WEAK SOLUTIONS AND INCOMPRESSIBLE LIMITS OF MULTI-DIMENSIONAL MAGNETOHYDRODYNAMIC FLOWS

by

Xianpeng Hu

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This dissertation was presented

by

Xianpeng Hu

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and approved by

Professor Dehua Wang, Department of Mathematics.

Professor Giovanni P. Galdi, Department of Mechanical Engineering and Materials Science.

Professor Stuart P. Hastings, Department of Mathematics.

Professor William J. Layton, Department of Mathematics.

Professor Huiqiang Jiang, Department of Mathematics.

Dissertation Director: Professor Dehua Wang, Department of Mathematics.

WEAK SOLUTIONS AND INCOMPRESSIBLE LIMITS OF MULTI-DIMENSIONAL MAGNETOHYDRODYNAMIC FLOWS

Xianpeng Hu, PhD

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This dissertation addresses mathematical issues regarding the existence of global weak solutions of isentropic compressible magnetohydrodynamic flows (MHD), the limit behavior of isentropic compressible MHD as Mach number vanishes, and the hydrodynamic limit of Vlasov-Maxwell-Boltzemann equations. More precisely, in the first part, global existence of weak solutions with large initial data to the Cauchy problem of the three-dimensional compressible MHD is established through an invading method for the adiabatic exponent $\gamma > \frac{3}{2}$ and constant viscosity coefficients. In the second part, we focus on the connection between the incompressible MHD and the compressible isentropic MHD; it is showed that as Mach number vanishes, the compressible isentropic MHD will converge to the incompressible MHD. In the third part, using relative entropy estimate about an absolute Maxwellian, we establish an incompressible Electron-Magnetohydrodynamics-Fourier limit for solutions of the Vlasov-Maxwell-Blotzmann equation considered over any periodic spatial domain in $\mathbb{R}^3$. It is shown that any properly scaled sequence of renormalized solutions of Vlasov-Maxwell-Boltzmann equations has fluctuations that (in the weak L^2 topology) converge to an infinitesimal Maxwellian with fluid variables that satisfy the incompressibility and Boussinesq relations. It is shown that every limit point and the magnetic field are governed by a weak solution of an incompressible electron-magnetohydrodynamics system for all time.

Keywords: Cauchy problem, Global weak solutions, Renormalized solutions, Incompressible limit, Hydrodynamic limit, Magnetohydrodynamics, Vlasov-Maxwell-Boltzmann.

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PREFACE

The aim of this dissertation is to present some mathematical results of both the compressible magnetohydrodynamics (MHD) and the incompressible MHD. The history of attempts to describe rigorously the flow of a compressible fluid covered a long time beginning from the observations of L. Euler in the middle of the 18th century and of C. Navier, H. Poisson and G. Stokes in the first half of the 19th century, and continues up to now. Despite the fact that the governing equations have been known for a very long time, we are far from being satisfied with the completeness of their mathematical analysis. Nevertheless, the considerable effort of outstanding analysis cited throughout this dissertation brought its fruits, and a great number of nontrivial results for compressible fluids has been achieved.

For that purpose, our starting point in this dissertation is to consider the fundamental question about the global existence of weak solution to the Cauchy problem of isentropic compressible MHD with large initial data, and we present it in the second chapter. The sophisticated techniques are used in this chapter; and from the vast of the literature, we can see the vitality of the study of compressible flows.

The third chapter is devoted to the mathematical study of zero Mach number limit; it establishes a connection between the isentropic compressible MHD and the incompressible MHD. Recall that the general informal derivation from compressible flows to incompressible flows assumes that the density of the flow is a constant. The aim of this chapter is to present a rigorous mathematical description of that transition in terms of the Mach number.

The fourth chapter is focused on the hydrodynamic limit of Vlasov-Maxwell-Boltzmann equations (VMB). We start by recalling some results about the limiting system and VMB, in particular their weak stability result. We then explain the strategy used to establish the convergence result of the renormalized solutions to the suitably scaled Boltzmann equation, as well as the main difficulties to be overcomed.

I owe my gratitude to all the people who have made this dissertation possible and because of whom my graduate experience has been one that I will cherish forever. First and foremost I'd like to thank my advisor, Professor Dehua Wang for giving me an invaluable opportunity to work on challenging and extremely interesting problems over the past four years. Professor Wang introduced me into this fantastic research model. He always encouraged me and supported me. He has always made himself available for help and advice. Without his extraordinary insight and expertise, this dissertation would have been a distant dream. It has been a pleasure to work with and learn from such an extraordinary individual. Special thanks are also due to Professors Giovanni P. Galdi, Stuart P. Hastings, William J. Layton, and Huiqiang Jiang for agreeing to serve on my committee and for taking their invaluable time to review the dissertation. In addition, Professor Galdi provided me many suggestions on my research project. I also want to thank my uncle who always encourages me to do good mathematics.

1.0 INTRODUCTION

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids in an electromagnetic field with a very broad range of applications. The dynamic motion of the fluid and the magnetic field interact strongly on each other. The hydrodynamic and electrodynamic effects are coupled. The equations of three-dimensional compressible magnetohydrodynamic flows in the isentropic case have the following form ([6, 35, 37, 39, 46, 48, 61]):

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\mathrm{div}\mathbf{u}), \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \quad \mathrm{div}\mathbf{H} = 0, \end{cases} \tag{1.0.1}$$

where ρ denotes the density, $\mathbf{u} \in \mathbb{R}^3$ the velocity, $\mathbf{H} \in \mathbb{R}^3$ the magnetic field, $p(\rho) = a\rho^\gamma$ the pressure with constant $a > 0$ and the adiabatic exponent $\gamma > 1$; the viscosity coefficients of the flow satisfy $2\mu + 3\lambda > 0$ and $\mu > 0$; $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and all these kinetic coefficients and the magnetic diffusivity are independent of the magnitude and direction of the magnetic field. The symbol $\otimes$ denotes the Kronecker tensor product. Usually, we refer to the first equation in (1.0.1) as the continuity equation, and the second equation as the momentum balance equation. It is well-known that the electromagnetic fields are governed by the Maxwell's equations. In magnetohydrodynamics, the displacement current can be neglected ([46, 48]). As a consequence, the last equation in (1.0.1) is called the induction equation, and the electric field can be written in terms of the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$,

$$\mathbf{E} = \nu \nabla \times \mathbf{H} - \mathbf{u} \times \mathbf{H}.$$

Although the electric field $\mathbf{E}$ does not appear in (1.0.1), it is indeed induced according to the above relation by the moving conductive flow in the magnetic field.

1.0.1 Existence of weak solutions

In this dissertation, we are interested in the global existence of solutions to the Cauchy problem of the three-dimensional isentropic compressible MHD (1.0.1) with the behavior at infinity

$$\rho \to \rho_\infty, \quad \mathbf{u} \to 0, \quad \mathbf{H} \to 0, \quad \text{as} \quad |x| \to \infty,$$

and the initial conditions

$$\begin{cases} \rho(x,0) = \rho_0(x), \quad \text{such that } (\rho_0^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho_0 - \rho_\infty)) \in L^1(\mathbb{R}^3), \quad \text{and} \quad \rho_0(x) \geq 0, \\ \rho(x,0)\mathbf{u}(x,0) = \mathbf{m}_0(x) \in L^1(\mathbb{R}^3), \ \mathbf{m}_0 = 0 \text{ if } \rho_0 = 0, \ \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\mathbb{R}^3), \\ \mathbf{H}(x,0) = \mathbf{H}_0(x) \in L^2(\mathbb{R}^3), \quad \mathrm{div}\mathbf{H}_0 = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \end{cases} \tag{1.0.2}$$

where ρ_∞ is a positive constant. There have been a lot of studies on MHD by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see [8, 9, 15, 18, 22, 34, 48, 67] and the references cited therein. In particular, the one-dimensional problem has been studied in many papers, for examples, [8, 9, 18, 34, 44, 54, 67] and so on. However, many fundamental problems for MHD are still open. For example, even for the one-dimensional case, the global existence of classical solution to the full perfect MHD equations with large data remains unsolved when all the viscosity, heat conductivity, and diffusivity coefficients are constant, although the corresponding problem for the Navier-Stokes equations was solved in [43] long time ago. The reason is that the presence of the magnetic field and its interaction with the hydrodynamic motion in the MHD flow of large oscillation cause serious difficulties. In this paper we consider the global weak solution to the three-dimensional MHD problem with large data, and investigate the fundamental problems such as global existence. A multi-dimensional nonisentropic MHD system for gaseous stars coupled with the Poisson equation is studied in [15], where all the viscosity coefficients depend on temperature, and the pressure depends on density asymptotically like the isentropic case $p(\rho) = a\rho^{\frac{5}{3}}$. In this dissertation, we study the multi-dimensional isentropic problem (1.0.1)-(1.0.2) with $\gamma > \frac{3}{2}$, where all the viscosity coefficients μ, λ, ν are constant. We remark that $\gamma = \frac{5}{3}$ for the monoatomic gases.

When there is no electromagnetic field, system (1.0.1) reduces to the compressible Navier-Stokes equations. See [19, 32, 51] and their references for the studies on the multi-dimensional Navier-Stokes equations. In particular, to overcome the difficulties of large oscillations of solutions, especially of density, the concept of a renormalized solutions is used in [19, 51]. Based on this idea, we study the Cauchy problem (1.0.1)-(1.0.2) for the MHD system in $\mathbb{R}^3$. The goal of this section is to establish the existence of global weak solutions for large initial data in certain functional spaces for $\gamma > \frac{3}{2}$ when the magnetic field and interaction present. The existence of global weak solutions is proved by using an existence result for bounded domains in [36] and an invading domain method. We first obtain *a priori* estimates directly from (1.0.1), which is the backbone of our result. In the proof of the existence, we use the invading domain method to find a sequence of approximate solutions as in [20, 36]. Then, motivated by the work in [17], we show that an improvement on the integrability of density can ensure the effectiveness and convergence of our approximation scheme. More specifically, we show that the uniform bound of $\rho^\gamma \ln(1+\rho)$ in L^1, rather than the uniform bound of $\rho^{\gamma+\theta}$ in L^1 for some $\theta > 0$ as used in [19, 20, 51], ensures the strong convergence of the density. To overcome the difficulty arising from the possible large oscillations of the density ρ, we adopt the method in Lions [51] and Feireisl [19] which is based on the celebrated weak continuity of the effective viscous flux $p - (\lambda + 2\mu)\text{div}\mathbf{u}$ (see also Hoff [31]). To achieve our goal for the MHD problem, we also need to develop estimates to deal with the magnetic field and its coupling and interaction with the fluid variables. The nonlinear term $(\nabla \times \mathbf{H}) \times \mathbf{H}$ will be dealt with by the idea arising in incompressible Navier-Stokes equations.

1.0.2 Low Mach number limit

From the physical point of view, the compressible flow behaves asymptotically like an incompressible flow when the density is almost constant, and the velocity and the magnetic field are small, in a large time scale. More precisely, we scale $\widetilde{\rho}$, $\widetilde{\mathbf{u}}$, and $\widetilde{\mathbf{H}}$ in the following way:

$$\widetilde{\rho} = \rho(x, \varepsilon t), \quad \widetilde{\mathbf{u}} = \varepsilon \mathbf{u}(x, \varepsilon t), \quad \widetilde{\mathbf{H}} = \varepsilon \mathbf{H}(x, \varepsilon t), \tag{1.0.3}$$

and we assume that the coefficients $\widetilde{\mu}$, $\widetilde{\lambda}$, and $\widetilde{\nu}$ are small and scaled as:

$$\widetilde{\mu} = \varepsilon \mu_\varepsilon, \quad \widetilde{\lambda} = \varepsilon \lambda_\varepsilon, \quad \widetilde{\nu} = \varepsilon \nu_\varepsilon, \tag{1.0.4}$$

where $\varepsilon \in (0,1)$ is a small parameter and the normalized coefficients μ_ε, λ_ε and ν_ε satisfy

$$\mu_\varepsilon \to \mu, \quad \lambda_\varepsilon \to \lambda, \quad \nu_\varepsilon \to \nu, \text{ as } \varepsilon \to 0+, \tag{1.0.5}$$

with $\mu > 0$, $2\mu + 3\lambda > 0$, and $\nu > 0$. Such a scaling as (1.0.4) ensures that the limit equation as $\varepsilon \to 0$ is not an Euler type system. Also notice that the parameter ε in the front of the magnetic field $\mathbf{H}$ in (1.0.3) can be understood as the reciprocal of Alfv́en number([58]). Under those scalings, system (1.0.1) yields

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu_\varepsilon \Delta \mathbf{u} - \lambda_\varepsilon \nabla \operatorname{div} \mathbf{u} + \frac{a}{\varepsilon^2} \nabla \rho^\gamma = (\nabla \times \mathbf{H}) \times \mathbf{H}, \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu_\varepsilon \nabla \times \mathbf{H}), \quad \operatorname{div} \mathbf{H} = 0. \end{cases} \tag{1.0.6}$$

As $\varepsilon \to 0$, the first equation in (1.0.6) yields the limit: $\operatorname{div}\mathbf{u} = 0$, which is the incompressible condition of a fluid, and the first two terms in the second equation of (1.0.6) become

$$\mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) = \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

On the other hand, the incompressible MHD equations read

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H}, \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \\ \operatorname{div} \mathbf{u} = 0, \quad \operatorname{div} \mathbf{H} = 0. \end{cases} \tag{1.0.7}$$

Thus, roughly speaking, it is also reasonable to expect from the mathematical point of view that weak solutions of (1.0.6) converge in certain suitable functional spaces to the weak solutions of (1.0.7) as ρ goes to a constant such as 1 and ε goes to 0, and the hydrostatic pressure p in (1.0.7) contains the "limit" of $(\rho^\gamma - 1)/\varepsilon^2$ in (1.0.6).

As for the low Mach number limit of compressible Navier-Stokes equations, there are a lot of publications in the literature which are considered during the last thirty years, see [1, 5, 10, 12, 21, 33, 45, 64, 65, 68].

1.0.3 Hydrodynamic limit

The third goal in this direction is to establish a theorem of hydrodynamic limits that should need only *a priori* estimates coming from physics, i.e. from the conservation laws of mass, momentum, energy. In spite of significant difficulties relating to our limited understanding of renormalized solutions, the progress in the derivation of macroscopic systems from the microscopic system is striking.

Hydrodynamic models such as the Euler or Navier-Stokes equations were first established by applying Newton's second law of motion to infinitesimal volume elements of the fluid under consideration, while kinetic equations are mathematical models used to describe the dilute particle gases at an intermediate scale between microscopic and macroscopic level. They appear in a variety of sciences such as plasma, astrophysics, aerospace engineering, nuclear engineering, particle−fluid interactions, semiconductor technology, social sciences, and biologies. Perhaps the most fundamental model for dynamics of dilute charged particles is described by the Vlasov-Maxwell-Boltzmann equations (VMB, for short).

The state of a fluid composed of identical point particles confined to a spatial domain $\Omega \subset \mathbb{R}^3$ is described at the kinetic level by a mass density F over the single-particle phase space $\mathbb{R}^3 \times \Omega$. More specifically, $F(x, \xi, t)d\xi dx$ gives the mass of the particles that occupy any infinitesimal volume $d\xi dx$ centered at the point $(\xi, x) \in \mathbb{R}^3 \times \Omega$ at the instant of time $t \geq 0$. To remove complications due to boundaries, we take Ω to be the periodic domain $\mathcal{T}^3 = \mathbb{R}^3/\text{Ł}^3$, where $\text{Ł}^3 \subset \mathbb{R}^3$ is any 3-dimensional lattice. If the particles interact only through a repulsive conservative interparticle force with finite range, then at low enough densities this range will be much smaller than the interparticle spacing. In that regime, all but binary collisions can be neglected in three dimensional space, and the evolution of $F = F(x, \xi, t)$ is governed by the classical Vlasov-Maxwell-Boltzmann equaitons [24, 30, 40]:

$$\frac{\partial F}{\partial t} + \xi \cdot \nabla_x F + e(E + \xi \times B) \cdot \nabla_\xi F = \mathcal{Q}(F,F), \quad x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3, \quad t \geq 0, \tag{1.0.8a}$$

$$\frac{1}{c^2}\frac{\partial E}{\partial t} - \nabla \times B = -\mu_0 j, \quad \mathrm{div} B = 0, \quad \text{on} \quad \mathbb{R}^3_x \times (0,\infty), \tag{1.0.8b}$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0, \quad \mathrm{div} E = \frac{\rho}{\eta_0}, \quad \text{on} \quad \mathbb{R}^3_x \times (0,\infty), \tag{1.0.8c}$$

$$\rho = e\int_{\mathbb{R}^3} F d\xi, \quad j_k = e\int_{\mathbb{R}^3} F\xi_k d\xi, \quad \text{for} \quad 1 \leq k \leq 3, \quad \text{on} \quad \mathbb{R}^3_x \times (0,\infty), \tag{1.0.8d}$$

where $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$ and the collison operator $\mathcal{Q}(F,F)$, which acts only on the velocity dependence of f (this reflects the physical assumption that collisions are localized in space and time), is defined as

$$\mathcal{Q}(F,F) = \int_{\mathbb{R}^3} d\xi^* \int_{S^2} d\omega b(\xi - \xi_*, \omega)(F' F'_* - F F_*).$$

Here the function $b(\xi,\omega)$ is nonnegative and $F_* = F(t,x,\xi_*)$, $F' = F(t,x,\xi')$, $F'_* = F(t,x,\xi'_*)$, with

$$\xi' = \xi - (\xi - \xi_*, \omega)\omega,$$

$$\xi'_* = \xi_* + (\xi - \xi_*, \omega)\omega,$$

for all $\omega \in S^2$, the unit sphere in $\mathbb{R}^3$, which yield one convenient parametrization of the set of solutions to the law of elastic collisions

$$\begin{cases} \xi' + \xi'_* = \xi + \xi_*, \\ |\xi'|^2 + |\xi'_*|^2 = |\xi|^2 + |\xi_*|^2. \end{cases} \tag{1.0.9}$$

The interpretation of ξ, ξ_*, ξ', ξ'_* is the following: ξ, ξ_* are the velocities of two colliding molecules immediately before collision while ξ', ξ'_* are the velocities immediately after the collision.

The coefficients μ_0 and η_0 are the magnetic permeability and the electric permittivity of the plasma in the vacuum, see [4, 28]. The constant c is the speed of light and satisfies

$$\mu_0 \eta_0 c^2 = 1.$$

The magnetic field $\mathbf{H}$ and the electrical displacement D have been eliminated from Maxwell's equations by the use of

$$\mu_0 \mathbf{H} = B, \qquad D = \eta_0 E.$$

Since $\mathbf{H}$ will not be used in what follows, we shall refer to B as the magnetic field. The constant e is the charge of the electron.

The nonnegative function $F(t, x, \xi)$ is the density of particles which at time t and position x move with velocity ξ under Lorentz force $f = E + \xi \times B$ where E is the electric field and B is the magnetic field. The function j is called the current density, while the function ρ is the charge density. Those unknown functions F, E, and B are strongly coupled, and the constraint on the divergence of E will be ensured provided that the conservation of charge holds, that is,

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_x j = 0,$$

since

$$\begin{aligned} \frac{\partial}{\partial t}(\mathrm{div}_x E - \rho) &= \mathrm{div}_x E_t - \rho_t \\ &= \mathrm{div}_x(\nabla_x \times B - j) - \rho_t \\ &= -\rho_t - \mathrm{div}_x j = 0, \end{aligned}$$

due to the fact $\mathrm{div}(\nabla \times v) = 0$ for any vector-valued function v. Similarly, the field B remains divergence free if it is so initially.

On the other hand, the incompressible Electron-Magnetohydrodynamics-Fourier equations describe the evolution of the velocity field $\mathbf{u} = \mathbf{u}(t, x)$ of an idealized fluid over a given spatial domain in $\mathbb{R}^3$ under the magnetic field $\mathbf{B} = \mathbf{B}(t, x)$, and it takes the form

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p - \alpha e E = -B \times (\nabla \times B), \tag{1.0.10a}$$

$$j = \nabla \times B \quad \text{and} \quad j = e\mathbf{u}; \tag{1.0.10b}$$

$$\partial_t B + \nabla \times E = 0, \tag{1.0.10c}$$

$$\nabla_x (h + \theta) = 0, \tag{1.0.10d}$$

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta, \tag{1.0.10e}$$

$$\mathrm{div}\mathbf{u} = 0, \qquad \text{and} \qquad \mathrm{div} B = 0, \tag{1.0.10f}$$

where $\alpha = \frac{1}{3(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |\xi|^2 \exp(-\frac{|\xi|^2}{2}) d\xi$.

Consider the Cauchy problem for (1.0.10) with initial data

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \theta(0,x) = \theta_0(x), \quad B(0,x) = B_0(x), \quad x \in \mathbb{R}^3, \tag{1.0.11}$$

where $\mathbf{u}_0, B_0 \in \{f \in L^2(\mathbb{R}^3) : \mathrm{div} f = 0 \quad \text{in} \quad \mathcal{D}'\}$ and $\theta_0 \in L^2(\mathbb{R}^3)$. A *weak solution* of the Cauchy problem (1.0.10)-(1.0.11) is a triple $(\mathbf{u}, \theta, B)$ where $(\mathbf{u}, B)$ is a weak solution of the incompressible electron-magnetohydrodynamic equation and θ is a solution in the sense of distributions of the Cauchy problem (1.0.10e) with initial data in (1.0.11).

The motivation of this work is to extend the program laid out in [2, 3] for the Boltzmann equation to the Vlasov-Maxwell-Boltzmann equation. One of the main objectives of that program is to connect the DiPerna-Lions theory of global renormalized solutions of Boltzmann equations to the Leray theory of global weak solutions of the incompressible fluid equations. The main result of [3] for the Navier-Stokes limit is to recover the motion equation with some restrictions, such as:

- the local conservation law of momentum and energy are assumed;

- a weak compactness assumption was required to pass to the limit.

Without making any nonlinear weak compactness hypothesis, in [27], Golse and Saint-Raymond had made a breakthrough in that project, and they can show the incompressible Navier-Stokes limit of Boltzmann equations with bounded kernels. In addition to building on the ideas in [3], their proof uses the entropy dissipation rate to decompose the collision operator in a new way and uses a new L^1 averaging theory to prove the compactness assumption. Recently, Levermore and Masmoudi [52] extended those results to a much wider class of collision kernels. For more references concerning the different scaling region of the classical Boltzmann equations, we refer the reader to [25, 53, 56, 57, 59, 60, 62, 63]. Also, Jang considered the incompressible hydrodynamic limit of the classical solution of VMB equations in [42].

2.0 GLOBAL EXISTENCE OF THE CAUCHY PROBLEM

The aim of this chapter is to prove the global existence of weak solutions to the Cauchy problem of the isentropic compressible MHD by an invading domain method. The global existence of weak solutions with large initial data to compressible flows is always a fundamental problem of the mathematical study, since the isentropic compressible MHD is a classical model to characterize the flow, with the effect from the electric and magnetic field.

In Section 2.1, we derive the classical *a priori* estimates which are fundamental bricks for the analysis followed. Also, we give the definition of the weak solutions, state our main results, and explain our strategy of the argument. In Section 2.2, following the method in [19, 20, 36], we show the existence of solutions to the Cauchy problem of (1.0.1) and the strong convergence of ρ in $L^1((0,T)\times\mathbb{R}^3)$.

2.1 STATEMENT OF EXISTENCE

In this section, we obtain *a priori* estimates, reformulate the Cauchy problem (1.0.1)-(1.0.2), and state the main results.

We first formally derive the energy equation and some *a priori* estimates. Multiplying the second equation in (1.0.1) by $\mathbf{u}$, integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho\mathbf{u}^2+\frac{a}{\gamma-1}(\rho^\gamma-\rho_\infty^\gamma-\gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty))\right)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\mu|D\mathbf{u}|^2+(\lambda+\mu)(\mathrm{div}\mathbf{u})^2\right)\mathrm{d}x\\&=\int_{\mathbb{R}^3}((\nabla\times\mathbf{H})\times\mathbf{H})\cdot\mathbf{u}\,\mathrm{d}x.\end{aligned} \tag{2.1.1}$$

The term on the right hand side of (2.1.1) can be rewritten as

$$\int_{\mathbb{R}^3}((\nabla\times\mathbf{H})\times\mathbf{H})\cdot\mathbf{u}\,\mathrm{d}x=-\int_{\mathbb{R}^3}\left(\mathbf{H}^\top\nabla\mathbf{u}\,\mathbf{H}+\frac{1}{2}\nabla(|\mathbf{H}|^2)\cdot\mathbf{u}\right)\mathrm{d}x.$$

Hence, (2.1.1) becomes

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho\mathbf{u}^2+\frac{a}{\gamma-1}(\rho^\gamma-\rho_\infty^\gamma-\gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty))\right)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\mu|D\mathbf{u}|^2+(\lambda+\mu)(\mathrm{div}\mathbf{u})^2\right)\mathrm{d}x\\&=-\int_{\mathbb{R}^3}\left(\mathbf{H}^\top\nabla\mathbf{u}\,\mathbf{H}+\frac{1}{2}\nabla(|\mathbf{H}|^2)\cdot\mathbf{u}\right)\mathrm{d}x.\end{aligned}\tag{2.1.2}$$

Multiplying the third equation in (1.0.1) by $\mathbf{H}$, integrating over Ω, and using the boundary condition in (1.0.2) and the condition $\mathrm{div}\mathbf{H}=0$, one has

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{H}|^2\mathrm{d}x+\int_{\mathbb{R}^3}(\nabla\times(\nu\nabla\times\mathbf{H}))\cdot\mathbf{H}\,\mathrm{d}x=\int_{\mathbb{R}^3}(\nabla\times(\mathbf{u}\times\mathbf{H}))\cdot\mathbf{H}\,\mathrm{d}x.\tag{2.1.3}$$

Direct calculations show that

$$\int_{\mathbb{R}^3}(\nabla\times(\nu\nabla\times\mathbf{H}))\cdot\mathbf{H}\,\mathrm{d}x=\nu\int_{\mathbb{R}^3}|\nabla\times\mathbf{H}|^2\mathrm{d}x,$$

$$\int_{\mathbb{R}^3}(\nabla\times(\mathbf{u}\times\mathbf{H}))\cdot\mathbf{H}\,\mathrm{d}x=\int_{\mathbb{R}^3}\left(\mathbf{H}^\top\nabla\mathbf{u}\,\mathbf{H}+\frac{1}{2}\nabla(|\mathbf{H}|^2)\cdot\mathbf{u}\right)\mathrm{d}x.$$

Thus (2.1.3) yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{H}|^2\mathrm{d}x+\nu\int_{\mathbb{R}^3}|\nabla\times\mathbf{H}|^2\mathrm{d}x=\int_{\mathbb{R}^3}\left(\mathbf{H}^\top\nabla\mathbf{u}\,\mathbf{H}+\frac{1}{2}\nabla(|\mathbf{H}|^2)\cdot\mathbf{u}\right)\mathrm{d}x.\tag{2.1.4}$$

Adding (2.1.2) and (2.1.4) gives

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho\mathbf{u}^2+\frac{a}{\gamma-1}(\rho^\gamma-\rho_\infty^\gamma-\gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty))+\frac{1}{2}|\mathbf{H}|^2\right)\mathrm{d}x\\&+\int_{\mathbb{R}^3}\left(\mu|D\mathbf{u}|^2+(\lambda+\mu)(\mathrm{div}\mathbf{u})^2+\nu|\nabla\times\mathbf{H}|^2\right)\mathrm{d}x=0.\end{aligned}\tag{2.1.5}$$

From Lemma 3.3 in [55], our assumptions on initial data, and (2.1.5), we have the following *a priori* estimates:

$$
\begin{aligned}
&\rho|\mathbf{u}|^2 \in L^\infty([0,T]; L^1(\mathbb{R}^3));\\
&\nabla\mathbf{u} \in L^2([0,T]; L^2(\mathbb{R}^3));\\
&\nabla\mathbf{H} \in L^2([0,T]; L^2(\mathbb{R}^3));\\
&\mathbf{H} \in L^\infty([0,T]; L^2(\mathbb{R}^3));\\
&(\rho^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty)) \in L^\infty([0,T]; L^1(\mathbb{R}^3)).
\end{aligned}
$$

In order to define weak solutions in the whole space, the following special type of Orlicz spaces $L_q^p(\mathbb{R}^3)$ are needed (see Appendix A in [51]):

$$
L_q^p(\mathbb{R}^3) = \left\{ f \in L^1_{loc}(\mathbb{R}^3): \ f\chi_{\{|f|<\eta\}} \in L^q(\mathbb{R}^3),\ f\chi_{\{|f|\geq\eta\}} \in L^p(\mathbb{R}^3),\ \text{for some } \eta > 0 \right\},
$$

where χ denotes the characteristic function of a set. Hence, we deduce from the previous estimate that $\rho - \rho_\infty \in L^\infty([0,T]; L_2^\gamma(\mathbb{R}^3))$ by using the following estimates:

$$
\begin{cases}
\rho^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty) \geq \nu|\rho-\rho_\infty|^2 \quad \text{if} \quad \gamma \geq 2,\\
\rho^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty) \geq \nu|\rho-\rho_\infty|^2 \quad \text{if} \quad \gamma < 2 \quad \text{and} \quad \rho \leq A,\\
\rho^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty) \geq \nu|\rho-\rho_\infty|^\gamma \quad \text{if} \quad \gamma < 2 \quad \text{and} \quad \rho \geq A,
\end{cases}
\tag{2.1.6}
$$

where ν depends on $A > 0$, γ, and ρ_∞.

Multiplying the continuity equation (i.e., the first equation in (1.0.1)) by $b'(\rho)$, we obtain the renormalized continuity equation:

$$
b(\rho)_t + \mathrm{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\mathrm{div}\mathbf{u} = 0, \tag{2.1.7}
$$

for some suitable function $b \in C^1(\mathbb{R}^+)$.

Following the strategy in [19, 36, 51], we introduce the concept of *finite energy weak solution* $(\rho, \mathbf{u}, \mathbf{H})$ to the Cauchy problem (1.0.1)-(1.0.2) in the following sense:

- The density ρ is a non-negative function,

$$\rho \in C([0,T]; L^1_{loc}(\mathbb{R}^3)) \cap L^\infty([0,T]; L^\gamma_2(\mathbb{R}^3)), \quad \rho(x,0) = \rho_0,$$

and the momentum $\rho\mathbf{u}$ satisfies

$$\rho\mathbf{u} \in C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\mathbb{R}^3));$$

- The velocity $\mathbf{u}$ and the magnetic field $\mathbf{H}$ satisfy the following:

$$\nabla\mathbf{u} \in L^2([0,T]; L^2(\mathbb{R}^3)), \quad \nabla\mathbf{H} \in L^2([0,T]; L^2(\mathbb{R}^3)), \quad \mathbf{H} \in C([0,T]; L^2_{weak}(\mathbb{R}^3)),$$

$\rho\mathbf{u} \otimes \mathbf{u}$, $\nabla \times (\mathbf{u} \times \mathbf{H})$, and $(\nabla \times \mathbf{H}) \times \mathbf{H}$ are integrable on $(0,T) \times \mathbb{R}^3$, and

$$\rho\mathbf{u}(x,0) = \mathbf{m}_0, \quad \mathbf{H}(x,0) = \mathbf{H}_0, \quad \mathrm{div}\mathbf{H} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3);$$

- The system (1.0.1) is satisfied in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$;
- The continuity equation in (1.0.1) is satisfied in the sense of renormalized solutions, that is, (2.1.7) holds in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$ for any $b \in C^1(\mathbb{R}^+)$ satisfying

$$b'(z) = 0 \text{ for all } z \in \mathbb{R}^+ \text{ large enough, say, } z \geq z_0, \tag{2.1.8}$$

where the constant z_0 depends on the choice of function b;
- The energy inequality

$$E(t) + \int_0^t \int_{\mathbb{R}^3} \left(\mu|D\mathbf{u}|^2 + (\lambda+\mu)(\mathrm{div}\mathbf{u})^2 + \nu|\nabla \times \mathbf{H}|^2\right) \mathrm{d}x\mathrm{d}s \leq E(0),$$

holds for a.e $t \in [0,T]$, where

$$E(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}\rho\mathbf{u}^2 + \frac{a}{\gamma-1}(\rho^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho-\rho_\infty)) + \frac{1}{2}|\mathbf{H}|^2\right) \mathrm{d}x,$$

and

$$E(0) = \int_{\mathbb{R}^3} \left(\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \frac{a}{\gamma-1}(\rho_0^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho_0-\rho_\infty)) + \frac{1}{2}|\mathbf{H}_0|^2\right) \mathrm{d}x.$$

Remark 2.1.1. As a matter of fact, the function b does not need to be bounded. By Lebesgue Dominated convergence theorem, we can show that if $\rho, \mathbf{u}$ is a pair of finite energy weak solutions in the renormalized sense, they also satisfy (2.1.7) for any $b \in C^1(0,\infty) \cap C[0,\infty)$ satisfying

$$|b'(z)z| \le cz^{\frac{\gamma}{2}} \text{ for z larger than some positive constant } z_0. \tag{2.1.9}$$

Now our main result on the existence of finite energy weak solutions reads as follows.

Theorem 2.1.1. *Assume that $\gamma > \frac{3}{2}$. Then for any given $T > 0$, the Cauchy problem (1.0.1)-(1.0.2) has a finite energy weak solution $(\rho, \mathbf{u}, \mathbf{H})$ on $\mathbb{R}^3 \times (0,T)$.*

Remark 2.1.2. The fluid density ρ as well as the momentum $\rho\mathbf{u}$ should be recognized in the sense of instantaneous values (cf. Definition 2.1 in [19]) for any time $t \in [0,T]$.

The strategy to prove Theorem 2.1.1 is to find a sequence of approximate solutions $\{(\rho_R, \mathbf{u}_R, \mathbf{H}_R)\}$, and then we show that the limit of such an sequence as $R \to \infty$ will be a solution to (1.0.1)-(1.0.2). That such kind of sequences of approximate solutions exist can be ensured by Theorem 2.1 in [36] by setting $\Omega = B_R$, where B_R is a ball with radius R and center at the origin. More precisely, if we assume that the boundary condition is given as

$$\mathbf{u}_R = \mathbf{H}_R = 0, \quad \text{on} \quad \partial B_R, \tag{2.1.10}$$

and the initial condition is given by

$$\begin{cases} \rho_R(x,0) = \rho_0(x)|_{B_R} \in L^\gamma(B_R), \quad \rho_R(x,0) \ge 0, \\ \rho_R(x,0)\mathbf{u}_R(x,0) = \mathbf{m}_0(x)|_{B_R} \in L^1(B_R), \ \mathbf{m}_0 = 0 \text{ if } \rho_0 = 0, \ \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(B_R), \\ \mathbf{H}_R(x,0) = \mathbf{H}_0(x)|_{B_R} \in L^2(B_R), \quad \mathrm{div}\mathbf{H}_0 = 0 \text{ in } \mathcal{D}'(B_R), \end{cases} \tag{2.1.11}$$

then Theorem 2.1 in [36] gives a finite energy weak solution $\{(\rho_R, \mathbf{u}_R, \mathbf{H}_R)\}$ with the following uniform estimate in R:

$$E_R(t) + \int_0^t \int_{B_R} \left(\mu|D\mathbf{u}_R|^2 + (\lambda+\mu)(\mathrm{div}\mathbf{u}_R)^2 + \nu|\nabla \times \mathbf{H}_R|^2\right) \mathrm{d}x\mathrm{d}s \le E(0), \tag{2.1.12}$$

where

$$E_R(t) = \int_{B_R} \left(\frac{1}{2}\rho_R\mathbf{u}_R^2 + \frac{a}{\gamma-1}(\rho_R^\gamma - \rho_\infty^\gamma - \gamma\rho_\infty^{\gamma-1}(\rho_R - \rho_\infty)) + \frac{1}{2}|\mathbf{H}_R|^2\right) \mathrm{d}x.$$

2.2 ON THE INTEGRABILITY OF THE DENSITY

We first derive an improved estimate of the density ρ_R uniform in R on $K \times (0,T)$ for any compact subset $K \subset \mathbb{R}^3$. The technique is similar to that in [20]. For this purpose, there exists a constant $R_0 > 0$ such that $K \subset B_{R_0}$.

Noting that the function $b(\rho) = \ln(1+\rho)$ satisfies the condition (2.1.9), and $\rho_R, \mathbf{u}_R, \mathbf{H}_R$ are the solution to (1.0.1) with the initial condition (2.1.11) and the boundary condition (2.1.10) in the sense of renormalized solutions, we have

$$(\ln(1+\rho_R))_t + \operatorname{div}\left(\ln\left(1+\rho_R\right)\mathbf{u}_R\right) + \left(\frac{\rho_R}{1+\rho_R} - \ln\left(1+\rho_R\right)\right)\operatorname{div}\mathbf{u}_R = 0. \tag{2.2.1}$$

Now we introduce an auxiliary operator

$$\mathfrak{B} : \left\{ f \in L^p(B_{R_0}) : \int_{B_{R_0}} f = 0 \right\} \mapsto [W_0^{1,p}(B_{R_0})]^3$$

which is a bounded linear operator, i.e.,

$$\|\mathfrak{B}[f]\|_{W_0^{1,p}(B_{R_0})} \le c(p)\|f\|_{L^p(B_{R_0})} \text{ for any } 1 < p < \infty; \tag{2.2.2}$$

and the function $W = \mathfrak{B}[f] \in \mathbb{R}^3$ solves the problem

$$\operatorname{div}\mathrm{W} = f \text{ in } \mathbb{R}^3. \tag{2.2.3}$$

Moreover, if f can be written in the form $f = \operatorname{div} g$ for some $g \in L^r$ with $g \cdot n = 0$ on ∂B_{R_0}, then

$$\|\mathfrak{B}[f]\|_{L^r(B_{R_0})} \le c(r)\|g\|_{L^r(B_{R_0})} \tag{2.2.4}$$

for arbitrary $1 < r < \infty$.

For $i = 1, 2, 3$, we define the functions:

$$\varphi_i = \begin{cases} \psi(t)\mathfrak{B}_i\left[\ln(1+\rho_R) - \frac{1}{|B_{R_0}|}\int_{B_{R_0}} \ln(1+\rho_R)\,\mathrm{d}x\right],, \text{ for } |x| \le R_0; \\ 0, \text{ for } |x| > R_0, \end{cases}$$

where $\psi \in \mathcal{D}(0,T)$. By virtue of (2.1.12) and (2.2.1), we get

$$\ln(1+\rho_R) \in C([0,T]; L^p(B_{R_0})) \text{ for any finite } p > 1.$$

Therefore, from (2.2.2), we have

$$\varphi_i \in C([0,T]; W_0^{1,p}(B_{R_0})) \text{ for any finite } p > 1.$$

In particular, $\varphi_i \in C(\mathbb{R}^3 \times [0,T])$ by the Sobolev embedding theorem. Consequently, φ_i can be used as test functions for the momentum balance equation in (1.0.1). After a little bit lengthy but straightforward computation, we obtain:

$$\int_0^T\!\!\int_{B_{R_0}} \psi a\rho_R^\gamma \ln(1+\rho_R)\,\mathrm{d}x\mathrm{d}t = \sum_{j=1}^{7} I_j, \tag{2.2.5}$$

where

$$\begin{aligned}
I_1 =& \frac{1}{|B_{R_0}|}\int_0^T \psi \int_{B_{R_0}} a\rho_R^\gamma\,\mathrm{d}x \int_{B_{R_0}} \ln(1+\rho_R)\,\mathrm{d}x\mathrm{d}t,\\
I_2 =& (\lambda+\mu)\int_0^T\!\!\int_{B_{R_0}} \psi \ln(1+\rho_R)\mathrm{div}\mathbf{u}_R\,\mathrm{d}x\mathrm{d}t,\\
I_3 =& -\int_0^T\!\!\int_{B_{R_0}} \psi_t\rho_R u_R^i \mathfrak{B}_i\left[\ln(1+\rho_R) - \frac{1}{|B_{R_0}|}\int_{B_{R_0}}\ln(1+\rho_R)\,\mathrm{d}x\right]\mathrm{d}x\mathrm{d}t,\\
I_4 =& \int_0^T\!\!\int_{B_{R_0}} \psi(\mu\partial_{x_j}u_R^i - \rho_R u_R^i u_R^j)\partial_{x_j}\mathfrak{B}_i\left[\ln(1+\rho_R) - \frac{1}{|B_{R_0}|}\int_{B_{R_0}}\ln(1+\rho_R)\,\mathrm{d}x\right]\mathrm{d}x\mathrm{d}t,\\
I_5 =& \int_0^T\!\!\int_{B_{R_0}} \psi\rho_R u_R^i\mathfrak{B}_i\left[\left(\ln(1+\rho_R) - \frac{\rho_R}{1+\rho_R}\right)\mathrm{div}\mathbf{u}_R\right.\\
&\left. -\frac{1}{|B_{R_0}|}\int_{B_{R_0}}\left(\ln(1+\rho_R) - \frac{\rho_R}{1+\rho_R}\right)\mathrm{div}\mathbf{u}_R\,\mathrm{d}x\right]\mathrm{d}x\mathrm{d}t,\\
I_6 =& \int_0^T\!\!\int_{B_{R_0}} \psi\rho_R u_R^i\mathfrak{B}_i[\mathrm{div}(\ln(1+\rho_R)\mathbf{u}_R)]\,\mathrm{d}x\mathrm{d}t,\\
I_7 =& \int_0^T\!\!\int_{B_{R_0}} \psi(\nabla\times\mathbf{H}_R)\times\mathbf{H}_R\cdot\mathfrak{B}_i\left[\ln(1+\rho_R) - \frac{1}{|B_{R_0}|}\int_{B_{R_0}}\ln(1+\rho_R)\,\mathrm{d}x\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now, we can estimate the integrals $I_1 - I_7$ as follows.

(1) First, we see that I_1 is bounded uniformly in R, from (2.1.12), and the following property:

$$\lim_{t\to\infty}\frac{\ln(1+t)}{t^\gamma} = 0.$$

(2) As for the second term, we also have

$$|I_2| \le \int_0^T\!\!\int_{B_{R_0}} |\psi\ln(1+\rho_R)\mathrm{div}\mathbf{u}_R|\,\mathrm{d}x\mathrm{d}t \le c,$$

by the Hölder inequality, (2.1.12), and the following property:

$$\lim_{t\to\infty}\frac{\ln^2(1+t)}{t^\gamma}=0,$$

where and throughout the rest of the paper, $c>0$ denotes a generic constant.

(3) Similarly, for the third term, we have

$$|I_3|\le\int_0^T\!\!\int_{B_{R_0}}\left|\psi_t\rho_R u_R^i\mathfrak{B}_i\left[\ln(1+\rho_R)-\frac{1}{|B_{R_0}|}\int_{B_{R_0}}\ln(1+\rho_R)\,\mathrm{d}x\right]\right|\,\mathrm{d}x\mathrm{d}t\le c.$$

Here, we have used (2.1.12), and the embedding $W^{1,p}(B_{R_0})\hookrightarrow L^\infty(B_{R_0})$ for $p>3$, since $\ln(1+\rho_R)-\frac{1}{|B_{R_0}|}\int_{B_{R_0}}\ln(1+\rho_R)\,\mathrm{d}x\in L^p(B_{R_0})$ for any $1<p<\infty$.

(4) Similarly to (3), we have

$$\left|\int_0^T\!\!\int_{B_{R_0}}\psi\partial_{x_j}u_R^i\partial_{x_j}\mathfrak{B}_i\left[\ln(1+\rho_R)-\frac{1}{|B_{R_0}|}\int_{B_{R_0}}\ln(1+\rho_R)\,\mathrm{d}x\right]\,\mathrm{d}x\mathrm{d}t\right|\le c,$$

and, by (2.1.12), and Hölder inequality, we have

$$\left|\int_0^T\!\!\int_{B_{R_0}}\psi\rho_R u_R^i u_R^j\partial_{x_j}\mathfrak{B}_i\left[\ln(1+\rho_R)-\frac{1}{|B_{R_0}|}\int_{B_{R_0}}\ln(1+\rho_R)\,\mathrm{d}x\right]\,\mathrm{d}x\mathrm{d}t\right|\le c.$$

Here, we used the restriction $\gamma>\frac{3}{2}$. Therefore, we obtain

$$|I_4|\le c.$$

(5) Next, by Hölder inequality and (2.2.2), we have,

$$|I_5|\le c\int_0^T|\psi|\|\rho_R\|_{L^\gamma(\mathbb{R}^3)}^{\frac{1}{2}}\|\sqrt{\rho_R}u_R\|_{L^2(\mathbb{R}^3)}\|\mathfrak{B}_i[w]\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^3)}\mathrm{d}t\le c,$$

since

$$w:=\left(\ln(1+\rho_R)-\frac{\rho_R}{1+\rho_R}\right)\mathrm{div}\mathbf{u}_R-\frac{1}{|B_{R_0}|}\int_{B_{R_0}}\left(\ln(1+\rho_R)-\frac{\rho_R}{1+\rho_R}\right)\mathrm{div}\mathbf{u}_R\,\mathrm{d}x\in L^r(B_{R_0}),$$

for some $1<r<2$, and here we have used the estimates (2.1.12).

(6) Similarly to (5), using (2.1.12), and we have

$$|I_6|\le\int_0^T\!\!\int_{B_{R_0}}\left|\psi\rho_R u_R^i\mathfrak{B}_i[\mathrm{div}(\ln(1+\rho_R)\mathbf{u}_R)]\right|\,\mathrm{d}x\mathrm{d}t\le c.$$

Here, we have also used the property (4.1.15).

(7) Finally, using Hölder inequality again, we have

$$|I_7| \leq c \int_0^T |\psi| \|\nabla \times \mathbf{H}_R\|_{L^2(B_{R_0})} \|\mathbf{H}_R\|_{L^2(B_{R_0})} \mathrm{d}t \leq c.$$

Here we used the result $\varphi_i \in C([0,T] \times \mathbb{R}^3)$, (2.1.12).

Consequently, we have proved the following result:

Lemma 2.2.1. *The solutions ρ_R of system (1.0.1) on $B_R \times (0,T)$ also satisfies the following estimate*

$$\int_0^T \int_{B_{R_0}} a\psi \rho_R^\gamma \ln(1+\rho_R) \, \mathrm{d}x\mathrm{d}t \leq c,$$

where the constant c is independent of $R > 0$.

Remark 2.2.1. Lemma 4.1.14 yields

$$\int_0^T \int_K a\rho_R^\gamma \ln(1+\rho_R) \, \mathrm{d}x\mathrm{d}t \leq c$$

for any compact subset $K \subset \mathbb{R}^3$, where c does not depend on R. Using the similar method to Lemma 4.1 in [20], it can be shown (cf. [19, 20, 36, 51]) that the optimal estimate for the density ρ_R is the following:

$$\int_0^T \int_K a\rho_R^{\gamma+\theta} \, \mathrm{d}x\mathrm{d}t \leq c$$

for any compact subset $K \subset \mathbb{R}^3$, where the constant c is independent of $R > 0$, and $\theta > 0$ is a constant which depends on γ. But as shown later, our estimate in Lemma 4.1.14 is enough for our purpose.

2.2.1 Passing to the limit

The uniform estimates on ρ_R in Lemma 4.1.14, and estimates in (2.1.12) imply, as $R \to \infty$,

$$\rho_R \to \rho \text{ in } L^\infty(0, T; L^\gamma_{loc}(\mathbb{R}^3)), \tag{2.2.6}$$

$$\rho_R - \rho_\infty \to \rho - \rho_\infty, \quad \text{weakly-}* \text{ in } \quad L^\infty(0, T; L^\gamma_2(\mathbb{R}^3)),$$

$$\nabla \mathbf{u}_R \to \nabla \mathbf{u} \text{ weakly in } L^2([0, T]; L^2(\mathbb{R}^3)), \tag{2.2.7}$$

and

$$\nabla \mathbf{H}_R \to \nabla \mathbf{H} \text{ weakly in } L^2([0, T]; L^2(\mathbb{R}^3)), \tag{2.2.8}$$

$$\mathbf{H}_R \to \mathbf{H} \quad \text{weakly}^* \text{ in } \quad L^\infty([0, T]; L^2(\mathbb{R}^3)),$$

$$\text{div}\mathbf{H} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)),$$

and, from Lemma 4.1.14 and Proposition 2.1 in [19], we have, as $R \to \infty$,

$$\rho_R^\gamma \to \overline{\rho^\gamma} \text{ weakly in } L^1([0, T]; L^1(K)), \tag{2.2.9}$$

subject to a subsequence, for any compact subset $K \subset \mathbb{R}^3$.

By (4.1.19), (2.2.8) and the compactness of $H^1_0(K) \hookrightarrow L^2(K)$ for any compact subset $K \subset \mathbb{R}^3$, we obtain,

$$\nabla \times (\mathbf{u}_\delta \times \mathbf{H}_\delta) \to \nabla \times (\mathbf{u} \times \mathbf{H}) \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)), \tag{2.2.10}$$

and

$$(\nabla \times \mathbf{H}_R) \times \mathbf{H}_R \to (\nabla \times \mathbf{H}) \times \mathbf{H} \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)), \tag{2.2.11}$$

as $R \to \infty$. On the other hand, by virtue of the momentum balance in (1.0.1) and estimates (2.1.12), we have, as $R \to \infty$,

$$\rho_R \mathbf{u}_R \to \rho \mathbf{u} \text{ in } C([0, T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\mathbb{R}^3)). \tag{2.2.12}$$

Similarly, we have, as $R \to \infty$,

$$\mathbf{H}_R \to \mathbf{H} \text{ in } C([0,T]; L^2_{weak}(\mathbb{R}^3)).$$

Thus, the limits ρ, $\rho\mathbf{u}$, $\mathbf{H}$ satisfy the initial conditions of (1.0.2) in the sense of distributions.

Since $\gamma > \frac{3}{2}$, (2.2.12) and (4.1.19) combined with the compactness of $H^1(K) \hookrightarrow L^2(K)$ for any compact subset $K \subset \mathbb{R}^3$ imply, as $R \to \infty$,

$$\rho_R \mathbf{u}_R \otimes \mathbf{u}_R \to \rho\mathbf{u} \otimes \mathbf{u} \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0,T)).$$

Consequently, letting $R \to \infty$ in (1.0.1) and making use of (2.2.6)-(2.2.12), $(\rho, \mathbf{u}, \mathbf{H})$ satisfies

$$\partial_t \rho + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{2.2.13}$$

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) - \mu\triangle\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + a\nabla\overline{\rho^\gamma} = (\nabla \times \mathbf{H}) \times \mathbf{H}, \tag{2.2.14}$$

$$\mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu\nabla \times \mathbf{H}), \quad \operatorname{div}\mathbf{H} = 0, \tag{2.2.15}$$

in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$. Therefore the only thing left to complete the proof of Theorem 2.1.1 is to show the strong convergence of ρ_R in L^1 or, equivalently, $\overline{\rho^\gamma} = \rho^\gamma$.

Since ρ_R, $\mathbf{u}_R$ is a renormalized solution of the continuity equation (1.0.1) in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$, we have

$$T_k(\rho_R)_t + \operatorname{div}(T_k(\rho_R)\mathbf{u}_R) + (T_k'(\rho_R)\rho_R - T_k(\rho_R))\operatorname{div}\mathbf{u}_R = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0,T)), \tag{2.2.16}$$

where T_k is the cut-off functions defined as follows:

$$T_k(z) = kT\left(\frac{z}{k}\right) \quad \text{for } z \in R, \ k = 1, 2, \ldots$$

and $T \in C^\infty(R)$ is concave and is chosen such that

$$T(z) = \begin{cases} z, & z \leq 1, \\ 2, & z \geq 3. \end{cases}$$

Passing to the limit for $R \to \infty$, we obtain

$$\partial_t \overline{T_k(\rho)} + \mathrm{div}(\overline{T_k(\rho)}\mathbf{u}) + \overline{(T_k'(\rho)\rho - T_k(\rho))\mathrm{div}\mathbf{u}} = 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3),$$

where

$$(T_k'(\rho_R)\rho_R - T_k(\rho_R))\mathrm{div}\mathbf{u}_R \to \overline{(T_k'(\rho)\rho - T_k(\rho))\mathrm{div}\mathbf{u}} \text{ weakly in } L^2(\mathbb{R}^3 \times (0,T))),$$

and

$$T_k(\rho_R) \to \overline{T_k(\rho)} \text{ in } C([0,T]; L^p_{weak}(\mathbb{R}^3)) \text{ for all } 1 \le p < \infty.$$

2.2.2 The effective viscous flux

In this section, we discuss the effective viscous flux $p(\rho) - (\lambda + 2\mu)\mathrm{div}\mathbf{u}$. Similarly to [19, 20, 51], we prove the following auxiliary result:

Lemma 2.2.2. *Let ρ_R, $\mathbf{u}_R$ be the sequence of approximation solutions obtained in Theorem 2.1 in [36]. Then,*

$$\lim_{R\to\infty} \int_0^T \psi \int_{\mathbb{R}^3} \phi(a\rho_R^\gamma - (\lambda + 2\mu)\mathrm{div}\mathbf{u}_R)T_k(\rho_R)\,\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \psi \int_{\mathbb{R}^3} \phi(a\overline{\rho^\gamma} - (\lambda + 2\mu)\mathrm{div}\mathbf{u})\overline{T_k(\rho)}\,\mathrm{d}x\mathrm{d}t,$$

for any $\psi \in \mathcal{D}(0,T)$ and $\phi \in \mathcal{D}(\mathbb{R}^3)$.

Proof. As in [19, 20], we consider the operators

$$\mathcal{A}_i[v] = \triangle^{-1}[\partial_{x_i} v],\ i = 1, 2, 3$$

where $\triangle^{-1}$ stands for the inverse of the Laplace operator on $\mathbb{R}^3$. To be more specific, $\mathcal{A}_i$ can be expressed by their Fourier symbol

$$\mathcal{A}_i[\cdot] = \mathcal{F}^{-1}\left[\frac{-\mathrm{i}\xi_i}{|\xi|^2}\mathcal{F}[\cdot]\right],\ i = 1, 2, 3,$$

with the following properties (see [20]):

$$\|\mathcal{A}_i v\|_{W^{1,s}(\mathbb{R}^3)} \leq c(s)\|v\|_{L^s(R^3)},\ 1 < s < \infty,$$
$$\|\mathcal{A}_i v\|_{L^q(\mathbb{R}^3)} \leq c(q,s)\|v\|_{L^s(R^3)},\ q \text{ finite, provided } \frac{1}{q} \geq \frac{1}{s} - \frac{1}{3},$$
$$\|\mathcal{A}_i v\|_{L^\infty(\mathbb{R}^3)} \leq c(s)\|v\|_{L^s(R^3)},\ \ \text{if } s > 3.$$

Next, we use the quantities

$$\varphi_i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[T_k(\rho_R)],\ \psi \in \mathcal{D}(0,T),\ \phi \in \mathcal{D}(\mathbb{R}^3),\ i = 1,2,3,$$

as the test functions for the momentum balance equation in (1.0.1) to obtain,

$$\begin{aligned}
&\int_0^T\!\!\int_{\mathbb{R}^3} \psi\phi(a\rho_R^\gamma - (\lambda+2\mu)\mathrm{div}\mathbf{u}_R)T_k(\rho_R)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T\!\!\int_{\mathbb{R}^3} \psi\partial_{x_i}\phi((\lambda+\mu)\mathrm{div}\mathbf{u}_R - a\rho_R^\gamma)\mathcal{A}_i[T_k(\rho_R)]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \mu\int_0^T\!\!\int_{\mathbb{R}^3} \psi\left(\partial_{x_j}\phi\partial_{x_j}u_R^i\mathcal{A}_i[T_k(\rho_R)] - u_R^i\partial_{x_j}\phi\partial_{x_j}\mathcal{A}_i[T_k(\rho_R)] + u_R^i\partial_{x_i}\phi T_k(\rho_R)\right)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\!\!\int_{\mathbb{R}^3} \phi\rho_R u_R^i\left(\partial_t\psi\mathcal{A}_i[T_k(\rho_R)] + \psi\mathcal{A}_i[(T_k(\rho_R) - T_k'(\rho_R)\rho_R)\mathrm{div}\mathbf{u}_R]\right)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\!\!\int_{\mathbb{R}^3} \psi\rho_R u_R^i u_R^j\partial_{x_j}\phi\mathcal{A}_i[T_k(\rho_R)]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\!\!\int_{\mathbb{R}^3} \psi u_R^i\left(T_k(\rho_R)\mathcal{R}_{i,j}[\rho_R u_R^j] - \phi\rho_R u_R^j\mathcal{R}_{i,j}[T_k(\rho_R)]\right)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\!\!\int_{\mathbb{R}^3} \psi\phi(\nabla\times\mathbf{H}_R)\times\mathbf{H}_R\cdot\mathcal{A}[T_k(\rho_R)]\,\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{2.2.17}$$

where the operators $\mathcal{R}_{i,j} = \partial_{x_j}\mathcal{A}_i[v]$ and the summation convention is used to simplify notations.

Analogously, we can repeat the above arguments for equation (2.2.14) and the test functions

$$\varphi_i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\overline{T_k(\rho)}],\ i = 1,2,3,$$

to obtain

$$
\begin{aligned}
&\int_0^T\!\!\int_{\mathbb{R}^3} \psi\phi(a\overline{\rho^\gamma} - (\lambda+2\mu)\mathrm{div}\mathbf{u})\overline{T_k(\rho)}\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T\!\!\int_{\mathbb{R}^3} \psi\partial_{x_i}\phi((\lambda+\mu)\mathrm{div}\mathbf{u} - a\overline{\rho^\gamma})\mathcal{A}_i[\overline{T_k(\rho)}]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \mu\int_0^T\!\!\int_{\mathbb{R}^3} \psi\left(\partial_{x_j}\phi\partial_{x_j}u^i\mathcal{A}_i[\overline{T_k(\rho)}] - u^i\partial_{x_j}\phi\partial_{x_j}\mathcal{A}_i[\overline{T_k(\rho)}] + u^i\partial_{x_i}\phi\overline{T_k(\rho)}\right)\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\!\!\int_{\mathbb{R}^3} \phi\rho u^i\left(\partial_t\psi\mathcal{A}_i[\overline{T_k(\rho)}] + \psi\mathcal{A}_i[\overline{(T_k(\rho)-T_k'(\rho)\rho)\mathrm{div}\mathbf{u}}]\right)\mathrm{d}x\mathrm{d}t \qquad (2.2.18)\\
&\quad - \int_0^T\!\!\int_{\mathbb{R}^3} \psi\rho u^iu^j\partial_{x_j}\phi\mathcal{A}_i[\overline{T_k(\rho)}]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\!\!\int_{\mathbb{R}^3} \psi u^i\left(\overline{T_k(\rho)}\mathcal{R}_{i,j}[\phi\rho u^j] - \phi\rho u^j\mathcal{R}_{i,j}[\overline{T_k(\rho)}]\right)\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\!\!\int_{\mathbb{R}^3} \psi\phi(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\mathcal{A}[\overline{T_k(\rho)}]\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Similarly to [19, 20], it can be shown that all the terms on the right-hand side of (2.2.17) converge to their counterparts in (2.2.18). Indeed, with the relations (2.2.6)-(2.2.12) and the Sobolev embedding theorem in mind, it is easy to see that it is enough to show

$$
\begin{aligned}
&\int_0^T\!\!\int_{\mathbb{R}^3} \psi u_R^i\left(T_k(\rho_R)\mathcal{R}_{i,j}[\phi\rho_Ru_R^j] - \phi\rho_Ru_R^j\mathcal{R}_{i,j}[T_k(\rho_R)]\right)\mathrm{d}x\mathrm{d}t \\
&\to \int_0^T\!\!\int_{\mathbb{R}^3} \psi u^i\left(\overline{T_k(\rho)}R_{i,j}[\phi\rho u^j] - \phi\rho u^j\mathcal{R}_{i,j}[\overline{T_k(\rho)}]\right)\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

because the properties of $\mathcal{A}_i$ and the weak convergence of $\mathbf{u}$ in $L^2([0,T];H^1(\mathbb{R}^3))$ imply

$$\mathcal{A}_i(T_k(\rho_R)) \to \mathcal{A}_i(\overline{T_k(\rho)}) \text{ in } C(\overline{(0,T)\times\mathbb{R}^3}),$$

$$\mathcal{R}_{i,j}(T_k(\rho_R)) \to \mathcal{R}_{i,j}(\overline{T_k(\rho)}) \text{ weakly in } L^p([0,T]\times\mathbb{R}^3) \text{ for all } 1<p<\infty,$$

and

$$\mathcal{A}_i[(T_k(\rho_R) - T_k'(\rho)\rho)\mathrm{div}\mathbf{u}_R] \to \mathcal{A}_i[\overline{(T_k(\rho)-T_k'(\rho)\rho)\mathrm{div}\mathbf{u}}] \text{ weakly in } L^2([0,T];H^1(\mathbb{R}^3)).$$

From Lemma 3.4 in [20], we have

$$
\begin{aligned}
&T_k(\rho_R)\mathcal{R}_{i,j}[\phi\rho_Ru_R^j] - \phi\rho_Ru_R^j\mathcal{R}_{i,j}[T_k(\rho_R)] \\
&\to \overline{T_k(\rho)}R_{i,j}[\phi\rho u^j] - \phi\rho u^j\mathcal{R}_{i,j}[\overline{T_k(\rho)}] \text{ weakly in } L^r(\mathbb{R}^3),\ i,j=1,2,3,
\end{aligned}
$$

for some $r>1$. Hence, we complete the proof of Lemma 2.2.2. □

2.2.3 The amplitude of oscillations

The main result of this subsection reads as follows, and is essentially taken from [20] (cf. Lemma 4.3 in [20]):

Lemma 2.2.3. *There exists a constant c independent of k such that*

$$\limsup_{R\to\infty} \|T_k(\rho_R) - T_k(\rho)\|_{L^{\gamma+1}((0,T)\times\mathbb{R}^3)} \le c.$$

Proof. By the convexity of functions $t \to p(t)$, $t \to -T_k(t)$, one has

$$\begin{aligned}
&\limsup_{R\to\infty} \int_0^T\!\!\int_{\mathbb{R}^3} \left(\rho_R^\gamma T_k(\rho_R) - \overline{\rho^\gamma}(\overline{T_k(\rho)})\right) \mathrm{d}x\mathrm{d}t \\
&= \limsup_{R\to\infty} \int_0^T\!\!\int_{\mathbb{R}^3} (\rho_R^\gamma - \rho^\gamma)(T_k(\rho_R) - T_k(\rho))\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\!\!\int_{\mathbb{R}^3} (\overline{\rho^\gamma} - \rho^\gamma)(T_k(\rho) - \overline{T_k(\rho)})\,\mathrm{d}x\mathrm{d}t \\
&\ge \limsup_{R\to\infty} \int_0^T\!\!\int_{\mathbb{R}^3} (\rho_R^\gamma - \rho^\gamma)(T_k(\rho_R) - T_k(\rho))\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.2.19}$$

On one hand, we have

$$y^\gamma - z^\gamma = \int_z^y \gamma s^{\gamma-1}\mathrm{d}s \ge \gamma \int_z^y (s-z)^{\gamma-1}\mathrm{d}s = \gamma(y-z)^\gamma,$$

for all $y \ge z \ge 0$, and

$$|T_k(y) - T_k(z)|^\gamma \le |y-z|^\gamma,$$

thus,

$$\begin{aligned}
(z^\gamma - y^\gamma)(T_k(z) - T_k(y)) &\ge \gamma|T_k(z) - T_k(y)|^\gamma |T_k(z) - T_k(y)| \\
&= \gamma|T_k(z) - T_k(y)|^{\gamma+1},
\end{aligned}$$

for all $z, y \ge 0$. On the other hand,

$$\begin{aligned}
&\limsup_{R\to\infty} \int_0^T\!\!\int_{\mathbb{R}^3} \left(\mathrm{div}\mathbf{u}_R T_k(\rho_R) - \mathrm{div}\overline{\mathbf{u}T_k(\rho)}\right) \mathrm{d}x\mathrm{d}t \\
&= \limsup_{R\to\infty} \int_0^T\!\!\int_{\mathbb{R}^3} \left(T_k(\rho_R) - T_k(\rho) + T_k(\rho) - \overline{T_k(\rho)}\right) \mathrm{div}\mathbf{u}_R\,\mathrm{d}x\mathrm{d}t \\
&\le 2\sup_R \|\mathrm{div} u_R\|_{L^2((0,T)\times\mathbb{R}^3)} \limsup_{R\to\infty} \|T_k(\rho_R) - T_k(\rho)\|_{L^2((0,T)\times\mathbb{R}^3)} \\
&\le c \limsup_{R\to\infty} \|T_k(\rho_R) - T_k(\rho)\|_{L^2((0,T)\times\mathbb{R}^3)} \\
&\le c + \frac{1}{2} \limsup_{R\to\infty} \|T_k(\rho_R) - T_k(\rho)\|^{\gamma+1}_{L^{\gamma+1}((0,T)\times\mathbb{R}^3)}.
\end{aligned} \tag{2.2.20}$$

The relations (2.2.19), (2.2.20) combined with Lemma 2.2.2 yield the desired conclusion. □

2.2.4 The renormalized solutions

We now use Lemma 2.2.3 to prove the following crucial result:

Lemma 2.2.4. *The limit functions* $\rho, \mathbf{u}$ *solve* (2.2.13) *in the sense of renormalized solutions, i.e.,*

$$\partial_t b(\rho) + \text{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\text{div}\mathbf{u} = 0, \tag{2.2.21}$$

holds in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$ *for any* $b \in C^1(R)$ *satisfying* $b'(z) = 0$ *for all* $z \in R$ *large enough, say,* $z \geq M$*, where the constant* M *may depend on* b*.*

Proof. Regularizing (2.2.16), one gets

$$\partial_t S_m[\overline{T_k(\rho)}] + \text{div}(S_m[\overline{T_k(\rho)}]\mathbf{u}) + S_m[\overline{(T_k'(\rho)\rho - T_k(\rho))\text{div}\mathbf{u}}] = r_m, \tag{2.2.22}$$

where $S_m[v] = v_m * v$ are the standard smoothing operators and $r_m \to 0$ in $L^2([0,T]; L^2(\mathbb{R}^3))$ for any fixed k (see Lemma 2.3 in [50]). Now, we are allowed to multiply (2.2.22) by $b'(S_m[\overline{T_k(\rho)}])$. Letting $m \to \infty$, we obtain

$$\begin{aligned}&\partial_t b[\overline{T_k(\rho)}] + \text{div}(b[\overline{T_k(\rho)}]\mathbf{u}) + (b'(\overline{T_k(\rho)})\overline{T_k(\rho)} - b(\overline{T_k(\rho)}))\text{div}\mathbf{u}\\ &= b'(\overline{T_k(\rho)})[\overline{(T_k'(\rho)\rho - T_k(\rho))\text{div}\mathbf{u}}] \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3).\end{aligned} \tag{2.2.23}$$

At this stage, the main idea is to let $k \to \infty$ in (2.2.23). We have

$$\overline{T_k(\rho)} \to \rho \text{ in } L^p(B_{R_0} \times (0,T)) \text{ for any } 1 \leq p < \gamma, \quad \text{as } k \to \infty,$$

since

$$\|\overline{T_k(\rho)} - \rho\|_{L^p(B_{R_0}\times(0,T))} \leq \liminf_{R\to\infty} \|T_k(\rho_R) - \rho_R\|_{L^p(B_{R_0}\times(0,T))},$$

and

$$\|T_k(\rho_R) - \rho_R\|^p_{L^p(B_{R_0}\times(0,T))} \leq 2^p k^{p-\gamma}\|\rho_R\|^\gamma_{L^\gamma(B_{R_0}\times(0,T))} \leq ck^{p-\gamma}. \tag{2.2.24}$$

Thus (2.2.23) will imply (2.2.21) provided we show

$$b'(\overline{T_k(\rho)})[\overline{(T_k'(\rho)\rho - T_k(\rho))\text{div}\mathbf{u}}] \to 0 \text{ in } L^1(B_{R_0} \times (0,T)) \text{ as } k \to \infty.$$

To this end, let us denote

$$Q_{k,M} = \{(t,x) \in B_{R_0} \times (0,T) \mid \overline{T_k(\rho)} \leq M\},$$

then

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3} \left|b'(\overline{T_k(\rho)})[\overline{(T_k'(\rho)\rho - T_k(\rho))\mathrm{div}\mathbf{u}}]\right| \,\mathrm{d}x\mathrm{d}t \\
&\le \sup_{0\le z\le M} |b'(z)| \iint_{Q_{K,M}} \left|\overline{(T_k'(\rho)\rho - T_k(\rho))\mathrm{div}\mathbf{u}}\right| \,\mathrm{d}x\mathrm{d}t \\
&\le \sup_{0\le z\le M} |b'(z)| \liminf_{R\to\infty} \|(T_k'(\rho_R)\rho_R - T_k(\rho_R))\mathrm{div}\mathbf{u}_R\|_{L^1(Q_{k,M})} \\
&\le \sup_{0\le z\le M} |b'(z)| \sup_R \|\mathbf{u}_R\|_{L^2([0,T];H^1(\mathbb{R}^3))} \liminf_{R\to\infty} \|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|_{L^2(Q_{k,M})} \\
&\le c \liminf_{R\to\infty} \|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|_{L^2(Q_{k,M})}.
\end{aligned}$$

Now, by interpolation, one has

$$\begin{aligned}
&\|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|^2_{L^2(Q_{k,M})} \\
&\le \|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|^{\frac{\gamma-1}{\gamma}}_{L^1(B_{R_0}\times(0,T))} \|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|^{\frac{\gamma+1}{\gamma}}_{L^{\gamma+1}(Q_{k,M})}.
\end{aligned} \tag{2.2.25}$$

Similarly to (2.2.24), we have

$$\|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|_{L^1(B_{R_0}\times(0,T))} \le ck^{1-\gamma} \sup_R \|\rho_R\|^\gamma_{L^\gamma(B_{R_0})} \le ck^{1-\gamma},$$

and, using $T_k'(z)z \le T_k(z)$,

$$\begin{aligned}
&\frac{1}{2}\|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|_{L^{\gamma+1}(Q_{k,M})} \\
&\le \|T_k(\rho_R) - T_k(\rho)\|_{L^{\gamma+1}(B_{R_0}\times(0,T))} + \|T_k(\rho)\|_{L^{\gamma+1}(Q_{k,M})} \\
&\le \|T_k(\rho_R) - T_k(\rho)\|_{L^{\gamma+1}(B_{R_0}\times(0,T))} + \|\overline{T_k(\rho)}\|_{L^{\gamma+1}(Q_{k,M})} \\
&\qquad + \|\overline{T_k(\rho)} - T_k(\rho)\|_{L^{\gamma+1}(B_{R_0}\times(0,T))} \\
&\le \|T_k(\rho_R) - T_k(\rho)\|_{L^{\gamma+1}(B_{R_0}\times(0,T))} + Mc|B_{R_0}| \\
&\qquad + \|\overline{T_k(\rho)} - T_k(\rho)\|_{L^{\gamma+1}(B_{R_0}\times(0,T))}.
\end{aligned} \tag{2.2.26}$$

From Lemma 2.2.3 and (2.2.26), we obtain

$$\limsup_{R\to 0+} \|T_k'(\rho_R)\rho_R - T_k(\rho_R)\|_{L^{\gamma+1}(Q_{k,M})} \le 4c + 2Mc|B_{R_0}|,$$

which, together with (2.2.25)-(2.2.26), completes the proof of Lemma 2.2.4. □

2.2.5 Strong convergence of the density

Now, we can complete the proof of Theorem 2.1.1. To this end, we introduce a sequence of functions $L_k \in C^1(R)$:

$$L_k(z) = \begin{cases} z\ln z, & 0 \le z < k, \\ z\ln(k) + z\int_k^z \frac{T_k(s)}{s^2}\mathrm{d}s, & z \ge k. \end{cases}$$

Noting that L_k can be written as

$$L_k(z) = \beta_k z + b_k(z), \tag{2.2.27}$$

where b_k satisfies the conditions in Lemma 2.2.4, we can use the fact that $\rho_R, \mathbf{u}_R$ are renormalized solutions of (1.0.1) on $B_R \times (0,T)$ to deduce

$$\partial_t L_k(\rho_R) + \mathrm{div}(L_k(\rho_R)\mathbf{u}_R) + T_k(\rho_R)\mathrm{div}\mathbf{u}_R = 0. \tag{2.2.28}$$

Similarly, by (2.2.13) and Lemma 2.2.4, we have

$$\partial_t L_k(\rho) + \mathrm{div}(L_k(\rho)\mathbf{u}) + T_k(\rho)\mathrm{div}\mathbf{u} = 0, \tag{2.2.29}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$. By (2.2.28), we can assume, as $R \to \infty$,

$$L_k(\rho_R) \to \overline{L_k(\rho)} \text{ in } C([0,T]; L^\gamma_{weak}(\mathbb{R}^3)).$$

Taking the difference of (2.2.28) and (2.2.29) and integrating with respect to t, we get

$$\begin{aligned} &\int_{\mathbb{R}^3} (L_k(\rho_R) - L_k(\rho))\phi \,\mathrm{d}x \\ &= \int_0^t \int_{\mathbb{R}^3} ((L_k(\rho_R)\mathbf{u}_R - L_k(\rho)\mathbf{u})\cdot\nabla\phi + (T_k(\rho)\mathrm{div}\mathbf{u} - T_k(\rho_R)\mathrm{div}\mathbf{u}_R)\phi)\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.2.30}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^3)$. Passing to the limit for $R \to 0$ and making use of (2.2.30), one obtains

$$\begin{aligned} &\int_{\mathbb{R}^3} (\overline{L_k(\rho)} - L_k(\rho))\phi \,\mathrm{d}x \\ &= \int_0^t \int_{\mathbb{R}^3} (\overline{L_k(\rho)} - L_k(\rho))\mathbf{u}\cdot\nabla\phi \,\mathrm{d}x\mathrm{d}t \\ &\quad + \lim_{R\to 0+} \int_0^t \int_{\mathbb{R}^3} (T_k(\rho)\mathrm{div}\mathbf{u} - T_k(\rho_R)\mathrm{div}\mathbf{u}_R)\phi \,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.2.31}$$

for any $\phi \in \mathcal{D}(\mathbb{R}^3)$.

In the sequel, we need cut-off functions defined as

$$\phi_m(x) = \phi\left(\frac{x}{m}\right), \quad m \in \mathbb{N}, \quad \phi \in (\mathcal{D}(\mathbb{R}^3))^3,$$

$$0 \leq \phi(x) \leq 1, \quad \phi(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| > 2. \end{cases}$$

It is easy to verify that $\phi_m(x)$ satisfies

$$\begin{cases} \sup_{x\in\mathbb{R}^3} |\nabla\phi_m(x)| \leq \frac{C}{m}; \\ \lim_{m\to\infty} \int_{\mathbb{R}^3} |\mathbf{u}\cdot\nabla\phi_m|^p dx = 0, \quad \text{for any} \quad \mathbf{u} \in (D^{1,2}(\mathbb{R}^3))^3, \quad \text{and} \quad 2 \leq p < 6, \end{cases} \tag{2.2.32}$$

where

$$D^{1,2}(\mathbb{R}^3) \stackrel{def}{=} \{f \in L^1_{loc}(\mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3)\}.$$

Taking the sequence $\phi = \phi_m$ as the test functions in (2.2.31), making use of the boundary conditions in (1.0.2), and passing to the limit as $m \to \infty$, one has

$$\int_{\mathbb{R}^3} (\overline{L_k(\rho)} - L_k(\rho))\, \mathrm{d}x = \int_0^t\int_{\mathbb{R}^3} T_k(\rho)\mathrm{div}\mathbf{u}\, \mathrm{d}x\mathrm{d}t - \lim_{R\to\infty} \int_0^t\int_{\mathbb{R}^3} T_k(\rho_R)\mathrm{div}\mathbf{u}_R\, \mathrm{d}x\mathrm{d}t. \tag{2.2.33}$$

We observe that the term $\overline{L_k(\rho)} - L_k(\rho)$ is bounded by (2.2.27).

At this stage, the main idea is to let $k \to \infty$ in (2.2.33). By Lemma 2.2.1, we can assume

$$\rho_\varepsilon \ln(\rho_\varepsilon) \to \overline{\rho\ln(\rho)} \text{ weakly star in } L^\infty([0,T]; L^\alpha(B_{R_0})) \text{ for all } 1 \leq \alpha < \gamma.$$

We also have

$$\overline{L_k(\rho)} \to \overline{\rho\ln(\rho)} \text{ in } L^\infty([0,T]; L^\alpha(B_{R_0})) \text{ as } k \to \infty \text{ for all } 1 \leq \alpha < \gamma,$$

since, by Lemma 2.2.1,

$$\lim_{k\to\infty} r(k) = 0, \quad \text{where } r(k) := \mathrm{meas}\{(x,t) \in B_{R_0} \times (0,T) | \rho_R(x,t) \geq k\};$$

and because $L_k(z) \leq z\ln z$, we have

$$
\begin{aligned}
&\|\overline{L_k(\rho)} - \overline{\rho\ln(\rho)}\|_{L^\infty([0,T];L^\alpha(K))} \\
&\leq \sup_{t\in[0,T]} \liminf_{R\to\infty} \|L_k(\rho_R) - \rho_R\ln(\rho_R)\|_{L^\infty([0,T];L^\alpha(K))} \\
&\leq 2q(k) \sup_R \sup_{t\in[0,T]} \max\left\{1, \int_K M(\rho_R^\alpha|\ln\rho_R|^\alpha)\,\mathrm{d}x\right\} \\
&\leq 2q(k) \sup_R \sup_{t\in[0,T]} \max\left\{1, 2\int_K (1+\rho_R^\alpha|\ln\rho_R|^\alpha)\ln(1+\rho_R^\alpha|\ln\rho_R|^\alpha)\,\mathrm{d}x\right\} \\
&\leq 2q(k) \sup_R \sup_{t\in[0,T]} \max\left\{1, c(\alpha)\mathrm{meas}\{K\} + c(\alpha)\int_{K\cap\{\rho_R\geq e\}} \rho_R^\alpha|\ln\rho_R|^{\alpha+1}\,\mathrm{d}x\right\} \\
&\leq 2q(k) \sup_R \sup_{t\in[0,T]} \max\left\{1, c(\alpha)\mathrm{meas}\{K\} + c(\alpha,\gamma)\int_K \rho_R^\gamma\,\mathrm{d}x\right\} \\
&\leq cq(k) \to 0, \quad \text{as } k\to\infty,
\end{aligned}
$$

for any compact subset $K \subset \mathbb{R}^3$, where the function M is defined in (3.2.1), c is a constant independent of R and

$$
q(k) := \|\chi_{[\rho_R\geq k]}\|_{L_N(\mathbb{R}^3)} \leq \left(N^{-1}\left(\frac{1}{r(k)}\right)\right)^{-1}.
$$

Similarly, we have

$$
L_k(\rho) \to \rho\ln(\rho) \text{ in } L^\infty([0,T];L^\alpha(B_{R_0})) \text{ as } k\to\infty, \quad \text{for all } 1\leq\alpha<\gamma,
$$

and, by Lemma 2.2.3,

$$
T_k(\rho) \to \overline{T_k(\rho)} \text{ in } L^\alpha([0,T];L^\alpha(\mathbb{R}^3)) \text{ as } k\to\infty, \quad \text{for all } 1\leq\alpha<\gamma+1. \tag{2.2.34}
$$

Finally, making use of Lemma 2.2.2 and the monotonicity of the pressure (see (2.2.19)), we obtain the following estimate on the right hand side of (2.2.33):

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3} T_k(\rho)\mathrm{div}\mathbf{u}\,\mathrm{d}x\mathrm{d}t - \lim_{R\to\infty}\int_0^t\int_{\mathbb{R}^3} T_k(\rho_R)\mathrm{div}\mathbf{u}_R\,\mathrm{d}x\mathrm{d}t \\
&\leq \int_0^t\int_{\mathbb{R}^3} (T_k(\rho) - \overline{T_k(\rho)})\mathrm{div}\mathbf{u}\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.2.35}
$$

From (2.2.34) and the Sobolev embedding theorem, we see that the right hand side of (2.2.35) tends to zero as $k \to \infty$. Accordingly, one can pass to the limit for $k \to \infty$ in (2.2.33) to conclude

$$\int_{\mathbb{R}^3} \left(\overline{\rho \ln(\rho)} - \rho \ln(\rho)\right)(x,t)\, \mathrm{d}x = 0, \text{ for } t \in [0,T] \; a.e. \tag{2.2.36}$$

Because of the convexity of the function $z \to z\ln z$, we have

$$\overline{\rho \ln(\rho)} \geq \rho \ln(\rho), \quad \text{a.e. in } \mathbb{R}^3 \times (0,T),$$

which, combining with (2.2.36), implies

$$\overline{\rho \ln(\rho)}(t) = \rho \ln(\rho)(t), \quad \text{for } t \in [0,T] \; a.e. \tag{2.2.37}$$

Theorem 2.11 in [19], combined with (2.2.37), implies

$$\rho_\varepsilon \to \rho, \quad \text{a.e. in } \mathbb{R}^3 \times (0,T).$$

From the estimate in Lemma 4.1.14 on ρ, together with Proposition 2.1 in [19], again we know,

$$\rho_R \to \rho \text{ weakly in } L^1(\mathbb{R}^3 \times (0,T)),$$

subject to a subsequence. By Theorem 2.10 in [19], we know that for any $\eta > 0$, there exists $\sigma > 0$ such that for all $R > 0$,

$$\int_E \rho_R(t,x)\, \mathrm{d}x\mathrm{d}t < \eta,$$

for any measurable set $E \subset \mathbb{R}^3 \times (0,T)$ with $\text{meas}\{E\} < \sigma$.

On the other hand, by virtue of Egorov's Theorem, for a given compact subset $K \subset \mathbb{R}^3$ and for $\sigma > 0$ given above, there exists a measurable set $E_\sigma \subset K \times (0,T)$ such that

$$\text{meas}\{E_\sigma\} < \sigma, \text{ and } \rho_R(x,t) \to \rho(x,t) \text{ uniformly in } \mathbb{R}^3 \times (0,T) - E_\sigma.$$

Therefore, we have, for any compact subset $K \subset \mathbb{R}^3$,

$$\begin{aligned} &\iint_{K\times(0,T)} |\rho_R - \rho|\, \mathrm{d}x\mathrm{d}t \\ &\leq \iint_{E_\sigma} |\rho_R - \rho|\, \mathrm{d}x\mathrm{d}t + \iint_{K\times(0,T)-E_\sigma} |\rho_R - \rho|\, \mathrm{d}x\mathrm{d}t \\ &\leq 2\eta + T\text{meas}\{K\} \sup_{(x,t)\in E_\sigma^c} |(\rho_R - \rho)(x,t)|, \end{aligned} \tag{2.2.38}$$

which tends to zero if we first let $R \to 0+$, and then let $\eta \to 0+$. The strong convergence of the sequence ρ_R in $L^1(K \times (0,T))$ follows from (2.2.38). Finally, a standard diagonalization process gives the strong convergence of the sequence ρ_R in $L^1(0,T; L_2^\gamma(\mathbb{R}^3))$.

The proof of Theorem 2.1.1 is completed.

3.0 INCOMPRESSIBLE LIMIT

Studies on magnetohydrodynamic flows always involve a choice at the onset to describe the system entirely in the context of either incompressible magnetohydrodynamics, or compressible MHD. For example, theoretic studies on turbulences have a particular leaning toward the incompressible model. This preference has largely been based on the benefits and advantages of the similarity of incompressible MHD to its hydrodynamic counterparts, and the practical consideration of limited computational resources.

In this chapter, we will set up a connection between the isentropic compressible MHD and the incompressible MHD in terms of the Mach number (the ratio between the speed of the flow and the speed of sound). It turns out that as Mach number vanishes, the compressible MHD converges to the incompressible MHD.

In Section 3.1, we will state our main result. Section 3.2 is devoted to the proof of the main result. The result in this chapter is a part of the work in [38].

3.1 LOW MACH LIMIT

In this section, we describe the setting of our problem and state our main results. First, we denote by P the orthogonal projection onto incompressible vector fields, i.e.

$$v = Pv + Qv, \quad \text{with} \quad \text{div}(Pv) = 0,\ \text{curl}(Qv) = 0,$$

for all $v \in L^2$. Indeed, in view of results in [23], we know that the operators P and Q are linear bounded operators in $W^{s,p}$ for all $s \geq 0$ and $1 < p < \infty$ in the whole space or bounded domains with smooth boundaries. Second, let us explain the notation of weak solutions to the incompressible MHD equations as follows: Given the initial conditions $\mathbf{u}_0 \in L^2$, $\mathbf{H}_0 \in L^2$ such that $\mathrm{div}\mathbf{u}_0 = 0$ and $\mathrm{div}\mathbf{H}_0 = 0$, $(\mathbf{u}, \mathbf{H})$ is a weak solution of (1.0.7) satisfying

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0, \tag{3.1.1}$$

where

$$\mathbf{u} \in C([0,T]; L^2_{weak}) \cap L^2([0,T]; H^1(\mathbb{R}^3)),\ \mathbf{H} \in C([0,T]; L^2_{weak}) \cap L^2([0,T]; H^1(\mathbb{R}^3)),$$

if for all $T < \infty$, $\psi \in C_0^\infty(\mathbb{R}^3)$ with $\mathrm{div}\psi = 0$, and $\varphi \in C_0^\infty([0,T))$, we have

$$\begin{aligned}&\psi(0)\int_{\mathbb{R}}^3 \mathbf{u}_0\varphi dx + \int_0^t \psi'(t)\int_{\mathbb{R}}^3 \mathbf{u}\cdot\varphi dxdt + \int_0^t \psi(t)\int_{\mathbb{R}}^3 (\mathbf{u}_i\partial_i\varphi_j\mathbf{u}_j - \mu\nabla\mathbf{u} : \nabla\varphi)\, dxdt\\ &= -\int_0^t\int_{\mathbb{R}}^3 \psi(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\varphi dxdt,\end{aligned}$$

and

$$\begin{aligned}&\psi(0)\int_{\mathbb{R}}^3 \mathbf{H}_0\varphi dx + \int_0^t \psi'(t)\int_{\mathbb{R}}^3 \mathbf{H}\cdot\varphi dxdt + \int_0^t \psi(t)\int_{\mathbb{R}}^3 (\mathbf{u}\times\mathbf{H})\cdot(\nabla\times\varphi)dxdt\\ &= \nu\int_0^t \psi(t)\int_{\mathbb{R}}^3 (\nabla\times\mathbf{H})\cdot(\nabla\times\varphi)dxdt.\end{aligned}$$

For more details as to the existence and regularity of weak solutions to the incompressible MHD equations, we refer the readers to [16, 47, 66].

For the convenience of presentation, we only discuss the case when $a = 1$. We consider a sequence of global weak solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)$ of the compressible MHD equations (1.0.6) in $\mathbb{R}^3 \times (0,T)$ and assume that

$$\begin{gathered}\rho_\varepsilon - 1 \in L^\infty([0,T]; L_2^\gamma(\mathbb{R}^3)),\ \nabla\mathbf{u}_\varepsilon \in L^2([0,T]; L^2(\mathbb{R}^3)),\\ \rho_\varepsilon|\mathbf{u}_\varepsilon|^2 \in L^\infty([0,T]; L^1(\mathbb{R}^3)),\ \rho_\varepsilon\mathbf{u}_\varepsilon \in C\left([0,T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3)\right),\\ \mathbf{H}_\varepsilon \in L^2([0,T]; H^1(\mathbb{R}^3)) \cap C([0,T]; L^2_{weak}(\mathbb{R}^3)),\end{gathered}$$

for all $T \in (0,\infty)$, where $C([0,T]; L^p_{weak})$ denotes the functions which are continuous with respect to $t \in [0,T]$ with values in L^p endowed with the weak topology. We require (1.0.6) to hold in the sense of distributions. Finally, we prescribe initial conditions

$$\rho_\varepsilon|_{t=0} = \rho_\varepsilon^0, \quad \rho_\varepsilon\mathbf{u}_\varepsilon|_{t=0} = m_\varepsilon^0 = \rho_\varepsilon^0\mathbf{u}_\varepsilon^0, \quad \mathbf{H}_\varepsilon|_{t=0} = \mathbf{H}_\varepsilon^0, \tag{3.1.2}$$

where $\rho_\varepsilon^0 \geq 0$, $(\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + (\gamma-1) \in L^1(\mathbb{R}^3)$, $m_\varepsilon^0 \in L^{2\gamma/(\gamma+1)}(\mathbb{R}^3)$, $m_\varepsilon^0 = 0$ on $\{\rho_\varepsilon^0 = 0\}$, $\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 \in L^1(\mathbb{R}^3)$, and $\mathbf{H}_\varepsilon^0 \in L^2(\mathbb{R}^3)$. Furthermore, we assume that the weak solutions $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ satisfy the following conditions at infinity:

$$\rho_\varepsilon \to 1, \qquad \mathbf{u}_\varepsilon \to 0, \qquad \mathbf{H}_\varepsilon \to 0, \qquad \text{as } |x| \to \infty,$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3} \left(\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 + |\mathbf{H}_\varepsilon^0|^2\right) dx + \frac{1}{\varepsilon^2(\gamma-1)}\int_{\mathbb{R}^3} \left((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + (\gamma-1)\right) dx \leq C, \tag{3.1.3}$$

where and hereafter C denotes a generic positive constant independent of ε. We assume finally that the total energy is conserved in the sense:

$$E_\varepsilon(t) + \int_0^t D_\varepsilon(s)ds \leq E_\varepsilon^0, \quad \text{a.e} \quad t \in [0,T], \tag{3.1.4}$$

where

$$E_\varepsilon = \frac{1}{2}\int_{\mathbb{R}^3} \left(\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + |\mathbf{H}_\varepsilon|^2 + \frac{1}{\varepsilon^2(\gamma-1)}\left((\rho_\varepsilon)^\gamma - \gamma\rho_\varepsilon + (\gamma-1)\right)\right) dx,$$
$$D_\varepsilon = \int_{\mathbb{R}^3} \left(\mu_\varepsilon|D\mathbf{u}_\varepsilon|^2 + \lambda_\varepsilon(\mathrm{div}\mathbf{u}_\varepsilon)^2 + \nu_\varepsilon|\nabla\times\mathbf{H}_\varepsilon|^2\right) dx,$$

and

$$E_\varepsilon^0 = \frac{1}{2}\int_{\mathbb{R}^3} \left(\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2 + |\mathbf{H}_\varepsilon^0|^2 + \frac{1}{\varepsilon^2(\gamma-1)}\left((\rho_\varepsilon^0)^\gamma - \gamma\rho_\varepsilon^0 + (\gamma-1)\right)\right) dx.$$

We recall the results in the previous chapter and in [36] which yield the existence of such a solution with the above properties precisely as $\gamma > \frac{3}{2}$.

Now we are ready to state our result in the whole space as follows.

Theorem 3.1.1. *Assume that $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ is a sequence of weak solutions to the compressible MHD equations (1.0.6) in the whole space $\mathbb{R}^3$ with the initial data $\{(\rho_\varepsilon^0, \mathbf{u}_\varepsilon^0, \mathbf{H}_\varepsilon^0)\}_{\varepsilon>0}$, satisfying the conditions (3.1.2), (3.1.3), (3.1.4) and $\gamma > \frac{3}{2}$. Also assume that $(\mathbf{u}, \mathbf{H}) \in [L^2([0,T]; H^1(\mathbb{R}^3)) \cap L^\infty([0,T]; L^2(\mathbb{R}^3))]^6$ is a weak solution to the incompressible MHD equations (1.0.7) with initial data $\mathbf{u}|_{t=0} = P\mathbf{u}_0$ and $\mathbf{H}|_{t=0} = \mathbf{H}_0$. Then, for any finite number T, up to a subsequence, the global weak solutions $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)\}_{\varepsilon>0}$ converge to $(\mathbf{u}, \mathbf{H})$. More precisely, as $\varepsilon \to 0$,*

$$\rho_\varepsilon - 1 \text{ converges to } 0 \text{ in } C([0,T]; L_2^\gamma(\mathbb{R}^3));$$

$$P\mathbf{u}_\varepsilon \textit{ converges strongly to } \mathbf{u} \textit{ in } L^2([0,T];L^p_{loc}(\mathbb{R}^3)), \textit{ for all } 1\le p<6;$$

$$Q\mathbf{u}_\varepsilon \textit{ converges strongly to } 0 \textit{ in } L^2([0,T];L^q(\mathbb{R}^3)), \quad \textit{for all} \quad 2<q<6;$$

$$\mathbf{H}_\varepsilon \textit{ converges to } \mathbf{H} \textit{ strongly in } L^2([0,T];L^2_{loc}(\mathbb{R}^3)) \textit{ and weakly in } L^2([0,T];H^1(\mathbb{R}^3)).$$

In this chapter, we will prove Theorem 3.1.1 in the spirit of [11, 38]. Before we start, we introduce homogeneous Sobolev spaces for $1<p<\infty$ and $s\in R$ defined as usual by

$$\dot{W}^{s,p}(\mathbb{R}^N)=(-\Delta)^{-s/2}L^p(\mathbb{R}^N) \quad \text{and} \quad \dot{H}^s(\mathbb{R}^N)=\dot{W}^{s,2}(\mathbb{R}^N),$$

where Δ is the Laplace operator and N is the dimension of the space.

Let us denote by $\zeta\in C_0^\infty(\mathbb{R}^N)$ a smoothing kernel such that $\zeta\ge 0$, $\int_{\mathbb{R}^N}\zeta dx=1$, and define $\zeta_\alpha(x)=\alpha^{-N}\zeta(x/\alpha)$. The following estimate will be useful in this section (cf. [11]):

$$\|f-f*\zeta_\alpha\|_{L^q}\le C\alpha^{1-\sigma}\|\nabla f\|_{L^2}, \quad \text{for all} \quad f\in\dot{H}^1, \tag{3.1.5}$$

where

$$q\in\left[2,\frac{2N}{N-2}\right) \quad \text{and} \quad \sigma=N\left(\frac{1}{2}-\frac{1}{q}\right),$$

and for $1<p_2<p_1<\infty$, $s\ge 0$ and $\alpha\in(0,1)$, we have

$$\|g*\zeta_\alpha\|_{L^{p_1}(\mathbb{R}^N)}\le C\alpha^{-s-N(1/p_2-1/p_1)}\|g\|_{W^{-s,p_2}(\mathbb{R}^N)}. \tag{3.1.6}$$

3.2 PROOF OF THEOREM

The aim of this section is to prove Theorem 3.1.1. We will finish the proof via several steps.

3.2.1 *A priori* estimates and consequences

We first deduce from (1.0.6) and from the conservation of energy that we have for almost all $t \geq 0$,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}\left(\rho_\varepsilon|\mathbf{u}_\varepsilon|^2+|\mathbf{H}_\varepsilon|^2+\frac{1}{\varepsilon^2(\gamma-1)}\left((\rho_\varepsilon)^\gamma-\gamma\rho_\varepsilon+(\gamma-1)\right)\right)dx\\
&\qquad+\int_0^t\int_{\mathbb{R}^3}\left(\mu_\varepsilon|D\mathbf{u}_\varepsilon|^2+\lambda_\varepsilon(\mathrm{div}\mathbf{u}_\varepsilon)^2+\nu_\varepsilon|\nabla\times\mathbf{H}_\varepsilon|^2\right)dxds\\
&\leq\frac{1}{2}\int_{\mathbb{R}^3}\left(\rho_\varepsilon^0|\mathbf{u}_\varepsilon^0|^2+|\mathbf{H}_\varepsilon^0|^2+\frac{1}{\varepsilon^2(\gamma-1)}\left((\rho_\varepsilon^0)^\gamma-\gamma\rho_\varepsilon^0+(\gamma-1)\right)\right)dx\\
&\leq C.
\end{aligned} \tag{3.2.1}$$

From this inequality we see that $\rho_\varepsilon|\mathbf{u}_\varepsilon|^2$, $|\mathbf{H}_\varepsilon|^2$ and $\frac{1}{\varepsilon^2}((\rho_\varepsilon)^\gamma-\gamma\rho_\varepsilon+(\gamma-1))$ are bounded in $L^\infty([0,T];L^1(\mathbb{R}^3))$ and that $D\mathbf{u}_\varepsilon$ and $\nabla\times\mathbf{H}_\varepsilon$ are bounded in $L^2([0,T];L^2(\mathbb{R}^3))$. In particular, we see that $\rho_\varepsilon-1$ is bounded in $L^\infty([0,T];L^\gamma(\mathbb{R}^3))$ as $\gamma>2$ and in $L^\infty([0,T];L_2^\gamma(\mathbb{R}^3))$ as $\gamma\in(\frac{3}{2},2]$ for all $T\in(0,\infty)$ in view of (2.1.6). Indeed, we have

$$\int_{R^3}\left(|\rho_\varepsilon-1|^2\chi_{\{|\rho_\varepsilon-1|\leq 1/2\}}+|\rho_\varepsilon-1|^\gamma\chi_{\{|\rho_\varepsilon-1|\geq 1/2\}}\right)\leq C\varepsilon^2. \tag{3.2.2}$$

Hence, $\rho_\varepsilon-1$ converges to 0 in $C([0,T];L_2^\gamma(\mathbb{R}^3))$ as $\varepsilon\to 0$.

The bound on $\mathbf{u}_\varepsilon$ in $L^2([0,T];L^2(\mathbb{R}^N))$ follows from (2.1.6) and (3.2.2). In fact, we can deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3}|\mathbf{u}_\varepsilon|^2dx&\leq C+\int_{\mathbb{R}^3}|\mathbf{u}_\varepsilon|^2\chi_{\{\rho_\varepsilon\leq 1/2\}}dx\\
&\leq C+\left(\int_{\mathbb{R}^3}\chi_{\{\rho_\varepsilon\leq 1/2\}}dx\right)^{1/\gamma}\left(\int_{\mathbb{R}^3}|\mathbf{u}_\varepsilon|^{2\gamma'}dx\right)^{1/\gamma'}\\
&\leq C\left(1+(\mathrm{meas}(|\rho_\varepsilon-1|\geq 1/2))^{1/\gamma}\|\mathbf{u}_\varepsilon\|_{L^2}^{2\theta}\|D\mathbf{u}_\varepsilon\|_{L^2}^{2(1-\theta)}\right)\\
&\leq C\left(1+\varepsilon^{2/\gamma}\|\mathbf{u}_\varepsilon\|_{L^2}^{2\theta}\|D\mathbf{u}_\varepsilon\|_{L^2}^{2(1-\theta)}\right),
\end{aligned}$$

where

$$\theta=\frac{2\gamma-3}{6\gamma}.$$

We then complete the proof of our claim using the bound on $D\mathbf{u}_\varepsilon$ in $L^2([0,T];L^2(\mathbb{R}^N))$ and the classical Young's inequality. Moreover, if we define the density fluctuation as

$$\varphi_\varepsilon=\frac{\rho_\varepsilon-1}{\varepsilon},$$

then, it is bounded uniformly in ε in $L^\infty([0,T];L_2^\kappa)$ with $\kappa = \min\{2,\gamma\}$. Furthermore, if we write

$$\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon^1 + \mathbf{u}_\varepsilon^2,$$

where

$$\mathbf{u}_\varepsilon^1 = \mathbf{u}_\varepsilon \chi_{\{|\rho_\varepsilon - 1| \le 1/2\}} \quad \text{and} \quad \mathbf{u}_\varepsilon^2 = \mathbf{u}_\varepsilon \chi_{\{|\rho_\varepsilon - 1| > 1/2\}},$$

then, we have

$$\sup_{t\ge 0} \int_{\mathbb{R}^N} |\mathbf{u}_\varepsilon^1|^2 dx \le 2 \sup_{t\ge 0} \int_{\mathbb{R}^N} \rho_\varepsilon |\mathbf{u}_\varepsilon|^2 dx \le C,$$

and for $p = 2\kappa/3$,

$$\begin{aligned}\int_{\mathbb{R}^3} |\mathbf{u}_\varepsilon^2|^2 dx &\le C \int_{\mathbb{R}^3} |\rho_\varepsilon - 1|^p \chi_{\{|\rho_\varepsilon - 1| > 1/2\}} |\mathbf{u}_\varepsilon|^2 dx \\ &\le C \|(\rho_\varepsilon - 1)\chi_{\{|\rho_\varepsilon - 1| > 1/2\}}\|^p_{L^\infty([0,T];L^\kappa(\mathbb{R}^3))} \|\mathbf{u}_\varepsilon\|^2_{L^{2\kappa/(\kappa - p)}} \\ &\le C\varepsilon^{2p/\kappa} \|\mathbf{u}_\varepsilon\|^{2-3p/\kappa}_{L^2(\mathbb{R}^3)} \|\nabla \mathbf{u}_\varepsilon\|^{3p/\kappa}_{L^2(\mathbb{R}^3)},\end{aligned}$$

hence, by Young's inequality, $\mathbf{u}_\varepsilon^1$ is bounded in $L^\infty([0,T];L^2(\mathbb{R}^3))$ and $\mathbf{u}_\varepsilon^2 \varepsilon^{-2/3}$ is bounded in $L^2([0,T];L^2(\mathbb{R}^3))$.

Recalling that $\gamma > 3/2$, we deduce that $\mathbf{u}_\varepsilon$ is bounded in

$$L^2([0,T];L^4(\mathbb{R}^3) \cap L^{2\gamma/(\gamma-1)}(\mathbb{R}^3)).$$

Hence, we have

$$\|\varphi_\varepsilon \mathbf{u}_\varepsilon\|_{L^2([0,T];L^{4/3}(\mathbb{R}^3) + L^{2\kappa/(\kappa+1)}(\mathbb{R}^3)} \le C.$$

Therefore, using Sobolev's imbedding, we deduce

$$\|\varphi_\varepsilon \mathbf{u}_\varepsilon\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \le C.$$

Finally, we already know that φ_ε^0 is bounded in $L_2^\kappa(\mathbb{R}^3)$, hence in $H^{-1}(\mathbb{R}^3)$, since $\gamma > 3/2$. On the other hand, m_ε^0 can be rewritten as

$$m_\varepsilon^0 = \frac{m_\varepsilon^0}{\sqrt{\rho_\varepsilon^0}} \sqrt{\rho_\varepsilon^0} \chi_{\{|\rho_\varepsilon^0 - 1| \le 1/2\}} + \frac{m_\varepsilon^0}{\sqrt{\rho_\varepsilon^0}} \frac{\sqrt{\rho_\varepsilon^0}}{\sqrt{|\rho_\varepsilon^0 - 1|}} \sqrt{|\rho_\varepsilon^0 - 1|} \chi_{\{|\rho_\varepsilon^0 - 1| > 1/2\}}.$$

This implies that m_ε^0 is bounded in $L^2(\mathbb{R}^3) + L^{2\kappa/(\kappa+1)}(\mathbb{R}^3)$, and hence in $H^{-1}(\mathbb{R}^3)$. Therefore, $\binom{\varphi_\varepsilon^0}{m_\varepsilon^0}$ is bounded in $H^{-1}(\mathbb{R}^3)$ uniformly in ε.

3.2.2 Strong convergence of $Q\mathbf{u}_\varepsilon$ to 0

We now prove that the gradient part of the velocity $Q\mathbf{u}_\varepsilon$ converges strongly to 0. More precisely, we claim that $Q\mathbf{u}_\varepsilon$ converges strongly to 0 in $L^2([0,T];L^p(\mathbb{R}^3))$ for all $p\in(2,6)$. Indeed, let us first observe that the compressible MHD equations can be rewritten in terms of the density fluctuation φ_ε, the momentum $m_\varepsilon=\rho_\varepsilon\mathbf{u}_\varepsilon$ and $\phi_\varepsilon=\binom{\varphi_\varepsilon}{m_\varepsilon}$ as follows

$$\partial_t\phi_\varepsilon+\frac{L\phi_\varepsilon}{\varepsilon}=F_\varepsilon^1+F_\varepsilon^2,$$

where the wave operator L is defined on $(\mathcal{D}'(\mathbb{R}^3))^4$ with values in $(\mathcal{D}'(\mathbb{R}^3))^4$ by

$$L\phi=\begin{pmatrix}\operatorname{div} m\\ \nabla\psi\end{pmatrix},\quad\text{with}\quad \phi=\begin{pmatrix}\psi\\ m\end{pmatrix},$$

and

$$F_\varepsilon^1=\begin{pmatrix}0\\ \mu_\varepsilon\Delta\mathbf{u}_\varepsilon^1+\lambda_\varepsilon\nabla\mathrm{div}\mathbf{u}_\varepsilon^1-\mathrm{div}(m_\varepsilon\otimes\mathbf{u}_\varepsilon)-\frac{a}{\varepsilon^2}\nabla(\rho_\varepsilon^\gamma-1-\gamma(\rho_\varepsilon-1))+(\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon\end{pmatrix},$$
$$F_\varepsilon^2=\begin{pmatrix}0\\ \mu_\varepsilon\Delta\mathbf{u}_\varepsilon^2+\lambda_\varepsilon\nabla\mathrm{div}\mathbf{u}_\varepsilon^2\end{pmatrix}.$$

Using *Duhamel*'s formula, we deduce that

$$Q\phi_\varepsilon(t)=\mathcal{L}\left(\frac{t}{\varepsilon}\right)Q\phi_\varepsilon^0+\int_0^t\mathcal{L}\left(\frac{t-s}{\varepsilon}\right)(QF_\varepsilon^1(s)+QF_\varepsilon^2(s))ds.$$

Here we used the fact that Q and $\mathcal{L}$ commute, since Q and L do.

At this stage, the following Strichartz's estimates from [11] are useful:

Lemma 3.2.1. *For all $s\geq 0$, we have*

$$\left\|\mathcal{L}\left(\frac{t}{\varepsilon}\right)Q\psi_0\right\|_{L^q((0,\infty);W^{-s-\sigma,p}(\mathbb{R}^3))}\leq C\varepsilon^{1/q}\|\psi_0\|_{H^{-s}(\mathbb{R}^3)},\tag{3.2.3}$$

$$\left\|\int_0^t\mathcal{L}\left(\frac{t-s}{\varepsilon}\right)Q\psi(s)ds\right\|_{L^q([0,T];W^{-s-\sigma,p}(\mathbb{R}^3))}\leq C(1+T)\varepsilon^{1/q}\|\psi\|_{L^q([0,T];H^{-s}(\mathbb{R}^3))},\tag{3.2.4}$$

for all $(p,q)\in(2,\infty)\times(2,\infty)$ and $\sigma\in(0,\infty)$ such that

$$\frac{2}{q}=2\left(\frac{1}{2}-\frac{1}{p}\right)\quad\text{and}\quad\sigma q=2.\tag{3.2.5}$$

Now, we choose $p \in (2,6)$, $q \in (2,\infty)$ and $\sigma \in (0,\infty)$ given by (3.2.5). One can deduce that

$$|Q\mathbf{u}_\varepsilon| \le |Q\mathbf{u}_\varepsilon - Q\mathbf{u}_\varepsilon * \zeta_\alpha| + \varepsilon |Q(\mathbf{u}_\varepsilon \varphi_\varepsilon) * \zeta_\alpha| + |Qm_\varepsilon * \zeta_\alpha|.$$

Hence,

$$\begin{aligned}\|Q\mathbf{u}_\varepsilon\|_{L^2([0,T];L^p(\mathbb{R}^3))} \le C\alpha^{1-3(1/2-1/p)} &\|\nabla \mathbf{u}_\varepsilon\|_{L^2([0,T];L^2(\mathbb{R}^3))} \\ &+ \varepsilon \alpha^{-1-3(1/2-1/p)} \|\varphi_\varepsilon \mathbf{u}_\varepsilon\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\ &+ \|Qm_\varepsilon * \zeta_\alpha\|_{L^2([0,T];L^p(\mathbb{R}^3))}.\end{aligned}$$

From the estimates in previous subsection, we know that F_ε^1 is bounded in $L^\infty([0,T];H^{-s_0})$ for all $s_0 > \frac{5}{2}$. On the other hand, we deduce from the uniform bound on $\mathbf{u}_\varepsilon^2 \varepsilon^{-\beta}$ in $L^2([0,T];L^2)$ that $\varepsilon^{-\beta} F_\varepsilon^2$ is bounded in $L^2([0,T];H^{-2}(\mathbb{R}^3))$. Then, using Lemma 3.2.1, we obtain, for all $\eta > 0$ small enough,

$$\begin{aligned}&\|Qm_\varepsilon * \zeta_\alpha\|_{L^2([0,T];L^p(\mathbb{R}^3))} \\ &\le C_T \alpha^{-1-\sigma} \left\| \mathcal{L}\left(\frac{t}{\varepsilon}\right) \psi_\varepsilon^0 \right\|_{L^q([0,T];W^{-1-\sigma,p}(\mathbb{R}^3))} \\ &\quad + C_T \alpha^{-5/2-\sigma-\eta} \left\| \int_0^T ds \mathcal{L}\left(\frac{t-s}{\varepsilon}\right) QF_\varepsilon^1(s) \right\|_{L^q([0,T];W^{-\eta-5/2-\sigma,p}(\mathbb{R}^3))} \\ &\quad + C\alpha^{-7/2+3/p} \left\| \int_0^t ds \mathcal{L}\left(\frac{t-s}{\varepsilon}\right) QF_\varepsilon^2(s) \right\|_{L^2([0,T];H^{-2})} \\ &\le C_T \alpha^{-1-\sigma} \varepsilon^{1/q} \|\psi_\varepsilon^0\|_{H^{-1}} + C_T \alpha^{-5/2-\sigma-\eta} \varepsilon^{1/q} \|F_\varepsilon^1\|_{L^\infty([0,T];H^{-\eta-5/2})} \\ &\quad + C\alpha^{-7/2+3/p} \varepsilon^\beta \|\varepsilon^{-\beta} F_\varepsilon^2\|_{L^2([0,T];H^{-2})}.\end{aligned}$$

Next, fixing $\alpha > 0$ and letting ε go to zero, we obtain

$$\limsup_{\varepsilon \to 0} \|Q\mathbf{u}_\varepsilon\|_{L^2([0,T];L^p(\mathbb{R}^3))} \le C\alpha^{-1/2+3/p},$$

where C is independent of ε and α. Noticing that $-1/2 + 3/p > 0$, we finally get, by letting $\alpha \to 0$,

$$\limsup_{\varepsilon \to 0} \|Q\mathbf{u}_\varepsilon\|_{L^2([0,T];L^p(\mathbb{R}^3))} = 0.$$

This implies that $Q\mathbf{u}_\varepsilon$ strongly converges to 0 in $L^2([0,T];L^p(\mathbb{R}^3))$ for all $2 < p < 6$.

3.2.3 Strong convergences of $P\mathbf{u}_\varepsilon$ and $\mathbf{H}_\varepsilon$

To this end, we show from the previous bounds that $\mathrm{div}\mathbf{u}_\varepsilon$ converges weakly to 0 in $L^2([0,T];L^2(\mathbb{R}^3))$ and that $P\mathbf{u}_\varepsilon$ converges to $\mathbf{u} = P\mathbf{u}$ strongly in $L^2([0,T];L^2(\mathbb{R}^3))$, and thus by Sobolev imbedding in $L^2([0,T];L^q)$ for all $2 \le q < 6$. These facts imply that $Q\mathbf{u}_\varepsilon$ converges weakly to 0 in $L^2([0,T];H^1(\mathbb{R}^3))$, and converges strongly to 0 in $L^2([0,T];L^p(\mathbb{R}^3))$ for all $2 < p < 6$. Indeed, since ρ_ε converges to 1 in $C((0,\infty);L_2^\gamma(\mathbb{R}^3))$ and $\gamma > \frac{3}{2}$, we deduce from (1.0.6) that $\mathrm{div}\mathbf{u}_\varepsilon$ converges weakly to 0 in $L^2([0,T];L^2(\mathbb{R}^3))$. The second part is proven by observing first that we project (1.0.6) onto divergence-free vector-fields:

$$\partial_t P(\rho_\varepsilon \mathbf{u}_\varepsilon) + P[\mathrm{div}(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] - \mu_\varepsilon \Delta P\mathbf{u}_\varepsilon = P((\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon). \tag{3.2.6}$$

Noticing the fact that the operator P is bounded in all Sobolev space $W^{s,p}$ for all $s \in [0,\infty)$ and $1 < p < \infty$ and the preceding bounds, (3.2.6) yields a bound on $\partial_t P(\rho_\varepsilon \mathbf{u}_\varepsilon)$ in $L^1([0,T];H^{-1}(\mathbb{R}^3)) + L^2([0,T];L^1(\mathbb{R}^3)) + L^2([0,T];H^{-1}(\mathbb{R}^3))$, hence, in $L^1([0,T];H^{-1}(\mathbb{R}^3))$. In addition, $P(\rho_\varepsilon \mathbf{u}_\varepsilon)$ is bounded in $L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3)) \cap L^2([0,T];L^r(\mathbb{R}^3))$ with

$$\frac{1}{r} = \frac{1}{\gamma} + \frac{1}{6}.$$

Next, we will need the following compactness Lemma (cf. Lemma 5.1 in [51]):

Lemma 3.2.2. *Let g_n, h_n converge weakly to g, h respectively in $L^{p_1}(0,T;L^{p_2})$, $L^{q_1}(0,T;L^{q_2})$ where $1 \le p_1, p_2 \le \infty$,*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Assume in addition that

$$\frac{\partial g_n}{\partial t} \quad \textit{is bounded in} \quad L^1(0,T;W^{-m,1}) \quad \textit{for some} \quad m \ge 0 \quad \textit{independent of } n,$$

and

$$\|h_n - h_n(\cdot + \xi, t)\|_{L^{q_1}(0,T;L^{q_2})} \to 0 \quad \textit{as} \quad |\xi| \to 0, \quad \textit{uniformly in } n.$$

Then $g_n h_n$ converges to gh in the sense of distributions in $\Omega \times (0,T)$.

Applying this lemma with the previous bounds, we deduce that $P(\rho_\varepsilon \mathbf{u}_\varepsilon) \cdot P\mathbf{u}_\varepsilon$ converges in the sense of distributions to $|\mathbf{u}|^2$. We then conclude easily that $P\mathbf{u}_\varepsilon$ converges in $L^2([0,T]; L^2(\mathbb{R}^3))$ to $\mathbf{u}$ upon using the weak convergence of $P\mathbf{u}_\varepsilon$ to $\mathbf{u}$ in $L^2([0,T]; L^2(\mathbb{R}^3))$ and remarking that we have

$$\left| \int_0^T \int_{\mathbb{R}^3} \big(|P\mathbf{u}_\varepsilon|^2 - P(\rho_\varepsilon \mathbf{u}_\varepsilon) \cdot P\mathbf{u}_\varepsilon \big) \, dxdt \right| \le C \|\rho_\varepsilon - 1\|_{C([0,T];L^\gamma)} \|\mathbf{u}_\varepsilon\|^2_{L^2([0,T];L^s)},$$

with $s = \frac{2\gamma}{\gamma - 1} < 6$ since $\gamma > \frac{3}{2}$.

Finally, the bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T]; L^2(\mathbb{R}^3))$ and the bound on $\nabla \mathbf{H}_\varepsilon$ in $L^2([0,T]; L^2(\mathbb{R}^3))$, combining Sobolev's inequality and interpolation theorem, we know that $\mathbf{H}_\varepsilon$ is bounded in $L^{8/3}([0,T]; L^4(\mathbb{R}^3))$, and also we can assume that $\mathbf{H}_\varepsilon$ converges weakly to some $\mathbf{H}$ in $L^2([0,T]; H^1(\mathbb{R}^3))$ with $\mathrm{div}\mathbf{H} = 0$. Finally, from the induction equation in (1.0.6), we deduce that $\partial_t \mathbf{H}_\varepsilon$ is bounded in $L^{8/7}([0,T]; H^{-1}(\mathbb{R}^3))$, due to the fact that $\mathbf{u}_\varepsilon$ is bounded in $L^2([0,T]; L^4(\mathbb{R}^N))$. This property, combining Aubin-Lions compactness Lemma, implies that $\mathbf{H}_\varepsilon$ converges strongly to $\mathbf{H}$ in $L^{8/7}([0,T]; L^2_{loc}(\mathbb{R}^3))$. Moreover, the uniform bound on $\mathbf{H}_\varepsilon$ in $L^\infty([0,T]; L^2(\mathbb{R}^3))$ implies that $\mathbf{H}_\varepsilon$ converges strongly to $\mathbf{H}$ in $L^2([0,T]; L^2_{loc}(\mathbb{R}^3))$. Therefore, by a standard argument, we deduce that the limits $\mathbf{u}$ and $\mathbf{H}$ satisfy the induction equation in (1.0.7) in the sense of distributions, and also the nonlinear term $(\nabla \times \mathbf{H}_\varepsilon) \times \mathbf{H}_\varepsilon$ in the second equation of (1.0.6) converges to $(\nabla \times \mathbf{H}) \times \mathbf{H}$ in the sense of distributions. For a detailed statement of the above argument, we refer it to the argument of the convergence of the magnetic field in [38].

The proof of Theorem 3.1.1 is complete.

4.0 HYDRODYNAMIC LIMIT

The kinetic theory, introduced by Boltzmann at the end of the nineteenth century, provides a description of gases at an intermediate level between the hydrodynamic description which does not allow to take into account phenomena far from thermodynamic equilibrium, and the atomistic description which is often too complex. For a detailed presentation of the model for the dilute gases and their derivation from the fundamental laws of physics, we refer to the book of Cercignani, Illner, and Pulvirenti [7]. That kinetic theory aims at describing a gas (or a plasma), that is a system constructed of a large number of electrically neutral (or charged) particles from a *microscopic point of view*.

The aim of this chapter is to describe the state of art about the hydrodynamic limit of the Vlasov-Maxwell-Boltzmann equations, which is not so complete as its counterpart of incompressible Navier-Stokes equations from the Boltzmann equations. Due to the strong coupling between the fluid variables, the electric field, and the magnetic field for Vlasov-Maxwell-Boltzmann equations, the convergence results describing the hydrodynamic limit of VMB require additional regularity assumptions on the solutions to VMB.

In section 4.1, we will derive a dimensionless analysis, prove the relative entropy inequality, and state the main result. In Section 4.2, we focus on the convergence of the functions as the mean free path vanishes. In Section 4.3, we discuss the limit equation of the Maxwell equation. Section 4.4 is devoted to the analysis of the vanishing conservation effect. Finally, we prove our main result in Section 4.5. We remark that the result reported in this chapter is collected in [41].

4.1 DIMENSIONLESS ANALYSIS AND PRELIMINARY

Our starting point is the Vlasov-Maxwell-Boltzmann equation. In this section we collect the basic facts we will need. These will include its nondimensionalization and its formal conservation and dissipation laws.

In this dissertation, we will focus on the nondimensional form of the Vlasov-Maxwell-Boltzmann equation. This form is motivated by the fact the incompressible electron-magneto-hydrodynamics-fourier system (1.0.10) can be formally derived from the Boltzmann equation through a scaling in which the density F is close to a spatially homogeneous Maxwellian $M = M(\xi)$ that has the same total mass, momentum, and energy as the initial data F^{in}. To this end, we introduce

$$t = t_* \hat{t}, \qquad \xi = x_* \hat{x}, \qquad \xi = \xi_* \hat{\xi},$$
$$F = \frac{1}{\mu_0 \xi_*^3 x_*^2} \hat{F}, \qquad E = \frac{\xi_*}{t_*} \hat{E}, \qquad B = \frac{1}{t_*} \hat{B},$$

and

$$b = \frac{x_*}{\eta_0 \xi_*} \hat{b}.$$

In this paper, we are interested in the scaling of non-relativistic domain, which means that we are concentrated in the case

$$\xi_* = \frac{x_*}{t_*}, \quad \text{and} \quad \varepsilon = \left(\frac{\xi_*}{c} \right)^2 \ll 1.$$

Substituting those new variables back to (1.0.8), and dropping hats, we obtain

$$\frac{\partial F}{\partial t} + \xi \cdot \nabla_x F + e(E + \xi \times B) \cdot \nabla_\xi F = \frac{1}{\varepsilon} \mathcal{Q}(F, F), \tag{4.1.1a}$$
$$\varepsilon \frac{\partial E}{\partial t} - \nabla \times B = -j, \tag{4.1.1b}$$
$$\frac{\partial B}{\partial t} + \nabla \times E = 0, \tag{4.1.1c}$$
$$\text{div} B = 0, \quad \text{and} \quad \text{div} E = \frac{\rho}{\varepsilon}, \tag{4.1.1d}$$

where the coefficient ε in the right hand side of (4.1.1a) is usually refereed as the dimensionless mean free path or Knudsen number.

Since the incompressible magnetohydrodynamic flow is the large-scale low-frequency fluid-like behavior of a plasma system ([4, 28]), we need to further scale the time to the order of ε^{-1} while setting the distance between the absolute Maxwellian M and the density F to be of order ε. For this purpose, in the system (4.1.1), we introduce a further scaling as

$$\begin{aligned}
&\hat{t} = \varepsilon t, \qquad \hat{x} = \varepsilon x, \qquad \hat{\xi} = \varepsilon \xi, \\
&\hat{F} = \frac{1}{\varepsilon^5} F, \qquad \hat{E} = \frac{1}{\varepsilon} E, \qquad \hat{B} = \frac{1}{\varepsilon} B, \\
&\text{and} \\
&\hat{b} = \varepsilon^2 b.
\end{aligned}$$

Then substituting the scaling as above back into (4.1.1), and dropping hats, we obtain

$$\varepsilon \frac{\partial F}{\partial t} + \xi \cdot \nabla_x F + e\varepsilon(\varepsilon E + \xi \times B) \cdot \nabla_\xi F = \frac{1}{\varepsilon} \mathcal{Q}(F, F), \tag{4.1.2a}$$

$$\varepsilon \frac{\partial E}{\partial t} - \nabla \times B = -\frac{j}{\varepsilon}, \tag{4.1.2b}$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0, \tag{4.1.2c}$$

$$\text{div} B = 0, \quad \text{and} \quad \text{div} E = \frac{\rho}{\varepsilon}, \tag{4.1.2d}$$

The incompressible Electron-Magnetohydrodynamics-Fourier equation is obtained with a scaling in which F is considered close to the absolute Maxwellian M. As in [3, 27, 52], it is natural to introduce the relative density, $G = G(t, x, \xi)$, defined by $F = MG$, where the dimensionless equilibrium Maxwellian is now

$$M = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\left(-\frac{1}{2}|\xi|^2\right). \tag{4.1.3}$$

This, the so-called absolute Maxwellian, corresponds to the spatially homogeneous fluid state with its density and temperature equal to 1, bulk velocity equal to 0 and no effect from the electric field and the magnetic field. This state is consistent with the form of the incompressible Electron-Magnetohydrodynamics-Fourier system. Recasting the initial-value problem (4.1.2) for G yields

$$\varepsilon\frac{\partial G}{\partial t} + \xi\cdot\nabla_x G + e\varepsilon(\varepsilon E + \xi\times B)\cdot\nabla_\xi G - e\varepsilon^2 E\cdot\xi G = \frac{1}{\varepsilon}Q(G,G), \tag{4.1.4a}$$

$$\varepsilon\frac{\partial E}{\partial t} - \nabla\times B = -\frac{j}{\varepsilon}, \tag{4.1.4b}$$

$$\frac{\partial B}{\partial t} + \nabla\times E = 0, \tag{4.1.4c}$$

$$\mathrm{div}B = 0, \quad \text{and} \quad \mathrm{div}E = \frac{\rho}{\varepsilon}. \tag{4.1.4d}$$

where the collision operator is now given by

$$Q(G,G) = \int_{\mathbb{R}^3}\int_{S^2}(G'_*G' - G_*G)b(\xi_* - \xi,\omega)d\omega M_* d\xi_*.$$

4.1.1 Relative Entropy

For any pair of measurable functions f and g defined a.e. on $\mathbb{R}^3\times\mathbb{R}^3$ and satisfying $f\geq 0$ and $g>0$ a.e., we use the following notation for the relative entropy

$$H(f|g) = \int_{\mathcal{T}}\int_{\mathbb{R}^3}\left[f\ln\left(\frac{f}{g}\right) - f + g\right]d\xi dx \in [0,\infty], \tag{4.1.5}$$

which is a way to measure how far f is away from g. We are interested in the evolution of

$$\mathcal{H}_\varepsilon(t) = \varepsilon H(F_\varepsilon|M) + \frac{\varepsilon^3}{2}\int_{\mathcal{T}}(\varepsilon|E_\varepsilon|^2 + |B_\varepsilon|^2)dx \tag{4.1.6}$$

where $(F_\varepsilon, E_\varepsilon, B_\varepsilon)_{\{\varepsilon>0\}}$ are renormalized solutions of Vlasov-Maxwell-Boltzmann equations. This quantity contains the information from the standard (rescaled) L^2 norm of the electromagnetic field and from the relative entropy between the renormalized solution $F_\varepsilon(t,x,\xi)$ and the absolute Maxwellian M.

The following lemma is devoted to the study of the evolution of the relative entropy, deduced from

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_\varepsilon = \varepsilon\int_{\mathcal{T}}\int_{\mathbb{R}^3}\partial_t F_\varepsilon(\ln F_\varepsilon - \ln M)d\xi dx \\ + \frac{\varepsilon^3}{2}\frac{d}{dt}\int_{\mathcal{T}}(\varepsilon|E_\varepsilon|^2 + |B_\varepsilon|^2)dx.\end{aligned} \tag{4.1.7}$$

Lemma 4.1.1. *$\mathcal{H}_\varepsilon(t)$ satisfies the differential inequality:*

$$\frac{d}{dt}\mathcal{H}_\varepsilon(t) + \frac{1}{4\varepsilon}\int_{\mathcal{T}}\int_{\mathbb{R}^3} \ln\left(\frac{F_{\varepsilon*}'F_\varepsilon'}{F_{\varepsilon*}F_\varepsilon}\right)(F_{\varepsilon*}'F_\varepsilon' - F_{\varepsilon*}F_\varepsilon)d\xi dx \leq 0. \tag{4.1.8}$$

Proof. Observing that

$$\partial_t F_\varepsilon \ln F_\varepsilon = \partial_t(F_\varepsilon \ln F_\varepsilon) - \partial_t F_\varepsilon,$$

we obtain

$$\varepsilon\int_{\mathcal{T}}\int_{\mathbb{R}^3} \partial_t F_\varepsilon \ln F_\varepsilon d\xi dx = -\frac{1}{4\varepsilon}\int_{\mathcal{T}}\int_{\mathbb{R}^3} \ln\left(\frac{F_{\varepsilon*}'F_\varepsilon'}{F_{\varepsilon*}F_\varepsilon}\right)(F_{\varepsilon*}'F_\varepsilon' - F_{\varepsilon*}F_\varepsilon)d\xi dx,$$

and, by (4.1.2)

$$\varepsilon\int_{\mathcal{T}}\int_{\mathbb{R}^3} \partial_t F_\varepsilon \ln M d\xi dx = -\varepsilon^2 e\int_{\mathcal{T}}\int_{\mathbb{R}^3} F_\varepsilon E_\varepsilon\cdot\xi d\xi dx.$$

Here, we used the following identity twice (see [27])

$$\int_{\mathcal{T}} Q(f,f)\zeta(\xi)d\xi = \frac{1}{4}\int_{\mathcal{T}}\int_{\mathbb{R}^3} d\xi d\xi_* \int_{S^2} d\omega B(f'f_*' - ff_*)[\zeta + \zeta_* - \zeta' - \zeta_*'].$$

Hence,

$$\begin{aligned}\varepsilon\int_{\mathcal{T}}\int_{\mathbb{R}^3} \partial_t F_\varepsilon(\ln F_\varepsilon - \ln M)d\xi dx = &-\frac{1}{4\varepsilon}\int_{\mathcal{T}}\int_{\mathbb{R}^3} \ln\left(\frac{F_{\varepsilon*}'F_\varepsilon'}{F_{\varepsilon*}F_\varepsilon}\right)(F_{\varepsilon*}'F_\varepsilon' - F_{\varepsilon*}F_\varepsilon)d\xi dx\\ &+ e\varepsilon^2\int_{\mathcal{T}}\int_{\mathbb{R}^3} F_\varepsilon E_\varepsilon\cdot\xi d\xi dx.\end{aligned} \tag{4.1.9}$$

On the other hand, multiplying equation (4.1.2b) by E_ε, equation (4.1.2c) by B_ε, integrating them in x over $\mathbb{R}^3$ and then summing them together, we obtain,

$$\frac{d}{dt}\int_{\mathcal{T}}(\varepsilon|E_\varepsilon|^2 + |B_\varepsilon|^2)dx = -\frac{2}{\varepsilon}\int_{\mathcal{T}} E_\varepsilon\cdot j_\varepsilon dx = -e\frac{2}{\varepsilon}\int_{\mathcal{T}}\int_{\mathbb{R}^3} E_\varepsilon\cdot\xi F_\varepsilon d\xi dx. \tag{4.1.10}$$

Substituting (4.1.10) back into (4.1.9), we obtain

$$\begin{aligned}\varepsilon\int_{\mathcal{T}}\int_{\mathbb{R}^3} \partial_t F_\varepsilon(\ln F_\varepsilon - \ln M)d\xi dx = &-\frac{1}{4\varepsilon}\int_{\mathcal{T}}\int_{\mathbb{R}^3} \ln\left(\frac{F_{\varepsilon*}'F_\varepsilon'}{F_{\varepsilon*}F_\varepsilon}\right)(F_{\varepsilon*}'F_\varepsilon' - F_{\varepsilon*}F_\varepsilon)d\xi dx\\ &- e\frac{\varepsilon^3}{2}\frac{d}{dt}\int_{\mathcal{T}}(\varepsilon|E_\varepsilon|^2 + |B_\varepsilon|^2)dx,\end{aligned}$$

which is exactly the differential inequality (4.1.8). □

In order to avoid unnecessary constants in the sequel, we will assume that the nondimensionalization has the following normalizations:

$$\int_{S^2} d\omega = 1, \qquad \int_{\mathbb{R}^3} M d\xi = 1, \qquad \int_{\mathcal{T}} dx = 1,$$

associated with the domain S^2, $\mathbb{R}^3$, and $\mathcal{T}^3$ respectively;

$$\int_{\mathcal{T}} \int_{\mathbb{R}^3} G^{in} M d\xi dx = 1, \qquad \int_{\mathcal{T}} \int_{\mathbb{R}^3} \xi G^{in} M d\xi dx = 0,$$

$$\int_{\mathcal{T}} \int_{\mathbb{R}^3} \frac{1}{2} |\xi|^2 G^{in} M d\xi dx = \frac{3}{2},$$

associated with the initial data; and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\xi_* - \xi, \omega) d\omega M_* d\xi_* M d\xi = 1,$$

associated with the Boltzmann kernel.

Since $M d\xi$ is a positive unit measure on $\mathbb{R}^3$, we denote by $< \eta >$ the average over this measure of any integrable function $\eta = \eta(\xi)$,

$$< \eta >= \int_{\mathbb{R}^3} \eta M d\xi.$$

Since $d\mathcal{M} = b(\xi_* - \xi, \omega) d\omega M_* d\xi_* M d\xi$ is a non-negative unit measure on $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$, we denote by $\ll \tau \gg$ the average over this measure of any integrable function $\tau = \tau(\xi, \xi_*, \omega)$,

$$\ll \tau \gg = \int_{\mathbb{R}^3} \tau d\mathcal{M}.$$

The collision measure $d\mathcal{M}$ is invariant under the transformations

$$(\omega, \xi_*, \xi) \to (\omega, \xi, \xi_*), \quad (\omega, \xi_*, \xi) \to (\omega, \xi', \xi'_*).$$

These, and compositions of these, are called collisional symmetries (cf. [3, 27]).

Further, we can explain Lemma 4.1.1 in terms of G_ε as follows:

$$\begin{aligned} &\varepsilon \frac{d}{dt} \int_{\mathcal{T}} < G_\varepsilon \ln G_\varepsilon - G_\varepsilon + 1 > dx + \frac{\varepsilon^3}{2} \frac{d}{dt} \int_{\mathcal{T}} (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx \\ &\quad + \frac{1}{4\varepsilon} \int_{\mathcal{T}} \ll \ln \left(\frac{G_{\varepsilon *}' G_\varepsilon'}{G_{\varepsilon *} G_\varepsilon} \right) (G_{\varepsilon *}' G_\varepsilon' - G_{\varepsilon *} G_\varepsilon) \gg dx \le 0. \end{aligned} \tag{4.1.11}$$

This yields, if G_ε solves the VMB equation (4.1.4), then inequality (4.1.11) implies

$$\mathcal{H}_\varepsilon(t) + \frac{1}{\varepsilon}\int_0^t R(G_\varepsilon(s))ds = \mathcal{H}_\varepsilon(0), \tag{4.1.12}$$

where $\mathcal{H}_\varepsilon(t)$ is the entropy functional

$$\mathcal{H}_\varepsilon(t) = \varepsilon \int_{\mathbb{R}^3} < G_\varepsilon \ln G_\varepsilon - G_\varepsilon + 1 > dx + \frac{\varepsilon^3}{2}\int_{\mathbb{R}^3} (\varepsilon|E_\varepsilon|^2 + |B_\varepsilon|^2)dx, \tag{4.1.13}$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int_{\mathbb{R}^3} \frac{1}{4} \ll \ln\left(\frac{G'_* G'}{G_* G}\right)(G'_* G' - G_* G) \gg dx. \tag{4.1.14}$$

This choice of $\mathcal{H}_\varepsilon$ as the entropy functional (4.1.13) is based on the fact that its integrand is a non-negative strictly convex function of G with a minimum value of zero at $G = 1$. Thus for any G,

$$H(G) \geq 0, \qquad \text{and} \qquad H(G) = 0, \qquad \text{iff} \quad G = 1. \tag{4.1.15}$$

This is the so-called relative entropy of G with respect to the absolute equilibrium $G = 1$; it provides a natural measure of the proximity of G to that equilibrium.

We can expect that, the terms involving the entropy $\mathcal{H}_\varepsilon$ measure the proximity of G_ε and G_ε^0 to the absolute equilibrium value of 1. On the other hand, the terms involving the dissipation rate R, can be understood to measure the proximity of G_ε to any Maxwellian through their characterization.

4.1.2 Assumptions

In this subsection, we state our assumptions. To begin with, we give our additional assumptions regarding the collision kernel b. These assumptions are satisfied by many classical collision kernels. We have already stated that the collision kernel b is nonnegative almost everywhere. The three additional assumptions on the kernel b are technical in nature; that is, they are required by our mathematical argument. In applications, we therefore examine which of the commonly studied physical collisional kernels satisfy these assumptions. The three additional assumptions are

- **(H0)** $b \in L^1(B_R \times S^2) \quad \text{for all} \quad R \in (0, \infty)$, where $B_R = \{z \in \mathbb{R}^3 : |z| < R\}$, and

$$\begin{cases} b(z, w) \quad \text{depends only on} \quad |z| \quad \text{and} \quad |(z, \omega)|, \\ (1 + |z|^2)^{-1} \left(\int_{z+B_R} A(\xi) d\xi \right) \to 0, \quad \text{as} \quad |z| \to \infty, \quad \text{for all} \quad R \in (0, \infty). \end{cases}$$

- **(H1)** $\frac{1}{b_\infty} \le b(z, \omega) \le b_\infty, \quad z \in \mathbb{R}^3, \quad \omega \in S^2, \quad \text{for some} \quad b_\infty > 0;$
- **(H2)** $\frac{|A(\xi)|+|B(\xi)|}{1+|\xi|^p} \in L^\infty_\xi \quad \text{for some} \quad p \ge 0.$

The assumption (**H0**) is assumed to make possible the global existence of renormalized solutions to Vlasov-Maxwell-Boltzmann equations, see [14, 40]. The class of collision kernels satisfying (**H0**), (**H1**) and (**H2**) is not empty since it contains at least all collision kernels of the form $b(z, \omega) = b(|\cos(z, \omega)|)$ satisfying (**H0**). These collision kernels correspond to cutoff Maxwellian molecules and satisfy (**H2**) with $p = 3$ (see [27]).

Next, we impose one more technical assumption on the sequence of fluctuations $\{g_\varepsilon\}_{\{\varepsilon>0\}}$ (which is not implied by the weak stability result in [40]).

- **(H3)** The family $(1 + |\xi|^2)\frac{g_\varepsilon^2}{N_\varepsilon}$ is relatively compact in $w - L^1(dtMd\xi dx)$, where $N_\varepsilon = 1 + \frac{\varepsilon}{3} g_\varepsilon$.

This assumption is the same as $(A2)$ of [53] and similar to $(H2)$ of [3], with the only difference being that we had to add the time variable, since we are dealing with the nonstationary case, when compared with the stationary case in [3].

4.1.3 Main Result

The incompressible Electron-Magnetohydrodynamics-Fourier system is recovered by setting the distance to the absolute Maxwellian M to be of order ε while rescaling time to the order of ε^{-1}. Namely, we consider a sequence of solutions G_ε to the scaled Vlasov-Maxwell-Boltzmann equation

$$\varepsilon\partial_t G_\varepsilon + \xi \cdot \nabla_x G_\varepsilon + e\varepsilon(\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \cdot \nabla_\xi G_\varepsilon - e\varepsilon^2 E_\varepsilon \cdot \xi G_\varepsilon = \frac{1}{\varepsilon} Q(G_\varepsilon, G_\varepsilon), \tag{4.1.16}$$

in the form

$$G_\varepsilon = 1 + \varepsilon g_\varepsilon. \tag{4.1.17}$$

We expect that as ε tends to zero, the leading behavior of the fluctuations g_ε is formally consistent with the incompressible Electron-Magnetohydrodynamics-Fourier equations. Indeed, formally, substituting (4.1.17) into (4.1.16) and using Taylor's expansion of the collision operator, we obtain

$$\varepsilon\partial_t g_\varepsilon + \xi \cdot \nabla_x g_\varepsilon + e\varepsilon(\varepsilon E + \xi \times B) \cdot \nabla_\xi g_\varepsilon - e\varepsilon E_\varepsilon \cdot \xi - e\varepsilon^2 E \cdot \xi g_\varepsilon + \frac{1}{\varepsilon} L g_\varepsilon = Q(g_\varepsilon, g_\varepsilon), \tag{4.1.18}$$

where L, the linearized collision operator, is given by

$$Lg = -2Q(1, g) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (g + g_* - g' - g'_*) b d\omega M_* d\xi_*.$$

Repeated application of the $d\mathcal{M}$-symmetries yields the identity

$$\begin{aligned} < vLg > &= \ll v(g + g_* - g' - g'_* \gg \\ &= \frac{1}{4} \ll (v + v_* - v' - v'_*)(g + g_* - g' - g'_*) \gg, \end{aligned}$$

for every $v = v(\xi)$ and $g = g(\xi)$ for which the integral makes sense. This shows that L is formally self-adjoint and has a non-negative Hermitian form. Furthermore, using the $d\mathcal{M}$-characterization, it can be shown that for any $g = g(\xi)$ in the domain of L, the following statements are equivalent:

$$Lg = 0; \tag{4.1.19a}$$

$$g = \alpha + \beta \cdot \xi + \frac{1}{2}\gamma|\xi|^2, \qquad \text{for some} \qquad (\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}. \tag{4.1.19b}$$

This characterizes $N(L)$, the null space of L, as the set obtained by linearizing about $(\alpha, \beta, \gamma) = (0, 0, 0)$. From (4.1.18), we deduce formally that the limit of Lg_ε is zero and it can be expected that the limit of g_ε will belong to $N(L)$. Indeed, it was proved by H. Grad (see [27, 29]) that for any collision kernel b satisfying (**H1**), L is a bounded nonnegative self-adjoint Fredholm operator on $L^2(Md\xi)$ with null space

$$KerL = \text{span}\{1, \xi_1, \xi_2, \xi_3, |\xi|^2\}.$$

Because each entry of the tensor $\xi \otimes \xi - \frac{1}{3}|\xi|^2 I$ and of the vector $\frac{1}{2}\xi(|\xi|^2 - 5)$ is orthogonal to $KerL$, there exist a unique tensor Φ and a unique vector Ψ such that

$$L\Phi = \xi \otimes \xi - \frac{1}{3}|\xi|^2 I, \quad \Phi \in (KerL)^\perp \subset L^2(Md\xi); \tag{4.1.20}$$

$$L\Psi = \frac{1}{2}\xi(|\xi|^2 - 5), \quad \Psi \in (KerL)^\perp \subset L^2(Md\xi). \tag{4.1.21}$$

Then, our main result of this work goes as follows.

Theorem 4.1.1. *Let $G_\varepsilon(t, x, \xi)$ be a sequence of non-negative solutions to the scaled Vlasov-Maxwell-Boltzmann equation (4.1.4) such that, when it is written according to formula (4.1.17), the sequence g_ε converges in the sense of distributions and almost everywhere to a function g as ε tends to zero. Furthermore, assume that the moments*

$$< g_\varepsilon >, \quad < \xi g_\varepsilon >, \quad < \xi \otimes \xi g_\varepsilon >, \quad < \xi|\xi|^2 g_\varepsilon >,$$

$$< \Phi \otimes \xi g_\varepsilon >, \quad < \Phi Q(g_\varepsilon, g_\varepsilon) >, \quad < \Psi \otimes \xi g_\varepsilon >, \quad < \Psi Q(g_\varepsilon, g_\varepsilon) >,$$

converges in the sense of distributions to the corresponding moments

$$< g >, \quad < \xi g >, \quad < \xi \otimes \xi g >, \quad < \xi|\xi|^2 g >,$$

$$< \Phi \otimes \xi g >, \quad < \Phi Q(g, g) >, \quad < \Psi \otimes \xi g >, \quad < \Psi Q(g, g) >,$$

and that all formally small terms in ε vanish. Then the limiting form of g is that of an infinitesimal Maxwellian,

$$g = h + \mathbf{u} \cdot \xi + \theta \left(\frac{1}{2}|\xi|^2 - \frac{3}{2} \right), \tag{4.1.22}$$

where the velocity $\mathbf{u}$ *satisfies the incompressibility relation, while the density and temperature functions,* h *and* θ*, satisfy the Boussinesq relation:*

$$\mathrm{div}\mathbf{u} = 0, \qquad \nabla_x(h + \theta) = 0. \tag{4.1.23}$$

Moreover, the functions h*,* $\mathbf{u}$*, and* θ *are weak solutions of the equations*

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla P - \alpha e E = (\nabla \times B) \times B; \tag{4.1.24a}$$

$$j = \nabla \times B \quad \text{and} \quad j = e\mathbf{u}; \tag{4.1.24b}$$

$$\partial_t B + \nabla \times E = 0; \tag{4.1.24c}$$

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \nabla \theta, \tag{4.1.24d}$$

$$\mathrm{div} B = 0. \tag{4.1.24e}$$

In these equations the coefficients μ *and* κ *are given by*

$$\mu = \frac{1}{10} < \Phi : L\Phi >, \qquad \kappa = \frac{2}{15} < \Psi \cdot L\Psi > . \tag{4.1.25}$$

Remark 4.1.1. Our result can be extended to a much larger class of collision kernel as [52].

4.1.4 Global Solutions

In order to mathematically justify the incompressible Electron-Magnetohydrodynamics-Fourier limit of the Vlasov-Maxwell-Boltzmann equation, we must make precise:

- the notion of solutions for the Vlasov-Maxwell-Boltzmann equations;
- the notion of solutions for the incompressible Electron-Magnetohydrodynamics-Fourier system (1.0.10).

Ideally, these solutions should be global while the bounds should be physical natural. We therefore work in the setting of DiPerna-Lions renormalized solutions for the Vlasov-Maxwell-Boltzmann equations, and in the setting of Leray solutions for the incompressible Electron-Magnetohydrodynamics-Fourier system. These theories have the virtues of considering physically natural classes of initial data, and consequently, of yielding global solutions.

DiPerna-Lions renormalized solutions are not known to satisfy many properties that one would formally expect to be satisfied by solutions of the Vlasov-Maxwell-Boltzmann equations. In particular, the theory does not assert either the local conservation of momentum, the global conservation of energy, the global entropy equality, or even a local entropy inequality; nor does it assert the uniqueness of the solution. Nevertheless, as shown in [40], it provides enough control to establish an incompressible Electron-Magnetohydrodynamics-Fourier limit theory for bounded collision kernels and, as shown in [52], to do so for a much larger class of collision kernels.

4.1.4.1 Renormalized Solutions to the Vlasov-Maxwell-Boltzmann Equations

In the spirit of the DiPerna-Lions theory for Boltzmann equation and the idea in [40], modified slightly for the periodic box, it is possible to show the weak stability of a global weak solution to a whole class of formally equivalent initial-value problems. More precisely, let $G_\varepsilon \geq 0$ be a sequence of DiPerna-Lions renormalized solutions to the scaled Vlasov-Maxwellian-Boltzmann initial-value problem

$$\varepsilon\partial_t G_\varepsilon + \xi\cdot\nabla_x G_\varepsilon + e\varepsilon\left(\varepsilon E_\varepsilon + \xi\times B_\varepsilon\right)\cdot\nabla_\xi G_\varepsilon - e\varepsilon^2 E_\varepsilon\cdot\xi G_\varepsilon = \frac{1}{\varepsilon}Q(G_\varepsilon, G_\varepsilon), \tag{4.1.26a}$$

$$\varepsilon\frac{\partial E_\varepsilon}{\partial t} - \nabla\times B_\varepsilon = -\frac{j_\varepsilon}{\varepsilon}, \tag{4.1.26b}$$

$$\frac{\partial B_\varepsilon}{\partial t} + \nabla\times E_\varepsilon = 0, \tag{4.1.26c}$$

$$\mathrm{div} B = 0, \quad \text{and} \quad \mathrm{div} E = \frac{\rho_\varepsilon}{\varepsilon}. \tag{4.1.26d}$$

with

$$G_\varepsilon(0,x,\xi) = G^0_\varepsilon(x,\xi) \geq 0, \quad E_\varepsilon(0,x) = E^0_\varepsilon(x), \quad B_\varepsilon(0,x) = B^0_\varepsilon(x).$$

Assume that the initial data G^0_ε satisfies the normalizations and the entropy bound

$$\mathcal{H}_\varepsilon(0) \leq C\varepsilon^3, \tag{4.1.27}$$

for some fixed $C > 0$.

A *Renormalized Solution Relative to M* of (1.0.8) is a triple $(F_\varepsilon, E_\varepsilon, B_\varepsilon)$ such that $F_\varepsilon \in C(\mathbb{R}_+; L^1_{loc}(R^3; L^1(\mathbb{R}^3)))$, $E_\varepsilon, B_\varepsilon \in C_w(\mathbb{R}_+; L^2(\mathbb{R}^3))$ and satisfies

$$\Gamma'\left(\frac{F_\varepsilon}{M}\right)\mathcal{Q}(F_\varepsilon, F_\varepsilon) \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathcal{T}) \tag{4.1.28}$$

for all $\Gamma \in C^1(\mathbb{R}_+)$ such that

$$\Gamma(0) = 0, \quad \text{and} \quad z \mapsto (1+z)\Gamma'(z) \quad \text{is bounded on} \quad \mathbb{R}_+, \tag{4.1.29}$$

has finite relative entropy for all positive times:

$$\mathcal{H}_\varepsilon(t) + \frac{1}{\varepsilon}\int_0^t R(G(s))ds \le \mathcal{H}_\varepsilon(0), \tag{4.1.30}$$

and finally satisfies

$$\begin{aligned}
&\int_0^\infty \int_{\mathcal{T}} \int_{\mathbb{R}^3} \Gamma\left(\frac{F_\varepsilon}{M}\right)\left(\partial_t \chi + \frac{1}{\varepsilon}\xi \cdot \nabla_x \chi\right) M d\xi dx dt \\
&\quad + e\int_0^\infty \int_{\mathcal{T}} \int_{\mathbb{R}^3} \Gamma\left(\frac{F_\varepsilon}{M}\right)(\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \cdot \nabla_\xi \chi M d\xi dx dt \\
&\quad - e\int_0^\infty \int_{\mathcal{T}} \int_{\mathbb{R}^3} \Gamma\left(\frac{F_\varepsilon}{M}\right)\varepsilon E_\varepsilon \cdot \xi \chi M d\xi dx dt \\
&\quad + e\int_0^\infty \int_{\mathcal{T}} \int_{\mathbb{R}^3} \Gamma'\left(\frac{F_\varepsilon}{M}\right)\varepsilon E_\varepsilon \cdot \xi F_\varepsilon \chi d\xi dx dt \\
&\quad + \frac{1}{\varepsilon^2}\int_0^\infty \int_{\mathcal{T}} \int_{\mathbb{R}^3} \Gamma'\left(\frac{F_\varepsilon}{M}\right)\mathcal{Q}(F_\varepsilon, F_\varepsilon)\chi M d\xi dx dt = 0
\end{aligned} \tag{4.1.31}$$

for each test function $\chi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$.

4.1.4.2 The Limiting System (1.0.10) The DiPerna-Lions theory has many similarities with the Leray theory of global weak solutions of the initial-value problem for incompressible magnetohydrodynamic flow (for instance, [66]). For the limiting system (1.0.10) with mean zero initial data, the Leray theory is set in the following Hilbert spaces of vector- and scalar-valued functions:

$$\mathbf{H}_v = \left\{ w \in L^2(dx;\mathbb{R}^3) : \quad \mathrm{div} w = 0, \quad \int_{\mathcal{T}} w dx = 0 \right\},$$
$$\mathbf{H}_s = \left\{ \chi \in L^2(dx;\mathbb{R}) : \quad \int_{\mathcal{T}} \chi dx = 0 \right\},$$
$$\mathbf{V}_v = \left\{ w \in \mathcal{H}_v : \quad \int_{\mathcal{T}} |\nabla w|^2 dx < \infty \right\},$$
$$\mathbf{V}_s = \left\{ \chi \in \mathbf{H}_s : \quad \int_{\mathcal{T}} |\nabla \chi| dx < \infty \right\}.$$

Let $\mathbf{H} = \mathbf{H}_v \bigoplus \mathbf{H}_v \bigoplus \mathbf{H}_s$ and $\mathbf{H} = \mathbf{V}_v \bigoplus \mathbf{V}_v \bigoplus \mathbf{V}_s$. Leray theory yields: given any $(\mathbf{u}^{in}, B^{in}, \theta^{in}) \in \mathbf{H}$, there exists a $(\mathbf{u}, B, \theta) \in C([0,\infty); w - \mathbf{H}) \cap L^2_{loc}(0,\infty;\mathbf{V})$ which equals initially $(\mathbf{u}^{in}, B^{in}, \theta^{in}) \in \mathbf{H}$ and satisfies the incompressible magnetohydrodynamic system in the sense that for every $(\phi, \psi, \chi) \in \mathbf{H} \cap C^1(\mathcal{T})$

$$\begin{aligned}
&\int_{\mathcal{T}} \phi \cdot \mathbf{u}(t) dx - \int_{\mathcal{T}} \phi \cdot \mathbf{u}(s) dx - \int_s^t \int_{\mathcal{T}} \nabla_x \phi : (\mathbf{u} \otimes \mathbf{u}) dx d\tau \\
&\quad = -\mu \int_s^t \int_{\mathcal{T}} \nabla_x \phi : \nabla_x \mathbf{u} dx d\tau - \alpha e \int_s^t \int_{\mathcal{T}} E \phi dx d\tau + \int_s^t \int_{\mathcal{T}} \nabla_x \phi : (B \otimes B) dx d\tau; \\
&\int_{\mathcal{T}} \psi \cdot B(t) dx - \int_{\mathcal{T}} \psi \cdot B(s) dx + \int_s^t \int_{\mathcal{T}} E \cdot (\nabla_x \times \psi) dx d\tau = 0; \\
&\int_{\mathcal{T}} \chi \theta(t) dx - \int_{\mathcal{T}} \chi \theta(s) dx - \int_s^t \int_{\mathcal{T}} \nabla_x \chi \cdot (\mathbf{u}\theta) dx d\tau \\
&\quad = -\kappa \int_s^t \int_{\mathcal{T}} \nabla_x \chi \cdot \nabla_x \theta dx d\tau,
\end{aligned}$$

for every $0 \le s < t$. Moreover, $(\mathbf{u}, B, \theta)$ satisfies the dissipation inequalities

$$\int_{\mathcal{T}} \frac{1}{2} \left(|\mathbf{u}(t)|^2 + \alpha |B(t)|^2 \right) + \int_0^t \mu |\nabla_x \mathbf{u}|^2 dx ds \le \int_{\mathcal{T}} \frac{1}{2} \left(|\mathbf{u}^{in}|^2 + \alpha |B^{in}|^2 \right) dx, \tag{4.1.32a}$$
$$\int_{\mathcal{T}} \frac{1}{2} |\theta(t)|^2 dx + \int_0^t \int_{\mathcal{T}} \kappa |\nabla_x \theta|^2 dx ds \le \int_{\mathcal{T}} \frac{1}{2} |\theta^{in}|^2 dx, \tag{4.1.32b}$$

for every $t > 0$.

A global existence theory, similar to Leray theory of incompressible Navier-Stokes equations, can be established via Garlerkin's method, the dissipation inequalities (4.1.32) and Ohm's law which expresses the electric field E in terms of the magnetic field and the velocity as, see [4, 28]

$$j = \sigma(E + \mathbf{u} \times B),$$

where $\sigma > 0$ is the electrical conductivity. To obtain the dissipation inequality (4.1.32a), we first multiply (1.0.10a) by $\mathbf{u}$ to obtain, using (1.0.10b),

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\mathcal{T})} + \mu\|\nabla\mathbf{u}\|^2_{L^2(\mathcal{T})} - \alpha\int_{\mathcal{T}} E\cdot(\nabla\times B)dx = 0. \tag{4.1.33}$$

Here, we used the identity

$$B\times(\nabla\times B)\cdot\mathbf{u} = \frac{1}{e}(B\times(\nabla\times B))\cdot j = \frac{1}{e}(B\times(\nabla\times B))\cdot(\nabla\times B) = 0,$$

according to (1.0.10b). Then, we multiply (1.0.10c) by αB to obtain

$$\frac{\alpha}{2}\frac{d}{dt}\|B\|^2_{L^2(\mathcal{T})} + \alpha\int_{\mathcal{T}} E\cdot(\nabla\times B) = 0. \tag{4.1.34}$$

Adding (4.1.33) and (4.1.34), and then integrating it over $(0,T)$ yield the energy inequality (4.1.32a).

In summarize, we have the following existence theory for the incompressible system (1.0.10).

Proposition 4.1.1. *For each* $\mathbf{u}_0, B_0 \in \{f\in L^2(\mathbb{R}^3) : \mathrm{div} f = 0 \quad in \quad \mathcal{D}'\}$ *and* $\theta_0 \in L^2(\mathbb{R}^3)$, *there exists at least one weak solution* $(\mathbf{u},\theta,B)$ *of* (1.0.10)-(1.0.11) *that satisfies the energy inequality*

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{T}}(|\mathbf{u}(t,x)|^2 + \alpha|B(t,x)|^2 + \frac{5}{2}|\theta(t,x)|^2)dx + \int_0^t\int_{\mathcal{T}}\left(\mu|\nabla\mathbf{u}|^2 + \frac{5}{2}\kappa|\nabla\theta|^2\right)dxds\\ &\quad\le \frac{1}{2}\int_{\mathcal{T}}(|\mathbf{u}_0|^2 + \alpha|B_0|^2 + \frac{5}{2}|\theta_0|^2)dx\end{aligned}$$

for all $t > 0$.

4.2 IMPLICATIONS OF THE ENTROPY INEQUALITY

As stated before, authors in [40] had proposed the possibility of establishing the global existence of a renormalized solution G_ε to the scaled Vlasov-Maxwell-Boltzmann equation, upon constructing an approximating sequence with the required integrability on the magnetic field and the electric field, satisfying the entropy inequality

$$\begin{aligned}
&\varepsilon \int_{\mathcal{T}} < G_\varepsilon(t) \ln G_\varepsilon(t) - G_\varepsilon(t) + 1 > dx + \frac{\varepsilon^3}{2} \int_{\mathcal{T}} (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx \\
&\quad + \frac{1}{\varepsilon} \int_0^t \int_{\mathcal{T}} \frac{1}{4} \ll (G'_{*\varepsilon} G'_\varepsilon - G_{*\varepsilon} G_\varepsilon) \ln \left(\frac{G'_{*\varepsilon} G'_\varepsilon}{G_{*\varepsilon} G_\varepsilon} \right) \gg dxds \\
&\leq \varepsilon \int_{\mathcal{T}} < G^0_\varepsilon \ln G^{in}_\varepsilon - G^{in}_\varepsilon + 1 > dx + \frac{\varepsilon^3}{2} \int_{\mathcal{T}} (\varepsilon |E^{in}_\varepsilon|^2 + |B^{in}_\varepsilon|^2) dx.
\end{aligned} \tag{4.2.1}$$

Consider a sequence of such solutions G_ε indexed by a vanishing positive sequence ε such that for some constant $C > 0$, the initial data G^0_ε satisfies the entropy bound:

$$\varepsilon \int_{\mathcal{T}} < G^{in}_\varepsilon \ln G^{in}_\varepsilon - G^{in}_\varepsilon + 1 > dx + \frac{\varepsilon^3}{2} \int_{\mathcal{T}} (\varepsilon |E^{in}_\varepsilon|^2 + |B^{in}_\varepsilon|^2) dx \leq C\varepsilon^3. \tag{4.2.2}$$

This then implies bounds on the sequence G_ε through the entropy inequality (4.2.1). This section contains results that follow directly from the convexity of the integrands in the entropy inequality (4.2.1) and the entropy bound (4.2.2).

Since the entropy integrand, $G \ln G - G + 1$, is a strictly convex function of G with a quadratic minimum of zero at $G = 1$, the integral approximately measures the square of the derivations from this minimum. Now, we introduce a convex function $h = h(z)$ defined over $z > -1$ by

$$h(z) = (1 + z) \ln(1 + z) - z. \tag{4.2.3}$$

The entropy inequality (4.2.1) and entropy bound (4.2.2) then give

$$\begin{aligned}
&\int_{\mathcal{T}} \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon(t)) \right\rangle dx + \frac{1}{2} \int_{\mathcal{T}} (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx \\
&\quad \leq \int_{\mathcal{T}} \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g^0_\varepsilon) \right\rangle dx + \frac{1}{2} \int_{\mathcal{T}} (\varepsilon |E^{in}_\varepsilon|^2 + |B^{in}_\varepsilon|^2) dx \leq C.
\end{aligned} \tag{4.2.4}$$

The second integral on the left hand side of the entropy inequality (4.2.1) is the entropy dissipation. Since $q_\varepsilon = \varepsilon^{-2}(G'_{*\varepsilon}G'_\varepsilon - G_{*\varepsilon}G_\varepsilon)$ and the convex function $r = r(z)$ defined over $z > -1$ by

$$r(z) = z\ln(1+z), \tag{4.2.5}$$

the entropy inequality (4.2.1) and the entropy bound (4.2.2) can then be recast in the form

$$\begin{aligned} &\int_{\mathcal{T}} \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon(t)) \right\rangle dx + \int_{\mathcal{T}} (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx \\ &+ \int_0^t \int_{\mathcal{T}} \frac{1}{4} \ll \frac{1}{\varepsilon^4} r\left(\frac{\varepsilon^2 q_\varepsilon}{G_{*\varepsilon}G_\varepsilon} \right) G_{*\varepsilon}G_\varepsilon \gg dxds \le C. \end{aligned} \tag{4.2.6}$$

Some of results in [3, 27] were established in the greatest possible generality, relied only on the *a priori* estimates and in particular has nothing to do with the equations. We will record them below without proof; they are used in various places in the present work.

Theorem 4.2.1. *Under assumptions (H0)-(H2), let F_ε be a family of renormalized solutions to (4.1.2) with initial data $(F_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in})$ satisfying $\mathcal{H}_\varepsilon(0) \le C\varepsilon^3$, and define the associated family of fluctuations by*

$$g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}.$$

Then

- *g_ε is relatively compact in $w - L^1_{loc}(dtdx; L^1((1+|\xi|^2)Md\xi))$ and for almost every $t \in [0,\infty)$ g satisfies*

$$\int_{\mathcal{T}} \frac{1}{2} < g^2(t) > dx \le \liminf_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon(t)) \right\rangle dx \le C. \tag{4.2.7}$$

  Moreover, for almost every (t,x), $g(t,x,\cdot) \in N(L)$, which means that g is of the form

$$g(t,x,\cdot) = h(t,x) + \mathbf{u}(t,x)\cdot\xi + \theta(t,x)\left(\frac{1}{2}|\xi|^2 - \frac{3}{2}\right), \tag{4.2.8}$$

  where $(h, \mathbf{u}, \theta) \in L^\infty(dt; L^2(dx; \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}))$.

- *the rescaled collision integrands*

$$q_\varepsilon = \frac{1}{\varepsilon^2}(G'_\varepsilon G'_{\varepsilon *} - G_\varepsilon G_{\varepsilon *}) \tag{4.2.9}$$

are such that the renormalized family $\gamma(G_\varepsilon)q_\varepsilon$ *is relatively compact in* $w - L^1_{loc}(dtdx; L^1((1+|\xi|^2)d\mathcal{M}))$*; further, any of the limit points* q *of* $\gamma(G_\varepsilon)q_\varepsilon$ *as* $\varepsilon \to 0$ *satisfies the* $d\mathcal{M}$*-symmetry relations*

$$\ll \phi(\xi) q \gg = \ll \frac{1}{4}(\phi + \phi_* - \phi' - \phi'_*) q \gg, \tag{4.2.10}$$

and, $q \in L^2(0,T; L^2(d\mathcal{M}dx))$*.*

- *for any subsequence* $\varepsilon_n \to 0$ *such that*

$$g_{\varepsilon_n} \to g, \quad \textit{and} \quad \gamma(G_{\varepsilon_n})q_{\varepsilon_n} \to q$$

in $w - L^1_{loc}(dtdx; L^1((1+|\xi|^2)Md\xi))$ *and in* $w - L^1_{loc}(dtdx; L^1((1+|\xi|^2)Md\mathcal{M}))$ *respectively.*

- *denoting* $N_\varepsilon = \frac{2}{3} + \frac{1}{3}G_\varepsilon$*,* $\frac{g_\varepsilon}{N_\varepsilon}$ *is bounded in* $L^\infty_t(L^2(Md\xi dx))$ *and* $\frac{q_\varepsilon}{N_\varepsilon}$ *is relatively compact in* $w - L^1_{loc}(dtdx; L^1((1+|\xi|^2)d\mathcal{M}))$*.*

Notice that the statement above does not involve the fact that g_ε will eventually represent fluctuations of the number density in the Vlasov-Maxwell-Boltzmann equation; the only features of the Vlasov-Maxwell-Boltzmann equation used in these results are the entropy and entropy dissipation bounds resulting from the entropy inequality (4.2.1) and bound (4.2.2). More precisely, the entropy and entropy dissipation bounds provide the weak compactness statements regarding g_ε and q_ε respectively.

Lemma 4.2.1. *Under the same conditions as Theorem 4.2.1, for almost every* $t \in [0, \infty)$ *the function* g *and* q *satisfy*

$$\begin{aligned} &\int_{\mathcal{T}} \frac{1}{2} < g^2(t) > dx + \frac{1}{2}\int_{\mathcal{T}} (|\chi|^2 + |B|^2) dx + \int_0^t \int_{\mathcal{T}} \frac{1}{4} \ll g^2 \gg dxds \\ &\le \liminf_{\varepsilon \to 0} \int_{\mathcal{T}} \left\langle \frac{1}{\varepsilon^2} h(\varepsilon g^0_\varepsilon) \right\rangle dx \le C. \end{aligned} \tag{4.2.11}$$

Proof. Taking the lim inf on the both sides of the entropy inequality (4.2.6), we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_{\mathcal{T}}\left\langle\frac{1}{\varepsilon^2}h(\varepsilon g_\varepsilon(t))\right\rangle dx+\frac{1}{2}\liminf_{\varepsilon\to 0}\int_{\mathcal{T}}(\varepsilon|E_\varepsilon|^2+|B_\varepsilon|^2)dx\\
&\quad+\liminf_{\varepsilon\to 0}\int_0^t\int_{\mathcal{T}}\frac{1}{4}\ll\frac{1}{\varepsilon^4}r\left(\frac{\varepsilon^2 q_\varepsilon}{G_{*\varepsilon}G_\varepsilon}\right)G_{*\varepsilon}G_\varepsilon\gg dxds\\
&\le\liminf_{\varepsilon\to 0}\left(\int_{\mathcal{T}}\left\langle\frac{1}{\varepsilon^2}h(\varepsilon g_\varepsilon^{in}(t))\right\rangle dx+\frac{1}{2}\int_{\mathcal{T}}(\varepsilon|E_\varepsilon^{in}|^2+|B_\varepsilon^{in}|^2)dx\right)\\
&\le C.
\end{aligned}\tag{4.2.12}$$

Due to the lower semi-continuity of the weak convergence, we deduce that

$$\int_{\mathcal{T}}(|\chi|^2+|B|^2)dx\le\liminf_{\varepsilon\to 0}\int_{\mathcal{T}}(\varepsilon|E_\varepsilon|^2+|B_\varepsilon|^2)dx,\tag{4.2.13}$$

while, from the second assertion of Proposition 3.1 in [3], we have

$$\begin{aligned}
&\int_{\mathcal{T}}\frac{1}{2}<g^2(t)>dx+\int_0^t\int_{\mathcal{T}}\frac{1}{4}\ll g^2\gg dxds\\
&\le\liminf_{\varepsilon\to 0}\int_{\mathcal{T}}\left\langle\frac{1}{\varepsilon^2}h(\varepsilon g_\varepsilon(t))\right\rangle dx+\liminf_{\varepsilon\to 0}\int_0^t\int_{\mathcal{T}}\frac{1}{4}\ll\frac{1}{\varepsilon^4}r\left(\frac{\varepsilon^2 q_\varepsilon}{G_{*\varepsilon G_\varepsilon}}\right)G_{*\varepsilon}G_\varepsilon\gg dxds.
\end{aligned}\tag{4.2.14}$$

Substituting (4.2.13) and (4.2.14) back into (4.2.12), we finish the proof of (4.2.11). □

To better understand the behavior of the fluctuation $\{g_\varepsilon\}_{\{\varepsilon>0\}}$, we first, as in [27], introduce a class of bump functions

$$\Upsilon=\left\{\gamma:\mathbb{R}_+\to[0,1]|\gamma\in C^1,\quad\gamma\left(\left[\frac{3}{4},\frac{5}{4}\right]\right)=\{1\},\quad supp\gamma\subset\left[\frac{1}{2},\frac{3}{2}\right]\right\}.\tag{4.2.15}$$

We decompose g_ε as

$$g_\varepsilon=g_\varepsilon^b+\varepsilon g_\varepsilon^c\tag{4.2.16}$$

with

$$g_\varepsilon^b=\frac{1}{\varepsilon}(G_\varepsilon-1)\gamma(G_\varepsilon),\quad g_\varepsilon^c=\frac{1}{\varepsilon^2}(G_\varepsilon-1)(1-\gamma(G_\varepsilon)),$$

where $\gamma\in\Upsilon$.

Then we have the following entropy controls (Proposition 2.1 and Proposition 2.7 in [27]):

Lemma 4.2.2 (Entropy controls). *Assume that the bump function $\gamma\in\Upsilon$ as in (4.2.15). The relative fluctuation g_ε of number density satisfies the following estimates:*

- $\varepsilon|g_\varepsilon^b| \leq \frac{1}{2}$ *and*

$$g_\varepsilon^b = O(1) \quad in \quad L_t^\infty(L^2(Md\xi dx));$$

- $(1-\gamma(G_\varepsilon)) \leq 4\varepsilon^2|g_\varepsilon^c|$, *which implies that* $\frac{1}{\varepsilon}(1-\gamma(G_\varepsilon)) \leq 2|g_\varepsilon^c|^{\frac{1}{2}}$, *and*

$$g_\varepsilon^c = O(1) \quad in \quad L_t^\infty(L^1(Md\xi dx));$$

- $(1-\gamma(G_\varepsilon))G_\varepsilon \leq 5\varepsilon^2|g_\varepsilon^c|$, *and* $(1-\gamma(G_\varepsilon)) \leq 4\varepsilon^2|g_\varepsilon^c|$.

4.3 THE IMPLICATION OF THE MAXWELL EQUATIONS

In the consideration of the asymptotic behavior of the solutions under the hypothesis $\mathcal{H}_\varepsilon(0) \leq C\varepsilon^3$, one of the difficulties when we deal with the magnetic field and the electric field comes from the fact that the relative entropy does not provide useful information on the electric field E_ε due to the ε in the front of the electric field in the definition of the relative entropy $\mathcal{H}_\varepsilon$, while from the relative entropy, we can obtain the uniform bound $\|B_\varepsilon\|_{L_t^\infty(L^2(dx))}$, and hence we can assume

$$B_\varepsilon \to B \quad \text{weakly}^* \quad \text{in} \quad L_t^\infty(L^2(dx)), \tag{4.3.1}$$

with $\mathrm{div}B = 0$. Furthermore, from the relative entropy, it is found that $\varepsilon^{\frac{1}{2}}\|E_\varepsilon\|_{L_t^\infty(L^2(dx))}$ is uniformly bounded, and hence, we can assume that

$$\varepsilon^{\frac{1}{2}}E_\varepsilon \to \chi, \quad \text{weakly}^* \quad \text{in} \quad L_t^\infty(L^2(dx)) \tag{4.3.2}$$

for some function $\chi \in L_t^\infty(L^2(dx))$, and hence,

$$\varepsilon\frac{\partial E_\varepsilon}{\partial t} \to 0, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3).$$

Next, since g_ε converges to g in $w-L^1_{loc}(dtdx; L^1((1+|\xi|^2)Md\xi dx))$, thus, by Cauchy-Schwartz inequality, we can deduce that g_ε converges to g in $w-L^1_{loc}(dtdx; L^1(|\xi|Md\xi dx))$. Notice that due to the fact $<\xi>=0$, $\frac{j_\varepsilon}{\varepsilon}=<\xi g_\varepsilon>$ and hence $\frac{j_\varepsilon}{\varepsilon}$ converges to j in $w-L^1_{loc}(dtdx)$. Then we take the limit as $\varepsilon\to 0$ in the equation (4.1.2b) to get

$$\nabla\times B=j \tag{4.3.3}$$

in the sense of distributions. Furthermore,

$$\begin{aligned}\left\|\frac{j_\varepsilon}{\varepsilon}\right\|_{L^\infty_t(L^2(\mathcal{T}))} &= \|<\xi g_\varepsilon>\|_{L^\infty_t(L^2(\mathcal{T}))}\\ &\le |\mathcal{T}|^{\frac12}\|g_\varepsilon\|_{L^\infty_T(L^2(Md\xi dx))}<|\xi|^2>^{\frac12}\\ &<\infty.\end{aligned}$$

Here $|\mathcal{T}|$ stands for the Lebesgue measure of $\mathcal{T}$. This implies that $\frac{j_i}{\varepsilon}$ converges weakly* to j in $L^\infty_t(L^2(\mathcal{T}))$.

On the other hand, for the electric field E_ε, we have

Lemma 4.3.1. *The family $\{E_\varepsilon\}_{\{\varepsilon>0\}}$ formally satisfies*

$$\begin{aligned}E_\varepsilon=\partial_t(\varepsilon E_\varepsilon\times B_\varepsilon)-(\nabla\times B_\varepsilon)\times B_\varepsilon+\frac{j_\varepsilon}{\varepsilon}\times B_\varepsilon+\varepsilon\mathrm{div}(E_\varepsilon\otimes E_\varepsilon)\\ -\varepsilon\frac12\nabla|E_\varepsilon|^2-\varepsilon E_\varepsilon\int_{\mathbb{R}^3}g_\varepsilon Md\xi,\end{aligned}\tag{4.3.4}$$

in the sense of distributions. Hence, $\{E_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $(W^{1,\infty}_0((0,T)\times\mathcal{T}))'$.

Proof. Indeed, multiplying (4.1.2b) by B_ε, multiplying (4.1.2c) by $\varepsilon E_\varepsilon$, and adding them together to yield

$$\partial_t(\varepsilon E_\varepsilon\times B_\varepsilon)-(\nabla\times B_\varepsilon)\times B_\varepsilon+\varepsilon E_\varepsilon\times(\nabla\times E_\varepsilon)=-\frac{j_\varepsilon}{\varepsilon}\times B_\varepsilon. \tag{4.3.5}$$

Note that

$$E\mathrm{div}E+(\nabla\times E)\times E=\mathrm{div}(E\otimes E)-\frac12\nabla|E|^2. \tag{4.3.6}$$

The identity (4.3.5) can be rewritten as, using (4.1.2d)

$$\begin{aligned}E_\varepsilon\rho_\varepsilon&=\varepsilon E_\varepsilon\mathrm{div}E_\varepsilon\\ &=\partial_t(\varepsilon E_\varepsilon\times B_\varepsilon)-(\nabla\times B_\varepsilon)\times B_\varepsilon+\frac{j_\varepsilon}{\varepsilon}\times B_\varepsilon+\varepsilon\mathrm{div}(E_\varepsilon\otimes E_\varepsilon)-\varepsilon\frac12\nabla|E_\varepsilon|^2.\end{aligned}\tag{4.3.7}$$

Because

$$\rho_\varepsilon = \int_{\mathbb{R}^3} (1+\varepsilon g_\varepsilon) M d\xi = \int_{\mathbb{R}^3} M d\xi + \varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi = 1 + \varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi,$$

one obtains, according to (4.3.7),

$$\begin{aligned} E_\varepsilon = {} & \partial_t(\varepsilon E_\varepsilon \times B_\varepsilon) - (\nabla \times B_\varepsilon) \times B_\varepsilon + \frac{j_\varepsilon}{\varepsilon} \times B_\varepsilon + \varepsilon \mathrm{div}(E_\varepsilon \otimes E_\varepsilon) \\ & - \varepsilon \frac{1}{2} \nabla |E_\varepsilon|^2 - \varepsilon E_\varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi. \end{aligned} \tag{4.3.8}$$

Next, due to the uniform bounds

$$\|\sqrt{\varepsilon} E_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C, \qquad \|B_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C,$$

we have

$$\partial_t(\varepsilon E_\varepsilon \times B_\varepsilon) \to 0$$

in $(W^{1,\infty}((0,T)\times\mathcal{T}))'$ as $\varepsilon \to 0$, and $-(\nabla \times B_\varepsilon) \times B_\varepsilon + \frac{j_\varepsilon}{\varepsilon} \times B_\varepsilon$ is uniformly bounded in $(W^{1,\infty}((0,T)\times\mathcal{T}))'$ by using the identity (4.3.6) for B.

Also, we can control the term $\varepsilon E_\varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi$ as follows

$$\begin{aligned} & \left\| \varepsilon E_\varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi \right\|_{L^1((0,T)\times\mathcal{T})} \\ & \leq \sqrt{\varepsilon} \|\sqrt{\varepsilon} E_\varepsilon\|_{L^2((0,T)\times\mathcal{T})} \left(\int_{\mathbb{R}^3} M d\xi \right)^{\frac{1}{2}} \| < g_\varepsilon^2 > \|_{L^1((0,T)\times\mathcal{T})}^{\frac{1}{2}} \\ & \leq C\sqrt{\varepsilon} \to 0 \end{aligned}$$

as $\varepsilon \to 0$. Hence, according to (4.3.8), we deduce that $\{E_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $(W_0^{1,\infty}((0,T)\times\mathcal{T}))'$. □

According to Lemma 4.3.1, we have

Lemma 4.3.2. *$E_\varepsilon \to E$ weakly in $(W_0^{2,p})'$, for some function $E \in (W_0^{2,p})'$ with $p > 4$, and (E, B) satisfies*

$$\partial_t B + \nabla \times E = 0 \tag{4.3.9}$$

in $(W_0^{2,p})'$.

Proof. Indeed, the uniform bound on E_ε in $(W_0^{1,\infty})'$ and Sobolev embedding $W_0^{2,p}((0,T)\times\mathcal{T})\hookrightarrow W_0^{1,\infty}((0,T)\times\mathcal{T})$ for any $4<p<\infty$ imply that E_ε is uniformly bounded in $(W_0^{2,p}((0,T)\times\mathcal{T}))'$ and hence is weakly convergent in $(W_0^{2,p}((0,T)\times\mathcal{T}))'$ since $(W_0^{2,p}((0,T)\times\mathcal{T}))'$ with $4<p<\infty$ is a reflexive space.

Next, since

$$\frac{\partial B_\varepsilon}{\partial t}+\nabla\times E_\varepsilon=0,$$

holds in $\mathcal{D}'(\mathbb{R}_+\times\mathcal{T})$, we take an arbitrarily text function $\Phi\in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^3)$ to obtain

$$-\int_0^t\int_{\mathcal{T}} B_\varepsilon\cdot\frac{\partial\Phi}{\partial t}dxds+\int_0^t\int_{\mathcal{T}} E_\varepsilon\cdot\nabla\times\Phi dxds=0. \tag{4.3.10}$$

Hence, from (4.3.10), we obtain

$$\begin{aligned}\int_0^t\int_{\mathcal{T}} E\cdot\nabla\times\Phi dxds&=\lim_{\varepsilon\to 0}\int_0^t\int_{\mathcal{T}} E_\varepsilon\cdot\nabla\times\Phi dxds\\&=\lim_{\varepsilon\to 0}\int_0^t\int_{\mathcal{T}} B_\varepsilon\cdot\frac{\partial\Phi}{\partial t}dxds\\&=\int_0^t\int_{\mathcal{T}} B\cdot\frac{\partial\Phi}{\partial t}dxds\\&=-\int_0^t\int_{\mathcal{T}}\Phi\cdot\frac{\partial B}{\partial t}dxds\end{aligned} \tag{4.3.11}$$

Hence, from (4.3.11), we deduce that the limits (E,B) satisfy (4.3.9). □

Observe that, since E_ε is convergent at least in the sense of distributions, we can conclude that $\chi=0$.

4.4 VANISHING OF CONSERVATION DEFECTS

The controls stated in Section 4 establish the local conservation laws of momentum and energy in the limit as $\varepsilon \to 0$, in the spirit of the argument in the proof of the Navier-Stokes-Fourier limit of Boltzmann equations (cf. [27]).

Before stating the main result of the present section, we need to introduce a new class of bump functions as in [27]. For each $C > 0$, set

$$\Upsilon_C = \{\gamma \in \Upsilon | \|\gamma'\|_{L^\infty} \leq C\}.$$

Consider the transformation $\mathfrak{T}$ defined by $\mathfrak{T}\gamma = 1 - (1-\gamma)^2$; clearly $\mathfrak{T}$ maps Υ_C into Υ_{2C}. Define

$$\tilde{\Upsilon} = \mathfrak{T}\Upsilon_8 \subset \Upsilon_{16}, \tag{4.4.1}$$

and notice that $\tilde{\Upsilon} \neq \emptyset$ since $\Upsilon_8 \neq \emptyset$. For each $\gamma \in \tilde{\Upsilon}$, define

$$\hat{\gamma}(z) = \gamma(z) + (z-1)\frac{d\gamma}{dz}. \tag{4.4.2}$$

Notice that

$$supp \hat{\gamma} \subset \left[\frac{1}{2}, \frac{3}{2}\right], \qquad \hat{\gamma}\left(\left[\frac{3}{4}, \frac{5}{4}\right]\right) = \{1\}. \tag{4.4.3}$$

On the other hand, let $\tilde{\gamma} \in \Upsilon_8$ be such that $\gamma = \mathfrak{T}\tilde{\gamma}$. One has

$$1 - \hat{\gamma}(z) = (1-\tilde{\gamma})\left[(1-\tilde{\gamma}) - 2(z-1)\frac{d\tilde{\gamma}}{dz}\right], \qquad z \geq 0$$

so that

$$|1 - \hat{\gamma}| \leq 9(1-\tilde{\gamma}), \qquad z \geq 0. \tag{4.4.4}$$

Theorem 4.4.1 (Vanishing of conservation defects). *Let $\gamma \in \tilde{\Upsilon}$, and denote by $\eta \equiv \eta(\xi)$ any collision invariant (i.e. $\eta(\xi) = 1$ or $\eta(\xi) = \xi_1, ..., \xi_3$ or else $\eta(\xi) = |\xi|^2$) or any linear combination thereof. Then*

$$\partial_t < \eta g_\varepsilon^b > + \frac{1}{\varepsilon}\nabla_x \cdot < \xi\eta g_\varepsilon^b > + eB_\varepsilon \cdot < \xi \times \nabla_\xi \eta g_\varepsilon^b > - eE_\varepsilon \cdot < \xi\eta > \to 0, \tag{4.4.5}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})$ as $\varepsilon \to 0$.

Proof. We begin with the renormalized form (4.1.31) of the Vlasov-Maxwell-Boltzmann equation (4.1.2) with $\Gamma(z) = (z-1)\gamma(z)$

$$\begin{aligned}
&\left(\partial_t + \frac{1}{\varepsilon}\xi \cdot \nabla_x\right)(Mg_\varepsilon^b) + e(\varepsilon E_\varepsilon + \xi \times B_\varepsilon)\cdot \nabla_\xi (Mg_\varepsilon^b) \\
&+ e\varepsilon E_\varepsilon \cdot \xi M g_\varepsilon^b - e\left(\gamma(G_\varepsilon) + (G_\varepsilon - 1)\frac{d\gamma}{dz}(G_\varepsilon)\right) E_\varepsilon \cdot \xi F_\varepsilon \\
&= \frac{1}{\varepsilon^3}\int_{S^2}\int_{\mathbb{R}^3} (F'_\varepsilon F'_{\varepsilon *} - F_\varepsilon F_{\varepsilon *})\left(\gamma(G_\varepsilon) + (G_\varepsilon - 1)\frac{d\gamma}{dz}(G_\varepsilon)\right) b d\omega M_* d\xi_*.
\end{aligned} \tag{4.4.6}$$

Here, we used the decomposition (4.2.16).

From (4.4.6), we deduce that

$$\begin{aligned}
&\partial_t < \eta g_\varepsilon^b > + \frac{1}{\varepsilon}\nabla_x \cdot < \xi \eta g_\varepsilon^b > + e\int_{\mathbb{R}^3} (\varepsilon E_\varepsilon + \xi \times B_\varepsilon)\cdot \nabla_\xi (Mg_\varepsilon^b)\eta d\xi \\
&+ e\int_{\mathbb{R}^3} \varepsilon E_\varepsilon \cdot \xi M g_\varepsilon^b \eta d\xi - e\int_{\mathbb{R}^3} \hat{\gamma}_\varepsilon E_\varepsilon \cdot \xi F_\varepsilon \eta d\xi \\
&= \frac{1}{\varepsilon} \ll q_\varepsilon \hat{\gamma}_\varepsilon \eta \gg,
\end{aligned} \tag{4.4.7}$$

where the notation

$$\hat{\gamma}_\varepsilon = \hat{\gamma}(G_\varepsilon),$$

the function $\hat{\gamma}$ being defined in terms of γ by (4.4.2)

Observing that

$$(X \times Y)\cdot Z = Y \cdot (Z \times X) = X \cdot (Y \times Z),$$

we have

$$\begin{aligned}
\int_{\mathbb{R}^3} (\varepsilon E_\varepsilon + \xi \times B_\varepsilon)\cdot \nabla_\xi (Mg_\varepsilon^b)\eta d\xi &= -\left(\varepsilon E_\varepsilon \cdot < \nabla_\xi \eta g_\varepsilon^b > + \int_{\mathbb{R}^3} (\xi \times B_\varepsilon)\cdot \nabla_\xi \eta g_\varepsilon^b M d\xi\right) \\
&= -\left(\varepsilon E_\varepsilon \cdot < \nabla_\xi \eta g_\varepsilon^b > - B_\varepsilon \cdot < \xi \times \nabla_\xi \eta g_\varepsilon^b >\right).
\end{aligned} \tag{4.4.8}$$

On the one hand, notice that following the same line of the argument of Proposition 4.1 in [27], it can be shown that

$$\left\| \frac{1}{\varepsilon} \ll q_\varepsilon \hat{\gamma}_\varepsilon \eta \gg \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} \to 0 \tag{4.4.9}$$

as $\varepsilon \to 0$.

In order to estimate the L^1-norm of the conservation defects, for the last two terms in the left-hand side of (4.4.7), we claim

$$\left\|\varepsilon E_\varepsilon\cdot <\nabla_\xi \eta g_\varepsilon^b>\right\|_{L^1_{loc}(\mathbb{R}_+\times\mathcal{T})} \longrightarrow 0; \tag{4.4.10}$$

$$\left\|\int_{\mathbb{R}^3}\varepsilon E_\varepsilon\cdot\xi M g_\varepsilon^b \eta d\xi\right\|_{L^1_{loc}(\mathbb{R}_+\times\mathcal{T})} \longrightarrow 0; \tag{4.4.11}$$

and

$$\left\|\int_{\mathbb{R}^3}\hat{\gamma}_\varepsilon E_\varepsilon\cdot\xi F_\varepsilon \eta d\xi - \int_{\mathbb{R}^3} E_\varepsilon\cdot\xi\eta M d\xi\right\|_{L^1_{loc}(\mathbb{R}_+\times\mathcal{T})} \longrightarrow 0 \tag{4.4.12}$$

as $\varepsilon \longrightarrow 0$.

Indeed, using the elementary bounds

$$|\hat{\gamma}_\varepsilon| \leq 9, \qquad |1-\hat{\gamma}_\varepsilon| \leq 9, \qquad 0 \leq G_\varepsilon|\hat{\gamma}_\varepsilon| \leq \frac{27}{2}, \tag{4.4.13}$$

for the inequality (4.4.10), we have, ,

$$\left|<\nabla_\xi\eta g_\varepsilon^b>\right| \leq \left(\int_{\mathbb{R}^3}(\nabla_\xi\eta)^2 M d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(g_\varepsilon^b)^2 M d\xi\right)^{\frac{1}{2}} \leq C\left(\int_{\mathbb{R}^3}(g_\varepsilon^b)^2 M d\xi\right)^{\frac{1}{2}},$$

since

$$\int_{\mathbb{R}^3}(\nabla_\xi\eta)^2 M d\xi \leq C$$

for all $\eta \in N(L)$ and where C is a positive constant. Hence, by Cauchy-Schwartz inequality and the first statement in Lemma 4.2.2

$$\begin{aligned}\left\|\varepsilon E_\varepsilon\cdot <\nabla_\xi\eta g_\varepsilon^b>\right\|_{L^1_{loc}(\mathbb{R}_+\times\mathcal{T})} &\leq C\left\|\varepsilon|E_\varepsilon|\left(\int_{\mathbb{R}^3}(g_\varepsilon^b)^2 M d\xi\right)^{\frac{1}{2}}\right\|_{L^1_{loc}(\mathbb{R}_+\times\mathcal{T})}\\ &\leq C\varepsilon^{\frac{1}{2}}\|\varepsilon^{\frac{1}{2}}E_\varepsilon\|^{\frac{1}{2}}_{L^\infty_t(L^2(\mathcal{T}))}\|g_\varepsilon^b\|^{\frac{1}{2}}_{L^\infty_t(L^2(Mdxd\xi))}\\ &\leq C\varepsilon^{\frac{1}{2}} \longrightarrow 0,\end{aligned}$$

as $\varepsilon \longrightarrow 0$.

Similarly, for the inequality (4.4.11), we have, ,

$$\left|\int_{\mathbb{R}^3}\xi\eta g_\varepsilon^b M d\xi\right| \leq \left(\int_{\mathbb{R}^3}(\xi\eta)^2 M d\xi >\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(g_\varepsilon^b)^2 M d\xi\right)^{\frac{1}{2}} \leq C\left(\int_{\mathbb{R}^3}(g_\varepsilon^b)^2 M d\xi\right)^{\frac{1}{2}},$$

since

$$\int_{\mathbb{R}^3} (\xi\eta)^2 M d\xi \leq C$$

for all $\eta \in N(L)$ and where C is a positive constant. Hence, by Cauchy-Schwartz inequality and the first statement in Lemma 4.2.2

$$\begin{aligned}
\left\| \int_{\mathbb{R}^3} \varepsilon E_\varepsilon \cdot \xi\eta g_\varepsilon^b M d\xi \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} &\leq C \left\| \varepsilon |E_\varepsilon| \left(\int_{\mathbb{R}^3} (g_\varepsilon^b)^2 M d\xi \right)^{\frac{1}{2}} \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} \\
&\leq C\varepsilon^{\frac{1}{2}} \|\varepsilon^{\frac{1}{2}} E_\varepsilon\|^{\frac{1}{2}}_{L^\infty_t(L^2(\mathcal{T}))} \|g_\varepsilon^b\|^{\frac{1}{2}}_{L^\infty_t(L^2(Mdxd\xi))} \\
&\leq C\varepsilon^{\frac{1}{2}} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$.

We are only left to deal with (4.4.12). To this end, we rewrite

$$\begin{aligned}
\int_{\mathbb{R}^3} \hat{\gamma}_\varepsilon E_\varepsilon \cdot \xi F_\varepsilon \eta d\xi - \int_{\mathbb{R}^3} E_\varepsilon \cdot \xi\eta M d\xi &= \int_{\mathbb{R}^3} (\hat{\gamma}_\varepsilon - 1) E_\varepsilon \cdot \xi F_\varepsilon \eta d\xi + \varepsilon \int_{\mathbb{R}^3} E_\varepsilon \cdot \xi\eta g_\varepsilon M d\xi \\
&= I_1 + I_2.
\end{aligned} \tag{4.4.14}$$

Notice that from (4.4.4), we have

$$|\hat{\gamma}_\varepsilon - 1| \leq 9(1 - \tilde{\gamma}(G_\varepsilon)) \leq 9(1 - \tilde{\gamma}(G_\varepsilon))^{\frac{1}{2}}$$

for some $\tilde{\gamma} \in \Upsilon_8$ and hence we can control I_1 as, using $F_\varepsilon = MG_\varepsilon$, Lemma 4.2.2 and the fact $0 \leq 1 - \tilde{\gamma}(G_\varepsilon) \leq 1$,

$$\begin{aligned}
\|I_1\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} &\leq 9 \left\| \int_{\mathbb{R}^3} |E_\varepsilon||\xi\eta||1 - \tilde{\gamma}(G_\varepsilon)|^{\frac{1}{2}} G_\varepsilon M d\xi \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} \\
&\leq 9 \left\| \int_{\mathbb{R}^3} |E_\varepsilon||\xi\eta||1 - \tilde{\gamma}(G_\varepsilon)|^{\frac{1}{2}} M d\xi \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} \\
&\quad + 9\varepsilon \left\| \int_{\mathbb{R}^3} |E_\varepsilon||\xi\eta||1 - \tilde{\gamma}(G_\varepsilon)|^{\frac{1}{2}} g_\varepsilon M d\xi \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} \\
&\leq 18\sqrt{\varepsilon} \left\| \sqrt{\varepsilon} \int_{\mathbb{R}^3} |E_\varepsilon||\xi\eta||g_\varepsilon^c|^{\frac{1}{2}} M d\xi \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} \\
&\quad + 9\varepsilon \left\| \int_{\mathbb{R}^3} |E_\varepsilon||\xi\eta| g_\varepsilon M d\xi \right\|_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})} \\
&\leq 18\sqrt{\varepsilon} \|\sqrt{\varepsilon} E_\varepsilon\|_{L^2_{loc}(\mathbb{R}_+ \times \mathcal{T})} < |\xi\eta|^2 >^{\frac{1}{2}} \; \||g_\varepsilon^c|\|^{\frac{1}{2}}_{L^1_{loc}(\mathbb{R}_+ \times \mathcal{T}; L^1(Md\xi))} \\
&\quad + 9\varepsilon \|E_\varepsilon\|_{L^2_{loc}(\mathbb{R}_+ \times \mathcal{T})} < |\xi\eta|^2 >^{\frac{1}{2}} \; \|g_\varepsilon\|_{L^2_{loc}(\mathbb{R}_+ \times \mathcal{T}; L^2(Md\xi))} \\
&\leq C\sqrt{\varepsilon} + C\varepsilon \to 0
\end{aligned} \tag{4.4.15}$$

as $\varepsilon \to 0$.

For I_2, we have

$$\begin{aligned} \|I_2\|_{L^1_{loc}(\mathbb{R}_+\times\mathcal{T})} &\le \sqrt{\varepsilon}\|\sqrt{\varepsilon}E_\varepsilon\|_{L^2_{loc}(\mathbb{R}_+\times\mathcal{T})} < |\xi\eta| >^{\frac{1}{2}} \|g_\varepsilon\|_{L^2_{loc}(\mathbb{R}_+;L^2(Md\xi dx))} \\ &\le C\varepsilon \to 0 \end{aligned} \tag{4.4.16}$$

as $\varepsilon \to 0$.

Adding (4.4.14), (4.4.15) and (4.4.16) together gives (4.4.12).

Combining (4.4.7)–(4.4.12), the proof of (4.4.5) is finished. □

It is worthwhile to explain what we have obtained now in Theorem 4.4.1. In fact, if $\eta = 1$ or $\eta = |\xi|^2$, then the last term in the left hand side of (4.4.5) will vanish; that is,

$$E_\varepsilon \cdot < \xi >= E_\varepsilon \cdot < \xi|\xi|^2 >= 0,$$

because

$$< \xi >=< \xi|\xi|^2 >= 0.$$

This implies that the term $E_\varepsilon \cdot < \xi\eta >$ will only possibly appear in the conservation law of momentum. Hence,

$$\partial_t < g^b_\varepsilon \xi_k > + \frac{1}{\varepsilon}\nabla_x \cdot < \xi\xi_k g^b_\varepsilon > + eB_\varepsilon \cdot < \xi \times \nabla_\xi \xi_k g^b_\varepsilon > - \alpha e(E_\varepsilon)_k \to 0, \tag{4.4.17}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathcal{T})$ for all $1 \le k \le 3$ since $< \xi_k^2 >= \alpha = \frac{1}{3} < |\xi|^2 >$.

4.5 PROOF OF THEOREM 4.1.1

4.5.1 The Incompressibility and Boussinesq Relations

Let us start with considering the normalized Vlasov-Maxwell-Botlzmann equation written in the form

$$\varepsilon\partial_t G_\varepsilon + \xi\cdot\nabla_x G_\varepsilon + e\varepsilon\left(\varepsilon E_\varepsilon + \xi\times B_\varepsilon\right)\cdot\nabla_\xi G_\varepsilon - e\varepsilon^2 E_\varepsilon\cdot\xi G_\varepsilon = \frac{1}{\varepsilon}Q(G_\varepsilon, G_\varepsilon). \tag{4.5.1}$$

Hence, the renormalized form of Vlasov-Maxwell-Boltzman equation reads

$$\varepsilon\partial_t h_\varepsilon + \xi\cdot\nabla_x h_\varepsilon + e\varepsilon\left(\varepsilon E_\varepsilon + \xi\times B_\varepsilon\right)\cdot\nabla_\xi h_\varepsilon - e\varepsilon E_\varepsilon\cdot\xi\frac{G_\varepsilon}{N(G_\varepsilon)} = \frac{1}{\varepsilon^2}\frac{1}{N(G_\varepsilon)}Q(G_\varepsilon, G_\varepsilon),$$

where

$$h'_\varepsilon = \frac{1}{\varepsilon}\frac{1}{N(G_\varepsilon)} = \frac{1}{\varepsilon}\frac{1}{\frac{2}{3}+\frac{1}{3}G_\varepsilon},$$

and

$$h_\varepsilon = \frac{3}{\varepsilon}\ln\left(1+\frac{1}{3}\varepsilon g_\varepsilon\right).$$

Since h_ε formally behaves like g_ε for small ε, it should be thought of as the normalized form of the fluctuations g_ε. This means that for every $\chi\in C^1(\mathcal{T};L^\infty(Md\xi))$ and every $0\le s\le t<\infty$

$$\begin{aligned}
&\varepsilon\int_{\mathcal{T}} < h_\varepsilon(t)\chi > dx - \varepsilon\int_{\mathcal{T}} < h_\varepsilon(s)\chi > dx - \int_s^t\int_{\mathcal{T}} < h_\varepsilon\xi\cdot\nabla_x\chi > dxd\tau \\
&\quad + e\int_s^t\int_{\mathcal{T}}\varepsilon^2 E_\varepsilon\cdot < \xi h_\varepsilon\chi > dxd\tau - e\int_s^t\int_{\mathcal{T}}\int_{\mathbb{R}^3}\varepsilon(\varepsilon E_\varepsilon + \xi\times B_\varepsilon)\cdot\nabla_\xi\chi h_\varepsilon M d\xi dxd\tau \\
&\quad - e\int_s^t\int_{\mathcal{T}}\varepsilon E_\varepsilon\cdot < \xi\frac{G_\varepsilon}{N(G_\varepsilon)} > dxd\tau \\
&\quad = \int_s^t\int_{\mathcal{T}} \ll \frac{q_\varepsilon}{N_\varepsilon}\chi \gg dxd\tau.
\end{aligned} \tag{4.5.2}$$

Due to the fact

$$\frac{G_\varepsilon}{N(G_\varepsilon)}\le 3$$

and the entropy control

$$\|\varepsilon^{\frac{1}{2}}E_\varepsilon\|_{L^\infty_t(L^2(dx))}\le C,$$

one obtain

$$\int_s^t \int_{\mathcal{T}} \varepsilon E_\varepsilon \cdot < \xi \frac{G_\varepsilon}{N(G_\varepsilon)} > dx d\tau \to 0,$$

as $\varepsilon \to 0$. On the other hand, since as stated in the last statement of Theorem 4.2.1 (cf. also Corollary 3.2 in [3]) that h_ε has the same limit g as the sequence g_ε in $w - L^2_{loc}(dt; w^2_L(Md\xi dx))$, one deduce that

$$\varepsilon \int_{\mathcal{T}} < h_\varepsilon(t)\chi > dx - \varepsilon \int_{\mathcal{T}} < h_\varepsilon(s)\chi > dx \to 0;$$

$$\int_s^t \int_{\mathcal{T}} \varepsilon^2 E_\varepsilon \cdot < \xi h_\varepsilon \chi > dx d\tau \to 0;$$

and

$$\int_s^t \int_{\mathcal{T}} \int_{\mathbb{R}^3} \varepsilon(\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \cdot \nabla_\xi \chi h_\varepsilon M d\xi dx d\tau \to 0,$$

as $\varepsilon \to 0$, thanks to the uniform bounds

$$\|\varepsilon^{\frac{1}{2}} E_\varepsilon\|_{L^\infty_{\mathbb{R}_+}(L^2(\mathcal{T}))} \leq C, \qquad \|B_\varepsilon\|_{L^\infty_{\mathbb{R}_+}(L^2(\mathcal{T}))} \leq C.$$

Taking the limit in (4.5.2) as ε tends to zero while using Theorem 4.2.1 to establish the limits of the terms involving h_ε and q_ε respectively yields

$$-\int_s^t \int_{\mathcal{T}} < g\xi \cdot \nabla_x \chi > dx d\tau = \int_s^t \int_{\mathcal{T}} \ll q\chi \gg dx d\tau;$$

hence, the limiting form of the normalized Vlasov-Maxwell-Boltzmann equation is

$$\xi \cdot \nabla_x g = \int \int q b(\xi_* - \xi, \omega) d\omega M_* d\xi_*. \tag{4.5.3}$$

Since q is in $L^2(d\mathcal{M}dx)$, then for every $\eta = \eta(\xi)$ in $L^2(d\mathcal{M})$, an application of the Cauchy-Schwarz inequality shows that $\ll \eta q \gg$ is in $L^2(dx)$. By a repeated application of the $d\mathcal{M}$-symmetries in Theorem 4.2.1, one has that for any η in $L^2(d\mathcal{M})$

$$\ll \eta q \gg = \frac{1}{4} \ll (\eta + \eta_* - \eta'_* - \eta')q \gg . \tag{4.5.4}$$

Successively apply the identity (4.5.4) for $\eta = 1, \xi, \frac{1}{2}|\xi|^2$ and use the microscopic conservation laws (1.0.9) to obtain

$$\ll q \gg = 0, \quad \ll \xi q \gg = 0, \quad \ll \frac{1}{2}|\xi|^2 q \gg = 0.$$

Since these η are also in $L^2(Md\xi)$, it then follows from the limiting Vlasov-Maxwell-Boltzmann equation (4.5.3) that g satisfies the local conservation laws of mass, momentum, and energy:

$$\mathrm{div}_x < \xi g >= 0, \qquad \mathrm{div}_x < \xi \otimes \xi g >= 0, \qquad \mathrm{div}_x \left\langle \xi \frac{1}{2}|\xi|^2 g \right\rangle = 0. \tag{4.5.5}$$

Theorem 4.2.1 states that g has the form of the infinitesimal Maxwellian

$$g = h + \mathbf{u} \cdot \xi + \theta \left(\frac{1}{2}|\xi|^2 - \frac{3}{2} \right).$$

Substituting this into (4.5.5), the local mass and energy conservation laws yiled the incompressibility relation for the velocity field $\mathbf{u}$ while that of momentum yields the Boussinesq relation between ρ and θ:

$$\mathrm{div}_x \mathbf{u} = 0, \qquad \nabla_x (h + \theta) = 0.$$

4.5.2 Proof of Convergence to Incompressible Electron-Magnetohydrodynamic-Fourier Equations

Throughout this subsection, it is assumed that the bump function γ belongs to $\tilde{\Upsilon}$ (defined by (4.4.1)). Using Theorem 4.4.1, the classical Sobolev embedding theorems, and the continuity of pseudo-differential operators of order 0 on $W^{s,p}$ for $1 < p < \infty$, one sees that, for all $s > 0$

$$\begin{aligned} &\partial_t P < \xi g_\varepsilon^b > + P \nabla_x \cdot \frac{1}{\varepsilon} \left\langle \left(\xi \otimes \xi - \frac{1}{3}|\xi|^2 I \right) g_\varepsilon^b \right\rangle \\ &+ eP \left(B_\varepsilon \cdot < \xi \times \nabla_\xi \eta g_\varepsilon^b > \right) - \alpha e P E_\varepsilon \\ &\to 0 \end{aligned} \tag{4.5.6}$$

in $L^1_{loc}(dt; W^{-s,1}_{loc}(\mathbb{R}^3))$, and

$$\begin{aligned} &\partial_t \left\langle \left(\frac{1}{5}|\xi|^2 - 1 \right) g_\varepsilon^b \right\rangle + \nabla_x \cdot \frac{1}{\varepsilon} \left\langle \xi \left(\frac{1}{5}|\xi|^2 - 1 \right) g_\varepsilon^b \right\rangle \\ &\to 0 \end{aligned} \tag{4.5.7}$$

in $L^1_{loc}(dtdx)$ as $\varepsilon \to 0$. Here, the operator P is the Leray projection, i.e. the $L^2(dx)$-orthogonal projection on the space of divergence-free vector fields. In (4.5.7), we used

$$\xi \times \nabla_\xi \left(\frac{1}{5}|\xi|^2 - 1 \right) = 0.$$

By Theorem 4.2.1 and Proposition 4.2.2, pick any sequence $\varepsilon_n \to 0$ such that

$$g^b_{\varepsilon_n} \to g \quad \text{in} \quad w^* - L^\infty_t(L^2(Md\xi dx)), \tag{4.5.8}$$

$$\gamma_{\varepsilon_n} q^b_{\varepsilon_n} \to q \quad \text{in} \quad w - L^1_{loc}(L^1(dtdx; L^1((1+|\xi|^2)d\mathcal{M})). \tag{4.5.9}$$

In this section, we deal exclusively with such extracted sequences, drop the index n and abuse the notations g_ε, g^b_ε, g^c_ε, q_ε and so on to designate the subsequences g_{ε_n}, $g^b_{\varepsilon_n}$, $g^c_{\varepsilon_n}$, q_{ε_n}. Set $\mathbf{u}$ and θ the limiting fluctuations of velocity and temperature fields defined by

$$< \xi g^b_\varepsilon > \to \mathbf{u}, \quad \text{in} \quad w^* - L^\infty_t(L^2_x); \tag{4.5.10}$$

$$\left\langle \left(\frac{1}{3}|\xi|^2 - 1 \right) g^b_\varepsilon \right\rangle \to \theta, \quad \text{in} \quad w^* - L^\infty_t(L^2_x). \tag{4.5.11}$$

The second entropy control in Proposition implies that g^b_ε and g_ε have the same limit g in $w - L^1_{loc}(dtdx; L^1(Md\xi))$; hence the Boussinesq relation and the incompressibility condition hold:

$$\text{div}_x \mathbf{u} = 0, \qquad \theta + < g >= 0. \tag{4.5.12}$$

Denote by ς either the tensor Φ or the vector Ψ. Since L is self-adjoint on $L^2(Md\xi)$ so that

$$\begin{aligned} \frac{1}{\varepsilon} < (L\varsigma) g^b_\varepsilon >= \frac{1}{\varepsilon} < \varsigma(Lg^b_\varepsilon) > = \frac{1}{\varepsilon} \ll \varsigma(g^b_\varepsilon + g^b_{\varepsilon*} - g^{b\prime}_\varepsilon - g^{b}_{\varepsilon*}{}') \gg \\ = \ll \varsigma \left[\frac{1}{\varepsilon}(g^b_\varepsilon + g^b_{\varepsilon*} - g^{b\prime}_\varepsilon - g^{b}_{\varepsilon*}{}') + (g^b_\varepsilon g^b_{\varepsilon*} - g^{b\prime}_\varepsilon g^{b}_{\varepsilon*}{}') \right] \gg \\ + < \varsigma Q(g^b_\varepsilon, g^b_\varepsilon) > . \end{aligned} \tag{4.5.13}$$

The first term in the last right hand side of (4.5.13) converges to the diffusion term while the second term converges to the convection term in the incompressible MHD system. These limits are analyzed in the next two lemmas. The convergence to the diffusion term is obtained by an argument that closely follows [27], except that the present work should pay additional attention to the Maxwell effect. This apparently minor difference makes our analysis slightly more difficult than that in [27].

Lemma 4.5.1. *Define*

$$\nu = \frac{1}{10} < \Phi : L\Phi >, \qquad \kappa = \frac{2}{15} < \Psi \cdot L\Psi > . \tag{4.5.14}$$

Then, as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon} < (L\Phi) g_\varepsilon^b > - < \Phi Q(g_\varepsilon^b, g_\varepsilon^b) > \to -\nu(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^\top);$$

$$\frac{1}{\varepsilon} < (L\Psi) g_\varepsilon^b > - < \Psi Q(g_\varepsilon^b, g_\varepsilon^b) > \to -\frac{5}{2}\kappa \nabla_x \theta$$

in $w - L^1_{loc}(dtdx)$.

The convection term is the nonlinear part of the limiting system and its convergence is therefore the most difficult to establish. The analysis below rests not only on all *a priori* estimates and the arguments in [27], but also the compactness of the moment of g_ε^b in ξ which is stated in Lemma 4.5.3.

Lemma 4.5.2. *The following convergence hold in the sense of distributions on* $\mathbb{R}_+ \times \mathbb{R}^3$*:*

$$P\nabla_x \cdot < \Phi Q(g_\varepsilon^b, g_\varepsilon^b) > \to P\nabla_x \cdot (\mathbf{u} \otimes \mathbf{u}),$$

$$\nabla_x < \Psi Q(g_\varepsilon^b, g_\varepsilon^b) > \to \frac{5}{2}\nabla_x \cdot (\mathbf{u}\theta),$$

as $\varepsilon \to 0$.

4.5.3 The Lorentz Force Term

The key result of this subsection is to deal with the convergence of the Lorentz force term. To this end, we first state the following compactness about the moment of g_ε in ξ (see Lemma 3.7 in [27]).

Lemma 4.5.3. *Let $\gamma \in \Upsilon$ be the same as in (4.2.15) and the hypothesis (H3) hold. Then, the family g^b_ε has the following property: for each sequence $\varepsilon_n \to 0$, each function $\chi = \chi(\xi)$ such that $\frac{|\chi(\xi)|}{1+|\xi|^2} \to 0$ as $|\xi| \to \infty$, each $T > 0$, there exists a function $\eta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that* $\lim_{z\to 0^+} \eta(z) = 0$ *and*

$$\int_0^T \int_{\mathcal{T}} \left|\left\langle g^b_{\varepsilon_n}\chi\right\rangle(t, x + y) - \left\langle g^b_{\varepsilon_n}\chi\right\rangle(t, x)\right|^2 dxdt \leq \eta(|y|)$$

for each $y \in \mathbb{R}^3$ such that $|y| \leq 1$, uniformly in n.

Proof. For any $\gamma \in \Upsilon$, since F_ε is a renormalized solution of (4.1.4) relatively to M, using the nonlinear function $\Gamma(z) = (z-1)\gamma(z)$ in the renormalized formulation (4.1.31), we obtain

$$\begin{aligned}(\varepsilon\partial_t + \xi\cdot\nabla_x)g^b_\varepsilon = &\int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon\hat{\gamma}_\varepsilon bd\mathbb{R}^3 megaM_* d\xi_* - e\mathrm{div}_\xi\left(\varepsilon(\varepsilon E_\varepsilon + \xi\times B_\varepsilon)g^b_\varepsilon\right)\\ &+ e\hat{\gamma}_\varepsilon\varepsilon E_\varepsilon\cdot\xi G_\varepsilon,\end{aligned} \tag{4.5.15}$$

with $\hat{\gamma}$ defined in terms of the truncation γ by (4.4.2). Denoting

$$f\wedge L = \begin{cases} f, & \text{if} \quad |f| \leq L;\\ L, & \text{if} \quad f \geq L;\\ -L, & \text{if} \quad f \leq -L\end{cases}$$

for every $L > 1$, we deduce from (4.5.15) that

$$\begin{aligned}(\varepsilon\partial_t + \xi\cdot\nabla_x)(g^b_\varepsilon\wedge L) = &\left(\int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon\hat{\gamma}_\varepsilon bd\omega M_* d\xi_*\right)1_{\{|g^b_\varepsilon|\leq L\}}\\ &- e\mathrm{div}_\xi\left(\varepsilon(\varepsilon E_\varepsilon + \xi\times B_\varepsilon)\left(g^b_\varepsilon\wedge L\right)\right)\\ &+ e\hat{\gamma}_\varepsilon\varepsilon E_\varepsilon\cdot\xi G_\varepsilon 1_{\{|g^b_\varepsilon|\leq L\}},\end{aligned} \tag{4.5.16}$$

Furthermore, for every $N > 1$, we decompose $g_\varepsilon^b \wedge L$ as

$$g_\varepsilon^b \wedge L = \overline{g_\varepsilon^b} + \hat{g_\varepsilon^b}, \qquad \overline{g_\varepsilon^b}^{in} = 0,$$

with

$$(\varepsilon\partial_t + \xi \cdot \nabla_x)\overline{g_\varepsilon^b} = \left(\int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon \hat{\gamma}_\varepsilon b d\omega M_* d\xi_*\right) 1_{\{|g_\varepsilon^b| \le L\}} 1_{\{|A_\varepsilon| > N\}}, \tag{4.5.17}$$

and

$$\begin{aligned}(\varepsilon\partial_t + \xi \cdot \nabla_x)\hat{g_\varepsilon^b} = &\left(\int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon \hat{\gamma}_\varepsilon b d\omega M_* d\xi_*\right) 1_{\{|g_\varepsilon^b| \le L\}} 1_{\{|A_\varepsilon| \le N\}} \\ &- e\mathrm{div}_\xi \left(\varepsilon(\varepsilon E_\varepsilon + \xi \times B_\varepsilon)\left(g_\varepsilon^b \wedge L\right)\right) \\ &+ e\hat{\gamma}_\varepsilon \varepsilon E_\varepsilon \cdot \xi G_\varepsilon 1_{\{|g_\varepsilon^b| \le L\}},\end{aligned} \tag{4.5.18}$$

where

$$A_\varepsilon = \int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon \hat{\gamma}_\varepsilon b d\omega M_* d\xi_*.$$

Step 1: Control of $\mathbb{R}^3 v g_\varepsilon^b$. From (4.5.17), if we denote

$$S_\varepsilon = \left(\int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon \hat{\gamma}_\varepsilon b d\omega M_* d\xi_*\right) 1_{\{|A_\varepsilon| > N\}} 1_{\{|g_\varepsilon^b| \le L\}},$$

then we obtain

$$\overline{g_\varepsilon^b}(t, x, \xi) = \int_0^{\frac{t}{\varepsilon}} S_\varepsilon(t - \varepsilon s, x - s\xi, \xi) ds. \tag{4.5.19}$$

Notice that, since $|\hat{\gamma}_\varepsilon| \le 9$ and q_ε is weakly compact in $L^1(dtdxd\mathcal{M})$, S_ε is uniformly bounded in $L^1(dtdxMd\xi)$. Therefore,

$$\left\|\overline{g_\varepsilon^b}(t, x, \xi)\right\|_{L^1(dtdxMd\xi)} \le \|S_\varepsilon\|_{L^\infty_t(L^1(dxd\mathcal{M}))}. \tag{4.5.20}$$

Step 2: Compactness of $\hat{g_\varepsilon^b}$. Setting

$$\begin{aligned}\hat{\mathcal{S}}_\varepsilon = &\left(\int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon \hat{\gamma}_\varepsilon b d\omega M_* d\xi_*\right) 1_{\{|A_\varepsilon| \le N\}} 1_{\left\{|\hat{g_\varepsilon^b}| \le L\right\}} \\ &- e\mathrm{div}_\xi \left(\varepsilon(\varepsilon E_\varepsilon + \xi \times B_\varepsilon)\left(g_\varepsilon^b \wedge L\right)\right) \\ &+ e\hat{\gamma}_\varepsilon \varepsilon E_\varepsilon \cdot \xi G_\varepsilon\end{aligned}$$

Notice that $|\hat{\gamma}_\varepsilon G_\varepsilon| \leq \frac{27}{2}$, and hence, by the interpolation between L^1 and L^∞, we have

$$\left(\int_{\mathbb{R}^3}\int_{S^2} q_\varepsilon \hat{\gamma}_\varepsilon b d\omega M_* d\xi_*\right) 1_{\{|A_\varepsilon|\leq N\}} 1_{\{|\hat{g^b_\varepsilon}|\leq L\}} + e\hat{\gamma}_\varepsilon \varepsilon E_\varepsilon \cdot \xi G_\varepsilon \in L^2(dtdxMdx)$$

and

$$\operatorname{div}_\xi \left(\varepsilon(\varepsilon E_\varepsilon + \xi \times B_\varepsilon)\left(g^b_\varepsilon \wedge L\right)\right) \in L^2(dtdx; H^{-1}(d\xi)).$$

Thus, from (4.5.18), we obtain

$$(\varepsilon\partial_t + \xi\cdot\nabla_x)\hat{g^b_\varepsilon} = \hat{\mathcal{S}}_\varepsilon \in L^2(dtdxMdx) + L^2(dtdx; H^{-1}(d\xi). \tag{4.5.21}$$

Applying the averaging theorem in [13, 26], we deduce from (4.5.21) that, for all $\chi(\xi)$ such that $\frac{\chi(\xi)}{1+|\xi|^2} \to 0$ as $|\xi| \to \infty$,

$$\| < \hat{g^b_\varepsilon}\chi > \|_{L^2(0,T; H^{\frac{1}{4}}(\mathcal{T}))} \leq C_{N,L}, \tag{4.5.22}$$

where $C_{N,L}$ depends only on N, L. This yields the compactness of $< \hat{g^b_\varepsilon}\chi >$ in space; namely, there exists a function $\eta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\lim_{z\to 0^+}\eta(z) = 0$

$$\left\|\left\langle \hat{g^b_\varepsilon}\chi\right\rangle(t, \cdot + y) - \left\langle \hat{g^b_\varepsilon}\chi\right\rangle(t,\cdot)\right\|_{L^2((0,T)\times\mathcal{T})} \leq \eta(|y|). \tag{4.5.23}$$

Step 3: Compactness of $g^b_\varepsilon \wedge L$. From (4.5.21) and the weak compactness of q_ε in $L^1(dtdxd\mathcal{M})$, we have, for large enough N, $\left\|\overline{g^b_\varepsilon}(t,x,\xi)\right\|_{L^1(dtdxMd\xi)}$ can be as small as we like. Thus, this, combining with (4.5.23) that there exists a function $\eta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\lim_{z\to 0^+}\eta(z) = 0$

$$\left\|\left\langle\left(g^b_\varepsilon \wedge L\right)\chi\right\rangle(t, \cdot + y) - \left\langle\left(g^b_\varepsilon \wedge L\right)\chi\right\rangle(t,\cdot)\right\|_{L^1((0,T)\times\mathcal{T})} \leq \eta(|y|). \tag{4.5.24}$$

Then, using the hypothesis that $\left\{\left(g^b_\varepsilon\right)^2\right\}_{\{\varepsilon>0\}}$ is relatively compact in $w{-}L^1(dt(1{+}|\xi|^2)Md\xi dx)$, we deduce easily that there exists a function $\eta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\lim_{z\to 0^+}\eta(z) = 0$

$$\left\|\left\langle\left(g^b_\varepsilon \wedge L\right)\chi\right\rangle(t, \cdot + y) - \left\langle\left(g^b_\varepsilon \wedge L\right)\chi\right\rangle(t,\cdot)\right\|_{L^2((0,T)\times\mathcal{T})} \leq \eta(|y|). \tag{4.5.25}$$

Step 4: Compactness of g^b_ε. Due to the hypothesis that $\left\{\left(g^b_\varepsilon\right)^2\right\}_{\{\varepsilon>0\}}$ is relatively compact in $w - L^1(dt(1+|\xi|^2)Md\xi dx)$, for every $\beta > 0$, there exists an integer $L > 1$ such that

$$\left\|\left\langle\left(g^b_\varepsilon \wedge L\right)\chi\right\rangle(t,\cdot) - \left\langle\left(g^b_\varepsilon\right)\chi\right\rangle(t,\cdot)\right\|_{L^2((0,T)\times\mathcal{T})} \leq C\beta,$$

uniformly in ε. Thus, for such β and L, we have

$$\left\|\left\langle\left(g_\varepsilon^b \wedge L\right)\chi\right\rangle(t,\cdot+y)-\left\langle g_\varepsilon^b\chi\right\rangle(t,\cdot+y)\right\|_{L^2((0,T)\times\mathcal{T})}\leq C\beta,$$

and

$$\left\|\left\langle\left(g_\varepsilon^b \wedge L\right)\chi\right\rangle(t,\cdot)-\left\langle g_\varepsilon^b\chi\right\rangle(t,\cdot)\right\|_{L^2((0,T)\times\mathcal{T})}\leq C\beta,$$

uniformly in ε. Hence, the above two inequalities, combining together with (4.5.25), imply there exists a function $\eta:\mathbb{R}_+\mapsto\mathbb{R}_+$ such that $\lim_{z\to 0^+}\eta(z)=0$ and

$$\int_0^T\int_{\mathcal{T}}\left|\left\langle g_{\varepsilon_n}^b\chi\right\rangle(t,x+y)-\left\langle g_{\varepsilon_n}^b\chi\right\rangle(t,x)\right|^2dxdt\leq\eta(|y|)$$

for each $y\in\mathbb{R}^3$ such that $|y|\leq 1$, uniformly in n. □

Now, we are ready to prove the convergence of the term of Lorentz force.

Lemma 4.5.4. *The following convergence holds in the sense of distributions on $\mathbb{R}_+\times\mathbb{R}^3$:*

$$B_\varepsilon\cdot<\xi\times\nabla_\xi\xi_k g_\varepsilon^b>\to(B\times(\nabla\times B))_k,$$

as $\varepsilon\to 0$, for all $1\leq k\leq 3$. The notation a_k stands for the i-th component of the vector a. Further, we have $j=e\mathbf{u}$.

Proof. For any $1\leq k\leq 3$, $\nabla_\xi\xi_k=e_k$, where $\{e_k\}_{k=1}^3$ is the standard basis for $\mathbb{R}^3$. This implies

$$<\xi\times\nabla_\xi\xi_k g_\varepsilon^b>=<\xi g_\varepsilon^b>\times e_k.$$

Then, we can rewrite $B_\varepsilon\cdot<\xi\times\nabla_\xi\xi_k g_\varepsilon^b>$ as

$$B_\varepsilon\cdot<\xi\times\nabla_\xi\xi_k g_\varepsilon^b>=B_\varepsilon\cdot\left(<\xi g_\varepsilon^b>\times e_k\right). \tag{4.5.26}$$

Defining

$$j_\varepsilon^b=e\frac{<\xi(1+\varepsilon g_\varepsilon^b)>}{\varepsilon}=e<\xi g_\varepsilon^b>,$$

since $<\xi>=0$. Then, we have

$$\left\|j_\varepsilon^b-\frac{j_\varepsilon}{\varepsilon}\right\|_{L_t^\infty(L^1(dxMd\xi))}\to 0, \tag{4.5.27}$$

as $\varepsilon \to 0$. Indeed, from the definition of g_ε^c, we know that $\varepsilon g_\varepsilon^c$ is uniformly bounded in $L_t^\infty(L^2(dxMd\xi))$ while from the second statement of Lemma 4.2.2, g_ε^c is uniformly bounded in $L_t^\infty(L^1(dxMd\xi))$. Thus, by the interpolation between L^2 and L^1, we deduce that

$$\|\varepsilon g_\varepsilon^c\|_{L_t^\infty(L^{\frac{3}{2}}(dxMd\xi))} \le C\varepsilon^{\frac{1}{2}},$$

for some constant $C > 0$. Therefore, we have

$$\begin{aligned}
\left\| j_\varepsilon^b - \frac{j_\varepsilon}{\varepsilon} \right\|_{L_t^\infty(L^1(dxMd\xi))} &= \left\| g_\varepsilon^b \xi - g_\varepsilon \xi \right\|_{L_t^\infty(L^1(dxMd\xi))} \\
&= \|\varepsilon g_\varepsilon^c \xi\|_{L_t^\infty(L^1(dxMd\xi))} \\
&\le \|\varepsilon g_\varepsilon^c\|_{L_t^\infty(L^{\frac{3}{2}}(dxMd\xi))} |\mathcal{T}|^{\frac{1}{3}} < |\xi|^3 >^{\frac{1}{3}} \\
&\le C\varepsilon^{\frac{1}{2}} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$. Here the notation $|\mathcal{T}|$ stands for the Lebesgue measure of the set $\mathcal{T}$. Hence, (4.5.27), combining with the weak convergence of $\left\{\frac{j_i}{\varepsilon}\right\}_{\{\varepsilon>0\}}$ in $L_t^\infty(L^2(dxMd\xi))$ and the uniform bound of $\{j_\varepsilon^b\}_{\{\varepsilon>0\}}$ in $L_t^\infty(L^2(dxMd\xi))$, implies that j_ε^b converges weakly to j in $L_t^\infty(L^2(dxMd\xi))$. Note that $\frac{j_\varepsilon}{\varepsilon} = e < g_\varepsilon \xi >$, we have $j = e\mathbf{u}$.

Notice that, (4.1.2c) implies

$$\partial_t B_\varepsilon = -\nabla \times E_\varepsilon \in L^\infty(0, T; W^{-4,2}(\mathcal{T})) \subset L^1(0, T; W^{-s,1}(\mathcal{T})),$$

for some $s > 4$ large enough, and is bounded in $L^1(0, T; W^{-s,1}(\mathcal{T}))$ uniformly in ε. On the other hand, Lemma 4.5.3 with $\chi(\xi) = \xi_k$ implies that for each $T > 0$,

$$\int_0^T \int_{\mathcal{T}} | < \xi g_\varepsilon^b > (t, x + y) \times e_k - < \xi g_\varepsilon^b > (t, x) \times e_k|^2 dx dt \le \eta(|y|), \tag{4.5.28}$$

for each $y \in \mathbb{R}^3$ such that $|y| \le 1$, uniformly in ε, where η is a function $\mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\lim_{z\to 0^+} \eta(z) = 0$. Hence, by Lemma 5.1 in [51], one has

$$(B_\varepsilon) \cdot \left(< \xi g_\varepsilon^b > \times e_k\right) \to B \cdot (e\mathbf{u} \times e_k) = B \cdot (j \times e_k) = (B \times j)_k, \tag{4.5.29}$$

in the sense of distributions.

This finishes our proof. □

BIBLIOGRAPHY

[1] Alazard, T., *Low Mach number limit of the full Navier-Stokes equations.* Arch. Ration. Mech. Anal. 180 (2006), no. 1, 1-73.

[2] Bardos, C.; Golse, F.; Levermore, D., *Fluid dynamic limits of kinetic equations. I. Formal derivations.* J. Statist. Phys. 63 (1991), 323–344.

[3] Bardos, C.; Golse, F.; Levermore, C. D., *Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation.* Comm. Pure Appl. Math. 46 (1993), 667–753.

[4] Boris V. Somov, *Plasma astrophysics, Part 1:Fundamentals and practice.* Springer, 2006.

[5] Bresch, D.; Desjardins, B.; Grenier, E.; Lin, C. K., *Low mach number limit of viscous polytropic flows: Formal asymptotics in periodic case*, Stud. Appl. Math. 109 (2002), 125–149.

[6] Cabannes, H., *Theoretical Magnetofluiddynamics*, Academic Press, New York, 1970.

[7] Cercignani, C.; Illner, R.; Pulvirenti, M., *The mathematical theory of dilute gases.* Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994.

[8] Chen, G.-Q.; Wang, D., *Global solution of nonlinear magnetohydrodynamics with large initial data.* J. Differential Equations, 182 (2002), 344-376.

[9] Chen, G.-Q.; Wang, D., *Existence and continuous dependence of large solutions for the magnetohydrodynamic equations.* Z. Angew. Math. Phys. 54 (2003), 608–632.

[10] Danchin, R., *Zero Mach number limit for compressible flows with periodic boundary conditions.* Amer. J. Math. 124 (2002), no. 6, 1153–1219.

[11] Desjardins, B.; Grenier, E., *Low Mach number limit of viscous compressible flows in the whole space*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 455 (1999), 2271–2279.

[12] Desjardins, B.; Grenier, E.; Lions, P. L.; Masmoudi, N., *Incompressible limit for solutions of the isentropic Navier-Stokes equations with Dirichlet boundary conditions*, J. Math. Pures Appl. (9) 78 (1999), 461–471.

[13] DiPerna, R. J.; Lions, P.-L., *Global weak solutions of Vlasov-Maxwell systems.* Comm. Pure Appl. Math. 42 (1989), 729–757.

[14] DiPerna, R. J.; Lions, P.-L., *On the Cauchy problem for Boltzmann equations: global existence and weak stability.* Ann. of Math. (2) 130 (1989), 321–366.

[15] Ducomet, B.; Feireisl, E., *The equations of Magnetohydrodynamics: On the interaction between matter and radiation in the evolution of gaseous stars.* Commun. Math. Phys. 226 (2006), 595-629.

[16] Duvaut, G.; Lions, J. L., *Inéquations en thermoélasticité et magnéto-hydrodynamique*, Arch. Rational Mech. Anal. 46 (1972) 241-279.

[17] Erban, D., *On the existence of solutions to the Navier-Stokes equations of a two-dimensional compressible flow.* Math. Methods Appl. Sci. 26 (2003), 489–517.

[18] Fan, J.; Jiang, S.; Nakamura, G., *Vanishing shear viscosity limit in the magnetohydrodynamic equations*, Commun. Math. Phys. 270 (2007), 691-708.

[19] Feireisl, E., *Dynamics of viscous compressible fluids.* Oxford Lecture Series in Mathematics and its Applications, 26. Oxford University Press, Oxford, 2004.

[20] Feireisl, E.; Novotný, A.; Petzeltová, H., *On the existence of globally defined weak solutions to the Navier-Stokes equations.* J. Math. Fluid Mech. 3 (2001), 358–392.

[21] Feireisl, E.; Novotný, A., *The low Mach number limit for the full Navier-Stokes-Fourier system.* Arch. Ration. Mech. Anal. 186 (2007), no. 1, 77–107.

[22] Freistühler, H.; Szmolyan, P., *Existence and bifurcation of viscous profiles for all intermediate magnetohydrodynamic shock waves.* SIAM J. Math. Anal., 26 (1995), 112-128.

[23] Galdi, G. P., *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems*, Springer-Verlag, New York, 1994.

[24] Glassey, Robert T., *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[25] Golse, F.; Levermore, C. D., *Stokes-Fourier and acoustic limits for the Boltzmann equation: convergence proofs.* Comm. Pure Appl. Math. 55 (2002), 336–393.

[26] Golse, F.; Lions, P.-L.; Perthame, B.; Sentis, R., *Regularity of the moments of the solution of a transport equation.* J. Funct. Anal. 76 (1988), 110–125.

[27] Golse, F.; Saint-Raymond, L., *The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels.* Invent. Math. 155 (2004), 81–161.

[28] Goossens, M., *An introduction to plasma astrophysics and magnetohydrodynamics.* Kluwer Academic Publishers, The Netherlands, 2003.

[29] Grad, H., *Asymptotic theory of the Boltzmann equation. II. 1963 Rarefied Gas Dynamics* (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. I pp. 26–59 Academic Press, New York

[30] Guo, Y., *The Vlasov-Maxwell-Boltzmann system near Maxwellians.* Invent. Math. 153 (2003), 593–630.

[31] Hoff, D., *Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data.* Arch. Rational Mech. Anal. 132 (1995), 1–14.

[32] Hoff, D., *Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids.* Arch. Rational Mech. Anal. 139 (1997), 303–354.

[33] Hoff, D., *The zero-Mach limit of compressible flows*, Comm. Math. Phys. 192 (1998), 543–554.

[34] Hoff, D.; Tsyganov, E., *Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics.* Z. Angew. Math. Phys. 56 (2005), 791–804.

[35] Hu, X.; Wang, D., *Global solutions to the three-dimensional full compressible magnetohydrodynamic flows*, Comm. Math. Phys. 283 (2008), 255–284.

[36] Hu, X.; Wang, D., *Global existence and large-time behavior of solutions to the multi-dimensional MHD equations of compressible isentropic flows*, To appear in Arch. Rational Mech. Anal.

[37] Hu, X.; Wang, D., *Compactness of weak solutions to compressible magnetohydrodynamics equations with density-dependent viscosities*, J. Differential Equations 245 (2008), 2176-2198.

[38] Hu, X.; Wang, D., *Low Mach number limit of viscous compressible magnetohydrodynamic flows*, SIAM J. Math. Anal. 41 (2009), 1272-1294.

[39] Hu, X.; Wang, D., *Global existence and incompressible limit of weak solutions to the multi-dimensional compressible magnetohydrodynamics.* Proceedings of Hyperbolic problems: theory, numerics, applications, 2008.

[40] Hu, X.; Wang, D., *On the Cauchy problem of the Vlasov-Maxwell-Boltzmann equations: weak stability and large time behavior.* Preprint.

[41] Hu, X.; Wang, D., *Incompressible magnetohydrodynamic limit of the Vlasov-Maxwell-Boltzmann equations.* Preprint.

[42] Jang, J., *Vlasov-Maxwell-Boltzmann diffusive limit.* To appear Arch. Rational Mech. Anal.

[43] Kazhikhov, V.; Shelukhin, V. V., *Unique global solution with respect to time of initial-boundary-value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech. 41 (1977), 273-282.

[44] Kawashima, S.; Okada, M. *Smooth global solutions for the one-dimensional equations in magnetohydrodynamics.* Proc. Japan Acad. Ser. A Math. Sci., 58 (1982), 384-387.

[45] Klainerman, S.; Majda, A., *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids.* Comm. Pure Appl. Math. 34 (1981), no. 4, 481–524.

[46] Kulikovskiy, A. G.; Lyubimov, G. A., *Magnetohydrodynamics*, Addison-Wesley, Reading, Massachusetts, 1965.

[47] Ladyzenskaja, O. A.; Solonnikov,V. A., *The linearization principle and invariant manifolds for problems of magnetohydrodynamics.* Boundary value problems of mathematical physics and related questions in the theory of functions, 7.

[48] Laudau, L. D.; Lifshitz, E. M., *Electrodynamics of Continuous Media*, 2nd ed., Pergamon, New York, 1984.

[49] Levermore, C. D.; Masmoudi, N., *From the Boltzmann equation to an incompressible Navier-Stokes-Fourier system.* To appear Arch. Rational Mech. Anal.

[50] Lions, P. L., *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models.* Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[51] Lions, P. L., *Mathematical topics in fluid mechanics. Vol. 2. Compressible models.* Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[52] Lions, P. L.; Masmoudi, N., *Incompressible limit for a viscous compressible fluid*, J. Math. Pures Appl. (9) 77 (1998), 585–627.

[53] Lions, P.-L.; Masmoudi, N., *From the Boltzmann equations to the equations of incompressible fluid mechanics. I, II.* Arch. Ration. Mech. Anal. 158 (2001), 173–193, 195–211.

[54] Liu, T.-P.; Zeng, Y., *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws.* Memoirs Amer. Math. Soc. 599, 1997.

[55] Maremonti, P. *Existence and stability of time-periodic solutions to the Navier-Stokes equations in the whole space.* Nonlinearity 4 (1991), 503–529.

[56] Masmoudi, N.; Saint-Raymond, L., *From the Boltzmann equation to the Stokes-Fourier system in a bounded domain.* Comm. Pure Appl. Math. 56 (2003), 1263–1293.

[57] Masmoudi, N., *Incompressible, inviscid limit of the compressible Navier-Stokes system*, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001), 199–224.

[58] Matthaeus, H.; Brown, R. *Nearly incompressible magnetohydrodynamics at low Mach number.* Phys. Fluids 31 (1988), 3634–3644.

[59] Métivier, G.; Schochet, S.' *The incompressible limit of the non-isentropic Euler equations.* Arch. Ration. Mech. Anal. 158 (2001), no. 1, 61–90.

[60] Métivier, G.; Schochet, S.' *Averaging theorems for conservative systems and the weakly compressible Euler equations.* J. Differential Equations 187 (2003), no. 1, 106–183.

[61] Novotné, A.; Stra**v**skraba, I., *Introduction to the theory of compressible flow*, Oxford University Press: Oxford, 2004.

[62] Puel, M.; Saint-Raymond, L., *Quasineutral limit for the relativistic Vlasov-Maxwell system.* Asymptot. Anal. 40 (2004), 303–352.

[63] Saint-Raymond, L., *Convergence of solutions to the Boltzmann equation in the incompressible Euler limit.* Arch. Ration. Mech. Anal. 166 (2003), 47–80.

[64] Schochet, S., *The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit.* Comm. Math. Phys. 104 (1986), no. 1, 49–75.

[65] Schochet, S., *The mathematical theory of low Mach number flows.* M2AN Math. Model. Numer. Anal. 39 (2005), no. 3, 441–458.

[66] Sermange, M.; Temam, R., *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. 36 (1983), 635-664.

[67] Wang, D. *Large solutions to the initial-boundary value problem for planar magnetohydrodynamics.* SIAM J. Appl. Math. 63 (2003), 1424-1441.

[68] Wang, S.; Jiang, S., *The convergence of the Navier-Stokes-Poisson system to the incompressible Euler equations*, Comm. Partial Differential Equations 31 (2006), 571–591.