(x),\ldots,\mathrm{ord}\,f_k(x),\alpha),$$

with ψ a $\mathcal{L}_{Res}$-formula and ϑ a $\mathcal{L}_{Pres}$-formula and $f_1,\ldots,f_k$ polynomials in $S[X]$.

In other words, for any extension $\overline{\mathbf{k}}$ of $\mathbf{k}$, a subset of $\overline{\mathbf{k}}^n$ defined by a formula in $\mathcal{L}_{DP}\cup\mathbf{k}$ can also be defined by a formula that does not have any quantifiers over the valued field sort.

5.2 DEFINABLE SETS ON SCHEMES

From this point onwards, we will use the following conventions.

Convention. 1. By a *Denef-Pas formula over k*, we will always mean a formula $\phi(x)$ in the language $\mathcal{L}_{DP}\cup k[[t]]$ where x denotes variables of the Val-sort.

2. We also adopt the notation of Section 4. In other words, $\mathbb{D}$ will denote the spectrum of the ring $k[[t]]$, etc.

3. An *Artin stack over* $\mathbb{D}$ will always mean an Artin stack over $\mathbb{D}$ that is *reduced, flat and strongly finitely generated over* $\mathbb{D}$. In particular, a *scheme over* $\mathbb{D}$ will be assumed to have these properties.

Let X be an affine scheme over $\mathbb{D}$. Choose an embedding of X into $\mathbb{A}^n_{\mathbb{D}} \subset \mathbb{A}^n_{k((t))}$. Then a subset C of $|Gr(X)|$ is said to be a *definable set of arcs on* X if there exists a Denef-Pas formula $\phi(x)$ over k satisfying the following condition:

For any extension K of k, let i_K denote the map $Gr(X)(K) \to |Gr(X)|$. Then the subset $\Theta^{-1}(i_K^{-1}(C)) \subset X(\mathbb{D}) \subset \mathbb{A}^n_{\mathbb{D}}(\mathbb{D})$ is defined by $\phi(x)$.

It is easy to see that this notion is independent of the choice of the embedding of X into $\mathbb{A}^n_{\mathbb{D}}$. If X is a general scheme, we say that a subset C of $|Gr(X)|$ is definable if for any affine subscheme X' of X, the set $C \cap |Gr(X')|$ is definable.

We recall some properties of definable sets on schemes. The following lemmas are proved in [DL] when X is of the form $T \times_{\mathbb{D}_0} \mathbb{D}$ for some variety T over $\mathbb{D}_0$. The proofs in the general situation are essentially the same, but we present them in detail anyway, for the sake of convenience.

Lemma 5.2.1. *([DL], Proposition 2.3) Let X be a scheme over $\mathbb{D}$. Let C be a definable subset on X. Then for any $n \geq 0$, the set $\tau_n(C)$ is a constructible subset of $|Gr_n(X)|$.*

Proof. Without loss of generality, we may assume that $X = \mathbb{A}^m_{\mathbb{D}}$.

The morphism $\tau_n : |Gr(X)| \to |Gr_n(X)|$ has a section which is defined as follows:

Let $x \in |Gr_n(X)|$ be of the form $\Theta_n(\widetilde{x})$ where $\widetilde{x}$ is given by an m-tuple of n-truncated power series of the form $(a_1(t), \ldots, a_m(t))$. Let $b_i(t)$ be the power series obtained by "extending $a_i(t)$ by zeroes", i.e. if $a_i(t) = \sum_{j=0}^n a_{ij}t^j \in K[t]/(t^{n+1})$ for some extension K of k, $b_i(t) = \sum_{j=0}^n a_{ij}t^j \in K[[t]]$. The m-tuple $(b_1(t), \ldots, b_m(t))$ defines a morphism $\widetilde{y}$: $\mathrm{Spec}(K) \times_{\mathbb{D}_0} \mathbb{D} \to X$. Then we define $s(x) = \Theta(\widetilde{y})$.

Then $y \in \tau_n(C)$ if and only if $s(y) \in C' = \tau_n^{-1}\tau_n(C)$. Thus it is enough to prove that the condition $s(y) \in \tau_n^{-1}\tau_n(C)$ defines a constructible subset of $|Gr_n(X)|$. This can be easily deduced from Theorem 5.1.1 as follows.

The set $\tau_n^{-1}\tau_n(C)$ is clearly definable. Using Theorem 5.1.1, we may assume that it is given by a finite disjunction of formulas of the form

$$\psi(\overline{\mathrm{ac}}\, f_1(x), \overline{\mathrm{ac}}\, f_2(x), \ldots, \overline{\mathrm{ac}}\, f_p(x)) \wedge \vartheta(\mathrm{ord}\, f_1(x), \ldots, \mathrm{ord}\, f_p(x))$$

where $f_i \in k[[t]][x]$, ψ is a formula in $\mathcal{L}_{Res}$ and ϑ is a formula in $\mathcal{L}_{Pres}$. Since a finite union of constructible subsets is constructible, we may assume that C' is given by a single formula of this sort which we denote by $\phi(x)$.

Consider the set I of p-tuples $r = (r_1, \ldots r_p)$ such that $0 \leq r_i \leq n$. Let J be the subset of I defined by the formula ϑ. For each $r \in J$, let ϕ_r denote the formula

$$\phi_r(x) = (\bigwedge_{i=1}^{m} (\mathrm{ord}\, f_i(x) = r_i)) \wedge \psi(\overline{\mathrm{ac}}\, f_1(x), \ldots \overline{\mathrm{ac}}\, f_p(x)).$$

For each integer i such that $1 \leq i \leq p$, let $\alpha_i(x)$ denote the formula

$$\alpha_i(x) = (\mathrm{ord}\, f_i(x) \geq n + 1).$$

Clearly, C' can be defined by the formula

$$\phi = (\bigvee_{r \in J} \phi_r(x)) \vee (\bigvee_{i=1}^{p} \alpha_i(x)).$$

Each $\phi_r(x)$ and each $\alpha_i(x)$ defines a locally closed condition in the variables a_{ij}. This completes the proof. □

Definition 5.2.2. *(Weakly stable sets) We say that a definable set of arcs C on a scheme X is* weakly stable at level n *if $\tau_n^{-1}\tau_n(C) = C$. We say that C is* weakly stable *if it is weakly stable at level n for some integer n.*

Lemma 5.2.3. *Let $f : X \to Y$ be a smooth morphism of schemes. Let $C \subset |Gr(Y)|$ be a definable set of arcs on Y. Let $C' = Gr(f)^{-1}(C)$. Then C is weakly stable if and only if C' is weakly stable.*

Proof. Since f is smooth, it has the right lifting property with respect to the morphism $\mathbb{D}_n \to \mathbb{D}_{n+1}$ for all n. This easily implies the result. □

Weakly stable sets have a certain "compactness" property:

Lemma 5.2.4. *([DL], Lemma 2.4) For each $i \in N$, let C_i denote a weakly stable definable set of arcs on X. Suppose $C = \bigcup_{i \in \mathbb{N}} C_i$ is definable and weakly stable. Then C is the union of a finite number of the C_i's.*

Proof. Without loss of generality, we may assume that X is an affine scheme. By the remark following Proposition 4.2.3, $Gr(X)$ is affine, hence quasi-compact. We saw in the proof of Lemma 5.2.1 that a weakly stable set is defined by a locally closed condition in finitely many variables. In other words, for any weakly stable set C, there exists a ring A which is finitely generated over k and a morphism $Gr(X) \to \mathrm{Spec}(A)$ such that C is the preimage of a locally closed subset of $\mathrm{Spec}(A)$. Any locally closed subset of a noetherian space is constructible. Thus, by Lemma 3.7.9, C is pro-constructible as well as ind-constructible. Thus the result follows from Lemma 3.7.10. □

Remark. Alternatively, one could employ the techniques used in the proof of Lemma 2.4 of [DL]. The proof presented above is hinted at in [DL] as an alternative argument.

5.3 DEFINABLE SETS OF ARCS ON ARTIN STACKS

We now generalize these ideas to Artin stacks.

We note that given a morphism $f : X \to Y$ of *schemes* over $\mathbb{D}$, a set of arcs C on Y is definable if and only if $f^{-1}(C)$ is a definable set of arcs on X. Using this, the notion of a definable set of arcs on an Artin stack is defined by the familiar inductive technique:

Definition 5.3.1. *Let X be an Artin stack over $\mathbb{D}$. Let $p : X' \to X$ be an atlas for X with X' being an affine scheme over $\mathbb{D}$. Then a set of arcs of $C \subset |Gr(X)|$ is said to be* definable with respect to p *if its preimage in $|Gr(X')|$ is a definable set of arcs on X'.*

As one might expect, this notion is independent of p. The proof of the following proposition is simply a routine induction argument.

Proposition 5.3.2. *Let X be an Artin stack over $\mathbb{D}$. Let $p : X' \to X$ be an atlas for X with X' being an affine scheme over $\mathbb{D}$. Then a set of arcs $C \subset |Gr(X)|$ is definable with respect to p if and only if it is definable with respect to any other atlas of X.*

Proof. For each $k \geq -1$, consider the statements $A(k)$ and $B(k)$:

$A(k)$: Suppose X is a k-geometric stack over $\mathbb{D}$. Then a set of arcs $C \subset |Gr(X)|$ is definable with respect to a certain atlas of X if and only if it is definable with respect to *any* atlas of X.

$B(k)$: Suppose $f : X \to Y$ is a k-representable morphism between k-geometric stacks over $\mathbb{D}$. Then a set of arcs $C \subset |Gr(Y)|$ is definable with respect to some atlas of Y if and only if $f^{-1}(C)$ is definable with respect to some atlas of X.

We use these two statements as our induction hypotheses. We already know $A(-1)$ and $B(-1)$ to be true.

First we show that $B(k-1)$ implies $A(k)$. Suppose that X is a k-geometric stack over $\mathbb{D}$. Let $p : U \to X$ and $q : V \to X$ be atlases where U and V are affine schemes. Consider the following homotopy cartesian square:

$$\begin{array}{ccc} U \times^h_X V & \xrightarrow{r} & U \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{q} & X. \end{array}$$

Suppose C is a subset of $|Gr(X)|$ that is definable with respect to p. Then $p^{-1}(C)$ is definable. The stack $U \times^h_X V$ is $(k-1)$-representable and thus $r^{-1}(p^{-1}(C))$ is definable with respect to some atlas of $U \times^h_X V$. Thus $s(r^{-1}(p^{-1}(C))) = q^{-1}(C)$ is definable with respect to some atlas of V. As V is a scheme, this means that $q^{-1}(C)$ is definable. Since q was arbitrary, this proves $A(k)$.

Now we show that $A(k)$ implies $B(k)$. Let $f : X \to Y$ be a k-representable morphism between k-geometric stacks over $\mathbb{D}$. Let $C \in |Gr(Y)|$ be a set of arcs on Y that is definable (with respect to any atlas, since we are assuming $A(k)$). Choose atlases $Y' \to Y$ and $X' \to X \times^h_Y Y'$. Thus we have the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{r} & X \times^h_Y Y' & \xrightarrow{p} & X \\ & {\scriptstyle f''}\searrow & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & Y' & \xrightarrow{q} & Y. \end{array}$$

Then $q^{-1}(C)$ is definable due to $A(k)$. $(f'')^{-1}(q^{-1}(C))$ is definable since X' and Y' are both affine schemes. The morphism $p \circ r : X' \to X$ is an atlas of X. Thus $p(r((f'')^{-1}(C))) = f^{-1}(C)$ is definable due to $A(k)$. This is completes the induction. □

Definition 5.3.3. *Let X be an Artin stack over $\mathbb{D}$. A definable set of arcs C on X is* weakly stable at level n *if $\tau_n^{-1}(\tau_n(C)) = C$. We say that C is* weakly stable *if it is weakly stable at level n for some n.*

Let $f : X' \to X$ be a smooth morphism of Artin stacks over $\mathbb{D}$. A definable set C of arcs on X is weakly stable if and only if $Gr(f)^{-1}(C)$ is weakly stable. Indeed, this follows by an easy adaptation of Lemma 5.2.3.

Lemma 5.3.4. *Lemma 5.2.1 and Lemma 5.2.4 are true when X is an Artin stack over $\mathbb{D}$.*

Proof. This is a trivial consequence of the definitions. Indeed, let $X' \to X$ be an atlas for X where X' is an affine scheme. Then if C is a definable subet of $|Gr(X)|$, let C' be its preimage in $|Gr(X')|$. Then C' is definable by the above definition. Thus, by applying Lemma 5.2.1, it follows that for any n, $\tau_n(C)$ is a constructible subset of $|Gr_n(X')|$. Its image in $|Gr_n(X)|$ is precisely the set $\tau_n(C)$. Thus Lemma 5.2.1 is proved for Artin stacks.

The proof of the generalization of Lemma 5.2.4 is similar. □

Lemma 5.3.5. *Let X be an Artin stack over $\mathbb{D}$ and let C be a definable subset of arcs on X. There exists a closed substack X' of X such that $\dim(X/\mathbb{D}) > \dim(X'/\mathbb{D})$ and $C \backslash Gr(X')$ is the disjoint union of a countable collection of weakly stable sets.*

Proof. For any X and C as in the statement of the lemma, let $\mathcal{W}_C$ denote the collection of closed substacks X' of X such that $C \backslash Gr(X')$ is the disjoint union of a countable collection of weakly stable sets. Clearly the intersection of finitely many elements of $\mathcal{W}_C$ is in $\mathcal{W}_C$. As X is noetherian, it follows that $\mathcal{W}_C$ has a unique minimal element. We denote this closed substack of X by X^C. We wish to check that $\dim(X^C/\mathbb{D}) < \dim(X/\mathbb{D})$. By choosing X to be irreducible, it will suffice to prove that $X^C \neq X$.

Let $f : X \to Y$ be a smooth, surjective morphism of Artin stacks over $\mathbb{D}$. Let D be a definable subset of arcs on Y and let $C = Gr(f)^{-1}(D)$. Since the preimage of a weakly stable subset of $Gr(Y)$ with respect to $Gr(f)$ is weakly stable, it follows that $f^{-1}(Y^D)$ is an element of $\mathcal{W}_C$. Thus, X^C is a closed substack of $f^{-1}(Y^D)$. Let $\overline{f(X^C)}$ denote the reduced closed substack of Y such that $|\overline{f(X^C)}|$ is the closure of $f(|X^C|)$ under the Zariski topology. The image of a weakly stable set of arcs on X gets mapped to a weakly stable set of arcs on Y. Thus, $\overline{f(X^C)}$ is an element of $\mathcal{W}_D$ and hence Y^D is a closed substack of $\overline{f(X^C)}$. Since we already know that $X^C \subset f^{-1}(Y^D)$, it follows that $\overline{f(X^C)} = Y^D$.

Now suppose X is an irreducible Artin stack over $\mathbb{D}$ and let C be a definable set of arcs on X. Suppose $X = X^C$. Then choose any atlas $p : U \to X$ where U are affine scheme of finite type over $\mathbb{D}$. Let $W \to U \times^h_X U$ be an atlas such that W is an affine scheme of finite type over $\mathbb{D}$. Thus we have the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{pr_2} & U \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{p} & X. \end{array}$$

Let C_U, C_W denote the preimages of C in $|Gr(U)|$ and $|Gr(W)|$ respectively.

By the above arguments, $X = \overline{p(U^{C_U})} = \overline{p \circ pr_1(W^{C_W})}$. We claim that $pr_1(W^{C_W})$ (the scheme theoretic image of W^{C_W} with respect to the map pr_1) is dense in $|U|$. Indeed, if we assume that this is not the case, there exists an open subscheme V of U such that V does not intersect $\overline{pr_1(W^{C_W})}$. Then the image of V in X is an open substack X' of X such that X' does not intersect $\overline{p \circ pr_1(W^{C_W})}$, which is impossible. Thus $\overline{pr_1(W^{C_W})} = U$. However, applying the above arguments the map $pr_1 : W \to U$, we see that $\overline{pr_1(W^{C_W})} = U^{C_U}$. Thus $U = U^{C_U}$.

Thus in order to complete the proof, it will suffice to prove that if X is *an affine scheme*, then $X \neq X^C$. (The rest of this argument is from [DL].)

So, now we suppose that that X is an affine scheme over $\mathbb{D}$. As above, we may assume that X is irreducible. Choose a closed immersion of X into $\mathbb{A}^m$ for some m. Suppose C is defined by a Denef-Pas formula ϕ. Let F be the product of all the polynomials in the variables in the Val-sort which occur in ϕ and which do not vanish identically on X. Let Y be the closed subscheme of X cut out by F. Let C_i denote the set of arcs x in C such that if $\widetilde{x} : \mathbb{D} \to X$ is the corresponding morphism (i.e. $\Theta(\widetilde{x}) = x$), then $\mathrm{ord}(\widetilde{x}^*(F)) = i$. Clearly, C_i is weakly stable at level i. Since F does not vanish identically on X and since X

is irreducible, Y is a proper subscheme of X. It follows that X^C is a proper subscheme of X. □

5.4 DIMENSIONS OF ARC-SPACES

As in the case of Lemma 5.2.1 and Lemma 5.2.4, the following in this section are proved in [DL] for schemes of the form $T \times_{\mathbb{D}_0} \mathbb{D}$ where T is a variety over $\mathbb{D}_0$. The proofs can be adapted to the case in which X is a flat scheme of finite type over $\mathbb{D}$ without any change. However, we present the proofs in detail for the sake of completeness.

Suppose X be a closed subscheme of $\mathbb{A}^p_{\mathbb{D}}$ which is the complete intersection of the q-tuple of polynomials $f_1(x), \ldots, f_q(x)$ where $x = (x_1, \ldots, x_p)$ is a p-tuple of variables. We assume that X is *smooth* over $\mathbb{D}$. A point P of $Gr_n(X)$ is given by a p-tuple $a(t) = (a_1(t), \ldots, a_p(t))$ of polynomials of degree n with coefficients in some extension of the base field such that $f(a(t)) = t^{n+1}h(t)$ for some polynomial $h(t)$. Any point of $Gr_{n+1}(X)$ which maps to P under the map τ_n^{n+1} must of the form $a(t) + t^{n+1}\alpha$ with $\alpha \in K^p$ (for some extension K of k) such that $f(a(t) + \alpha t^{n+1}) \equiv 0 \pmod{t^{n+1}}$. Using the Taylor expansion, we see

$$f(a(t) + \alpha t^{n+1}) \equiv t^{n+1}h(0) + t^{n+1}D(\overline{f})(a(0)) \cdot \alpha \pmod{t^{n+2}}$$

where $\overline{f}$ is the q-tuple of polynomials obtained by reducing f modulo t. Since X is non-singular, $D(\overline{f})(a(0))$ has rank q. Thus that the the equation $h(0) = D(\overline{f})(a(0)) \cdot \alpha$ has a $(p-q)$-dimensional solution space. Thus we have the following result for the case when X is a scheme.

Lemma 5.4.1. *1. If X is a smooth stack over $\mathbb{D}$:*

- *$Gr_n(X) = \tau_n(Gr(X))$.*
- *$Gr_{n+1}(X)$ is a $\mathbb{A}^d$-bundle over $Gr_n(X)$ where $d = \dim(X/\mathbb{D})$.*
- *$\dim(Gr_n(X)) = (n+1)\dim(X/\mathbb{D})$.*

2. If X and Y are schemes over $\mathbb{D}$ and $X \to Y$ is a smooth morphism of relative dimension d, the morphism $Gr_n(X) \to Gr_n(Y)$ has relative dimension $(n+1)d$.

The proof of this lemma for a general Artin stack can be obtained by the usual inductive argument. We omit the details of the argument which is entirely routine.

The following lemma generalizes the above bounds on dimension for non-smooth morphisms between stacks.

Lemma 5.4.2. *([DL], Lemma 4.3) Let X be an irreducible Artin stack over $\mathbb{D}$ and let $\dim(X/\mathbb{D}) = d$.*

1. *For any $n \in \mathbb{N}$,*
$$\dim \tau_n(|Gr(X)|) \leq (n+1)d.$$
2. *For any n, m in $\mathbb{N}$, with $m \geq n$, the fibers of $\tau_m(|Gr(X)|) \to \tau_n(|Gr(X)|)$ are of dimension $\leq (m-n)d$.*

Proof. It will suffice to prove (2) since (1) follows immediately from it. Also, it suffices to prove (2) for $m = n+1$.

First, we suppose that X is a scheme (flat and of finite type over $\mathbb{D}$). Without loss of generality, we may assume that X is affine. We will make use of the description of $Gr_n(X)$ presented after Proposition 4.1.1. Thus, suppose $X = \operatorname{Spec}(A)$ where A is a ring of the form $A = R[x]/I$ where $x = (x_1, \ldots, x_p)$ is a p-tuple of variables and I is an ideal in $R[x]$. Choose variables $(a_{ij})_{1\leq i\leq p, 0\leq j<\infty}$ and let $a(t) = (a_i(t))_{1\leq i\leq n}$ be the p-tuple of power series defined by $a_i(t) = \sum_{j=0}^{\infty} a_{ij}t^j$. Suppose f is a polynomial in I. Let $f(a(t)) = \sum_{j=0}^{\infty} \theta_{f,j}t^j$ where $\theta_{f,j}$ is a polynomial in $k[\{a_{ij}\}_{i,j}]$. It is easy to see that if $j \leq n$, $\theta_{f,j}$ only involves the variables in the set $A_n := (a_{ij})_{1\leq i\leq p, 0\leq j\leq n}$. Let I'_n be the ideal of $k[A_n]$ generated by the polynomials $0 \leq j \leq n$ as f varies over all elements of I. Then $Gr_n(X)$ is represented by the closed subscheme of $\mathbb{A}_k^{(n+1)p}$ defined by the ideal I'_n.

A point P of $Gr(X)$ is determined by a p-tuple of power series $(\alpha_i(t))_{1\leq j\leq n}$ where $\alpha_i(t) = \sum_{j=0}^{\infty} \alpha_{ij}t^j$ where $\alpha_{ij} \in K$ for some extension K of k. Define a p-tuple of varibles $y = (y_j)_{1\leq j\leq p}$ by $x_j = \sum_{j=0}^{n} \alpha_{ij}t^j + t^{n+1}y_j$. Let $I_{P,n}$ be the ideal of $R[y]$ generated by polynomials of the form $(1/t^{n+1})f(x)$. Then if $X_{P,n}$ is the closed subscheme of $\mathbb{A}_R^p$ defined by $I_{P,n}$. Then the fiber of $\tau_{n+1}(Gr(X)) \to \tau_n(Gr(X))$ over $\tau_n(P)$ is isomorphic to the closed fiber of $X_{P,n}$. The schemes $X_{P,n}$ and X have isomorphic fibers over the point $\operatorname{Spec}(K) \to \mathbb{D}_0 \to \mathbb{D}$ of $\mathbb{D}$. Thus it suffices to show that $X_{P,n}$ is flat over $\mathbb{D}$. A scheme defined by an ideal J of $R[x]$ is flat over $\mathbb{D}$ if and only if t is not a zero-divisor in $R[x]/J$. It is easy to see that $I_{P,n}$ has this property since I has it by hypothesis. This proves the result when X is a scheme.

Now suppose X is an Artin stack over $\mathbb{D}$. Choose an atlas $U \to X$ where U is an affine scheme. U may be chosen to be of pure dimension over $\mathbb{D}$. Suppose $\dim(U/X) = p$.

We have the commutative square

$$\begin{array}{ccc} Gr_{n+1}(U) & \longrightarrow & Gr_{n+1}(X) \\ \downarrow & & \downarrow \\ Gr_n(U) & \longrightarrow & Gr_n(X). \end{array}$$

Let P be a point of $\tau_n(Gr(X))$. Let C be the preimage of P in $|Gr_n(U)|$. Clearly, C is a subset of $\tau_n(|Gr(U)|)$. Let C' be the preimage of C in $\tau_{n+1}(Gr(U))$. By the previous lemma, $\dim(C) = (n+1)p$. Since we have proved the result for schemes, $\dim(C') \leq (n+1)p + (p+d)$. By the previous lemma, the image of C' in $|Gr_{n+1}(X)|$ has dimension $\leq (n+1)p + (p+d) - (n+2)p = d$. $\square$

We recall a theorem of Greenberg (see [Gr2]).

Theorem 5.4.3. *Let $\mathcal{R}$ be an excellent discrete valuation ring with maximal ideal $\mathfrak{m}$. Let $f_1, \ldots, f_p$ be polynomials in the n-variables $x_1, \ldots, x_q$. Then, there exist integers $N \geq 1, c \geq 1$ and $s \geq 0$ depending on the ideal generated by $f_1, \ldots, f_p$ such that for any $\nu \geq N$ and any $a \in \mathcal{R}^q$ such that*

$$f(x) \equiv 0 \ (mod\ \mathfrak{m}^{\nu})$$

there exists y in $\mathcal{R}^p$ such that $y \equiv x$ (mod $\mathfrak{m}^{[\nu/c]-s}$) and $f(y) = 0$.

The following lemma says that "the set of arcs on X having a sufficiently high intersection number with a (proper) closed substack of X is sufficiently small."

Lemma 5.4.4. *Let X be an irreducible Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let S be a closed substack of dimension $< d$. Let $e \geq 0$ be any integer. Then for sufficiently large n and i,*

$$\dim(\tau_{n,X}(\tau_{i,X}^{-1}(Gr_i(S)))) \leq (n+1)d - e - 1.$$

Proof. First consider the case that X is a scheme. Let c,s,N be as in the statement of Greenberg's theorem above. Let $n \geq i \geq c(e+s)+N$.

Applying Lemma 5.4.2 to the morphism $\tau_{n,X}(Gr(X)) \to \tau_{e,X}(Gr(X))$, we see that

$$\dim(\tau_{n,X}(\tau_{i,X}^{-1}(Gr_i(S)))) \leq \dim(\tau_{e,X}(\tau_{i,X}^{-1}(Gr_i(S)))) + (n-e)d.$$

For any subset C of $Gr_i(X)$, we have $\tau_{e,X}(\tau_{i,X}^{-1}(C)) \subset \tau_{e,X}^{i}(C)$. By choosing $C = \tau_{i,X}(S)$, we get the inclusion $\tau_{e,X}(\tau_{i,X}^{-1}(Gr_i(S))) \subset \tau_{e,X}^{i}(Gr_i(S))$. On the other hand, $Gr(S) \subset \tau_{i,X}^{-1}(Gr_i(S))$. Thus, $\tau_{e,X}(Gr(S)) \subset \tau_{e,X}(\tau_{i,X}^{-1}(Gr_i(S)))$. By Greenberg's theorem and our choice of i, $\tau_{e,X}(Gr(S)) = \tau_{e,X}^{i}(Gr_i(S))$. Thus, $\tau_{e,X}(Gr(S)) = \tau_{e,X}(\tau_{i,X}^{-1}(Gr_i(S)))$.

Finally, by applying Lemma 5.4.2 to the set $\tau_e(Gr(S))$, we see that

$$\dim(Gr(X)) \leq (e+1)\dim(S/\mathbb{D}) \leq (e+1)(d-1).$$

Thus, $\dim(\tau_{n,X}(\tau_{i,X}^{-1}(Gr_i(S)))) \leq (e+1)(d-1)+(n-e)d = (n+d)-e-1$. This proves the result for schemes.

Now suppose that X is an Artin stack over $\mathbb{D}$. Let $U \to X$ be an atlas with U an affine scheme which is of pure dimension over $\mathbb{D}$ with $\dim(U/X) = p$. Let S be a closed substack of X. Let $T = S \times_X U$. Then T is a closed substack of U. $\dim(T) = \dim(S)+p < \dim(U)$. Thus T does not contain any of the components of U. Thus if $\{U_i\}_{i=1}^{k}$ is the set of components of U, applying the above arguments to the pair $(U_i, U_i \cap T)$ for each i, we get that for sufficiently large n, i,

$$\dim(\tau_{n,U}(\tau_{i,U}^{-1}(Gr_i(T)))) \leq (n+1)(p+d)-e-1.$$

Using Lemma 5.4.1, we see that

$$\begin{aligned} \dim(\tau_{n,X}(\tau_{i,X}^{-1}(Gr_i(S)))) &\leq (n+1)(p+d)-e-1-(n+1)p \\ &= (n+1)d-e-1 \end{aligned}$$

as desired. □

6.0 STABLE SETS

In this section, we introduce the notion of a *stable set of arcs* on a stack. These are weakly stable sets that are extremely well-behaved under truncation. Once this is achieved, the rest of the argument for the construction of motivic measure will be achieved by simply repeating the construction for motivic measure on varieties with a few modifications.

We reiterate our assumption from the previous section that all Artin stacks over $\mathbb{D}$ will be assumed to be flat, strongly finitely presented over $\mathbb{D}$.

6.1 SINGULARITY INDEX OF ARCS

To begin with, we need a few definitions.

Singular locus:

Let $f : X \to Y$ be a morphism between schemes of pure dimension over $\mathbb{D}$. Let $\dim(X/\mathbb{D}) = d$ and $\dim(Y/\mathbb{D}) = p$. Let $\Omega_{X/Y}$ denote the sheaf of relative differentials of X over Y. Let $\mathcal{J}_f$ denote the $(d-p)$-th Fitting ideal of $\Omega_{X/Y}$. If $Y = \mathbb{D}$, we write $\mathcal{J}(X/\mathbb{D})$ for $\mathcal{J}_h$. The closed subscheme of X defined by $\mathcal{J}_h$ (resp. $\mathcal{J}(X/\mathbb{D})$) is called the *singular locus of* h (*resp. singular locus of* X). We denote the singular locus of X by X_{sing}.

If $X \to Y$ is a smooth morphism, the sequence

$$0 \to f^*\Omega_{Y/\mathbb{D}} \to \Omega_{X/\mathbb{D}} \to \Omega_{X/Y} \to 0$$

is exact and splits (non-canonically).

Thus, if $F_d(\mathcal{M})$ denotes the d-th Fitting ideal sheaf of a coherent sheaf, we have (see [La], Chapter XIX, Proposition 2.8)

$$F_d(\Omega_{X/\mathbb{D}}) = F_p(\Omega_{Y/\mathbb{D}}) \cdot F_{d-p}(\Omega_{X/Y}).$$

Since $\Omega_{X/Y}$ is locally free of rank $d-p$, it follows that the ideal $F_{d-p}(\Omega_{X/Y})$ is simply R. Thus $X_{sing} = X \times_Y Y_{sing}$. Thus, by the remark following Definition 3.6.2, we can extend this definition to stacks. To be precise, if X is an Artin stack and $U \to X$ is an atlas of X, the image stack of the morphism $U_{sing} \to X$ is a closed substack of X which we will call as the *singular locus of* $X \to \mathbb{D}$ and denote by X_{sing}.

More generally, we can define the singular locus of a morphism $f : X \to Y$ of Artin stacks. We omit the details.

Order of the Jacobian:

Let X be a scheme over $\mathbb{D}$. Let x be a point of $Gr_n(X)$ for any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Suppose x is given by a morphism $\widetilde{x} : \mathbb{D}_n \times_{\mathbb{D}_0} \mathrm{Spec}(K) \to X$ for some extension K of k. Then we define *the order of the Jacobian ideal of* X *at* x to be $\mathrm{ord}_t(\widetilde{x}^*(\mathcal{J}(X/\mathbb{D})))$ and we denote it by $l(x, X)$.

More generally, if $f : X \to Y$ is a morphism of schemes over $\mathbb{D}$, then we define the *order of the Jacobian of* f *at* x to be $\mathrm{ord}_t(\widetilde{x}^*(\mathcal{J}_f))$ and denote it by $\ell(x, f)$.

Lemma 6.1.1. *Let* $f : X \to Y$ *be a smooth morphism of schemes. Let*

$$\mathrm{Spec}(K) \to Gr_n(X)$$

be an n*-arc on* X*. Then* $\ell(x, X) = \ell(Gr(f)(x), Y)$*.*

Proof. This follows from the fact that $X_{sing} = Y_{sing} \times_Y X$. □

Suppose X is an Artin stack and x is a point on $Gr_n(X)$ given by a morphism $\widetilde{x} : \mathbb{D}_n \times_{\mathbb{D}_0} \mathrm{Spec}(K) \to X$ (i.e. $x = \Theta_n(\widetilde{x})$). Let $U \to X$ be an atlas of X. Then there exists a lifting $u : \mathbb{D}_n \times_{\mathbb{D}_0} \mathrm{Spec}(K') \to U$ of the arc x to U. Then we define the *order of the Jacobian of* X *at* x as $\ell(x, X) := \ell(u, U)$. It follows easily from the above lemma that this definition does not depend on the choice of U or u. Also, if $X \to Y$ is a smooth morphism of Artin stacks and x an n-arc on X, we have the equality $\ell(x, X) = \ell(Gr(f)(x), Y)$. More generally, if $f : X \to Y$ is a morphism of Artin stacks over $\mathbb{D}$, we can define $\ell(x, f)$ in a similar manner.

If x is an n-arc on X and $\mathbb{D}_n \times_{\mathbb{D}_0} \mathrm{Spec}(K) \to X$ is the corresponding morphism, $\ell(X, x) \leq e$, if and only if the morphism

$$\mathbb{D}_e \times_{\mathbb{D}_0} \mathrm{Spec}(K) \to \mathbb{D}_n \times_{\mathbb{D}_0} \mathrm{Spec}(K) \to X$$

factors through $X_{sing} \to X$, i.e. $x : \mathrm{Spec}(K) \to Gr_n(X)$ factors through $Gr_e(X_{sing})$. Thus we define

$$Gr_n^{(e)}(X) := Gr_n(X) \backslash (Gr_n(X) \times^h_{Gr_n(X_{sing})} Gr_{e+1}(X_{sing})).$$

The points of $Gr_n^{(e)}(X)$ represent arcs on X on which the order of the Jacobian is $\leq e$.

Definition 6.1.2. *A morphism $f : X \to Y$ of Artin stacks over $\mathbb{D}$ is* pseudo-smooth *if for any point x in $Gr(X)$, we have $\ell(x, X) \geq \ell(Gr(f)(x), Y)$.*

Intuitively, a morphism is pseudo-smooth if it "does not increase the proximity of an arc to the singular locus".

We saw in Lemma 6.1.1 that smooth morphisms are pseudo-smooth. We record the following easy lemma for later use.

Lemma 6.1.3. *Let $X \to Y$ be a smooth morphism of Artin stacks over $\mathbb{D}$. Then the morphism $X \times^h_Y X \to X \times^h X$ is pseudo-smooth.*

Proof. Let $f : X \to Y$ be a smooth morphism of Artin stacks. Let x be a point of $X \times^h_Y X$ defined by two points x_1 and x_2 of X. Let x' be the image of x in $X \times^h X$. We wish to prove that $\ell(x, X \times^h_Y X) \geq \ell(x', X \times^h X)$.

Let $pr_1, pr_2 : X \times^h X \to X$ be the two projections. The two morphisms $X \times^h_Y X \to X$ are smooth. Thus $\ell(x, X \times^h_Y X) = \ell(x_1, X) = \ell(x_2, X)$. Hence it will suffice to show that $\ell(x', X \times^h X) \geq \ell(x_1, X)$. In other words, it suffices to prove that pr_1 is pseudo-smooth. This is obvious when X is a scheme. The generalization to stacks follows by the usual inductive argument. □

6.2 A REVIEW OF STABLE SETS FOR VARIETIES

We saw in the discussion preceding Lemma 5.4.1 that if X is a smooth scheme over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$, for each integer $n \in \mathbb{Z}_{\geq 0}$, $Gr_{n+1}(X) \to Gr_n(X)$ is a $\mathbb{A}^d$-bundle. In this section, we review results which generalize this result for singular schemes. Roughly speaking, this result continues to be true on singular schemes "away from the singular locus."

The main results we wish to review are ([DL], Lemma 4.1 and Lemma 3.4). The presentation we follow is from ([Lo]).

Let X be an affine (not necessarily smooth) scheme with relative dimension d over $\mathbb{D}$. Let n and e be integers with $n \geq e$. Let $x \in |Gr^{(e)}(X)|$, represented by $\mathrm{Spec}(K) \to Gr(X)$. Let $\widetilde{x} : \mathrm{Spec}(K) \times_{\mathbb{D}_0} \mathbb{D} \to X$ be such that $\Theta(\widetilde{x}) = x$.

Since $l(x, X) \leq e$, $\widetilde{x}$ maps the generic point of $\mathrm{Spec}(K) \times_{\mathbb{D}_0} \mathbb{D}$ into the regular locus of X. Thus $\widetilde{x}^*(\Omega_{X/\mathbb{D}})$ is an $R \otimes_k K$-module of rank d. Since $F_d(X) = (t^e)$, the torsion of $\widetilde{x}^*(\Omega_{X/\mathbb{D}})$ has length e.

Let

$$\widehat{T}_{X,\widetilde{x}} = Hom_{K[[t]]}(\widetilde{x}^*(\Omega_{X/\mathbb{D}}), K[[t]]) \otimes_{K[[t]]} K.$$

This is a d-dimensional K-vector space. If $n \geq e$, any $K[[t]]$-morphism

$$\widetilde{x}^*(\Omega_{X/\mathbb{D}}) \to K[t]/(t^{n+1})$$

kills the torsion of $\widetilde{x}^*(\Omega_{X/\mathbb{D}})$ and thus lifts to a a $K[[t]]$-homomorphism

$$\widetilde{x}^*(\Omega_{X/\mathbb{D}}) \to K[[t]].$$

Thus $\widehat{T}_{X,\widetilde{x}}$ depends only on $\tau_e(x)$.

Let $x_1 = \Theta(\widetilde{x}_1)$ and $x_2 = \Theta(\widetilde{x}_2)$ are points of $Gr(X)$ represented by two morphisms $\mathrm{Spec}(K) \to Gr(X)$. Suppose that $\tau_n(x_1) = \tau_n(x_2)$, it is easy to check that $(\widetilde{x}_1^* - \widetilde{x}_2^*)$ defines a $K[[t]]$ derivation

$$\mathcal{O}_X(X) \longrightarrow (t^{n+1})/(t^{2n+2}) \subset K[[t]]/(t^{2n+2}),$$

i.e. an element of $Hom_{K[[t]]}(\widetilde{x}_1^*(\Omega_{X/\mathbb{D}}), (t^{n+1}/t^{2n+2}))$. Its reduction modulo t^{n+2} lies in $\widehat{T}_{X,x} \otimes_K (t^{n+1})/(t^{n+2})$. It can be proved that every element of $\widehat{T}_{X,x}$ occurs in this manner.

Now suppose that X is a closed subscheme of $\mathbb{A}^m_{\mathbb{D}}$ for some integer m. Then for any integer n, $Gr_n(X)$ is a closed subscheme of $Gr_n(\mathbb{A}^m_{\mathbb{D}}) \cong \mathbb{A}_k^{m(n+1)}$. Then using Hensel's lemma, it can be proved that $\rho^{n+1}_{n,X}(\tau_n(x))$ is a closed subset of $\tau^{n+1}_{n,\mathbb{A}^m_{\mathbb{D}}}(\tau_n(x)) \cong \mathbb{A}^m_k$ cut out by $(m-d)$ independent linear equations whose coefficients are regular functions in the coordinates of $\tau_n(x) \in \mathbb{A}_k^{m(n+1)}$. (We will not reproduce the details of the proof. See [Lo], Lemma 9.1 or [DL], Lemma 4.1.)

Lemma 6.2.1. *([Lo], Lemma 9.1) Let X be a scheme over $\mathbb{D}$. Let $n \geq e$. Then the fiber of $(\rho^{n+1}_n)^{-1}(x)$ is an affine space with translation space $\widehat{T}_{X,x} \otimes_k (t^{n+1})/(t^{n+2})$. This defines an affine space bundle of rank d over $\tau_n(Gr^{(e)}(X))$.*

In particular, if A is a locally closed subset of $\tau_n(Gr(X))$, then $\rho^{n+1}_{n,X}(A)$ is a closed subset of $\tau^{n+1}_{n,\mathbb{A}^m_{\mathbb{D}}}(A)$ which is locally closed in $Gr_{n+1}(\mathbb{A}^m_{\mathbb{D}})$. In particular, $\rho^{n+1}_n(A)$ is a *locally closed subset* of $Gr_{n+1}(X)$.

Definition 6.2.2. *Let X be a scheme over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. A definable set of arcs on X is said to be* stable at level n *if it is weakly stable at level n and for locally closed subset $|Z|$ of $|Gr_n(X)|$ contained in $\tau_n(C)$ such that:*

1. *$(\rho^{n+1}_n)^{(-1)}(Z)$ is a locally closed subset $|Z'|$ of $Gr_{n+1}(X)$.*
2. *$Z' \to Z$ is a $\mathbb{A}^d$-bundle.*

6.3 VERY STABLE SETS

In this section we generalize the results of the previous section for Artin stacks.

Lemma 6.3.1. *Let X be an Artin stack over $\mathbb{D}$. Let Z be a locally closed subset of $|Gr_n(X)|$ contained in $\tau_n(|Gr^{(e)}(X)|)$. Then $Z' := (\rho^{n+1}_n)^{(-1)}(Z)$ is a locally closed subset of $|Gr_{n+1}(X)|$.*

Proof. Let $V \to X$ be an atlas of X such that V is an affine scheme. Then $Gr_n(V) \to Gr_n(X)$ is an atlas. Let $U = Gr_n(V) \times^h_{Gr_n(X)} F$. Then U is a locally closed subscheme of $Gr_n(V)$ such that $|U| \subset \tau_n(|Gr^{(e)}(V)|)$. Thus $(\rho^{n+1}_n)^{(-1)}(|U|)$ is a locally closed subset of $Gr_{n+1}(V)$. Let

U' be the reduced locally closed subscheme of $Gr_{n+1}(V)$ such that $|U'| = (\rho_n^{n+1})^{(-1)}(|U|)$. Let F' be the image of $U' \to Gr_{n+1}(X)$. Clearly $|F'| = (\rho_n^{n+1})^{(-1)}(Z)$. By the remark following Definition 3.6.2, the fact that U' is a locally closed subscheme of $Gr_{n+1}(V)$, implies that F' is a locally closed substack of $Gr_{n+1}(X)$. □

Definition 6.3.2. *Let $F \to G$ be a morphism of sheaves over some scheme S. Let T be a scheme over S. We say that $F \to G$ is a Zariski locally trivial T-bundle if for any affine scheme U over S and any morphism $U \to G$, the sheaf $F \times_G U$ is a scheme and the morphism $F \times_G U \to U$ is a Zariski locally trivial T-bundle.*

Definition 6.3.3. *Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let Z be a locally closed subset of $\tau_n(Gr(X))$. Let F be the reduced, locally closed substack of $Gr_n(X)$ such that $|F| = Z$. Let $Z' := (\rho_n^{n+1})^{(-1)}(Z)$. We say that Z is* very stable *if:*

1. *Z' is a locally closed subset of $|Gr_{n+1}(X)|$. Let F' be the reduced locally closed substack of $Gr_{n+1}(X)$ such that $|F'| = Z'$.*
2. *Let T be any affine scheme over $\mathbb{D}_0$ and $t' : T \to F'$ be any morphism. Let $t = \tau_n^{n+1} \circ t'$ and let $i > 0$ be an integer. Then*

$$\pi_i(F', t') \to \pi_i(F, t)$$

is a surjective homomorphism of group sheaves, the kernel of which is Zariski-locally (over T) isomorphic to $\mathbb{G}_a^{d_i}$ for some integer d_i independent of t'.

3. *$\pi_0(F')$ is a Zariski locally trivial $\mathbb{A}^{d_0}$-bundle over $\pi_0(F)$ where*

$$\sum_{i=0}^{\infty} (-1)^i d_i = d.$$

The following result generalizes Lemma 6.2.1.

Proposition 6.3.4. *Let m, n, e be integers such that $n \geq e$.*

1. *Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let Z be a locally closed subset of $|Gr_n(X)|$ contained in $\tau_n(|Gr^{(e)}(X)|)$. Then Z has a stratification*

$$Z := Z_0 \supset Z_1 \supset \ldots \supset Z_k$$

by subsets that are closed with respect to Z such that for each i, $Z_i \backslash Z_{i+1}$ is very stable.

2. *With the hypothesis of part (1), a subset Z is very stable if and only if it satisfies condition (1) and (2) in Definition 6.3.3.*
3. *If $f : X \to Y$ is a pseudo-smooth morphism of Artin stacks over $\mathbb{D}$, then for any x in $\tau_n(|Gr^{(e)}(X)|)$, the induced morphism*

$$(\rho_n^{n+1})^{(-1)}(x) \to (\rho_n^{n+1})^{(-1)}(Gr_n(f)(x))$$

is an affine map which is surjective if f is smooth.

Proof. Suppose X is m-geometric. The proof is by induction on m.

We already know that the result is true when $m = -1$ and $m = 0$. Suppose that the result has been proved for $m = k$. We now suppose that X is a $(k+1)$-stack. In order to prove (1), it will suffice to prove that Z has a non-empty open subset which is very stable. Let F be the reduced, locally closed substack of $Gr_n(X)$ such that $|F| = Z$. Let $V \to X$ be an atlas of X such that V is an affine scheme. Let $V_1 := V \times_X V$. Let $U := F \times^h_{Gr_n(X)} Gr_n(V)$ and $U_1 := U \times^h_F U$. Then $U \to Gr_n(V)$ is a closed immersion such that $|U| \subset \tau_n(|Gr^{(e)}(V)|)$. Let U' and U'_1 be the reduced, locally closed substacks of $Gr_{n+1}(V)$ and $Gr_{n+1}(V_1)$ such that $|U'| = (\rho_n^{n+1})^{(-1)}(|U|)$ and $|U'_1| = (\rho_n^{n+1})^{(-1)}(|U_1|)$. The maps $u : U \to F$ and $u' : U' \to F'$ are atlases for F and F'.

Clearly, V_1 is k-geometric. Thus, we may apply the induction hypothesis to $|U_1|$. Thus, there exists an open substack W_1 of U_1 such that $|W_1|$ is very stable. Let W_1 be the maximal open substack of U_1 with this property.

We start by reducing the problem to the case that $W_1 = U_1$. The morphism $W_1 \to F$ is smooth. Thus if we denote the image of this morphism by G, G is an open substack of F. Let $W := U \times^h_F G$ and let $W_2 := W \times^h_G W$. We claim that $W_2 = W_1$. Clearly W_1 is an open substack of W_2. Suppose Z is an affine scheme over $\mathbb{D}_0$ and suppose that $z_2 : Z \to W_2$ is any morphism. Then since $W_1 \to G$ is surjective, by replacing Z by an etale cover if necessary, we can construct a morphism $z_1 : Z \to W_1$ such that the two morphisms

$$Z \xrightarrow{z_1} W_1 \longrightarrow G$$

and

$$Z \xrightarrow{z_2} W_2 \longrightarrow G$$

are equivalent (i.e. they lie in the same component of $G(Z)$). Thus the group sheaves $\pi_i(W_2, z_1)$ and $\pi_i(W_2, z_2)$ are isomorphic.

Let $G' := G \times^h_F F'$. Let $W'_i := W_i \times^h_F F'$ for $i = 1, 2$. Then G' is the image of $W'_i \to F'$ for both $i = 1, 2$. Thus, by replacing Z by an etale cover if required, we may assume that there exist lifts $z'_1 : Z \to W'_1$ and $z'_2 : Z \to W'_1$ of z_1 and z_2 such that the morphisms

$$Z \xrightarrow{z'_1} W'_1 \longrightarrow G'$$

and

$$Z \xrightarrow{z'_2} W'_2 \longrightarrow G'$$

are equivalent. Thus the homotopy group sheaves $\pi_i(W'_2, z'_1)$ and $\pi_i(W'_2, z'_2)$ are isomorphic. Thus, since $|W_1|$ is very stable, W_2 satisfies property (2) in Definition 6.3.3. By the induction hypothesis, statement (2) in the statement of this proposition is known to be true for k-stacks. Thus it follows that W_2 is stable. Since W_1 was supposed to be maximal with this property, $W_1 = W_2$.

Thus, now we replace U by W and F by G and thus reduce the problem to the case in which $|U_1|$ is very stable. So now we may assume that U' is a Zariski locally trivial $\mathbb{A}^p$-bundle over U for some p and $\pi_0(U'_1)$ is a Zariski locally trivial $\mathbb{A}^q$-bundle over $\pi_0(U_1)$.

We have the commutative square

$$\begin{array}{ccc} \pi_0(U'_1) & \longrightarrow & U' \times U' \\ \downarrow & & \downarrow \\ \pi_0(U_1) & \longrightarrow & U \times U. \end{array}$$

We have the isomorphisms

$$\begin{aligned} \pi_0(U' \times^h_F U') &\cong \pi_0((U \times^h_F U) \times^h_{(U \times U)} (U' \times U')) \\ &\cong \pi_0(U \times^h_F U) \times_{\pi_0(U \times U)} \pi_0(U' \times U') \end{aligned}$$

where the first isomorphism follows from the above commutative square. The second isomorphism is a consequence of the fact that if $K \to L$ and $M \to L$ are two fibrations of fibrant simplicial sets and L is 0-truncated, then

$$\pi_0(K \times^h_L M) = \pi_0(K) \times_{\pi_0(L)} \pi_0(M).$$

Thus $\pi_0(U' \times^h_F U') \to \pi_0(U \times^h_F U)$ is the pullback of the $\mathbb{A}^{2p}$-bundle $(U' \times U') \to (U \times U)$. We will denote this morphism $\pi_0(U' \times^h_F U') \to \pi_0(U \times^h_F U)$ by α.

By Lemma 6.1.3, the morphism $V \times^h_X V \to V \times_{\mathbb{D}} V$ is pseudo-smooth. By applying part (3) of this proposition (which is known to be true for k-geometric stacks by the induction hypothesis), we see that if t is any point of $U \times^h_F U$ and $pr_1(t), pr_2(t) \in |U|$ are the images of t under the two projections, the morphism

$$\alpha_t : (\rho^{n+1}_n)^{(-1)}(t) \to (\rho^{n+1}_n)^{(-1)}((pr_1(t), pr_2(t)))$$

is an affine map. Let $d_1 := \min_{|U \times^h_F U|}(q - rank(\alpha_t))$.

Let $T \to U \times_F U$ be an atlas. Then $\pi_0(U'_1) \times_{\pi_0(U_1)} T \to T$ is a $\mathbb{A}^q$-bundle. The morphism

$$\pi_0(U'_1) \times_{\pi_0(U_1)} T \to (U' \times U') \times_{(U \times U)} T$$

is an affine map on each of the fibers over T. The rank of this map is equal to d_1 on an open subscheme T' of T. Thus the set of points t of U_1 such that $q - rank(\alpha_t) = d_1$ is an open subset of the form $|W_1|$ for an open substack W_1 of U_1.

Let G be the image of $W_1 \to F$. Let $W := G \times^h_F U$ and $W_2 := W \times^h_G W$. We claim that $W_1 = W_2$.

Let $t : \mathrm{Spec}(K) \to U \times^h_F U$ be a point on $U \times^h_F U$. We choose K to be algebraically closed (hence uncountable). Let $t_1 = u \circ pr_1 \circ t$ and $t_2 = u \circ pr_2 \circ t$ be morphisms $\mathrm{Spec}(K) \to F$. Let t' be a lift of t to U'_1 . Let $t'_1 = u' \circ pr_1 \circ t'$ and $t'_2 = u' \circ pr_2 \circ t'$. The fiber of α at $\alpha \circ t'$ is clearly is isomorphic to $\mathbb{A}^{q-rank(\alpha_t)}$.

The morphisms $pr_1 \circ t', pr_2 \circ t' : \mathrm{Spec}(K) \to U'$ induce morphisms

$$\pi_0(\mathrm{Spec}(K) \times^h_{t_1,F,t_2} \mathrm{Spec}(K)) \cong \pi_1(F, t_1, t_2) \longrightarrow \pi_0(U' \times^h_F U')$$

and

$$\pi_0(\mathrm{Spec}(K) \times^h_{t'_1,F',t'_2} \mathrm{Spec}(K)) \cong \pi_1(F', t'_1, t'_2) \longrightarrow \pi_0(U' \times^h_F U').$$

It is easy to see that the square

$$\begin{array}{ccc} \pi_1(F', t'_1, t'_2) & \longrightarrow & \pi_0(U' \times^h_{F'} U') \\ \downarrow & & \downarrow \\ \pi_1(F, t_1, t_2) & \longrightarrow & \pi_0(U' \times^h_F U'). \end{array} \tag{6.3.1}$$

is cartesian. The morphism $t' : \mathrm{Spec}(K) \to \pi_0(U' \times^h_{F'} U')$ factors through $\pi_1(F', t'_1, t'_2) \to \pi_0(U' \times^h_{F'} U')$ and the morphism $\alpha \circ t' : \mathrm{Spec}(K) \to \pi_0(U' \times^h_F U')$ factors through $\pi_1(F, t_1, t_2) \to \pi_0(U' \times^h_F U')$. The map $\pi_1(F', t'_1, t'_2) \to \pi_1(F, t_1, t_2)$ is isomorphic to the map $\pi_1(F', t'_1) \to \pi_1(F, t_1)$. We have already seen in Lemma 4.3.2 that the latter map is surjective. Since it is a homomorphism of group sheaves, all its fibers are isomorphic. Thus $\pi_1(F', t'_1, t'_2) \to \pi_1(F, t_1, t_2)$ is surjective and all its fibers are isomorphic. In particular, since the fiber at $\alpha \circ t'$ is isomorphic to $\mathbb{A}^{q - rank(\alpha_t)}$. Thus it follows that $rank(\alpha_t)$ is constant on the fibers of $\pi_0(U \times^h_F U) \to U \times U$ and in fact only depends on t_1.

Thus, if $s : \mathrm{Spec}(K) \to W_1$ is a point on W_1, and $t : \mathrm{Spec}(K) \to U_1$ are two points such that the two morphisms

$$\mathrm{Spec}(K) \xrightarrow{s} W_1 \longrightarrow F$$

and

$$\mathrm{Spec}(K) \xrightarrow{t} U_1 \longrightarrow F$$

are equivalent, then $rank(\alpha_s) = rank(\alpha_t)$. Thus $W_1 = W_2$.

Thus we may replace F by G and U by W to reduce the problem to the case in which $rank(\alpha_t)$ is constant.

Let R be the image sheaf of the morphism $\pi_0(U \times^h_F U) \to (U \times U)$. Let R' be the image sheaf of the morphism $\pi_0(U' \times^h_{F'} U') \to (U' \times U')$. Let $R^{(2)}$ be the image sheaf of $\alpha : \pi_0(U' \times^h_{F'} U') \to \pi_0(U' \times^h_F U')$. $R^{(2)}$ is the image of a morphism between two affine bundles over $\pi_0(U \times^h_F U)$, the rank of which is constant on all fibers. Thus $R^{(2)}$ is a $\mathbb{A}^{q-d_1}$-bundle over $\pi_0(U \times^h_F U)$ and $\pi_0(U' \times^h_{F'} U')$ is a $\mathbb{A}^{d_1}$-bundle over $R^{(2)}$.

Let T be any scheme, $t' : T \to F'$ be any morphism and let $t : T \to F$ be the morphism $T \to F' \to F$. Then, using diagram (6.3.1) we see that $\pi_1(F', t') \to \pi_1(F, t)$ is a $\mathbb{A}^{d_1}$-bundle. Thus, the kernel of the homomorphism $\pi_1(F', t') \to \pi_1(F, t)$ is a sheaf that is locally isomorphic to $\mathbb{A}^{d_1}$. The group law on the kernel is induced by the pseudo-smooth morphism

$$(V \times^h_X V) \times^h_V (V \times^h_X V) \to (V \times^h_X V)$$

and is hence an affine map on each fiber. It is easy to check that the only group law on $\mathbb{A}^{d_1}$ given by affine maps is the one defined by the vector space structure on $\mathbb{A}^{d_1}$. Thus the kernel is locally isomorphic to $\mathbb{G}_a^{d_1}$. This proves condition (2) in Definition 6.3.3 for $|F|$ with $i = 1$. Since $|U \times_F^h U|$ is very stable, the long exact sequence corresponding to $U \to F$ proves condition (2) in Definition 6.3.3 for $|F|$ with $i > 1$.

We claim that the morphism $R^{(2)} \to \pi_0(U \times_F^h U) \times_{(U\times U)} R'$ is surjective.

Let T be any scheme and let $t : T \to \pi_0(U \times_F^h U)$ be a morphism. Let $t_1, t_2 : T \to U$ be the compositions of t with the two projection morphisms $\pi_0(U \times_F^h U) \to U$. Let $t_1', t_2' : T \to U'$ be morphisms such that $(t_1', t_2') : T \to (U' \times U')$ factors through R'. It will suffice to prove that there exists a common lift $T \to \pi_0(U' \times_{F'} U')$ of the morphisms t and (t_1', t_2') with respect to the morphisms $\pi_0(U' \times_{F'}^h U') \to \pi_0(U \times_F^h U)$ and $\pi_0(U' \times_{F'}^h U') \to (U' \times U')$ respectively.

The morphism t factors through $\pi_1(F, t_1, t_2) \to \pi_0(U \times_F^h U)$. The fiber of the morphism $\pi_0(U' \times_{F'}^h U') \to U' \times U'$ at (t_1', t_2') is the image of $\pi_1(F', t_1', t_2') \to \pi_0(U' \times_{F'}^h U')$. Thus, it will suffice to prove that $\pi_1(F', t_1', t_2') \to \pi_1(F, t_1, t_2)$ is surjective. Since $(t_1, t_2) : T \to (U \times U)$ factors through R and $(t_1', t_2') : T \to (U' \times U')$ factors through R', the morphism $\pi_1(F', t_1', t_2') \to \pi_1(F, t_1, t_2)$ is isomorphic to the map $\pi_1(F', t_1') \to \pi_1(F, t_1)$. By Lemma 4.3.2, this morphism is known to be a surjection.

This proves that $R^{(2)} \to \pi_0(U \times_F^h U) \times_{U\times U} R'$ is surjective. Since both $R^{(2)}$ and $\pi_0(U \times_F^h U) \times_{(U\times U)} R'$ are subsheaves of $\pi_0(U' \times_F^h U')$, it follows that $R^{(2)} \cong \pi_0(U \times_F^h U) \times_{(U\times U)} R'$. Thus $R' \to R$ is an $\mathbb{A}^{q-d_1}$-bundle.

Let Z be any scheme over $\mathbb{D}_0$ and let $Z \to \pi_0(F)$ be any morphism. Let $Z \to U$ be a lift of $Z \to \pi_0(F)$ and let $Z \to R$ be a lift of $Z \to U$ (which we may assume to exist by replacing Z by a suitable etale cover). Then $\pi_0(F' \times_{\pi_0(F)} Z)$ is the quotient of

$$R' \times_R Z \rightrightarrows U' \times_U Z.$$

Replacing Z by a suitable cover, we may assume that the two bundles $R' \times_R Z \to Z$ and $U' \times_U Z \to Z$ are trivial over Z. Thus $F' \times_F Z \to Z$ is a $\mathbb{A}^{2q-(q-d_1)}$-bundle.

Let $d_0 = 2p - (q - d_1)$. Then by the induction hypothesis,

$$\begin{aligned} q + \sum_{i=2}^{\infty} (-1)^{i-1} &= \dim(V \times_X^h V) \\ &= \dim(V) + (\dim(V) - \dim(X)) \\ &= 2p - d \end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^i d_i &= 2p-(q-d_1)+\sum_{i=1}^{\infty}(-1)^i d_i \\ &= 2p-(q-d_1)-d_1+(-1)\sum_{i=2}^{\infty}(-1)^{i-1}d_i \\ &= 2p-q+d_1-d_1-(2p-d-q) \\ &= d.\end{aligned}$$

This completes the proof of (1) for $m = k + 1$.

The proof of (3) is an easy consequence of the proof of (1) and of the fact that given any morphism $X_1 \to X_2$ of Artin stacks, we can find atlases $V_1 \to X_1$ and $V_2 \to X_2$ such that the map $X_1 \to X_2$ lifts to $V_1 \to V_2$ which is pseudo-smooth if $X_1 \to X_2$ is pseudo-smooth.

The proof of (2) also follows from the proof of (1). Indeed, suppose F is a substack of $Gr_n(X)$ such that $|F| \subset \tau_n(|Gr^{(e)}(X)|)$ and assume that $|F|$ satisfies conditions (1) and (2) in Definition 6.3.3. Then, using the notation of the proof of part (1), it follows that $U \times^h_F U$ is stable (by the induction hypothesis applied to the π_i sheaves of F for $i \geq 2$). Also, by applying condition (2) in Definition 6.3.3 for $i = 1$ we see that $rank(\alpha_t)$ is constant. Then the argument in the proof of (1) can be repeated to show that F is stable. □

Corollary 6.3.5. *Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let Z be constructible subset of $|Gr_n(X)|$ contained in $\tau_n(|Gr^{(e)}(X)|)$. Let $Z' := (\rho_n^{n+1})^{(-1)}(Z)$. Then $[Z'] = \mathbb{L}^d \cdot [Z]$ in the ring $K_0(St^{fp}(k))$.*

Proof. It suffices to show that if F is a very stable subset of $|Gr_n(X)|$ contained in the set $\tau_n(|Gr^{(e)}(X)|)$, and if F' is the reduced, locally closed substack of $Gr_{n+1}(X)$ such that $|F'| = (\rho_n^{n+1})^{(-1)}(|F|)$, then $[F'] = \mathbb{L}^d \cdot [F]$. Suppose that F is an n-truncated stack.

Consider the relative truncations

$$t_{\leq k}(F'/F) = t_{\leq k}(F') \times^h_{t_{\leq k}(F)} F.$$

Then $t_{\leq n}(F/F') = F'$. Thus, the morphism $F' \to F$ can be factored as

$$F' \to t_{\leq n-1}(F'/F) \to \ldots \to t_{\leq 0}(F'/F) \to F.$$

Using Corollary 3.5.9, each of the stacks in the above sequence is geometric and the i-th map is a $K(\mathbb{G}_a^{d_{m-i+1}}, d_{m-i+1})$-fibration. Thus the result follows from the remarks following Definition 3.10.3 □

Definition 6.3.6. *Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. A definable set of arcs Z on X is said to be* stable at level n *if it is weakly stable at level n and if $\tau_m(Z)$ is very stable for all $m \geq n$. We say that Z is* stable *if it is stable at level n for some n.*

Thus we are able to prove ([DL], Lemma 3.1) for Artin stacks.

Lemma 6.3.7. *Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let C be a definable subset of arcs on X. Then there exists a closed subvariety Y of X with $\dim(Y/\mathbb{D}) < \dim(X/\mathbb{D})$ and a countable, family of disjoint definable sets $\{C_i\}_{i \in \mathbb{Z}_{\geq 0}}$ such that:*

1. *For each i, C_i is stable at level n_i for some integer n_i.*
2. *$C \backslash Gr(Y) = \bigcup_{i \in \mathbb{Z}_{\geq 0}} C_i$.*
3. *$\lim_{i \to \infty}(\dim(\tau_{n_i}(C_i)) - (n_i + 1)d) = -\infty$.*

Proof. We choose $Y = X^C \cup X_{sing}$ where X^C is as defined in the proof of Lemma 5.3.5. Then clearly, we can obtain a decomposition of $C \backslash Gr(Y)$ of the form $\bigcup_{i=0}^{\infty} C_i$ where each C_i is stable at level m_i for some integer m_i. We need to prove that (3) holds.

Choose a sequence of integers e_i such that $e_1 \to \infty$ as $i \to \infty$. By Lemma 5.4.4, for *sufficiently large* i, there exist integers n_i such that

$$\dim(\tau_{n_i}(C_i)) \leq (n + 1)d - e_i - 1.$$

This completes the proof because of our choice of the sequence $\{e_i\}_i$. □

7.0 MOTIVIC MEASURE

Finally we are able to construct the motivic measure on the Boolean algebra of definable subsets of arcs on an Artin stack X over $\mathbb{D}$.

7.1 DEFINITION OF MOTIVIC MEASURE

The following result is a generalization of Proposition 3.2 of [DL]. The proof is *verbatim the same* since it is essentially a formal consequence of the lemmas that we have generalized. We reproduce the proof from [DL] for the sake of completeness.

Theorem 7.1.1. *Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let $\mathbf{B}$ be the set of all definable subsets of arcs on X. Then there exists a unique map $\mu : \mathbf{B} \to \widehat{\mathcal{M}}$ satisfying the following three properties:*

(1) If C is stable at level n, then $\mu(A) = [\tau_n(C)]\mathbb{L}^{-nd}$.

(2) If $C \in \mathbf{B}$ is contained in $Gr(S) \subset Gr(X)$ with Y a closed substack of X with $\dim(Y/\mathbb{D}) < \dim(X/\mathbb{D})$, then $\mu(C) = 0$.

(3) Let C_i be in $\mathbf{B}$ for each $i \in \mathbb{N}$. Suppose that the C_i are mutually disjoint and that $C := \bigcup_{i\in\mathbb{N}} C_i$. Then $\sum_{i\in\mathbb{N}} \mu(C_i)$ converges in $\widehat{\mathcal{M}}$ to $\mu(A)$.

We call μ the motivic measure *on X (and denote it by μ_X if we wish to make the reference to μ precise). μ also satisfies the following property:*

(4) If C and D are in $\mathbf{B}$, $C \subset D$, and if $\mu(D)$ belongs to the closure $F^m(\widehat{\mathcal{M}})$ of $F^m(\mathcal{M})$ in $\mathcal{M}$, then $\mu(C) \in F^m(\mathcal{M})$.

Proof. For a subalgebra $\mathbf{B}'$ of $\mathbf{B}$ and map $\mu' : \mathbf{B}' \to \widehat{\mathcal{M}}$, we say that the pair $(\mu', \mathbf{B}')$ satisfies (1) (resp. (2), (3) or (4)) if the condition (1) (resp. (2), (3), (4)) holds with μ replaced by μ' and $\mathbf{B}$ replaced by $\mathbf{B}'$.

Let $\mathbf{B}_0$ be the set of all $C \in \mathbf{B}$ which are stable. Then $\mathbf{B}_0$ is closed under finite unions and finite intersections. If $C \in \mathbf{B}_0$ is stable of level n, we define $\mu_0(C) = [\tau_n(C)]\mathbb{L}^{-nd}$. Then $(\mu_0, \mathbf{B}_0)$ satisfies (1) and (4). Also, μ_0 is additive. Suppose $C \in \mathbf{B_0}$ and C is the disjoint union $C = \cup_{i\in\mathbb{N}} C_i$ with $C_i \in \mathbf{B_0}$. Then by Lemma 5.2.4, $C = \sum_{i=1}^{m} C_i$ for some m. Thus, $(\mu_0, \mathbf{B}_0)$ satisfies (3).

Let $\mathbf{B}_1$ be the set of all $C \in \mathbf{B}$ which can be written as as a disjoint union $C = \bigcup_{i=1}^{\infty} C_i$ with $C_i \in \mathbf{B}_0$ and $\lim_{i\to\infty} \mu_0(C_i) = 0$. Then for any such C, we set $\mu_1(C) = \sum_{i=1}^{\infty} \mu_0(C_i)$. We need to verify that this does not depend on the choice of the C_i. Suppose $C = \bigcup_{i=1}^{\infty} C_i'$ is another expression of C as a disjoint union with $C_i' \in \mathbf{B}_0$ and $\lim_{i=0}^{\infty} \mu_0(C_i') = 0$. For each C_i, $C_i = \bigcup_{j=0}(C_i \cap C_j')$, and thus by (3), $\mu_0(C_i) = \sum_{j=0}^{\infty} \mu_0(C_i \cap C_j')$. Then

$$\sum_{i=0}^{\infty} \mu_0(C_i) = \sum_{i=0}^{\infty} \mu_0(\bigcup_{j\in\mathbb{N}}(C_i \cap C_j')) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mu_0(C_i \cap C_j').$$

For every i, if $\mu_0(C_i) \in F^m(\widehat{\mathcal{M}})$, by (4), $\mu_0(C_i \cap C_j') \in F^m(\widehat{\mathcal{M}})$. Thus, we can switch the order of summation and get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mu_0(C_i \cap C_j') = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \mu_0(C_j' \cap C_i) = \sum_{j=0}^{\infty} \mu_0(C_j').$$

Thus $\mu_1(C)$ does not depend on the sequence $\{C_i\}_{i=0}^{\infty}$.

It is easy to check that $(\mu_1, \mathbf{B}_1)$ satisfies (1) and (4).

Claim. *If Y is a closed substack of X with $\dim(Y/\mathbb{D}) < \dim(X/\mathbb{D})$ and if $C \in \mathbf{B}_1$, then $(C \backslash Gr(Y)) \in \mathbf{B}_1$ and $\mu_1(C \backslash Gr(Y)) = \mu_1(C)$.*

Suppose $C \in \mathbf{B}_0$. Suppose C is stable at level m_0. Pick some m larger than m_0. (**Note:** In the following equation, A^c denotes the complement of a set A in $Gr(X)$.)

$$\begin{aligned} Gr(Y)^c &= \left(\bigcap_{n\geq m} \tau_n^{-1}\tau_n(Gr(Y))\right)^c \\ &= \bigcup_{n\geq m} (\tau_n^{-1}\tau_n(Gr(Y)))^c \\ &= (\tau_m^{-1}\tau_m(Gr(Y)))^c \cup \bigcup_{n\geq m} \left((\tau_{n+1}^{-1}\tau_{n+1}(Gr(Y)))^c \backslash (\tau_n^{-1}\tau_n(Gr(Y)))^c\right) \\ &= (\tau_m^{-1}\tau_m(Gr(Y)))^c \cup \bigcup_{n\geq m} \left(\tau_n^{-1}\tau_n(Gr(Y)) \backslash \tau_{n+1}^{-1}\tau_{n+1}(Gr(Y))\right). \end{aligned}$$

Taking the intersection with C, we see that $C\backslash Gr(Y)$ can be expressed as the disjoint union

$$(C\backslash \tau_m^{-1}\tau_m(Gr(Y)) \cup \bigcup_{n\geq m} \left((\tau_n^{-1}\tau_n(Gr(Y))\backslash \tau_{n+1}^{-1}\tau_{n+1}(Gr(Y))) \cap C\right).$$

It is clear that all the sets in the above decomposition are weakly stable. A weakly stable subset of a stable set is stable. Thus as C is stable, all the sets in the decomposition are stable. Also,

$$\lim_{i\to 0} \mu_0 \left((\tau_n^{-1}\tau_n(Gr(Y))\backslash \tau_{n+1}^{-1}\tau_{n+1}(Gr(Y))) \cap C\right) = 0$$

by Lemma 5.4.4. Thus $C\backslash Gr(Y)$ is in $\mathbf{B}_1$. Also,

$$C = (C\backslash \tau_m^{-1}\tau_m(Gr(Y))) \cup (\tau_m^{-1}\tau_m(Gr(Y)) \cap C)$$

and $\lim_{m\to\infty} \mu_0(\tau_m^{-1}\tau_m(Gr(Y))) = 0$, again by Lemma 5.4.4. Thus $\mu_1(C) = \mu_1(C\backslash Gr(Y))$. This proves the above claim for $C \in \mathbf{B}_0$.

Now suppose that C is in $\mathbf{B}_1$. We write C as a disjoint union $C = \bigcup_{i=0}^{\infty} C_i$ such that $C_i \in \mathbf{B}_0$ and $\lim_{i\to\infty} \mu_0(C_i) = 0$. Then $C_i\backslash Gr(Y)$ is in $\mathbf{B}_1$ and $\lim_{i\in\infty} \mu_1(C_i\backslash Gr(Y)) = 0$. Let $C_i\backslash Gr(Y)$ be the disjoint union $C_i\backslash Gr(Y) = \cup_{j=0}^{\infty} D_{ij}$ such that $D_{ij} \in \mathbf{B}_0$ and $\lim_{j\to\infty} \mu_0(D_{ij}) = 0$. Then

$$C\backslash Gr(Y) = \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty} D_{ij}.$$

It is easy to see that there exists a reindexing $\{E_i\}_{i=0}^{\infty}$ of $\{D_{ij}\}_{i,j}$ such that $\lim_{i\to\infty} \mu_0(E_j) = 0$. This proves the claim.

By Lemma 6.3.7, for every $C \in \mathbf{B}$, there exists a closed substack Y of X with $\dim(Y/\mathbb{D}) < \dim(X/\mathbb{D})$ such that $C\backslash Gr(Y)$ is in $\mathbf{B}_1$. We define $\mu : \mathbf{B} \to \widehat{\mathcal{M}}$ as $\mu(C) = \mu_1(C\backslash Gr(Y))$. If Y' is another closed substack of X such that $C\backslash Gr(Y')$ is in $\mathbf{B}_1$, then by the above claim,

$$\mu_1(C\backslash Gr(Y)) = \mu_1(C\backslash Gr(Y \cup Y')) = \mu_1(C\backslash Gr(Y')).$$

Thus $\mu(C)$ is independent of the choice of $Gr(Y)$. Also, this argument shows that $(\mu, \mathbf{B})$ satisfies (2). $(\mu, \mathbf{B})$ satisifes (1) and (4) since $(\mu_1, \mathbf{B}_1)$ satisfies (1) and (4).

We check that $(\mu, \mathbf{B})$ satisfies property (3). First we observe the following:

Suppose C is a definable set. Applying Lemma 6.3.7, to the set C^c, for any integer $m \in \mathbb{N}$ there exists a stable set $C' \subset C^c$ such that there exists a closed substack Y of X such that there exists a disjoint collection of stable sets $\{C_i\}_{i=0}^{\infty}$ and integers $\{n_i\}_{i=0}^{\infty}$ such that:

1. $\bigcup_{i=0}^{\infty} C_i = C^c\backslash C'$.
2. C_i is stable at level $m_i < n_i$.
3. $\dim(\tau_{n_i}(C_i)) - (n_i + 1)d < -m - d$.
4. $\lim_{i\to\infty} \dim(\tau_{n_i}(C_i)) - n_i d = 0$.

Thus, $[C_i]\mathbb{L}^{-n_i d} \in F^m(\widehat{\mathcal{M}})$, $C^c\backslash C' \in \mathbf{B}_1$ and $\mu_1(C^c\backslash C') \in F^m(\widehat{\mathcal{M}})$. Let $D = (C')^c$.

Since μ_1, and hence μ, is finitely additive, we have

$$\begin{aligned} \mu(D) - \mu(C) &= \mu(D\backslash C) \\ &= \mu(C^c\backslash C'). \end{aligned}$$

Thus $\mu(D) - \mu(C)$ is in $F^m(\widehat{M})$. Since C' is weakly stable, so is D. (Thus every definable set can be externally approximated by a weakly stable set.)

Now suppose C is in $\mathbf{B}$ and $C = \cup_{i=0}^{\infty} C_i$ is a disjoint union with $C_i \in \mathbf{B}$. We wish to prove that the sum $\sum_{i=0}^{\infty} \mu(C_i)$ converges to $\mu(C)$. Let $m \in \mathbb{N}$ be some integer.

Applying the above argument to C and each C_i, we can find weakly stable sets D and $\{D_i\}_{i=0}^{\infty}$ such that $C \subset D$, $C_i \subset D_i$, $\mu(D) - \mu(C) \in F^m(\widehat{\mathcal{M}})$ and $\mu(D_i) - \mu(C_i) \in F^m(\widehat{\mathcal{M}})$.

It will suffice to prove the result after replacing C_i by $C_i \cup (D\backslash C)$. We also replace D_i by $D \cap D_i$. Thus we may assume that

$$C_i \cup (D\backslash C) \subset D_i \subset D$$

for each i. Then $D \subset \bigcup_{i=0}^{\infty}(C_i \cup (D \backslash C)) \subset \bigcup_{i=0}^{\infty} D_i$ and thus $D = \bigcup_{i=0}^{\infty} D_i$. By Lemma 5.2.4, there exists a finite subcollection $\{D_i\}_{i=0}^{e}$ such that $D = \bigcup_{i=0}^{e} D_i$.

Since

$$D = \left(\bigcup_{i=0}^{e} C_i\right) \cup \left(\bigcup_{i=0}^{e} (D_i \backslash C_i)\right),$$

we see that

$$\mu(C) \equiv \mu(D) \equiv \sum_{i=0}^{e} \mu(C_i) \bmod(F^m(\widehat{\mathcal{M}})).$$

Since m was arbitrary, this shows that $\sum_{i=0}^{\infty} \mu(C_i)$ converges to $\mu(C)$. This proves property (3) for $(\mu, \mathbf{B})$ and completes the proof of the proposition.

□

Let X be an affine scheme over $\mathbb{D}$. Choose a closed immersion $X \subset \mathbb{A}^m_{\mathbb{D}}$. Let C be a definable set of arcs on X. Then a function $\alpha : C \to \mathbb{Z}$ is said to be *definable* if its graph, as a subset of $Gr(\mathbb{A}^m_{\mathbb{D}}) \times \mathbb{Z}$ is given by a formula in the language of Denef-Pas. For an arbitrary scheme X, we define a function on the set of arcs of X to be definable if its restriction to each affine open subscheme is definable.

Let X be an Artin stack over $\mathbb{D}$ and C be a definable set of arcs on X. Let $U \to X$ be an atlas and let C' be the preimage of C in $|Gr(U)|$. Then we say that a function $\alpha : C \to \mathbb{Z}$ is *definable* if the induced function on C' is definable. It is easy to check that this definition is independent of the choice of the atlas.

Definition 7.1.2. *Let X be an Artin stack over $\mathbb{D}$. Let C be a definable set of arcs on X. Let $\alpha : C \to \mathbb{Z}$ be definable. Then we define*

$$\int_C \mathbb{L}^{-\alpha} d\mu := \sum_{n \in \mathbb{Z}} \mu(C \cap \alpha^{-1}(n)) \mathbb{L}^{-n}$$

whenever the right hand side converges in $\widehat{\mathcal{M}}$ in which case we say that $\mathbb{L}^{-\alpha}$ is integrable on C.

If a definable function α is bounded from below, Theorem 7.1.1, (4) implies that $\mathbb{L}^{-\alpha}$ is integrable.

7.2 CHANGE OF VARIABLES FORMULA

The motivic measure on schemes admits a "change of base formula" which allows us to compare the measure on two different schemes with the same relative dimension over $\mathbb{D}$. The advantage, of course, is that this allows us to handle some integrals by shifting the calculation to a more convenient scheme (for instance, we may replace a scheme by a smooth scheme by using a resolution of singularities). We will obtain a modest generalization of this formula for the motivic measure on Artin stacks. To be precise, the only morphisms we will treat are those that induce isomorphisms of the π_i sheaves for all $i \geq 1$.

In the proof of the "change of variables" formula for schemes, the key result is ([Lo], Lemma 9.2). In order to obtain our generalization of the "change of variables" formula, we need to slightly modify the argument of this lemma and hence we present it in full detail. The following lemma is gleaned from the proof of ([Lo], Lemma 9.2).

Lemma 7.2.1. *Let $h : Y \to X$ be a morphism of schemes over $\mathbb{D}$. Let $\dim(Y/\mathbb{D}) = p$ and $\dim(X/\mathbb{D}) = q$. Suppose that Y is smooth over $\mathbb{D}$. Let $y \in |Gr(Y)|$ be an arc on Y represented by a morphism $\mathrm{Spec}(K) \to Y$ for some algebraically closed extension K of k. Suppose $\ell(y,h) = e < \infty$. Let n be an integer $\geq \sup\{2e, \ell(Gr(h)(y), X)\}$.*

(1) Let $y_1 \in Gr(Y)(K)$ be such that $\tau_n(Gr(h)(y_1)) = \tau_n(Gr(h)(y))$. Then there exists an arc $y_2 \in Gr(Y)(K)$ such that $\tau_{n-e}(y) = \tau_{n-e}(y_2)$ and $Gr(h)(y_1) = Gr(h)(y_2)$.

(2) The set of all $\tau_n(y') \in (\rho^n_{n-e})^{-1}(\tau_{n-e}(y))$ such that $\tau_n(Gr(h)(y')) = \tau_n(Gr(h)(y))$ is a $(e + e(p-q))$-dimensional subspace of $(\rho^n_{n-e})^{-1}(\tau_{n-e}(y))$ (which is known to be an affine K-space of dimension (ep) by Lemma 6.2.2).

Remark. This is essentially part of the proof of ([Lo], Lemma 9.2) without the condition $p = q$. The proof requires a very small modification.

Proof. It will suffice to prove this result under the assumption $k = K$. We will assume also that X and Y are affine.

(**Notation:** Recall the notation of Section 4.1 – $R := k[[t]]$, $R_n := k[t]/(t^{n+11})$.)

In the following proof, if y, y_1, etc. denote the elements of $|Gr(Y)|$, we will use the symbols $\widetilde{y}$, $\widetilde{y}_1$ etc to denote morphisms $\mathbb{D} \to Y$ such that $\Theta(\widetilde{y}) = y$, $\Theta(\widetilde{y}_1) = y_1$, etc.

The sequence $Y \xrightarrow{h} X \to \mathbb{D}$ induces the exact sequence

$$h^*(\Omega_{X/\mathbb{D}}) \longrightarrow \Omega_{Y/\mathbb{D}} \longrightarrow \Omega_{Y/X} \longrightarrow 0.$$

The pullback by $\widetilde{y}$ gives a sequence of R-modules

$$(h\widetilde{y})^*(\Omega_{X/\mathbb{D}}) \longrightarrow \widetilde{y}^*(\Omega_{Y/\mathbb{D}}) \longrightarrow \widetilde{y}^*(\Omega_{Y/X}) \longrightarrow 0.$$

Since $\ell(y, Y)$, $\ell(Gr(h)(y), X)$ and $\ell(y, h)$ are all less than n, by the discussion in Section 6.2, the modules $(h\widetilde{y})^*(\Omega_{X/\mathbb{D}})$, $\widetilde{y}^*(\Omega_{Y/\mathbb{D}})$ and $\widetilde{y}^*(\Omega_{Y/X})$ are of rank q, p and $p - q$. Since $F_{p-q}((h\widetilde{y})^*(\Omega_{X/\mathbb{D}}))$ is equal to the 0-th Fitting module of the torsion of $(\widetilde{y}^*(\Omega_{Y/X})$, the torsion of $\widetilde{y}^*(\Omega_{Y/X})$ is of length e.

Applying the functor $Hom_R(-, R_n)$ we get a sequence of R-modules that are annihilated by the ideal (t^{n+1}). Hence we may see them as R_n modules.

Consider the morphism

$$D_y^{(n)} : Hom_R(\widetilde{y}^*(\Omega_{Y/\mathbb{D}}), R_n) \longrightarrow Hom_R((h\widetilde{y})^*(\Omega_{X/\mathbb{D}}), R_n).$$

The kernel of this map can be identified with $Hom_R(\widetilde{y}^*(\Omega_{Y/X}), R_n)$. The ideal t^e annihilates the R_n-torsion of this module. Since Y is smooth, $\widetilde{y}^*(\Omega_{Y/\mathbb{D}})$ is a free module, and so the map $Hom_R(\widetilde{y}^*(\Omega_{Y/X}, R_n)) \to Hom_R(\widetilde{y}^*(\Omega_{Y/\mathbb{D}}), R_n)$ maps the R_n-torsion of $Hom_R(\widetilde{y}^*(\Omega_{Y/X}), R_n))$ into the module

$$Hom_R(\widetilde{y}^*(\Omega_{Y/\mathbb{D}}), (t^{n+1-e})/(t^{n+1})) \subset Hom_R(\widetilde{y}^*(\Omega_{Y/\mathbb{D}}), R_n)).$$

Proof of (1):

We construct an arc $\widetilde{z}_{n+1} : \mathbb{D} \to Y$ such that if $z_{n+1} = \Theta(\widetilde{z}_{n+1})$, then $\tau_{n-e}(y) = \tau_{n-e}(z_{n+1})$ and $\tau_{n+1}(Gr(h)(y_1)) = \tau_{n+1}(Gr(h)(z_{n+1}))$. Assuming that this can be done, repeating this argument with y replaced by z_{n+1} and proceeding inductively, we can obtain a sequence $\{z_s\}_{s\geq n}$ such that $\tau_{s-e}(z_s) = \tau_{s-e}(z_{s+1})$ and $\tau_s(Gr(h)(z_s)) = \tau_s(Gr(h)(y_1))$ for each $s > n$. Then if $w_s : \mathbb{D}_s \to Y$ is the morphism

$$\mathbb{D}_s \longrightarrow \mathbb{D} \xrightarrow{z_s} Y,$$

we denote the induced map $\operatorname{colim}_{n\to\infty} \mathbb{D}_s = \mathbb{D} \to Y$ by y_2. It is clear that it has the required properties.

Thus, now it remains to construct z_{n+1}. The morphism

$$(h\widetilde{y})^* - (h\widetilde{y}_1)^* : \mathcal{O}_X(X) \to (t^{n+1})/(t^{n+2})$$

is a R-derivation. Thus it defines an element

$$v \in Hom_R((h\widetilde{y})^*(\Omega_{X/\mathbb{D}}), (t^{n+1})/(t^{n+2})).$$

Since $n \geq$ the length of the torsion of $(h\widetilde{y})^*(\Omega_{X/\mathbb{D}})$, any morphism

$$(h\widetilde{y})^*(\Omega_{X/\mathbb{D}}) \to (t^{n+1})/(t^{n+2})$$

annihilates the torsion of $(h\widetilde{y})^*(\Omega_{X/\mathbb{D}})$.

We claim that there exists an R-homomorphism $u : \widetilde{y}^*(\Omega_{Y/\mathbb{D}}) \longrightarrow R_{n+1}$ such that the diagram

$$\begin{array}{ccc} (h\widetilde{y})^*(\Omega_{X/\mathbb{D}}) & \longrightarrow & \widetilde{y}^*(\Omega_{Y/\mathbb{D}}) \\ & \searrow^{v} & \downarrow u \\ & & R_{n+1}. \end{array}$$

commutes. Indeed, in order to prove this, it suffices to construct a homomorphism from the module

$$\text{cokernel}((h\widetilde{y})^*(\Omega_{X/\mathbb{D}}) \to \widetilde{y}^*(\Omega_{X/\mathbb{D}})) = (\widetilde{y})^*(\Omega_{Y/X})$$

into $R_{n+1}/(t^{n+1}) = R_n$. This can be done since $n \geq e$. It is also clear that the image of u must lie in the ideal $(t^{n+1-e})/(t^{n+2})$ of R_{n+1}. We interpret u as a derivation

$$\mathcal{O}_Y(Y) \longrightarrow (t^{n+1-e})/(t^{n+2}).$$

Since Y is smooth, by the discussion in Section 6.2, any such derivation is of the form $(\widetilde{y}' - \widetilde{y}^*)$ for some $\widetilde{y}' : \mathbb{D} \to Y$ such that $\tau_{n-e}(\Theta(\widetilde{y}')) = \tau_{n-e}(y)$. By construction,

$$(h\widetilde{y}')^* - (h\widetilde{y})^* \equiv (h\widetilde{y}_1)^* - (h\widetilde{y})^* \pmod{(t^{n+2})}$$

and thus $\tau_{n+1}(Gr(h)(\Theta(\widetilde{y}'))) = \tau_{n+1}(Gr(h)(y_1))$. Thus we may set $z_{n+1} = \Theta(\widetilde{y}')$. This completes the proof of (1).

Proof of (2):

Let $y' = \Theta(\widetilde{y}') : \mathbb{D}_0 \to Gr(Y)$ be such that $\tau_{n-e}(y) = \tau_{n-e}(y')$. Then the difference $\widetilde{y}^* - (\widetilde{y}')^*$ defines a derivation $\mathcal{O}_Y(Y) \to (t^{n-e+1})/(t^{n+1})$ which is trivial if and only if $\tau_n(y) = \tau_n(y')$. Similarly, $(h\widetilde{y})^* - (h\widetilde{y}')^*$ is a derivation $\mathcal{O}_X(X) \to (t^{n-e+1})/(t^{n+1})$ which is trivial if and only if $\tau_n(Gr(h)(y)) = \tau_n(Gr(h)(y')$. Thus the set of all $\tau_n(y') \in (\rho^n_{n-e})^{-1}(\tau_{n-e}(y))$ is an affine space having translation space equal to the intersection A_y of the the kernel of the map $D_y^{(n)}$ with the module

$$Hom_R(\widetilde{y}^*(\Omega_{Y/\mathbb{D}}), (t^{n-e+1})/(t^{n+1})) \subset Hom_R(\widetilde{y}^*(\Omega_{Y/\mathbb{D}}), R_n).$$

As we observed above, the torsion of the kernel of $D_y^{(n)}$ is already contained in the module $Hom_R(\widetilde{y}^*(\Omega_{Y/\mathbb{D}}), (t^{n-e+1})/(t^{n+1}))$ and is of length e. Since the rank of the kernel is $(p-q)$, we see that the length of A_y is $e + e(p-q)$ as desired. □

Remark. Alternatively, this lemma could be obtained as a consequence of [Lo], Lemma 9.2. Indeed, if we first prove this result for the case $p = q$ and for the case in which h is a smooth map (in which case it is a consequence of Hensel's lemma), we can prove the result in the case $p \neq q$ by constructing a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{r} & X' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow{h} & X \end{array}$$

such that Y' and X' are smooth, the morphisms r and u are smooth and $\dim(X'/\mathbb{D}) = \dim(X/\mathbb{D})$. However, this involves roughly the same amount of effort.

Definition 7.2.2. *A morphism $f : X \to Y$ of Artin stacks is said to be 0-truncated if it induces an isomorphism of π_i sheaves for all $i > 0$.*

Clearly, a morphism $f : X \to Y$ of Artin stacks is 0-truncated if and only if for any scheme Z and any morphism $Z \to Y$, the stack $X \times^h_Y Z$ is 0-truncated. In other words, $X \times^h_Y Z$ is a sheaf that is k-geometric for some k.

Lemma 7.2.3. *Lemma 7.2.1 continues to hold when $h : Y \to X$ is a 0-truncated morphism of Artin stacks.*

Proof. We will use the notation in the statement of Lemma 7.2.1. Also, we will assume that $K = k$. Choose atlases $U \to X$ and $V \to Y \times_X U$ such that U and V are affine schemes. Thus we have the diagram

$$\begin{array}{ccc} V & \xrightarrow{g} & Y \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ U & \xrightarrow[f]{} & X \end{array}$$

in which the horizontal morphisms are atlases.

Suppose $\dim(U/\mathbb{D}) = s$ and $\dim(V/\mathbb{D}) = r$. Let $v : \mathrm{Spec}(k) \to Gr(V)$ be a lift of y.

Proof of Lemma 7.2.1, (1) for Artin stacks:

Let $v_1 : \mathrm{Spec}(k) \to Gr(V)$ be a lift of y_1 such that $Gr(h')(v_1) = Gr(h')(v)$. (Such a lift exists since $V \to Y \times^h_X U$ is surjective.) Then by Lemma 7.2.1, there exists a $v_2 : \mathrm{Spec}(k) \to Gr(V)$ such that $\tau_{n-e}(v) = \tau_{n-e}(v_2)$ and $Gr(h')(v_1) = Gr(h')(v_2)$. Then choose $y_2 = Gr(g)(v_2)$.

Proof of Lemma 7.2.1, (2) for Artin stacks:

Let $Gr(h)(y) = x$ and $Gr(h')(v) = u$. Consider the commutative diagram

$$\begin{array}{ccc} (\rho^n_{n-e})^{-1}(\tau_{n-e}(v)) & \longrightarrow & (\rho^n_{n-e})^{-1}(\tau_{n-e}(y)) \\ \downarrow & & \downarrow \\ (\rho^n_{n-e})^{-1}(\tau_{n-e}(u)) & \longrightarrow & (\rho^n_{n-e})^{-1}(\tau_{n-e}(x)). \end{array}$$

All the sets in this diagram are affine spaces and all the maps are affine maps. The left vertical map has relative dimension $e + e(r - s)$ due to Lemma 7.2.1. By Proposition 6.3.4, the horizontal maps are surjective. The top horizontal map has relative dimension $e(r-p)+\alpha$ where α depends on $\{\pi_i(Gr(Y), v)\}_i$ and the lower horizontal map has relative dimension $e(s - q) + \beta$ where β depends on $\{\pi_i(Gr(X), u)\}_i$. Since h is 0-truncated, $\alpha = \beta$. Thus the right vertical map has relative dimension equal to

$$e + e(r - s) - e(r - p) + e(s - q) = e + e(p - q)$$

as desired. □

Lemma 7.2.4. *([Lo], Lemma 9.2 or [DL], Lemma 3.4)*

Let $h : Y \to X$ be a 0-truncated morphism of Artin stacks over $\mathbb{D}$ with pure relative dimension d. Suppose that Y is smooth over $\mathbb{D}$. Let $C \subset |Gr(Y)|$ be a definable stable set of level l and assume that $Gr(h)|_C$ is injective and $l(-,h)|_C$ is constant equal to $e < \infty$. If $n \geq \sup\{2l, l+e, l(Gr(h)(y), X)\}$, then $\tau_n(C) \to \tau_n(Gr(h)(C))$ has the structure of an affine linear bundle of dimension e.

Remark. By saying that $\tau_n(C) \to \tau_n(Gr(h)(C))$ has the structure of an affine linear bundle, we mean that there exists *some* decomposition of $\tau_n(Gr(h)(C))$ into locally closed sets on which this map defines an affine linear bundle.

Proof. This is an immediate consequence of Lemma 7.2.3. Indeed, if if y_1 and y_2 are arcs in C such that $Gr(h)(y_1) = Gr(h)(y_2)$, it follows that $\tau_{n-e}(y_1) = \tau_{n-e}(y_2)$. On the other hand, since $n - e > l$, $(\rho^n_{n-e})^{-1}(y)$ is contained in C. Thus, $Gr(h)^{-1}(Gr(h)(y))$ is isomorphic to $\mathbb{A}^e$. □

After this point, the derivation of the "change of variables" formula is by the same argument as in the case of schemes. Hence we will be brief with the proof of the following theorem (see [DL], Lemma 3.3.)

Theorem 7.2.5. *Let $h : Y \to X$ be a 0-truncated morphism of Artin stacks of pure dimension d over $\mathbb{D}$. Assume that Y is smooth over $\mathbb{D}$. Let C be a set of arcs on Y and assume that $Gr(h)|_C$ is injective. Let $\alpha : Gr(h)(C) \to \mathbb{Z}$ be a definable function. Then*

$$\int_{Gr(h)(C)} \mathbb{L}^{-\alpha} d\mu_X = \int_C \mathbb{L}^{-\alpha \circ Gr(h) - l(-,h)} d\mu_Y.$$

Proof. Since a definable set can be approximated by stable sets (see Lemma 6.3.7), it suffices to prove this formula on stable sets. In that case, it follows easily from Lemma 7.2.4. □

8.0 FURTHER QUESTIONS

Since this work runs parallel to [DL] which is one of the earliest pieces of literature in this area, it is only to be expected that there should be considerable room for further improvement in this theory. We comment briefly on some of these issues:

1. We have assumed that the base field is of characterstic zero. While this restriction is in place in some of the earlier literature, it is probably not essential. The work of Sebag treats the case of complete discrete valuation rings with *perfect* residue fields. Our proof of Lemma 4.3.2 relies on a result in [Si1] which assumes that the field is of characterstic zero. However, it seems quite plausible that the result should be true for positive characterstic and the flaw lies in our proof.
2. The work of Cluckers and Loeser (see [CL]), which we described in the Introduction, sets up a very robust machinery for motivic integration on varieties. It is only natural to ask if we can perform an analogous construction on Artin stacks. The construction in [DL] involves a very careful study of definable sets using Pas' theorem. A principle obstacle seems to be that the proofs are not "coordinate-free" and so it might prove a little difficult to generalize those techniques to stacks. However, I am optimistic about the prospects regarding this problem.
3. It is proved in [To1] that if we restrict ourselves to the category of stacks such that the group sheaves π_i are affine for $i > 0$ and unipotent for $i > 1$, then the Grothendieck ring of stacks is actually the same as the Grothendieck ring of varieties. (Also see, [Jo] for similar results, formulated for 1-stacks). This is, of course, extremely useful since the Grothendieck ring of varieties maps into well-known rings, such as the Grothendieck ring of Hodge structures, $\mathbb{Z}$, etc. and these ring homomorphisms provide interpretations

of motivic measure in terms of Hodge structures, Euler characterstic, etc. Thus, if we restrict ourselves to this special category of stacks, we are able to extend the well-known invariants of varieties to stacks. Having a corresponding theory of motivic integration should contribute towards studying these invariants.

4. We have obtained the "change of variables" formula only for 0-truncated morphisms. It seems unlikely that we can get a good result without *any* restriction on the behavior of the morphism with respect to the homotopy group sheaves. However, the problem of obtaining a good set of restrictions for this purpose might be of some interest.

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