THREE-DIMENSIONAL ANALYSIS OF BASE-ISOLATED STRUCTURES

by

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Base isolation has become a widely accepted method for earthquake resistant design of structures. However, the research in the field has been generally restricted to one-dimensional motion. Structural response is not limited to this one-dimensional motion, and the torsional effect of multidimensional motion contributes to the horizontal displacements. A three-dimensional structure can not be modeled with multiple one-dimensional analyses; rather, a complete three-dimensional analysis must be undertaken, as shown in this study.

Four separate analyses for the calculation of the dynamic response of a base-isolated structure will be presented in this study. The first two analysis procedures are for a single-story base-isolated structure. The last two procedures are for a multi-story base-isolated structure. The first procedure for each structure assumes a fully linear response, in which the bearings and the superstructure remain in the linear elastic range of response. The second procedure allows for a non-linear response from the bearings, in which each individual bearing may yield, changing the effective stiffness value.

To expand upon the four analysis procedures, additional considerations presented in this paper include an appendix on the effect of bearing friction and an appendix on plasticity. These two concepts further enhance the applicability of the solution procedures.

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NOMENCLATURE

Symbol	Description
A_{ni}	solution parameter first defined in equation (2-31)
B_{ni+1}	solution parameter first defined in equation (2-32)
$[C_b]$	damping matrix for bearing level
$[C_i]$	damping matrix for floor *i*
$[C_u]$	damping matrix of multi-story superstructure
c	constant characteristic of the transition from elastic to plastic behavior
$\{\ddot{d}_b\}$	acceleration vector for bearing level
$\{\dot{d}_b\}$	velocity vector for bearing level
$\{d_b\}$	displacement vector for bearing level
$\{\ddot{d}_g\}$	input ground acceleration vector due to earthquake
$\ddot{d}_{gz}$	input vertical ground acceleration due to earthquake
$\{\ddot{d}_i\}$	acceleration vector for floor *i* relative to the bearing level
$\{\dot{d}_i\}$	velocity vector for floor *i* relative to the bearing level
$\{d_i\}$	displacement vector for floor *i* relative to the bearing level
$\{\ddot{d}_u\}$	superstructure acceleration vector defined in equation (4-41)
$\{dU\}$	increment of total displacement

$\{dU_e\}$	increment of elastic displacement
$\{dU_p\}$	increment of plastic displacement
$\{dV\}$	increment of total force
$\{d\alpha\}$	increment of translation of yield surface
$d\lambda$	plastic flow parameter defined in equation (C-18)
$d\mu$	hardening parameter defined in equation (C-25)
e_i	eccentricity between G_iY_i and O_iY
e_b^1	eccentricity between G_bY_b and G_1Y_1
F_{in}^I	inertial force in the *n*-direction at floor *i*
F_{in}^S	resisting elastic force in the *n*-direction at floor *i*
F_{in}^D	dissipation force due to damping in the *n*-direction at floor *i*
F_{in}^D	frictional force in the *n*-direction at floor *i*
f_i	eccentricity between G_iX_i and O_iX
f_b^1	eccentricity between G_bX_b and G_1X_1
$f(V,\alpha)$	equation of yield surface
G_i	mass center of floor *i* (*b* for bearing floor, *1* for first floor)
g	vertical acceleration due to gravity
$[K_b]$	stiffness matrix for bearing level
$[K_e]$	elastic bearing stiffness
$[K_i]$	stiffness matrix for floor *i*

$[K_u]$	stiffness matrix of multi-story superstructure
$[M_i]$	mass matrix for floor *i*
$[M_t]$	total mass matrix, first defined in equation (2-7)
$[M_u]$	mass matrix of multi-story superstructure
$[M_{uc}]$	column mass matrix of multi-story superstructure
m_i	mass of floor *i*
$\{N\}$	vector normal to the yield surface
O_i	origin of arbitrary coordinate axis
$[Q]$	matrix used to solve for incremental modal accelerations
$\{P\}$	vector used to solve for incremental modal accelerations
$\{\mathrm{sgn}(\dot{d}_b)\}$	vector of absolute values of bearing accelerations
t_i	time at the beginning of a time step
t_{i+1}	time at the end of a time step
u_i	displacement of mass center G_i along $G_i X_i$
V_i	instantaneous shear force of a bearing in the *i*-direction
V_i^y	yield force in the *i*-direction
v_i	displacement of mass center G_i along $G_i Y_i$
x_i	displacement of floor *i* along $O_i X$
y_i	displacement of floor *i* along $O_i Y$
$\{\ddot{z}_i\}$	modal acceleration vector for floor *i*

$\{\dot{z}_i\}$	modal velocity vector for floor *i*
$\{z_i\}$	modal displacement vector for floor *i*
$\{\ddot{z}_u\}$	superstructure modal acceleration vector
$\{\alpha\}$	translation vector of yield surface
α	parameter used in Hilber's non-linear analysis method (Hilber, 1977)
α_{ij}	orientation of stiffness element *j* of floor *i*
$[\alpha_u]$	superstructure solution parameter
β	parameter used in Newmark's non-linear analysis method (Hilber, 1977)
γ	parameter used in Newmark's non-linear analysis method (Hilber, 1977)
$\{\Delta R\}$	residual forces in non-linear solution
Δt	interval of time steps
θ_i	rotational displacement of mass center G_i about $G_i Z_i$
μ	coefficient of friction
ξ_{in}	damping ratio of floor *i* in mode *n*
τ	time, measured between beginning and ending of a single time step
$[\Phi_i]$	modal matrix
$\{\phi_{ij}\}$	mode shape *j* of floor *i*
Ω_{in}	damped frequency of floor *i* in mode *n*
ω_{in}	natural frequency *n* of floor *i*

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1.0 INTRODUCTION

1.1 INTRODUCTION TO BASE ISOLATION

Base isolation is an important concept in earthquake engineering. Initially, base isolation was a very suspect process for design of earthquake resistant structures, and engineers were wary of its applications; however, it has since become a widely accepted approach. The goal of base isolation is to reduce the energy that is transferred from the ground motion to the structure by buffering it with a bearing layer at the foundation which has relatively low stiffness. The bearing level has a longer period than the superstructure, which reduces the force and displacement demands on the superstructure, allowing it to remain elastic and generally undamaged.

One of the important properties of a base-isolation system is that although it is designed to be significantly more flexible than the elements of the superstructure, it must still be stiff enough to resist typical wind loadings and similar low-amplitude horizontal forces. Therefore, the bearings may have a relatively high initial stiffness but will quickly reach yield, at which point the bearings have a greatly reduced stiffness, extending the natural period of the structure.

1.2 LITERATURE REVIEW

There have been numerous papers and books published regarding base isolation of structures. However, the three-dimensional performance of these structures has been generally overlooked in the literature.

James M. Kelly is an influential researcher in the area of base-isolation. His book, *Earthquake Resistant Design with Rubber* (1996), discusses the theory and application of base-isolation in detail. One chapter of his work that is particularly important for this study is Chapter 6, a discussion of the rotational effects of coupled motion of a base-isolated structure. This chapter considers three degrees of freedom – x and y horizontal motion and the torsional degree of freedom – in structural models. The three degree of freedom system was previously presented in an article by Pan and Kelly in the Journal of Earthquake Engineering and Structural Dynamics in 1983. The method used to treat the three degree-of-freedom system in Kelly is quite different from that presented in this study, as it focuses on the relationships of the three mode shapes to one another. The formulations presented here are independent of the relationships between the mode shapes.

Abe, et al (2004-a) performed tests on various bearing materials to determine their properties such as stiffness and multi-directional behavior. The tests performed were the biaxial load test, in which a constant vertical load and a variable horizontal load were applied; a triaxial load test, in which a second variable horizontal loading was applied perpendicular to the biaxial test load; and a small amplitude test, in which the horizontal loading is minimal to determine the resistance behavior of the bearings under small deflections. The test results were then used to ascertain the accuracy of mathematic models that were developed in tandem with the experiments.

Plastic behavior was evident in the response of the bearings in the experimental phase of the study, so Abe, et al. (2004-b), used a plasticity model based upon the work of Ozdemir (1973) to model the nonlinear behavior of the bearings. These models are shown to accurately portray the behavior of the bearings from the biaxial and triaxial test results. However, the models are very specific to the vertical load conditions applied to the bearings during the testing. The experiments were conducted at two separate vertical load levels, and exhibited different responses for each loading.

The experiments performed by Abe, et al. (2004-a), suggest that the vertical force acting through the bearings affects their stiffness and damping properties. This effect is particularly visible in the response of the lead-plug rubber bearing, due to a closing of the gap between the plug and the rubber. However, it should be noted that for large deformations the damping ratio and stiffness values become more stable, and less dependent upon the vertical loading. Further research must be undertaken to ascertain a relationship between changes in the vertical loading and the response of the bearings. For the purposes of this study, it is assumed that the vertical acceleration of the structure due to ground motion is small with respect to the gravitational acceleration *g*. This implies that the total vertical acceleration, $g + \ddot{d}_{gz}$, will be very close to the gravitational acceleration value; therefore, the vertical force acting through the bearings will not significantly affect their properties.

As mentioned, the paper by Abe, et al. (2004-b) used a plasticity model based upon the work of Ozdemir (1973). This study, however, will use a different plasticity formulation. Ziegler (1959) modified Prager's hardening rule to develop a plasticity theory to apply to kinematic hardening. This theory will be further modified for the purposes of this work to extend to force-displacement relationships instead of the default stress-strain relationship. However, the concepts proposed by Ziegler can easily be seen in the work presented in Appendix C.

Mostaghel and Khodaverdian (1988) wrote a paper on the dynamic response of base-isolated structures which formed a skeleton for many of the derivations presented in this study. Their paper focused on friction-based isolation systems, and therefore introduced the friction component to the derivations which appears in Appendix B. The work presented in their paper is, however, restricted to unidirectional motion, considering only one horizontal degree of freedom and the vertical ground motion, which is integral to the frictional effect.

The PhD dissertation of Ahmad El-Hajj (1993), published at the University of Pittsburgh, is the foundation upon which this thesis is built. The formulations presented in this study are nearly identical to El-Hajj's, though additions and corrections have been made to improve and clarify his work. His dissertation developed a multi-dimensional approach to base isolation, incorporating both horizontal axes and the rotational component as suggested by Pan and Kelly in their 1983 paper. The modified Ziegler (1959) plasticity is also adapted from this dissertation, which modified the stress-strain formulation to apply it to the more convenient force-displacement relationship.

The treatment of nonlinearities in the bearing response is not restricted to the plasticity theory found in Appendix C. In each of the chapters discussing nonlinear response, a method is used to increase the accuracy of the calculation. This method is the Hilber-α Method, which is an extension of Newmark's β-Method. Hilber's (1977) method modifies the stiffness value used in each time step to improve convergence on the actual structural response, and figures heavily in the nonlinear structural response, as can be seen in Chapters 3 and 5.

1.3 PROPOSED STUDY

The work presented herein represents a multifaceted treatment of base-isolation. Not only do the formulations in this study incorporate the effect of coupled motion and the torsional degree of freedom, as shown in Kelly's work, but these formulations also allow for the inclusion of frictional components and plastic analysis. Each of these concepts may contribute to the dynamic response of a base-isolated structure.

The first analysis procedure demonstrated in this study is a single-story linear base-isolated structure. This analysis is very important; it is the basis upon which the more complex analyses are derived. Both the bearing level and the first floor are considered to be linear in this case.

The second analysis procedure is a single-story non-linear base-isolated structure. The first floor is assumed to remain linear, in accordance with the concept of base isolation. However, non-linearity is considered in the behavior of the bearings. This formulation employs the plasticity procedure discussed in Appendix C.

These two analyses are then expanded to apply to multi-story structures. In each procedure, however, the superstructure is assumed to remain linear at all times. Under proper conditions for base-isolation, this is an appropriate assumption.

Individual base-isolation systems can be completely ineffective for certain types of earthquakes, a fact which demonstrates the necessity of research into the seismic properties of an area, such as earthquake history and soil characteristics, before applying base-isolation to a structure.

2.0 SINGLE-STORY LINEAR ANALYSIS

The calculation of the dynamic response of a structure to a specified ground motion is a complex process. It requires determination of the equations of motion of the structure and a time-history analysis with a small time step to achieve accurate results. This analysis will first be developed for a simple three-dimensional one-story isolated structure considering three degrees of freedom at each floor: two perpendicular horizontal motions and in-plane rotation, as shown in Figure 1 and Figure 2. Accounting for these three degrees of freedom at both the isolation level and the first floor creates a total of six degrees of freedom. For the purposes of this study, this is absolutely the simplest structure to be considered.

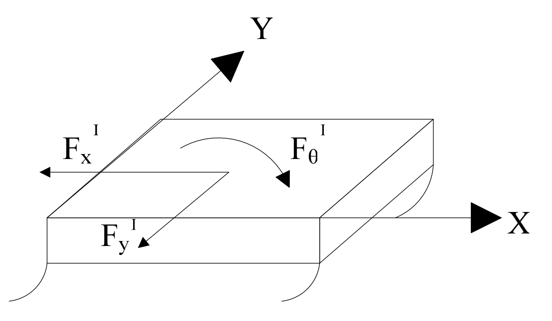

Figure 1 – First Floor Free Body Diagram

2.1 ANALYSIS PROCEDURE

The first step in the analysis is the determination of the equations of motion for each floor. Figure 1 represents a free-body diagram drawn by cutting the structure directly below the first floor, and considering only the first floor. The resisting elastic force and the dissipation force due to damping are not shown in the drawing, but act opposite to the direction of the displacement and velocity of the structure, respectively, directly below the floor level. With respect to Figure 1, the following summation of forces can be written in the *X*-direction:

$$F_{1x}^{I} + F_{1x}^{D} + F_{1x}^{S} = 0 \tag{2-1}$$

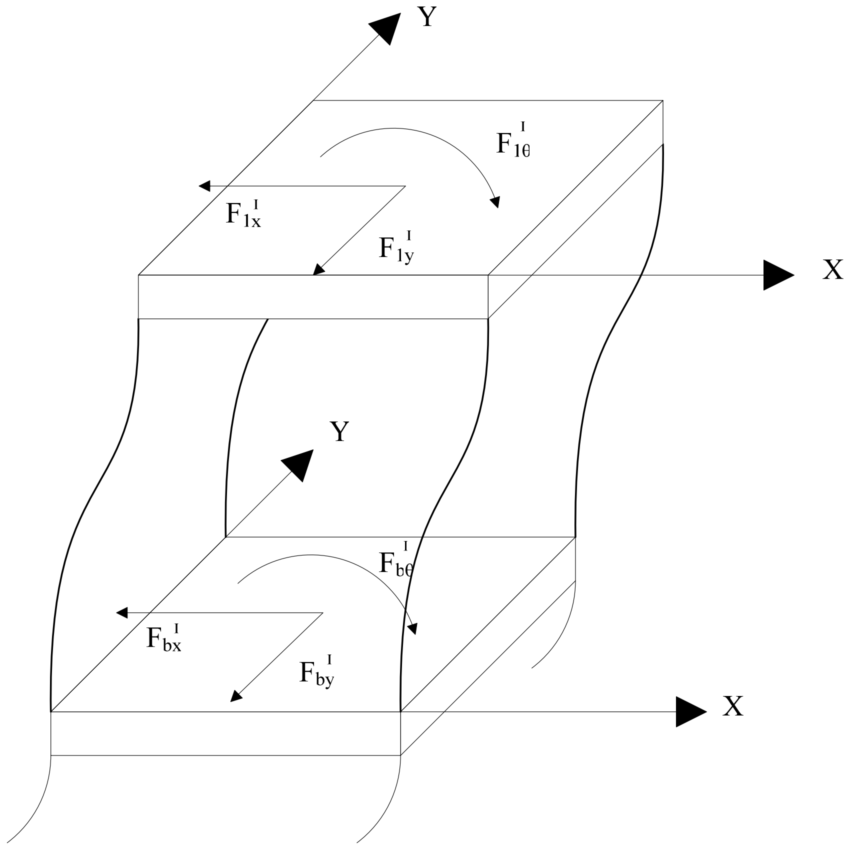

Figure 2 – Superstructure Free Body Diagram

Figure 2 represents a free-body diagram of the structure drawn by cutting the structure just below the bearing floor, and takes into account the entire structure. As was the case with Figure 1, the resisting elastic and damping forces are not shown. There is also a frictional force at the bearing level that is not shown. The frictional force acts opposite to the direction of velocity. With respect to Figure 2, the following summation of forces can be written in the *X*-direction:

$$F_{1x}^{I} + F_{bx}^{I} + F_{bx}^{D} + F_{bx}^{S} + F_{bx}^{F} = 0 \qquad (2\text{-}2)$$

in which

$F_{ix}^{I} \equiv$ the inertial force of floor i

$F_{ix}^{D} \equiv$ the damping force of floor i

$F_{ix}^{S} \equiv$ the elastic force of floor i

$F_{bx}^{F} \equiv$ the friction force at the bearing level

$i \equiv$ the floor: b for base, *1* for first floor (roof)

The formulations presented here allow for friction within the bearings to be considered; floor friction is negligible. To ignore the effects of friction at the bearing level, simply set the coefficient of friction, μ, to zero, and proceed with the solution.

The force summations presented in equations (2-1) and (2-2) can be applied in any of three directions: the two horizontal directions and a torsional summation, which represents the summation of moments. By writing out these equations in each of the three degrees of freedom, the following matrix equations can be written in a form similar to equations (2-1) and (2-2), respectively:

$$\begin{Bmatrix} F_{1x}^{I} \\ F_{1y}^{I} \\ F_{1\theta}^{I} \end{Bmatrix} + \begin{Bmatrix} F_{1x}^{D} \\ F_{1y}^{D} \\ F_{1\theta}^{D} \end{Bmatrix} + \begin{Bmatrix} F_{1x}^{S} \\ F_{1y}^{S} \\ F_{1\theta}^{S} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2\text{-}3)$$

$$\begin{Bmatrix} F_{bx}^{I} \\ F_{by}^{I} \\ F_{b\theta}^{I} \end{Bmatrix} + \begin{Bmatrix} F_{bx}^{D} \\ F_{by}^{D} \\ F_{b\theta}^{D} \end{Bmatrix} + \begin{Bmatrix} F_{bx}^{S} \\ F_{by}^{S} \\ F_{b\theta}^{S} \end{Bmatrix} = -\begin{Bmatrix} F_{1x}^{I} \\ F_{1y}^{I} \\ F_{1\theta}^{I} \end{Bmatrix} - \begin{Bmatrix} F_{bx}^{F} \\ F_{by}^{F} \\ F_{b\theta}^{F} \end{Bmatrix} \tag{2-4}$$

Equation (2-3) can be expanded via the derivations shown in Appendix A:

$$[M_1]\{\ddot{d}_1\} + [C_1]\{\dot{d}_1\} + [K_1]\{d_1\} = -[M_1]\{\ddot{d}_b\} - [M_1]\{\ddot{d}_g\} \tag{2-5}$$

2.1.1 Bearing Level Equations of Motion

Similarly, equation (2-4) can be expanded using the derivations from Appendix A. However, the frictional terms were not considered in the appendix. By definition, the friction force is equal to the normal force times the frictional constant μ. The normal forces in this formulation will be taken as the mass matrix times the total vertical acceleration of the structure, $(g + \ddot{d}_{gz})$. The direction of the frictional force is determined from the direction of the velocity of the bearing level, as seen in equation (2-11). Equation (2-4) becomes:

$$\begin{gathered} [M_t]\{\ddot{d}_b\} + [C_b]\{\dot{d}_b\} + [K_b]\{d_b\} = \\ -[M_1]\{\ddot{d}_1\} - [M_t]\{\ddot{d}_g\} - \mu(g + \ddot{d}_{gz})[M_t]\{\mathrm{sgn}(\dot{d}_b)\} \end{gathered} \tag{2-6}$$

in which

$$\begin{aligned} [M_t] &= [M_b] + [M_1] \\ &= \begin{bmatrix} m_t & 0 & -(m_1 f_1 + m_b f_b) \\ 0 & m_t & m_1 e_1 + m_b e_b \\ -(m_1 f_1 + m_b f_b) & m_1 e_1 + m_b e_b & (J_1 + J_b) + m_1(f_1^2 + e_1^2) + m_b(f_b^2 + e_b^2) \end{bmatrix} \end{aligned} \tag{2-7}$$

$$[M_1] = \begin{bmatrix} m_1 & 0 & -m_1 f_1 \\ 0 & m_1 & m_1 e_1 \\ -m_1 f_1 & m_1 e_1 & J_1 + m_1(f_1^2 + e_1^2) \end{bmatrix} \tag{2-8}$$

$$[C_i]=\begin{bmatrix} C_{ixx} & C_{ixy} & C_{ix\theta} \\ C_{iyx} & C_{iyy} & C_{iy\theta} \\ C_{i\theta x} & C_{i\theta y} & C_{i\theta\theta} \end{bmatrix} \qquad [K_i]=\begin{bmatrix} K_{ixx} & K_{ixy} & K_{ix\theta} \\ K_{iyx} & K_{iyy} & K_{iy\theta} \\ K_{i\theta x} & K_{i\theta y} & K_{i\theta\theta} \end{bmatrix} \tag{2-9}$$

$$\{d_b\}=\begin{Bmatrix} x_b \\ y_b \\ \theta_b \end{Bmatrix} \qquad \{d_1\}=\begin{Bmatrix} x_1 \\ y_1 \\ \theta_1 \end{Bmatrix} \qquad \{\ddot{d}_g\}=\begin{Bmatrix} \ddot{x}_g \\ \ddot{y}_g \\ 0 \end{Bmatrix} \tag{2-10}$$

$$\left.\begin{aligned} &\{\operatorname{sgn}(\dot{d}_b)\}=\begin{Bmatrix} \operatorname{sgn}(\dot{d}_{bx}) \\ \operatorname{sgn}(\dot{d}_{by}) \\ \operatorname{sgn}(\dot{d}_{b\theta}) \end{Bmatrix}=\begin{Bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{Bmatrix} \\ &\operatorname{sgn}(\dot{d}_{bi})=\frac{\dot{d}_{bi}}{|\dot{d}_{bi}|} \end{aligned}\right\} \tag{2-11}$$

$\ddot{d}_{gz} \equiv$ vertical acceleration of the ground due to the earthquake loading

Note that the stiffness matrices $[K_i]$ are determined via the process described in Appendix A.

The displacement vectors can be decomposed through the modal superposition method, in which a linear combination of the mode shapes will be used to define the displacements. The displacements can be written as a function of the mode shapes of the structure as such:

$$d_{bi}(t)=\sum_{j=1}^{3}\phi_{bij}z_{bj}(t) \tag{2-12}$$

$$d_{1i}(t)=\sum_{j=1}^{3}\phi_{1ij}z_{1j}(t) \tag{2-13}$$

The vectors $\{z_i\}$ represent a set of modal, or normal, coordinates. These modal coordinates represent the effects of each mode shape on the deformation of the structure, as seen in equations (2-12) and (2-13).

The mode shapes $\{\phi_i\}$ can be determined by solving the generalized eigenvalue problem

$$[K_1]\{d_1\}=\omega_{1n}^2[M_1]\{d_1\} \tag{2-14}$$

$$[K_b]\{d_b\} = \omega_{bn}^2 [M_t]\{d_b\} \tag{2-15}$$

The mode shapes are actually the eigenvectors from equations (2-14) and (2-15), and the natural frequencies are calculated from the eigenvalues. The modal matrices will be of the form

$$[\Phi_1] = \begin{bmatrix} \phi_{111} & \phi_{112} & \phi_{113} \\ \phi_{121} & \phi_{122} & \phi_{123} \\ \phi_{131} & \phi_{132} & \phi_{133} \end{bmatrix} \qquad [\Phi_b] = \begin{bmatrix} \phi_{b11} & \phi_{b12} & \phi_{b13} \\ \phi_{b21} & \phi_{b22} & \phi_{b23} \\ \phi_{b31} & \phi_{b32} & \phi_{b33} \end{bmatrix} \tag{2-16}$$

The modal matrix for the first floor, $[\Phi_1]$, is determined from equation (2-14) and the modal matrix for the bearings, $[\Phi_b]$, is determined from equation (2-15).

The columns of the modal matrices represent the mode shapes, with the first column representing the primary mode, which corresponds to the primary natural frequency. Each row of the modal matrix represents the way in which the modal displacements are combined to produce the three components of the actual structural response, as seen in equations (2-12) and (2-13). Therefore, each modal displacement contributes to each of the three degrees of freedom of the floor.

The mode shapes are mass-orthonormalized so that the following relation is obtained:

$$[M_t^*] = [\Phi_b]^T [M_t] [\Phi_b] = [I] \tag{2-17}$$

Equation (2-6) can be simplified, using equation (2-17), by first substituting equation (2-12) as follows:

$$\begin{aligned} &[M_t][\Phi_b]\{\ddot{z}_b\} + [C_b][\Phi_b]\{\dot{z}_b\} + [K_b][\Phi_b]\{z_b\} = \\ &\quad -[M_1][\Phi_1]\{\ddot{z}_1\} - [M_t]\{\ddot{d}_g\} - \mu(g + \ddot{d}_{gz})[M_t]\{\mathrm{sgn}(\dot{d}_b)\} \end{aligned} \tag{2-18}$$

Then, by premultiplying each side of the equation by the transpose of the modal matrix for the bearing level, the following equation is obtained:

$$\begin{aligned}&[\Phi_b]^T[M_t][\Phi_b]\{\ddot{z}_b\}+[\Phi_b]^T[C_b][\Phi_b]\{\dot{z}_b\}+[\Phi_b]^T[K_b][\Phi_b]\{z_b\}=\\&-[\Phi_b]^T[M_1][\Phi_1]\{\ddot{z}_1\}-[\Phi_b]^T[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[\Phi_b]^T[M_t]\left\{\mathrm{sgn}\left(\dot{d}_b\right)\right\}\end{aligned}\tag{2-19}$$

Next, the mass-orthonormalization shown in equation (2-17) is used to further simplify the expression:

$$\begin{aligned}&[I]\{\ddot{z}_b\}+diag[2\xi_{bi}\omega_{bi}]\{\dot{z}_b\}+diag\left[\omega_{bi}^2\right]\{z_b\}=\\&\qquad-[\Phi_b]^T[M_1][\Phi_1]\{\ddot{z}_1\}-[\Phi_b]^T[M_t]\{\ddot{d}_g\}\\&\qquad-\mu\left(g+\ddot{d}_{gz}\right)[\Phi_b]^T[M_t]\left\{\mathrm{sgn}\left(\dot{d}_b\right)\right\}\end{aligned}\tag{2-20}$$

This matrix equation consists of three separate equations of motion, one for each of the modal displacements. Each of the three equations is shown below in equation (2-21), with n = 1, 2, or 3, representing the modal displacement to be considered by the equation:

$$\begin{aligned}&\ddot{z}_{bn}(t)+2\xi_{bn}\omega_{bn}\dot{z}_{bn}(t)+\omega_{bn}^2 z_{bn}(t)=\\&\sum_{m=1}^{3}\lambda_{bnm}\ddot{z}_{1m}(t)+\sum_{m=1}^{3}\alpha_{bnm}\ddot{d}_{gm}(t)+\sum_{m=1}^{3}\mu\left(g+\ddot{d}_{gz}(t)\right)\alpha_{bnm}\,\mathrm{sgn}\left(\dot{d}_{bm}\right)\end{aligned}\tag{2-21}$$

in which the following substitutions were made:

$$\left[K_b^*\right]=[\Phi_b]^T[K_b][\Phi_b]=\begin{bmatrix}\omega_{b1}^2&0&0\\0&\omega_{b2}^2&0\\0&0&\omega_{b3}^2\end{bmatrix}\tag{2-22}$$

$$\left[C_b^*\right]=[\Phi_b]^T[C_b][\Phi_b]=\begin{bmatrix}2\xi_{b1}\omega_{b1}&0&0\\0&2\xi_{b2}\omega_{b2}&0\\0&0&2\xi_{b3}\omega_{b3}\end{bmatrix}\tag{2-23}$$

$$[\lambda_b]=-[\Phi_b]^T[M_1][\Phi_1]=\begin{bmatrix}\lambda_{b11}&\lambda_{b12}&\lambda_{b13}\\\lambda_{b21}&\lambda_{b22}&\lambda_{b23}\\\lambda_{b31}&\lambda_{b32}&\lambda_{b33}\end{bmatrix}\tag{2-24}$$

$$[\alpha_b]=-[\Phi_b]^T[M_t]=\begin{bmatrix}\alpha_{b11}&\alpha_{b12}&\alpha_{b13}\\\alpha_{b21}&\alpha_{b22}&\alpha_{b23}\\\alpha_{b31}&\alpha_{b32}&\alpha_{b33}\end{bmatrix}\tag{2-25}$$

Equation (2-22) is true because of the orthogonality property of modes. Note that equation (2-23) is the classical damping matrix. For simplicity in calculations, classical damping will be used throughout this paper. The ξ_{ij} terms represent the damping ratio of floor *i* in mode *j*. Equations (2-24) and (2-25) are products of the matrix multiplications required to simplify the equation of motion into its current state.

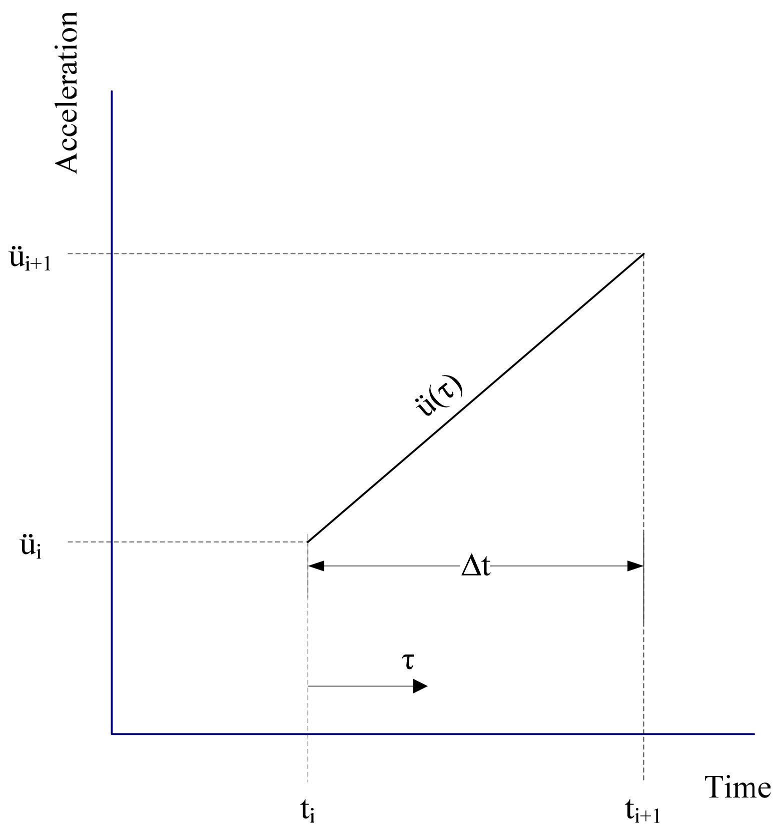

Figure 3 – Linear Acceleration Method

Equation (2-21) is not quite in a solvable form. To solve for the displacement of the structure as a function of time, a linear interpolation approach will be taken to approximate the change in accelerations. Figure 3 represents the linear acceleration method, in which accelerations are known at the beginning and end of each time step and a straight line approximates the unknown acceleration during the interval. This is reasonably accurate for a sufficiently small time step Δt. For example, the earthquake ground acceleration records from the Imperial Valley Irrigation District from the North-South motion of the 1940 El Centro, CA earthquake are recorded at an interval of $\Delta t = 0.02$ seconds (Chopra, 2001).

Implementing the linear acceleration method, expressions for the modal accelerations of the first floor and the ground accelerations can be written as follows:

$$\ddot{z}_{1k}(\tau) = \ddot{z}_{1k}(t_i) + \frac{\Delta \ddot{z}_{1k}(t_{i+1})}{\Delta t}\tau \qquad \text{(2-26)}$$

$$\ddot{x}_g(\tau) = \ddot{x}_g(t_i) + \frac{\Delta \ddot{x}_g(t_{i+1})}{\Delta t}\tau \qquad \text{(2-27)}$$

$$\ddot{y}_g(\tau) = \ddot{y}_g(t_i) + \frac{\Delta \ddot{y}_g(t_{i+1})}{\Delta t}\tau \qquad \text{(2-28)}$$

$$\ddot{d}_{gz}(\tau) = \ddot{d}_{gz}(t_i) + \frac{\Delta \ddot{d}_{gz}(t_{i+1})}{\Delta t}\tau \qquad \text{(2-29)}$$

As can be seen from Figure 3, $0 \le \tau \le \Delta t$.

Now, substituting equations (2-26) through (2-29) into equation (2-21) yields the following equation:

$$\ddot{z}_{bn}(\tau) + 2\xi_{bn}\,\omega_{bn}\,\dot{z}_{bn}(\tau) + \omega_{bn}^2\,z_{bn}(\tau) = A_{ni} + B_{n\,i+1}\frac{\tau}{\Delta t} \qquad \text{(2-30)}$$

in which n = 1, 2, 3, and

$$A_{ni} = \sum_{l=1}^{3}\left(\alpha_{bnl}\,\ddot{d}_{gl}(t_i) + \lambda_{bnl}\,\ddot{z}_{1l}(t_i) + \mu\left(\left(g + \ddot{d}_{gz}(t_i)\right)\alpha_{bnl}\,\mathrm{sgn}\left(\dot{d}_{bl}(t_i)\right)\right)\right) \qquad \text{(2-31)}$$

$$B_{n\,i+1} = \sum_{l=1}^{3}\left(\alpha_{bnl}\,\Delta\ddot{d}_{gl}(t_{i+1}) + \lambda_{bnl}\,\Delta\ddot{z}_{1l}(t_{i+1}) + \mu\left(\Delta\ddot{d}_{gz}(t_{i+1})\alpha_{bnl}\,\mathrm{sgn}\left(\dot{d}_{bl}(t_i)\right)\right)\right) \qquad \text{(2-32)}$$

Equation (2-30) is now in the form of a second-order non-homogeneous differential equation with two forcing functions. This type of problem has a solution that is written as a combination of the complementary solution and the particular solution. The complementary or homogeneous solution, or the solution to equation (2-30) if the right hand side were set to zero, is

$$z_{bn}^{c} = e^{-\xi_{bn}\,\omega_{bn}\,\tau}\left(C1_n \sin\Omega_{bn}\tau + C2_n \cos\Omega_{bn}\tau\right) \tag{2-33}$$

in which the damped natural frequency is represented by

$$\Omega_{bn} = \omega_{bn}\sqrt{1-\xi_{bn}^{2}} \tag{2-34}$$

The constants $C1_n$ and $C2_n$ in equation (2-33) are dependent upon initial conditions and will be determined below.

The particular solution to equation (2-30) is of the form

$$z_{bn}^{p} = C3_n + C4_n \frac{\tau}{\Delta t} \tag{2-35}$$

By substituting equation (2-35) and its derivatives into equation (2-30), the constants $C3_n$ and $C4_n$ can be determined as

$$C3_n = \frac{1}{\omega_{bn}^{2}}\left(A_{ni} - \frac{2\,\xi_{bn}}{\omega_{bn}\,\Delta t}B_{n\,i+1}\right) \tag{2-36}$$

$$C4_n = \frac{B_{n\,i+1}}{\omega_{bn}^{2}} \tag{2-37}$$

Combining the complementary solution from equation (2-33) and particular solution from equation (2-35) yields the following expression for z_{bn}:

$$\begin{aligned} z_{bn}(\tau) = {} & e^{-\xi_{bn}\,\omega_{bn}\,\tau}\left(C1_n \sin\Omega_{bn}\tau + C2_n \cos\Omega_{bn}\tau\right) \\ & + \frac{1}{\omega_{bn}^{2}}\left(A_{ni} + \left(\tau - \frac{2\,\xi_{bn}}{\omega_{bn}}\right)\frac{B_{n\,i+1}}{\Delta t}\right) \end{aligned} \tag{2-38}$$

As can be seen in Figure 3, as $\tau \to 0$, $t \to t_i$ and the following are true for $\tau = 0$:

$$z_{bn}(\tau = 0) = z_{bn}(t_i) \tag{2-39}$$

$$\dot{z}_{bn}(\tau = 0) = \dot{z}_{bn}(t_i) \tag{2-40}$$

These values can now be used to determine the constants $C1_n$ and $C2_n$. By applying equations (2-39) and (2-40) to equation (2-38), the following results are obtained:

$$C1_n = \frac{1}{\Omega_{bn}}\left(\dot{z}_{bn}(t_i) + \xi_{bn}\,\omega_{bn}\,z_{bn}(t_i) - \frac{\xi_{bn}}{\omega_{bn}}A_{ni} - \frac{\left(1-2\,\xi_{bn}^2\right)}{\omega_{bn}^2\,\Delta t}B_{n\,i+1}\right) \tag{2-41}$$

$$C2_n = z_{bn}(t_i) - \frac{A_{ni}}{\omega_{bn}^2} + \frac{2\,\xi_{bn}}{\omega_{bn}^3\,\Delta t}B_{n\,i+1} \tag{2-42}$$

Now by setting $\tau = \Delta t$ and substituting equations (2-41) and (2-42) back into the solution given by equation (2-38), the following expression for z_{bn} is given:

$$z_{bn}(t_{i+1}) = D_{ni} + R1_n\,B_{n\,i+1} + \frac{1}{\omega_{bn}^2}\left(A_{ni} + \left(\Delta t - \frac{2\,\xi_{bn}}{\omega_{bn}}\right)\frac{B_{n\,i+1}}{\Delta t}\right) \tag{2-43}$$

in which the following terms are defined for the purposes of simplification:

$$R1_n = e^{-\xi_{bn}\,\omega_{bn}\,\Delta t}\left(-\frac{1-2\,\xi_{bn}^2}{\omega_{bn}^2\,\Omega_{bn}\,\Delta t}\sin\Omega_{bn}\,\Delta t + \frac{2\,\xi_{bn}}{\omega_{bn}^3\,\Delta t}\cos\Omega_{bn}\,\Delta t\right) \tag{2-44}$$

$$D_{ni} = \frac{e^{-\xi_{bn}\,\omega_{bn}\,\Delta t}}{\Omega_{bn}}\left(E_{ni}\sin\Omega_{bn}\,\Delta t + F_{ni}\cos\Omega_{bn}\,\Delta t\right) \tag{2-45}$$

$$E_{ni} = \dot{z}_{bn}(t_i) + \xi_{bn}\,\omega_{bn}\,z_{bn}(t_i) - \frac{\xi_{bn}}{\omega_{bn}}A_{ni} \tag{2-46}$$

$$F_{ni} = \Omega_{bn}\left(z_{bn}(t_i) - \frac{A_{ni}}{\omega_{bn}^2}\right) \tag{2-47}$$

Similarly, the modal velocity can be derived from equation (2-38) by taking the derivative with respect to time and evaluating it at $\tau = \Delta t$. The modal velocity can then be written as:

$$\dot{z}_{bn}\left(t_{i+1}\right)=\left(G_{ni}-\xi_{bn}\,\omega_{bn}\,D_{ni}\right)+\left(R2_n-\xi_{bn}\,\omega_{bn}\,R1_n\right)B_{n\,i+1}+\frac{B_{n\,i+1}}{\omega_{bn}^2\,\Delta t} \tag{2-48}$$

in which

$$G_{ni}=e^{-\xi_{bn}\,\omega_{bn}\,\Delta t}\left(E_{ni}\cos\Omega_{bn}\,\Delta t-F_{ni}\sin\Omega_{bn}\,\Delta t\right) \tag{2-49}$$

$$R2_n=\Omega_{bn}e^{-\xi_{bn}\,\omega_{bn}\,\Delta t}\left(-\frac{1-2\,\xi_{bn}^2}{\omega_{bn}^2\,\Omega_{bn}\,\Delta t}\cos\Omega_{bn}\,\Delta t-\frac{2\,\xi_{bn}}{\omega_{bn}^3\,\Delta t}\sin\Omega_{bn}\,\Delta t\right) \tag{2-50}$$

The modal acceleration can also be derived from equation (2-38) by taking the second derivative with respect to time. The modal acceleration can be written as:

$$\ddot{z}_{bn}\left(t_{i+1}\right)=-H_{ni}-R3_n\,B_{n\,i+1} \tag{2-51}$$

in which

$$H_{ni}=2\,\xi_{bn}\,\omega_{bn}\,G_{ni}+\omega_{bn}^2\left(1-2\,\xi_{bn}^2\right)D_{ni} \tag{2-52}$$

$$R3_n=2\,\xi_{bn}\,\omega_{bn}\,R2_n+\omega_{bn}^2\left(1-2\,\xi_{bn}^2\right)R1_n \tag{2-53}$$

Recalling equation (2-32), equation (2-51) may be rewritten as

$$\ddot{z}_{bn}\left(t_{i+1}\right)=-H_{ni}-R3_n\sum_{l=1}^{3}\left(\alpha_{bnl}\,\Delta\ddot{d}_{gl}\left(t_{i+1}\right)+\lambda_{bnl}\,\Delta\ddot{z}_{1l}\left(t_{i+1}\right)\right) \tag{2-54}$$

Equation (2-54) is now in a form that can be solved using time-stepping methods to determine the response of the bearings over time, given a set of earthquake ground acceleration records and the response of the first floor. However, the first floor response is also unknown. Next, another formulation will be undertaken to determine a second equation with the response at the bearing level and the first floor unknown.

2.1.2 First Floor Equations of Motion

For the first floor equations of motion, the mode shapes will be mass-orthonormalized with respect to the mass matrix $[M_1]$. This assumption produces the following matrices for use in equation (2-5):

$$[M_1^*] = [\Phi_1]^T [M_1][\Phi_1] = [I] \tag{2-55}$$

$$[K_1^*] = [\Phi_1]^T [K_1][\Phi_1] = \begin{bmatrix} \omega_{11}^2 & 0 & 0 \\ 0 & \omega_{12}^2 & 0 \\ 0 & 0 & \omega_{13}^2 \end{bmatrix} \tag{2-56}$$

$$[C_1^*] = [\Phi_1]^T [C_1][\Phi_1] = \begin{bmatrix} 2\xi_{11}\,\omega_{11} & 0 & 0 \\ 0 & 2\xi_{12}\,\omega_{12} & 0 \\ 0 & 0 & 2\xi_{13}\,\omega_{13} \end{bmatrix} \tag{2-57}$$

Again, the damping is assumed to be classical, thus only a diagonal matrix is used. Following a procedure like that done to transform equation (2-6) into equation (2-21), equation (2-5) can be rewritten as:

$$\ddot{z}_{1n} + 2\xi_{1n}\,\omega_{1n}\,\dot{z}_{1n} + \omega_{1n}^2\,z_{1n} = \sum_{k=1}^{3}\left(\lambda_{1nk}\,\ddot{z}_{bk} + \alpha_{1nk}\,\ddot{d}_{gk}\right) \tag{2-58}$$

in which

$$[\lambda_1] = -[\Phi_1]^T [M_1][\Phi_b] = \begin{bmatrix} \lambda_{111} & \lambda_{112} & \lambda_{113} \\ \lambda_{121} & \lambda_{122} & \lambda_{123} \\ \lambda_{131} & \lambda_{132} & \lambda_{133} \end{bmatrix} \tag{2-59}$$

$$[\alpha_1] = -[\Phi_1]^T [M_1] = \begin{bmatrix} \alpha_{111} & \alpha_{112} & \alpha_{113} \\ \alpha_{121} & \alpha_{122} & \alpha_{123} \\ \alpha_{131} & \alpha_{132} & \alpha_{133} \end{bmatrix} \tag{2-60}$$

Now the solution to equation (2-58) can be written in the following form using the linear acceleration method:

$$\ddot{z}_{1n}\left(t_{i+1}\right)=\ddot{z}_{1n}\left(t_i\right)+\Delta\ddot{z}_{1n}\left(t_{i+1}\right) \tag{2-61}$$

This equation can be written as a function of time τ. When integrated, the modal acceleration function becomes a modal velocity function as follows:

$$\dot{z}_{1n}\left(t_{i+1}\right)=\dot{z}_{1n}\left(t_i\right)+\ddot{z}_{1n}\left(t_i\right)\Delta t+\Delta\ddot{z}_{1n}\left(t_{i+1}\right)\frac{\Delta t}{2} \tag{2-62}$$

By integrating that function with respect to time a function for the modal displacement is determined as follows:

$$z_{1n}\left(t_{i+1}\right)=z_{1n}\left(t_i\right)+\dot{z}_{1n}\left(t_i\right)\Delta t+\ddot{z}_{1n}\left(t_i\right)\frac{\Delta t^2}{2}+\Delta\ddot{z}_{1n}\left(t_{i+1}\right)\frac{\Delta t^2}{6} \tag{2-63}$$

By evaluating equation (2-58) at time t_{i+1} and substituting in equations (2-61), (2-62), and (2-63), the incremental form of equation (2-58) can be written as:

$$\begin{aligned}R4_n\Delta\ddot{z}_{1n}\left(t_{i+1}\right)+R5_n\ddot{z}_{1n}\left(t_i\right)+R6_n\dot{z}_{1n}\left(t_i\right)+\omega_{1n}^2 z_{1n}\left(t_i\right)=&\\ \sum_{k=1}^{3}\left(\lambda_{1nk}\ddot{z}_{bk}\left(t_{i+1}\right)+\alpha_{1nk}\ddot{d}_{gk}\left(t_{i+1}\right)\right)&\end{aligned} \tag{2-64}$$

in which

$$R4_n=1+\xi_{1n}\omega_{1n}\Delta t+\omega_{1n}^2\frac{\Delta t^2}{6} \tag{2-65}$$

$$R5_n=1+2\xi_{1n}\omega_{1n}\Delta t+\omega_{1n}^2\frac{\Delta t^2}{2} \tag{2-66}$$

$$R6_n=2\xi_{1n}\omega_{1n}+\omega_{1n}^2\Delta t \tag{2-67}$$

Equation (2-64) can be further expanded. By substituting equation (2-54) into equation (2-64), the unknown bearing accelerations drop out of the equation, which becomes

$$R4_n \Delta\ddot{z}_{1n}(t_{i+1}) + R5_n \ddot{z}_{1n}(t_i) + R6_n \dot{z}_{1n}(t_i) + \omega_{1n}^2 z_{1n}(t_i) = \\ -\sum_{k=1}^{3} \lambda_{1nk} H_{ki} - \sum_{k=1}^{3}\sum_{l=1}^{3} \lambda_{1nk}\lambda_{bkl} R3_k \Delta\ddot{z}_{1l}(t_{i+1}) \\ -\sum_{k=1}^{3}\sum_{l=1}^{3} \lambda_{1nk}\alpha_{bkl} R3_k \Delta\ddot{d}_{gl}(t_{i+1}) + \sum_{k=1}^{3} \alpha_{1nk}\ddot{d}_{gk}(t_{i+1}) \tag{2-68}$$

Dividing equation (2-68) through by $R4_n$ and grouping like terms creates the following generalized expression:

$$\Delta\ddot{z}_{1n}(t_{i+1}) + \sum_{m=1}^{3} Q_{nm}\Delta\ddot{z}_{1m}(t_{i+1}) = -\frac{1}{R4_n}\left(P1_n + P2_n + P3_n\right) \tag{2-69}$$

in which

$$Q_{nm} = \frac{1}{R4_n}\sum_{k=1}^{3} \lambda_{1nk}\lambda_{bkm} R3_k \tag{2-70}$$

$$P1_n = R5_n \ddot{z}_{1n}(t_i) + R6_n \dot{z}_{1n}(t_i) + \omega_{1n}^2 z_{1n}(t_i) \tag{2-71}$$

$$P2_n = \sum_{k=1}^{3}\left(\lambda_{1nk} H_{ki} - \alpha_{1nk}\ddot{d}_{gk}(t_{i+1})\right) \tag{2-72}$$

$$P3_n = \sum_{k=1}^{3}\sum_{l=1}^{3} \lambda_{1nk}\alpha_{bkl} R3_k \Delta\ddot{d}_{gl}(t_{i+1}) \tag{2-73}$$

As with the other equations in this chapter, n in equation (2-69) can be equal to 1, 2, or 3, depending upon the direction to be considered. By expanding this equation into its three components, the following equations are found:

$$\left(Q_{11}+1\right)\Delta\ddot{z}_{11}(t_{i+1}) + Q_{12}\,\Delta\ddot{z}_{12}(t_{i+1}) + Q_{13}\,\Delta\ddot{z}_{13}(t_{i+1}) = -\frac{1}{R4_1}\left(P1_1 + P2_1 + P3_1\right) \tag{2-74}$$

$$Q_{21}\,\Delta\ddot{z}_{11}(t_{i+1}) + \left(Q_{22}+1\right)\Delta\ddot{z}_{12}(t_{i+1}) + Q_{23}\,\Delta\ddot{z}_{13}(t_{i+1}) = -\frac{1}{R4_2}\left(P1_2 + P2_2 + P3_2\right) \tag{2-75}$$

$$Q_{31}\,\Delta\ddot{z}_{11}(t_{i+1}) + Q_{32}\,\Delta\ddot{z}_{12}(t_{i+1}) + \left(Q_{33}+1\right)\Delta\ddot{z}_{13}(t_{i+1}) = -\frac{1}{R4_3}\left(P1_3 + P2_3 + P3_3\right) \tag{2-76}$$

Equations (2-74), (2-75), and (2-76) can then be put into matrix form as follows:

$$[Q]\{\Delta\ddot{z}_1(t_{i+1})\} = \{P\} \tag{2-77}$$

Equation (2-77) can then be solved to determine the modal acceleration term by premultiplying each side of the equation by $[Q]^{-1}$:

$$\{\Delta\ddot{z}_1(t_{i+1})\} = [Q]^{-1}\{P\} \tag{2-78}$$

in which

$$\{\Delta\ddot{z}_1(t_{i+1})\} = \begin{Bmatrix} \Delta\ddot{z}_{11}(t_{i+1}) \\ \Delta\ddot{z}_{12}(t_{i+1}) \\ \Delta\ddot{z}_{13}(t_{i+1}) \end{Bmatrix} \tag{2-79}$$

These accelerations are then used in equations (2-61), (2-62), and (2-63) to determine the acceleration, velocity, and displacement of the first floor, respectively, at time t_{i+1}. The calculation of these values over time generates the overall structural response of the first floor, which is accurate given a small time step. Now the only remaining unknowns are the acceleration, velocity, and displacement of the bearing level.

To determine those three values, refer to equations (2-54), (2-48), and (2-43), respectively. Since the first floor accelerations are now known, these three equations can be solved for the response of the bearing level. However, these values for bearing level and first floor response are only preliminary values for the time step. To ensure equilibrium at each time step, iteration must be undertaken between the two equations of motion, as noted below. When the differences between two iterations are negligible, then the next time step can be considered. The structural response over the entire duration of the excitation is calculated in this manner, at which point the entire response is known.

The maximum displacement and maximum acceleration of the bearings are the most important values in the analysis. Displacement determines the free space required around the bearing level of the structure to avoid damage during the dynamic response. The acceleration values are important to determine the intensity of the motion induced in the structure.

2.2 SUMMARY OF SOLUTION STEPS

The process for solution of a single-story base-isolated structure is an iterative process based upon the equations outlined above. The steps of this process, to determine the response of the structure at time t_{i+1}, are as follows:

1. Assemble the mass and stiffness matrices as described in Appendix A.
2. Determine the modal matrices shown in equation (2-16) for each floor by solving equations (2-14) and (2-15).
3. Assemble the $[Q]$ matrix as described in equations (2-69) and (2-70). Assemble the $\{P\}$ vector as described in equations (2-69) and (2-71) through (2-73).
4. Solve equation (2-78), calculating the incremental modal acceleration values for the first floor. These will be taken as the initial values for these variables during the time step.
5. Substitute the values from step 4 into equations (2-61), (2-62), and (2-63). This substitution calculates the initial values of the modal acceleration, velocity, and displacement of the first floor, respectively, at time t_{i+1}.

6. Substitution of the values from step 5 into equations (2-31) and (2-32) will determine the parameters required to determine the bearing level response.
7. Substitute the parameters determined in step 6 into equations (2-43), (2-48), and (2-51) to determine the initial values for the modal displacement, velocity, and acceleration, respectively, of the bearing level.
8. Check equilibrium. Substitute the values for $\dot{z}_{1n}$, z_{1n}, and $\{\ddot{z}_b\}$ into equation (2-58) to determine a new value for $\ddot{z}_{1n}$. Using equation (2-61), determine the second iteration values for the first floor modal acceleration.
9. Repeat steps 5-8 until the change in modal response between iterations is negligible. To determine the actual response of the structure, equations (2-12) and (2-13), and their time derivatives, can be solved using the modal response. The values obtained represent the actual response of the bearing level and the first floor at time t_{i+1}.

3.0 SINGLE-STORY NON-LINEAR ANALYSIS

The method presented in the previous chapter assumes a fully linear response of the structure and the bearings. However, this is often not the case. In general, some elements of non-linearity enter the system via yielding and strain hardening. The isolation system is designed to minimize the motion of the superstructure, maintaining a linear response; however, the isolators themselves will often yield when subjected to ground motion. This yielding makes the response more difficult to calculate, since the stress-strain curve is no longer linear upon initiation of yielding. To compensate for this non-linear behavior, an effective stiffness will be introduced to account for both the linear elastic deformation prior to yielding and the plastic deformation that occurs after the yield limit has been reached. Appendix C offers a more complete discussion of the non-linearity of the bearings and the effective stiffness.

A more comprehensive analysis than that presented in Chapter 2 must be undertaken to truly solve for the non-linear structural response. To begin, recall equations (2-5) and (2-6).

$$[M_1]\{\ddot{d}_1\}+[C_1]\{\dot{d}_1\}+[K_1]\{d_1\}=-[M_1]\{\ddot{d}_b\}-[M_1]\{\ddot{d}_g\} \tag{3-1}$$

$$\begin{aligned}&[M_t]\{\ddot{d}_b\}+[C_b]\{\dot{d}_b\}+[K_b]\{d_b\}=\\&\quad-[M_1]\{\ddot{d}_1\}-[M_t]\{\ddot{d}_g\}-\mu(g+\ddot{d}_{gz})[M_t]\{\operatorname{sgn}(d_b)\}\end{aligned} \tag{3-2}$$

3.1 HILBER'S α METHOD

As mentioned in the introductory paragraph, these linear equations are inadequate when non-linearity occurs in the response. Therefore, a modification of the bearing equations is necessary to account for the non-linearities. One such modification is the Newmark Method, which modifies the stiffness of the structure to approximate the response over a time step. The Newmark Method introduces numerical damping, which is used to dampen the effects of the higher structural modes. Hilber (1977) further modified the Newmark Method with an α term which is used to enhance the results of the time-step solution by improving the numerical damping. Hilber's equation is presented here as it applies to equation (3-2), determined at time t_{i+1}:

$$\begin{aligned}[M_t]\{\ddot{d}_b(t_{i+1})\}+[C_b]\{\dot{d}_b(t_{i+1})\}+(1+\alpha)[K_b]\{d_b(t_{i+1})\}\\ -\alpha[K_b]\{d_b(t_i)\}=-[M_1]\{\ddot{d}_1(t_{i+1})\}-[M_t]\{\ddot{d}_g(t_{i+1})\}\\ -\mu\left(g+\ddot{d}_{gz}(t_{i+1})\right)[M_t]\{\mathrm{sgn}(\dot{d}_b(t_i))\}-\{\Delta R^{i+1}\}\end{aligned} \tag{3-3}$$

The same formula can then be applied to time t_i:

$$\begin{aligned}[M_t]\{\ddot{d}_b(t_i)\}+[C_b]\{\dot{d}_b(t_i)\}+(1+\alpha)[K_b]\{d_b(t_i)\}\\ -\alpha[K_b]\{d_b(t_{i-1})\}=-[M_1]\{\ddot{d}_1(t_i)\}-[M_t]\{\ddot{d}_g(t_i)\}\\ -\mu\left(g+\ddot{d}_{gz}(t_i)\right)[M_t]\{\mathrm{sgn}(\dot{d}_b(t_i))\}-\{\Delta R^i\}\end{aligned} \tag{3-4}$$

A time-step formulation can be created by subtracting equation (3-4) from equation (3-3). The time-step equation is:

$$\begin{aligned}[M_t]\{\Delta\ddot{d}_b(t_{i+1})\}+[C_b]\{\Delta\dot{d}_b(t_{i+1})\}+(1+\alpha)[K_b]\{\Delta d_b(t_{i+1})\}-\\ \alpha[K_b]\{\Delta d_b(t_i)\}=-[M_1]\{\Delta\ddot{d}_1(t_{i+1})\}-[M_t]\{\Delta\ddot{d}_g(t_{i+1})\}\\ -\mu\left(\Delta\ddot{d}_{gz}(t_{i+1})\right)[M_t]\{\mathrm{sgn}(\dot{d}_b(t_i))\}-\{\Delta R\}\end{aligned} \tag{3-5}$$

in which

$$\{\Delta R\} = \{\Delta R^{i+1}\} - \{\Delta R^{i}\} \tag{3-6}$$

$\{\Delta R^{i+1}\} \equiv$ residual forces at iteration i

$$\Delta \ddot{d}_{gz}(t_{i+1}) = \ddot{d}_{gz}(t_{i+1}) - \ddot{d}_{gz}(t_i) \tag{3-7}$$

3.2 NEWMARK'S β METHOD

Now that an iterative equation has been written with respect to displacement, velocity, and acceleration of the structure, Newmark's β-Method can then be used to calculate the velocity and displacement of the structure across the time step Δt. The following equations represent Newmark's method (Hilber, 1977) as it is applied to the bearing level velocity and displacement vectors:

$$\{\dot{d}_b(t_{i+1})\} = \{\dot{d}_b(t_i)\} + \left[(1-\gamma)\{\ddot{d}_b(t_i)\} + \gamma\{\ddot{d}_b(t_{i+1})\}\right]\Delta t_i \tag{3-8}$$

$$\begin{aligned}\{d_b(t_{i+1})\} = \{d_b(t_i)\} + \{\dot{d}_b(t_i)\}\Delta t_i + \\ \left[\left(\frac{1}{2}-\beta\right)\{\ddot{d}_b(t_i)\} + \beta\{\ddot{d}_b(t_{i+1})\}\right](\Delta t_i)^2\end{aligned} \tag{3-9}$$

in which

$\gamma \equiv$ factor accounting for algorithmic or numerical damping

$\beta \equiv$ factor accounting for time-step variation of acceleration

These parameters allow for a number of different methodologies for achieving accurate results. If the γ factor is set less than ½, negative damping is introduced. If the γ factor is set at ½, no additional damping is introduced and the method makes use of the trapezoidal rule. If the γ factor is set greater than ½, positive damping is introduced. Also, choosing β to be equal to zero utilizes the constant-acceleration method. Choosing β equal to ¼ utilizes the average-acceleration method. Choosing β equal to 1/6 utilizes the linear-acceleration method.

Considering equations (3-8) and (3-9), an incremental form is required to determine a solution for equation (3-5), since that equation is written in terms of incremental displacement, velocity, and acceleration. Rearranging the terms in equation (3-8), the following expression can be obtained:

$$\left\{\dot{d}_b\left(t_{i+1}\right)\right\}-\left\{\dot{d}_b\left(t_i\right)\right\}=\left[\left\{\ddot{d}_b\left(t_i\right)\right\}+\gamma\left(\left\{\ddot{d}_b\left(t_{i+1}\right)\right\}-\left\{\ddot{d}_b\left(t_i\right)\right\}\right)\right]\Delta t_i$$

This can then be written in incremental form by recalling that the incremental values are the change in velocity and acceleration over the time interval Δt, similar to equation (3-7):

$$\left\{\Delta\dot{d}_b\left(t_{i+1}\right)\right\}=\left[\left\{\ddot{d}_b\left(t_i\right)\right\}+\gamma\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}\right]\Delta t_i \qquad (3\text{-}10)$$

Equation (3-9) must also be transformed into an incremental equation. First it is necessary to group the displacement and acceleration terms as follows:

$$\left\{d_b\left(t_{i+1}\right)\right\}-\left\{d_b\left(t_i\right)\right\}=\left\{\dot{d}_b\left(t_i\right)\right\}\Delta t_i+\left[\frac{1}{2}\left\{\ddot{d}_b\left(t_i\right)\right\}+\beta\left(\left\{\ddot{d}_b\left(t_{i+1}\right)\right\}-\left\{\ddot{d}_b\left(t_i\right)\right\}\right)\right]\left(\Delta t_i\right)^2$$

Again, recall the form of equation (3-7). Applying that definition of the incremental terms gives the following incremental equation:

$$\left\{\Delta d_b\left(t_{i+1}\right)\right\}=\left\{\dot{d}_b\left(t_i\right)\right\}\Delta t_i+\left[\frac{1}{2}\left\{\ddot{d}_b\left(t_i\right)\right\}+\beta\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}\right]\left(\Delta t_i\right)^2 \qquad (3\text{-}11)$$

Now by substituting equations (3-10) and (3-11) into equation (3-5), the following equation is obtained:

$$
\begin{aligned}
&[M_t]\{\Delta\ddot{d}_b(t_{i+1})\} + [C_b](\{\ddot{d}_b(t_i)\} + \gamma\{\Delta\ddot{d}_b(t_{i+1})\})\Delta t_i + \\
&\quad (1+\alpha)[K_b]\left(\{\dot{d}_b(t_i)\}\Delta t_i + \left(\frac{1}{2}\{\ddot{d}_b(t_i)\} + \beta\{\Delta\ddot{d}_b(t_{i+1})\}\right)(\Delta t)^2\right) = \\
&\qquad \alpha[K_b]\{\Delta d_b(t_i)\} - [M_1]\{\Delta\ddot{d}_1(t_{i+1})\} - [M_t]\{\Delta\ddot{d}_g(t_{i+1})\} \\
&\qquad - \mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\{\mathrm{sgn}(\dot{d}_b(t_i))\} - \{\Delta R\}
\end{aligned}
$$

This equation still needs to be simplified. Grouping the incremental bearing acceleration terms yields the following:

$$
\begin{aligned}
&([M_t] + \gamma[C_b]\Delta t_i + \beta(1+\alpha)[K_b](\Delta t_i)^2)\{\Delta\ddot{d}_b(t_{i+1})\} = -[C_b]\{\ddot{d}_b(t_i)\}\Delta t_i - \\
&\quad [M_1]\{\Delta\ddot{d}_1(t_{i+1})\} - [M_t]\{\Delta\ddot{d}_g(t_{i+1})\} - \alpha[K_b]\{\Delta d_b(t_i)\} - \{\Delta R\} - \\
&\quad \mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\{\mathrm{sgn}(\dot{d}_b(t_i))\} - (1+\alpha)[K_b](\{\dot{d}_b(t_i)\}\Delta t + \tfrac{1}{2}\{\ddot{d}_b(t_i)\}(\Delta t)^2)
\end{aligned}
$$

This equation can be simplified by identifying the following quantities:

$$[\overline{K}(t_i)] = [M_t] + \gamma[C_b]\Delta t_i + \beta(1+\alpha)[K_b](\Delta t_i)^2 \quad (3\text{-}12)$$

$$\{D_1\} = \{\ddot{d}_b(t_i)\}\Delta t_i \quad (3\text{-}13)$$

$$\{D_2\} = (1+\alpha)(\{\dot{d}_b(t_i)\}\Delta t_i + \tfrac{1}{2}\{\ddot{d}_b(t_i)\}(\Delta t_i)^2) - \alpha\{\Delta d_b(t_i)\} \quad (3\text{-}14)$$

Now the incremental form of Hilber's equation can be written as:

$$
\begin{aligned}
&[\overline{K}(t_i)]\{\Delta\ddot{d}_b(t_{i+1})\} = -[C_b]\{D_1\} - [K_b]\{D_2\} - [M_i]\{\Delta\ddot{d}_1(t_{i+1})\} \\
&\quad - [M_t]\{\Delta\ddot{d}_g(t_{i+1})\} - \mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\{\mathrm{sgn}(\dot{d}_b(t_i))\} - \{\Delta R\}
\end{aligned}
\quad (3\text{-}15)
$$

Equation (3-15) can then be solved in terms of the incremental acceleration vector by premultiplying each side by the inverse of $[\overline{K}(t_i)]$:

$$\{\Delta\ddot{d}_b(t_{i+1})\} = -[\overline{K}(t_i)]^{-1}\begin{pmatrix} [C_b]\{D_1\} + [K_b]\{D_2\} + [M_1]\{\Delta\ddot{d}_1(t_{i+1})\} \\ + [M_t]\{\Delta\ddot{d}_g(t_{i+1})\} \\ + \mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\{\mathrm{sgn}(\dot{d}_b(t_i))\} + \{\Delta R\} \end{pmatrix} \quad (3\text{-}16)$$

Equation (3-16) has two unknown vectors in it. The primary unknown, $\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}$, acts as a dependent variable in equation (3-16). The secondary unknown, $\left\{\Delta\ddot{d}_1\left(t_{i+1}\right)\right\}$, acts as an independent variable here. Therefore, another set of equations must be used to determine the incremental acceleration of the first floor.

Once again it is assumed that floor friction is negligible and that the superstructure behavior is entirely linear. Therefore, the first floor equations of motion from Chapter 2 still apply to the non-linear solution. To solve for the first floor accelerations, recall equation (2-58), shown below.

$$\ddot{z}_{1n} + 2\,\xi_{1n}\,\omega_{1n}\,\dot{z}_{1n} + \omega_{1n}^2\,z_{1n} = \sum_{k=1}^{3}\left(\lambda_{1nk}\,\ddot{z}_{bk} + \alpha_{1nk}\,\ddot{d}_{gk}\right) \qquad (3\text{-}17)$$

Equation (2-58) was derived directly from equation (2-5). Note that the first term on the right hand side of equation (3-17) originated from the matrix expression

$$-\left[\Phi_1\right]^T\left[M_1\right]\left[\Phi_b\right]\left\{\ddot{z}_b\right\}$$

Now a different form of equation (3-17) is preferable, so the above expression must be altered. By reverting to the true displacement form instead of the modal displacement form [see equation (2-13)], the first term on the right hand side of the equation becomes:

$$-\left[\Phi_1\right]^T\left[M_1\right]\left\{\ddot{d}_b\right\} \equiv \left[\alpha_1\right]\left\{\ddot{d}_b\right\}$$

It can be seen from the above expression and equation (2-56) that equation (3-17) can be written in the following form at time t_{i+1}:

$$\begin{aligned}\ddot{z}_{1n}\left(t_{i+1}\right) + 2\xi_{1n}\omega_{1n}\dot{z}_{1n}\left(t_{i+1}\right) + \omega_{1n}^2 z_{1n}\left(t_{i+1}\right) = \\ \sum_{m=1}^{3}\alpha_{1nm}\ddot{d}_{bm}\left(t_{i+1}\right) + \sum_{m=1}^{3}\alpha_{1nm}\ddot{d}_{gm}\left(t_{i+1}\right)\end{aligned} \qquad (3\text{-}18)$$

As in Chapter 2, in which a linear analysis was derived, the linear acceleration method will be used and equation (2-60) is applicable here for a non-linear formulation, though the coefficient of the bearing acceleration has been changed to correspond to equation (3-18):

$$R4_n\,\Delta\ddot{z}_{1n}\left(t_{i+1}\right)+R5_n\,\ddot{z}_{1n}\left(t_i\right)+R6_n\,\dot{z}_{1n}\left(t_i\right)+\omega_{1n}^2 z_{1n}\left(t_i\right)= \sum_{k=1}^{3}\left(\alpha_{1nk}\ddot{d}_{bk}\left(t_{i+1}\right)+\alpha_{1nk}\ddot{d}_{gk}\left(t_{i+1}\right)\right) \tag{3-19}$$

However, it is desirable to write the equation in a form that allows the unknown value of $\left\{\Delta\ddot{z}_1\left(t_{i+1}\right)\right\}$ to be solved:

$$\Delta\ddot{z}_{1n}\left(t_{i+1}\right)=-\frac{1}{R4_n}\left(\begin{array}{l}R5_n\,\ddot{z}_{1n}\left(t_i\right)+R6_n\,\dot{z}_{1n}\left(t_i\right)+\omega_{1n}^2 z_{1n}\left(t_i\right)\\ \qquad -\sum_{m=1}^{3}\left(\alpha_{1nm}\ddot{d}_{bm}\left(t_{i+1}\right)+\alpha_{1nm}\ddot{d}_{gm}\left(t_{i+1}\right)\right)\end{array}\right) \tag{3-20}$$

in which

$$R4_n=1+\xi_{1n}\omega_{1n}\Delta t+\omega_{1n}^2\frac{\Delta t^2}{6} \tag{3-21}$$

$$R5_n=1+2\xi_{1n}\omega_{1n}\Delta t+\omega_{1n}^2\frac{\Delta t^2}{2} \tag{3-22}$$

$$R6_n=2\xi_{1n}\omega_{1n}+\omega_{1n}^2\Delta t \tag{3-23}$$

As previously mentioned, equation (3-16) had two unknown variables: the acceleration of the bearing level and the acceleration of the first floor. Equation (3-20) is another equation which depends on both the bearing level accelerations and the first floor modal accelerations. Therefore, using these two equations, both unknowns can be solved. Reexamining equation (3-16), a further simplification is possible by defining the following vectors and matrices:

$$\left\{\delta_b\right\}=\left[\overline{K}\left(t_i\right)\right]^{-1}\left(\left[C_b\right]\left\{D_1\right\}+\left[K_b\right]\left\{D_2\right\}+\left\{\Delta R\right\}\right)=\begin{Bmatrix}\delta_{b1}\\ \delta_{b2}\\ \delta_{b3}\end{Bmatrix} \tag{3-24}$$

$$[G]=\left[\overline{K}(t_i)\right]^{-1}[M_1] \tag{3-25}$$

$$[H]=\left[\overline{K}(t_i)\right]^{-1}[M_t] \tag{3-26}$$

$$\{\delta_g\}=[H]\{\Delta\ddot{d}_g(t_{i+1})\}=\begin{Bmatrix}\sum_{k=1}^{3}H_{1k}\Delta\ddot{d}_{gk}(t_{i+1})\\ \sum_{k=1}^{3}H_{2k}\Delta\ddot{d}_{gk}(t_{i+1})\\ \sum_{k=1}^{3}H_{3k}\Delta\ddot{d}_{gk}(t_{i+1})\end{Bmatrix}=\begin{Bmatrix}\delta_{g1}\\ \delta_{g2}\\ \delta_{g3}\end{Bmatrix} \tag{3-27}$$

$$\{\delta_f\}=\mu\,\Delta\ddot{d}_{gz}(t_{i+1})[H]\{\mathrm{sgn}(\dot{d}_b(t_i))\}=\mu\,\Delta\ddot{d}_{gz}(t_{i+1})\begin{Bmatrix}\sum_{k=1}^{3}H_{1k}\,\mathrm{sgn}(\dot{d}_{bk}(t_i))\\ \sum_{k=1}^{3}H_{2k}\,\mathrm{sgn}(\dot{d}_{bk}(t_i))\\ \sum_{k=1}^{3}H_{3k}\,\mathrm{sgn}(\dot{d}_{bk}(t_i))\end{Bmatrix}=\begin{Bmatrix}\delta_{f1}\\ \delta_{f2}\\ \delta_{f3}\end{Bmatrix} \tag{3-28}$$

Now a condensed form of equation (3-16) is written as

$$\Delta\ddot{d}_{bm}(t_{i+1})=-\left(\delta_{bm}+\delta_{gm}+\delta_{fm}+\sum_{k=1}^{3}G_{mk}\Delta\ddot{d}_{1k}(t_{i+1})\right) \tag{3-29}$$

Through the use of modal superposition, $\{\Delta\ddot{d}_1\}=[\Phi_1]\{\Delta\ddot{z}_1\}$ and equation (3-29) becomes

$$\Delta\ddot{d}_{bm}(t_{i+1})=-\left(\delta_{bm}+\delta_{gm}+\delta_{fm}+\sum_{k=1}^{3}\sum_{l=1}^{3}G_{mk}\phi_{1kl}\Delta\ddot{z}_{1l}(t_{i+1})\right) \tag{3-30}$$

By definition, the following identity is true:

$$\Delta\ddot{d}_{bm}(t_{i+1})=\ddot{d}_{bm}(t_{i+1})-\ddot{d}_{bm}(t_i) \tag{3-31}$$

which can be rewritten as

$$\ddot{d}_{bm}(t_{i+1})=\ddot{d}_{bm}(t_i)+\Delta\ddot{d}_{bm}(t_{i+1}) \tag{3-32}$$

Substitution of equations (3-30) and (3-32) into equation (3-20) gives the following result:

$$\Delta\ddot{z}_{1n}\left(t_{i+1}\right)=-\frac{1}{R4_n}\left(\begin{array}{r} R5_n\,\ddot{z}_{1n}\left(t_i\right)+R6_n\,\dot{z}_{1n}\left(t_i\right)+\omega_{1n}^2 z_{1n}\left(t_i\right) \\ -\sum_{m=1}^{3}\alpha_{1nm}\ddot{d}_{bm}\left(t_i\right)-\sum_{m=1}^{3}\alpha_{1nm}\ddot{d}_{gm}\left(t_{i+1}\right) \\ +\sum_{m=1}^{3}\alpha_{1nm}\left(\delta_{bm}+\delta_{gm}+\delta_{fm}+\sum_{k=1}^{3}\sum_{l=1}^{3}G_{mk}\phi_{1kl}\Delta\ddot{z}_{1l}\left(t_{i+1}\right)\right)\end{array}\right) \tag{3-33}$$

Grouping the incremental acceleration terms on the left hand side of the equation:

$$\left(\begin{array}{c}\Delta\ddot{z}_{1n}\left(t_{i+1}\right)+ \\ \frac{1}{R4_n}\left[\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\alpha_{1nm}G_{mk}\phi_{1kl}\Delta\ddot{z}_{1l}\left(t_{i+1}\right)\right]\end{array}\right)=-\frac{1}{R4_n}\left(\begin{array}{r} R5_n\,\ddot{z}_{1n}\left(t_i\right)+R6_n\,\dot{z}_{1n}\left(t_i\right)+ \\ \omega_{1n}^2 z_{1n}\left(t_i\right)+\sum_{m=1}^{3}\alpha_{1nm}\left(\delta_{bm}+\delta_{gm}+\delta_{fm}\right)- \\ \sum_{m=1}^{3}\alpha_{1nm}\left(\ddot{d}_{bm}\left(t_i\right)+\ddot{d}_{gm}\left(t_i\right)\right)\end{array}\right) \tag{3-34}$$

Defining the following identities allows equation (3-34) to be simplified into a much more palatable form:

$$P1_n=R5_n\,\ddot{z}_{1n}\left(t_i\right)+R6_n\,\dot{z}_{1n}\left(t_i\right)+\omega_{1n}^2\left(t_i\right) \tag{3-35}$$

$$P2_n=-\sum_{m=1}^{3}\alpha_{1nm}\left(\ddot{d}_{bm}\left(t_i\right)+\ddot{d}_{gm}\left(t_{i+1}\right)\right) \tag{3-36}$$

$$P3_n=\sum_{m=1}^{3}\alpha_{1nm}\left(\delta_{bm}+\delta_{gm}+\delta_{fm}\right) \tag{3-37}$$

$$Q_{nl}=\frac{1}{R4_n}\left[\sum_{k=1}^{3}\sum_{m=1}^{3}\alpha_{1nm}G_{mk}\phi_{1kl}\right] \tag{3-38}$$

Now equation (3-34) becomes

$$\Delta\ddot{z}_{1n}\left(t_{i+1}\right)+\sum_{l=1}^{3}Q_{nl}\Delta\ddot{z}_{1l}\left(t_{i+1}\right)=-\frac{1}{R4_n}\left[P1_n+P2_n+P3_n\right] \tag{3-39}$$

By expanding the equation for n = 1, 2, and 3,

$$\left(Q_{11}+1\right)\Delta\ddot{z}_{11}\left(t_{i+1}\right)+Q_{12}\Delta\ddot{z}_{11}\left(t_{i+1}\right)+Q_{13}\Delta\ddot{z}_{11}\left(t_{i+1}\right)=-\frac{1}{R4_1}\left(P1_1+P2_1+P3_1\right) \tag{3-40}$$

$$Q_{21}\Delta\ddot{z}_{12}(t_{i+1})+(Q_{22}+1)\Delta\ddot{z}_{12}(t_{i+1})+Q_{23}\Delta\ddot{z}_{12}(t_{i+1})=-\frac{1}{R4_2}(P1_2+P2_2+P3_2) \quad (3\text{-}41)$$

$$Q_{31}\Delta\ddot{z}_{13}(t_{i+1})+Q_{32}\Delta\ddot{z}_{13}(t_{i+1})+(Q_{33}+1)\Delta\ddot{z}_{13}(t_{i+1})=-\frac{1}{R4_3}(P1_3+P2_3+P3_3) \quad (3\text{-}42)$$

Writing equations (3-40), (3-41), and (3-42) in matrix form,

$$[Q]\{\Delta\ddot{z}_1(t_{i+1})\}=\{P\} \quad (3\text{-}43)$$

Equation (3-43) can then be rewritten as:

$$\{\Delta\ddot{z}_1(t_{i+1})\}=[Q]^{-1}\{P\} \quad (3\text{-}44)$$

Equation (3-44) can then be solved, as only the left hand side is unknown. From this result it is clear that the actual formulation for the non-linear response is very similar to that of the linear response. A comparison of equations (3-44) and (2-74) leads to the conclusion that except for a few minor changes in the parameters involved, from a calculation standpoint the non-linear method is not a much greater undertaking than a linear method.

Again, as was the case with the linear analysis, it is necessary to iterate the solution to obtain values which fully satisfy equilibrium. In the non-linear iteration process, the criterion for proceeding with the next time step is a negligible change in the effective stiffness of the bearing level, $[K_b]$. A slight difference in the effective stiffness over the time interval implies that the approximation of the behavior over that time step will be appropriate for the response calculations.

As results are obtained from a non-linear analysis, it should be noted that the choice of the non-linearity parameters will have an effect upon the accuracy of results. Therefore, before using these methods, it is important to refer to Hilber (1977) for appropriate variable values. It should also be noted that the obtained results are still non-linear approximations, as opposed to an exact solution for the structural response.

3.3 SUMMARY OF SOLUTION STEPS

The solution procedure, as enumerated in the above text, can be condensed into a stepwise process as follows.

1. Select values for the three parameters used in Hilber's modification of Newmark's Method – α, β, and γ. Hilber suggests using the values of -0.1, 0.3025, and 0.6, respectively.
2. Assemble the mass matrices as described in Appendix A. Determine the stiffness matrix for the first floor from Appendix A.
3. To determine the stiffness of the bearing level, transfer the bearing level displacements to each individual bearing. Then the stiffness for each bearing must be determined from Appendix C. Those individual bearing stiffness values are then combined as shown in Appendix A.
4. Solve the generalized eigenvalue problem shown in equation (2-14) to determine the mode shapes of the first floor.
5. Assemble the $[Q]$ matrix as shown in equations (3-38) and (3-39). Assemble the $\{P\}$ vector as shown in equations (3-35) through (3-37) and (3-39).

6. Solve equation (3-44) to determine the initial values of the incremental modal accelerations of the first floor. Using modal superposition, determine the incremental first floor accelerations from equation (2-13).
7. Substitute the values for $\{\Delta\ddot{z}_1(t_{i+1})\}$ into equation (3-30) to determine the incremental bearing level accelerations. Using these accelerations, determine the incremental bearing level velocity and displacement from equations (3-10) and (3-11), respectively.
8. Determine the displacement, velocity, and acceleration at time t_{i+1} from the previous values and the incremental values.
9. Substitute the values for bearing level displacement, velocity, and acceleration, along with the first floor acceleration, into equation (3-3) to determine the unknown residual force vector $\{\Delta R^{i+1}\}$.
10. From the bearing level displacements, determine the displacement of each individual bearing. From the bearing displacement, determine the force in that bearing. If a bearing has yielded, its lateral force must be reduced to the yield value and the amount of the reduction must be added to the residual force vector $\{\Delta R^{i+1}\}$.
11. Determine $\{\Delta R\}$ from equation (3-6), which will then be used in the next time step.
12. Assemble the effective bearing stiffness matrix. Compare with the previous value for the time step. If the difference is neglible, proceed to the next time increment, beginning with step 4 of this procedure. Otherwise, return to step 5 and perform another iteration of calculations for the current time step.

4.0 MULTI-STORY LINEAR ANALYSIS

The application of seismic isolation to single-story structures is very important as an introduction to the process and as an intermediate step toward multi-story base isolation. Base isolation is an extremely valuable tool when properly applied to a multi-story structure. The concept of base isolation is to eliminate the effect of the higher response modes, which tend to transmit high quantities of energy into the structure. Reducing the effect of the higher vibration modes from the response of a multi-story structure greatly decreases the likelihood of catastrophic structural failure in the event of an earthquake. Therefore, an analysis of a multi-story structure will be undertaken here to assess the precise effect of base-isolation on the overall response to dynamic loading.

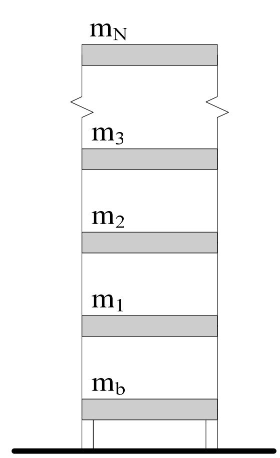

Figure 4 – Multistory Isolated Structure

An N-story structure is shown in Figure 4. The floors are numbered from 1 to N, with the first floor standing directly above the bearing floor and the *N*th floor acting as the roof of the structure. The relative displacements at each floor are shown in Figure 5.

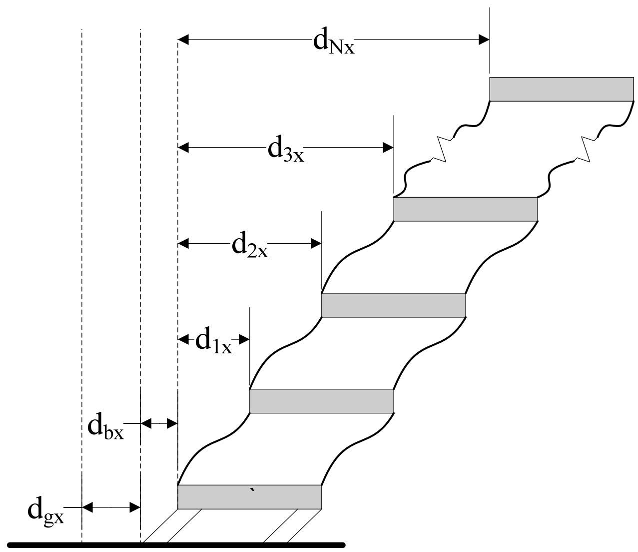

Figure 5 – Multistory Displacements

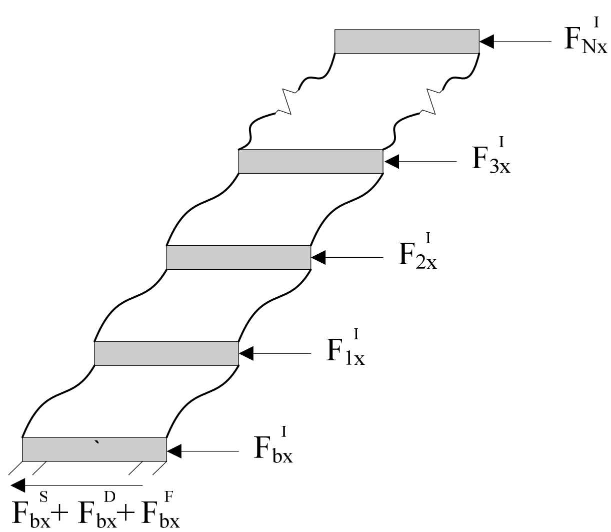

Figure 6 – Multistory Superstructure Free Body Diagram

Figure 6 represents a free-body diagram of the bearing level and the superstructure above, showing only the X-direction for simplicity. The resisting elastic and damping forces (superscripted with an S and D, respectively), as shown in the drawing, act opposite to the direction of the displacement and velocity of the structure, respectively, directly below the bearing floor level. A frictional force also acts directly below the bearing level, in the direction opposite that of the velocity.

4.1 ANALYSIS PROCEDURE

4.1.1 Bearing Level Equations of Motion

The free-body diagram shown in Figure 6 allows for a summation of forces to be written in the *X*-direction, which can be applied in each of the three degrees of freedom and written in matrix form as:

$$\left\{F_N^I\right\}+\ldots+\left\{F_i^I\right\}+\ldots+\left\{F_1^I\right\}+\left\{F_b^I\right\}+\left\{F_b^D\right\}+\left\{F_b^S\right\}+\left\{F_b^F\right\}=\{0\} \qquad (4\text{-}1)$$

$\left\{F_b^F\right\} \equiv$ frictional forces as defined in Chapter 2

The force vectors $\left\{F_i^I\right\}$ are determined from equation (A-9), which can be written in a more general form for floor *i* as:

$$\left\{F_i^I\right\}=\left[M_i\right]\left\{\ddot{d}_i\right\}+\left[M_i\right]\left\{\ddot{d}_b\right\}+\left[M_i\right]\left\{\ddot{d}_g\right\} \qquad (4\text{-}2)$$

in which

$$\left[M_i\right]=\begin{bmatrix} m_i & 0 & -m_i f_i \\ 0 & m_i & m_i e_i \\ -m_i f_i & m_i e_i & J_i+m_i\left(f_i^2+e_i^2\right) \end{bmatrix} \qquad (4\text{-}3)$$

$$\left\{\ddot{d}_i\right\}=\begin{Bmatrix}\ddot{x}_i\\ \ddot{y}_i\\ \ddot{\theta}_i\end{Bmatrix} \quad \left\{\ddot{d}_b\right\}=\begin{Bmatrix}\ddot{x}_b\\ \ddot{y}_b\\ \ddot{\theta}_b\end{Bmatrix} \quad \left\{\ddot{d}_g\right\}=\begin{Bmatrix}\ddot{x}_g\\ \ddot{y}_g\\ 0\end{Bmatrix} \tag{4-4}$$

The mass matrix $\left[M_i\right]$ is derived directly from equation (A-10) and generalized for floor i. The parameters e_i and f_i used in the mass matrix are defined in Appendix A. The inertial force presented in equation (4-2) can then be substituted for each floor in equation (4-1). By also substituting the stiffness, damping, and frictional vectors, equation (4-1) can now be written as:

$$\begin{aligned}
&\left([M_N]\{\ddot{d}_N\}+[M_N]\{\ddot{d}_b\}+[M_N]\{\ddot{d}_g\}\right)+\dots\\
&\quad+\left([M_i]\{\ddot{d}_i\}+[M_i]\{\ddot{d}_b\}+[M_i]\{\ddot{d}_g\}\right)+\dots\\
&\quad+\left([M_1]\{\ddot{d}_1\}+[M_1]\{\ddot{d}_b\}+[M_1]\{\ddot{d}_g\}\right)\\
&\quad+\left([M_b]\{\ddot{d}_b\}+[M_b]\{\ddot{d}_g\}+[C_b]\{\dot{d}_b\}+[K_b]\{d_b\}\right)\\
&\quad+\mu\left(g+\ddot{d}_{gz}\right)\left([M_N]+\dots+[M_i]+\dots+[M_b]\right)\left\{\operatorname{sgn}\left(\dot{d}_b\right)\right\}=0
\end{aligned} \tag{4-5}$$

Equation (4-5) is not in a manageable form; therefore, it is desirable to rewrite it in a more compact notation. By rearranging the terms, the following equation can be written:

$$\begin{aligned}
&\left([M_N]+\dots+[M_i]+\dots+[M_b]\right)\{\ddot{d}_b\}+[C_b]\{\dot{d}_b\}+[K_b]\{d_b\}=\\
&\quad-\left([M_N]\{\ddot{d}_N\}+\dots+[M_i]\{\ddot{d}_i\}+\dots+[M_1]\{\ddot{d}_1\}\right)\\
&\quad-\left([M_N]+\dots+[M_i]+\dots+[M_b]\right)\{\ddot{d}_g\}\\
&\quad-\mu\left(g+\ddot{d}_{gz}\right)\left([M_N]+\dots+[M_i]+\dots+[M_b]\right)\left\{\operatorname{sgn}\left(\dot{d}_b\right)\right\}
\end{aligned} \tag{4-6}$$

Then equation (4-6) can be further simplified by defining a "total mass" matrix

$$[M_t]=[M_b]+\sum_{i=1}^{N}[M_i] \tag{4-7}$$

Equation (4-6) now becomes:

$$\begin{aligned}
&[M_t]\{\ddot{d}_b\}+[C_b]\{\dot{d}_b\}+[K_b]\{d_b\}=-\sum_{i=1}^{N}\left([M_i]\{\ddot{d}_i\}\right)\\
&\quad-[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b\right)\right\}
\end{aligned} \tag{4-8}$$

Equation (4-8) represents the equations of motion for the three degrees of freedom of the bearing level. Notice that the right hand side of the equation shows a dependency upon the relative accelerations at each floor of the superstructure. Therefore, another set of equations with both the bearing displacements and the superstructure displacements unknown will be required to determine the overall response of the structure.

First, equation (4-8) will be solved for the bearing level relative accelerations as a function of the superstructure relative accelerations. Using modal superposition for the bearing level, equation (4-8) can be rewritten as follows:

$$\begin{aligned}&[M_t][\Phi_b]\{\ddot{z}_b\}+[C_b][\Phi_b]\{\dot{z}_b\}+[K_b][\Phi_b]\{z_b\}=\\&-\sum_{i=1}^{N}\left([M_i]\{\ddot{d}_i\}\right)-[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\mathrm{sgn}(\dot{d}_b)\}\end{aligned} \qquad (4\text{-}9)$$

in which

$$\{\ddot{d}_b\}=[\Phi_b]\{\ddot{z}_b\} \qquad (4\text{-}10)$$

Now if each side of equation (4-9) is premultiplied by $[\Phi_b]^T$, mass-orthonormalization (as presented in equation (2-17)) allows for further simplification. The process is similar to that undertaken in equations (2-18) through (2-21). After applying the linear acceleration technique demonstrated in equations (2-27) through (2-29), the following equation, which is similar to equation (2-30), is obtained:

$$\ddot{z}_{bn}(\tau)+2\xi_{bn}\omega_{bn}\dot{z}_{bn}(\tau)+\omega_{bn}^2 z_{bn}(\tau)=A_{ni}^t+B_{ni+1}^t\frac{\tau}{\Delta t} \qquad (4\text{-}11)$$

Equation (4-11) represents one of the three modal responses of equation (4-9) after applying mass-orthonormalization. This equation requires the following definitions:

$$A_{ni}^t=A_{ni}^N+A_{ni}^{N-1}+\ldots+A_{ni}^2+A_{ni}^1+A_{ni}^g+A_{ni}^f \qquad (4\text{-}12)$$

$$B_{ni+1}^t=B_{ni+1}^N+B_{ni+1}^{N-1}+\ldots+B_{ni+1}^2+B_{ni+1}^1+B_{ni+1}^g+B_{ni+1}^f \qquad (4\text{-}13)$$

Equations (4-12) and (4-13) represent "forcing functions" which act upon the bearings. The forces represented in equation (4-12) are from the superstructure accelerations, the ground acceleration, and the frictional forces at the bearing level, and are defined as such:

$$A_{ni}^{l} = \sum_{k=1}^{3} \lambda_{bnk}^{l} \ddot{d}_{u(3l-3+k)}\left(t_i\right) \tag{4-14}$$

$$A_{ni}^{g} = \sum_{k=1}^{3} \alpha_{bnk} \ddot{d}_{gk}\left(t_i\right) \tag{4-15}$$

$$A_{ni}^{f} = \sum_{k=1}^{3} \mu\left(g + \ddot{d}_{gz}\left(t_i\right)\right) \alpha_{bnk} \operatorname{sgn}\left(\dot{d}_{bk}\left(t_i\right)\right) \tag{4-16}$$

in which

$$\left[\lambda_b^l\right] = -\left[\Phi_b\right]^T \left[M_l\right] \qquad\qquad \left[\alpha_b\right] = -\left[\Phi_b\right]^T \left[M_t\right] \tag{4-17}$$

Equation (4-14) is shown in terms of the acceleration vector $\left\{\ddot{d}_u\right\}$, as opposed to the individual floor acceleration vectors $\left\{\ddot{d}_m\right\}$, to allow for calculation with a single superstructure acceleration vector. The global acceleration vector, in which the accelerations are relative to the bearing level, is defined in equation (4-41) as a $3N \times 1$ vector with $3N$ degrees of freedom for the N-story structure. The forces represented by equation (4-13) represent the same effective forces as those in equation (4-12) but are dependent upon the incremental acceleration values. Those quantities are defined as follows:

$$B_{ni+1}^{l} = \sum_{k=1}^{3} \lambda_{bnk}^{l} \Delta\ddot{d}_{u(3l-3+k)}\left(t_{i+1}\right) \tag{4-18}$$

$$B_{ni+1}^{g} = \sum_{k=1}^{3} \alpha_{bnk} \Delta\ddot{d}_{gk}\left(t_{i+1}\right) \tag{4-19}$$

$$B_{ni+1}^{f} = \sum_{k=1}^{3} \mu\left(\Delta\ddot{d}_{gz}\left(t_{i+1}\right)\right) \alpha_{bnk} \operatorname{sgn}\left(\dot{d}_{bk}\left(t_i\right)\right) \tag{4-20}$$

By substituting equations (4-14), (4-15), and (4-16) into equation (4-12), the following equation can be written:

$$A_{ni}^{t} = \sum_{l=1}^{N}\sum_{k=1}^{3}\lambda_{bnk}^{l}\ddot{d}_{u(3l-3+k)}\left(t_i\right)+\sum_{k=1}^{3}\alpha_{bnk}\ddot{d}_{gk}\left(t_i\right) \\ +\sum_{k=1}^{3}\mu\left(g+\ddot{d}_{gz}\left(t_i\right)\right)\alpha_{bnk}\,\mathrm{sgn}\left(\dot{d}_{bk}\left(t_i\right)\right) \tag{4-21}$$

Similarly, by substituting equations (4-18), (4-19), and (4-20) into equation (4-13), the following equation can be written:

$$B_{ni+1}^{t} = \sum_{l=1}^{N}\sum_{k=1}^{3}\lambda_{bnk}^{l}\Delta\ddot{d}_{u(3l-3+k)}\left(t_{i+1}\right)+\sum_{k=1}^{3}\alpha_{bnk}\Delta\ddot{d}_{gk}\left(t_{i+1}\right) \\ +\sum_{k=1}^{3}\mu\left(\Delta\ddot{d}_{gz}\left(t_{i+1}\right)\right)\alpha_{bnk}\,\mathrm{sgn}\left(\dot{d}_{bk}\left(t_i\right)\right) \tag{4-22}$$

Equations (4-21) and (4-22) are now representative of the values used in equation (4-11). A solution to equation (4-11) is now the next step. As in Chapter 2, the unknowns in this equation are the bearing modal response quantities and, through equations (4-21) and (4-22), the superstructure modal accelerations. Therefore, it is necessary to solve equation (4-11) for the bearing modal response in terms of the superstructure modal response. The superstructure response will be determined below. The process outlined here is identical to that of Chapter 2.

By inspection, equation (4-11) is functionally identical to equation (2-30). Therefore, the method used to determine the bearing modal response in Chapter 2 will be applicable here. The solution to equation (4-11) can be written in a form similar to equation (2-43), as follows:

$$z_{bn}\left(t_{i+1}\right)=D_{ni}+R1_{n}B_{ni+1}^{t}+\frac{1}{\omega_{bn}^{2}}\left[A_{ni}^{t}+\left(\Delta t-\frac{2\xi_{bn}}{\omega_{bn}}\right)\frac{B_{ni+1}^{t}}{\Delta t}\right] \tag{4-23}$$

in which

$$D_{ni}=\frac{e^{-\xi_{bn}\omega_{bn}\Delta t}}{\Omega_{bn}}\left(E_{ni}\sin\Omega_{bn}\Delta t+F_{ni}\cos\Omega_{bn}\Delta t\right) \tag{4-24}$$

$$R1_n = e^{-\xi_{bn}\,\omega_{bn}\,\Delta t}\left(-\frac{1-2\,\xi_{bn}^2}{\omega_{bn}^2\,\Omega_{bn}\,\Delta t}\sin\Omega_{bn}\,\Delta t + \frac{2\,\xi_{bn}}{\omega_{bn}^3\,\Delta t}\cos\Omega_{bn}\,\Delta t\right) \tag{4-25}$$

$$E_{ni} = \dot{z}_{bn}\left(t_i\right) + \xi_{bn}\omega_{bn} z_{bn}\left(t_i\right) - \frac{\xi_{bn}}{\omega_{bn}} A_{ni}^t \tag{4-26}$$

$$F_{ni} = \Omega_{bn}\left[z_{bn}\left(t_i\right) - \frac{A_{ni}^t}{\omega_{bn}^2}\right] \tag{4-27}$$

Again, as in the formulation from Chapter 2, the modal velocity of the bearings is derived from a time derivative of the modal displacement. Evaluating the velocity at time t_{i+1} gives an equation similar to equation (2-48):

$$\dot{z}_{bn}\left(t_{i+1}\right) = \left(G_{ni} - \xi_{bn}\omega_{bn}D_{ni}\right) + \left(R2_n - \xi_{bn}\omega_{bn}R1_n\right)B_{ni+1}^t + \frac{B_{ni+1}^t}{\omega_{bn}^2\Delta t} \tag{4-28}$$

in which

$$G_{ni} = e^{-\xi_{bn}\,\omega_{bn}\,\Delta t}\left(E_{ni}\cos\Omega_{bn}\,\Delta t - F_{ni}\sin\Omega_{bn}\,\Delta t\right) \tag{4-29}$$

$$R2_n = \Omega_{bn}e^{-\xi_{bn}\,\omega_{bn}\,\Delta t}\left(-\frac{1-2\,\xi_{bn}^2}{\omega_{bn}^2\,\Omega_{bn}\,\Delta t}\cos\Omega_{bn}\,\Delta t + \frac{2\,\xi_{bn}}{\omega_{bn}^3\,\Delta t}\sin\Omega_{bn}\,\Delta t\right) \tag{4-30}$$

The modal accelerations are then calculated from the time derivative of the modal velocities. Evaluating the acceleration at time t_{i+1} presents a solution for the modal accelerations similar to that presented in equation (2-54):

$$\ddot{z}_{bn}\left(t_{i+1}\right) = -H_{ni} - R3_n\,B_{ni+1}^t \tag{4-31}$$

in which

$$H_{ni} = 2\,\xi_{bn}\,\omega_{bn}\,G_{ni} + \omega_{bn}^2\left(1-2\,\xi_{bn}^2\right)D_{ni} \tag{4-32}$$

$$R3_n = 2\,\xi_{bn}\,\omega_{bn}\,R2_n + \omega_{bn}^2\left(1-2\,\xi_{bn}^2\right)R1_n \tag{4-33}$$

Equation (4-31) can be expanded by substituting in equation (4-22):

$$\ddot{z}_{bn}\left(t_{i+1}\right) = -H_{ni} - R3_n \left[\begin{array}{l} \sum_{k=1}^{3} \lambda_{bnk}^{N} \Delta\ddot{d}_{u(3N-3+k)}\left(t_{i+1}\right) + \ldots + \\ \sum_{k=1}^{3} \lambda_{bnk}^{1} \Delta\ddot{d}_{u(3-3+k)}\left(t_{i+1}\right) + \sum_{k=1}^{3} \alpha_{bnk} \Delta\ddot{d}_{gk}\left(t_{i+1}\right) \end{array} \right] \quad (4\text{-}34)$$

Equation (4-34) shows a solution for equation (4-8), solving for the modal accelerations of the bearing level with respect to the superstructure incremental relative accelerations.

4.1.2 Superstructure Equations of Motion

Now the superstructure equations of motion will be derived by first summing the forces as suggested in the free-body diagram of Figure 7.

$$\begin{aligned} [M_n]\{\ddot{d}_n\} + [C_n]\left(\{\dot{d}_n\} - \{\dot{d}_{n-1}\}\right) - [C_{n+1}]\left(\{\dot{d}_{n+1}\} - \{\dot{d}_n\}\right) \\ + [K_n]\left(\{d_n\} - \{d_{n-1}\}\right) - [K_{n+1}]\left(\{d_{n+1}\} - \{d_n\}\right) \\ = -[M_n]\{\ddot{d}_b\} - [M_n]\{\ddot{d}_g\} \end{aligned} \quad (4\text{-}35)$$

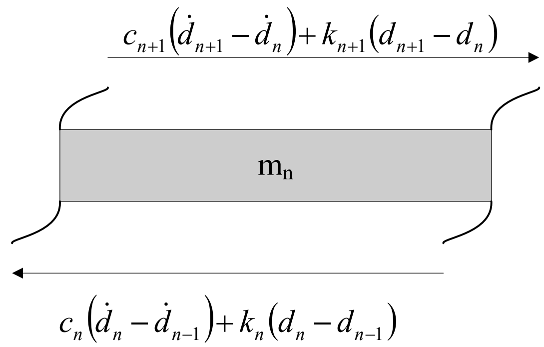

Figure 7 – Multistory Individual Floor Free Body Diagram

As shown in Figure 7 and equation (4-35), the equation of motion for each floor is dependent upon the displacement at that floor and the floors immediately above and below that floor. Also, the stiffness and damping matrices are required from the levels immediately above and below the considered floor. Note that the individual floor equations disregard the friction that was considered in the bearing level. The frictional component at each floor is considered negligible and therefore will be ignored in these formulations. By writing the matrix equations for the N floors of the superstructure, equation (4-35) can be written in the following matrix form:

$$[M_u]\{\ddot{d}_u\}+[C_u]\{\dot{d}_u\}+[K_u]\{d_u\}=-[M_{uc}]\{\ddot{d}_b\}-[M_{uc}]\{\ddot{d}_g\} \tag{4-36}$$

in which

$$[M_u]=\begin{bmatrix} [M_1] & [0] & [0] & \cdots & & [0] \\ [0] & [M_2] & [0] & & & \\ [0] & [0] & [M_3] & & & \vdots \\ \vdots & & & \ddots & & \\ & & & & [M_{N-1}] & [0] \\ [0] & & \cdots & & [0] & [M_N] \end{bmatrix} \tag{4-37}$$

$$[C_u]=\begin{bmatrix} [C_1+C_2] & [-C_2] & [0] & \cdots & & [0] \\ [-C_2] & [C_2+C_3] & [-C_3] & & & \\ [0] & [-C_3] & [C_3+C_4] & [-C_4] & & \vdots \\ \vdots & & & \ddots & & \\ & & & [-C_{N-1}] & [C_{N-1}+C_N] & [-C_N] \\ [0] & & \cdots & & [-C_N] & [C_N] \end{bmatrix} \tag{4-38}$$

$$[K_u]=\begin{bmatrix} [K_1+K_2] & [-K_2] & [0] & \cdots & & [0] \\ [-K_2] & [K_2+K_3] & [-K_3] & & & \\ [0] & [-K_3] & [K_3+K_4] & [-K_4] & & \vdots \\ \vdots & & & \ddots & & \\ & & & [-K_{N-1}] & [K_{N-1}+K_N] & [-K_N] \\ [0] & & \cdots & & [-K_N] & [K_N] \end{bmatrix} \tag{4-39}$$

$$\left[M_{uc}\right]=\begin{Bmatrix}\left[M_1\right]\\\left[M_2\right]\\\vdots\\\left[M_{N-1}\right]\\\left[M_N\right]\end{Bmatrix}\equiv 3N\times 3\ \text{matrix} \tag{4-40}$$

$$\left\{d_u\right\}=\begin{Bmatrix}\left\{d_1\right\}\\\left\{d_2\right\}\\\vdots\\\left\{d_{N-1}\right\}\\\left\{d_N\right\}\end{Bmatrix}\equiv 3N\times 1\ \text{vector} \tag{4-41}$$

$$\left\{d_n\right\}=\begin{Bmatrix}x_n\\y_n\\\theta_n\end{Bmatrix}\quad\left\{\ddot{d}_b\right\}=\begin{Bmatrix}\ddot{x}_b\\\ddot{y}_b\\\ddot{\theta}_b\end{Bmatrix}\quad\left\{\ddot{d}_g\right\}=\begin{Bmatrix}\ddot{x}_g\\\ddot{y}_g\\0\end{Bmatrix} \tag{4-42}$$

The method of modal superposition, as used in Chapter 2, can be used here to help simplify the format of equation (4-36). Expressing the accelerations of the superstructure floors and the bearing level in terms of their modal accelerations, the following relationships are evident:

$$\left\{\ddot{d}_u\right\}=\left[\Phi_u\right]\left\{\ddot{z}_u\right\}\qquad\qquad\left\{\ddot{d}_b\right\}=\left[\Phi_b\right]\left\{\ddot{z}_b\right\} \tag{4-43}$$

Then these values can be substituted into equation (4-36) to form the following equation:

$$\begin{aligned}\left[M_u\right]\left[\Phi_u\right]\left\{\ddot{z}_u\right\}+\left[C_u\right]\left[\Phi_u\right]\left\{\dot{z}_u\right\}+\left[K_u\right]\left[\Phi_u\right]\left\{z_u\right\}&=\\-\left[M_{uc}\right]\left[\Phi_b\right]\left\{\ddot{z}_b\right\}-\left[M_{uc}\right]\left\{\ddot{d}_g\right\}&\end{aligned} \tag{4-44}$$

Each side of the equation can then be premultiplied by $\left[\Phi_u\right]^T$. Recalling equations (2-55) through (2-57), which represent the mass-orthonormalization of the mode shapes as presented in Chapter 2, equation (4-44) can be rewritten as:

$$\left\{\ddot{z}_u\right\}+diag\left[2\xi_{un}\omega_{un}\right]\left\{\dot{z}_u\right\}+diag\left[\omega_{un}^2\right]\left\{z_u\right\}=\left[\lambda_u\right]\left\{\ddot{z}_b\right\}+\left[\alpha_u\right]\left\{\ddot{d}_g\right\} \tag{4-45}$$

in which $n=1,2,\ldots,3N$ and

$$\left[\lambda_u\right]=-\left[\Phi_u\right]^T\left[M_{uc}\right]\left[\Phi_u\right] \qquad \left[\alpha_u\right]=-\left[\Phi_u\right]^T\left[M_{uc}\right] \tag{4-46}$$

By writing one of the $3N$ equations from equation (4-45) and evaluating it at time t_{i+1}, the following expression is obtained:

$$\begin{aligned}\ddot{z}_{un}\left(t_{i+1}\right)+2\xi_{un}\omega_{un}\dot{z}_{un}\left(t_{i+1}\right)+\omega_{un}^2 z_{un}\left(t_{i+1}\right)=& \\ \sum_{k=1}^{3}\left(\lambda_{unk}\ddot{z}_{bk}\left(t_{i+1}\right)+\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right)\right)&\end{aligned} \tag{4-47}$$

As in the previous chapters, the linear acceleration approach will be used to approximate the behavior through the time interval. The equations for the linear acceleration method are repeated here for convenience:

$$\ddot{z}_{un}\left(t_{i+1}\right)=\ddot{z}_{un}\left(t_i\right)+\Delta\ddot{z}_{un}\left(t_{i+1}\right) \tag{4-48}$$

$$\dot{z}_{un}\left(t_{i+1}\right)=\dot{z}_{un}\left(t_i\right)+\ddot{z}_{un}\left(t_i\right)\Delta t+\Delta\ddot{z}_{un}\left(t_{i+1}\right)\frac{\Delta t}{2} \tag{4-49}$$

$$z_{un}\left(t_{i+1}\right)=z_{un}\left(t_i\right)+\dot{z}_{un}\left(t_i\right)\Delta t+\ddot{z}_{un}\left(t_i\right)\frac{\Delta t^2}{2}+\Delta\ddot{z}_{un}\left(t_{i+1}\right)\frac{\Delta t^2}{6} \tag{4-50}$$

Then by substituting equations (4-48), (4-49), and (4-50) into equation (4-47), the following result is obtained:

$$\begin{aligned}R4_n\,\Delta\ddot{z}_{un}\left(t_{i+1}\right)+R5_n\,\ddot{z}_{un}\left(t_i\right)+R6_n\,\dot{z}_{un}\left(t_i\right)+\omega_{un}^2 z_{un}\left(t_i\right)=& \\ -\sum_{k=1}^{3}\sum_{l=1}^{3}\lambda_{unk}R3_k\,\lambda_{bkl}^{N}\Delta\ddot{d}_{u(3N-3+l)}\left(t_{i+1}\right)-\ldots& \\ -\sum_{k=1}^{3}\sum_{l=1}^{3}\lambda_{unk}R3_k\,\lambda_{bkl}^{1}\Delta\ddot{d}_{u(3-3+l)}\left(t_{i+1}\right)& \\ -\sum_{k=1}^{3}\sum_{l=1}^{3}\lambda_{unk}R3_k\,\alpha_{bkl}\Delta\ddot{d}_{gl}\left(t_{i+1}\right)& \\ -\sum_{k=1}^{3}\lambda_{unk}H_{ki}+\sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right)&\end{aligned} \tag{4-51}$$

in which

$$R4_n = 1 + \xi_{un}\omega_{un}\Delta t + \omega_{un}^2 \frac{\Delta t^2}{6} \tag{4-52}$$

$$R5_n = 1 + 2\xi_{un}\omega_{un}\Delta t + \omega_{un}^2 \frac{\Delta t^2}{2} \tag{4-53}$$

$$R6_n = 2\xi_{un}\omega_{un} + \omega_{un}^2 \Delta t \tag{4-54}$$

Since the left hand side of equation (4-51) is in terms of modal accelerations of the superstructure and the right hand side is in terms of the actual accelerations, it is desired to convert equation (4-51) into a consistent format. To that end, the modal superposition method will be used here. Modal superposition, by definition, allows the following relationships to be written:

$$\Delta\ddot{d}_{u(3k-3+l)}\left(t_{i+1}\right) = \sum_{m=1}^{3N} \Phi_{u(3k=3+l)m} \Delta\ddot{z}_{um}\left(t_{i+1}\right) \tag{4-55}$$

Equation (4-55) can be written for each of the 3N components of the $\left\{\Delta\ddot{d}_u\left(t_{i+1}\right)\right\}$ vector. Substituting these results into equation (4-51) and grouping like terms yields the following expression:

$$\begin{aligned}
&\Delta\ddot{z}_{un}\left(t_{i+1}\right) + \frac{1}{R4_n}\left[\sum_{j=1}^{N}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3N} \lambda_{unk} R3_k \lambda_{bkl}^{j} \Phi_{u(3j-3+l)m} \Delta\ddot{z}_{um}\left(t_{i+1}\right)\right] = \\
&-\frac{1}{R4_n}\left\{\begin{aligned}&\left[R5_n \ddot{z}_{un}\left(t_i\right) + R6_n \dot{z}_{un}\left(t_i\right) + \omega_{un}^2 z_{un}\left(t_i\right)\right] + \sum_{k=1}^{3}\lambda_{unk} H_{ki} \\ &+ \sum_{k=1}^{3}\sum_{l=1}^{3}\lambda_{unk} R3_k \alpha_{bkl} \Delta\ddot{d}_{gl}\left(t_{i+1}\right) - \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right)\end{aligned}\right\}
\end{aligned} \tag{4-56}$$

in which $n = 1, 2, \ldots, 3N$. By writing these $3N$ equations in a simplified matrix form, equation (4-56) becomes:

$$[Q]\{\Delta\ddot{z}_u\} = \{P\} \tag{4-57}$$

in which

$[Q] \equiv 3N \times 3N$ matrix

$\{\Delta\ddot{z}_u\} \equiv 3N \times 1$ vector

$\{P\} \equiv 3N \times 1$ vector

The individual terms in the $[Q]$ matrix are defined as follows:

$$Q_{nm} = \frac{1}{R4_n}\left[\sum_{j=1}^{N}\sum_{k=1}^{3}\sum_{l=1}^{3}\lambda_{unk} R3_k \lambda_{bkl}^{j} \Phi_{u(3j-3+l)m}\right] \tag{4-58}$$

$$Q_{nn} = 1 + \frac{1}{R4_n}\left[\sum_{j=1}^{N}\sum_{k=1}^{3}\sum_{l=1}^{3}\lambda_{unk} R3_k \lambda_{bkl}^{j} \Phi_{u(3j-3+l)m}\right] \tag{4-59}$$

Equation (4-58) applies to the non-diagonal terms (i.e. $n \neq m$) and equation (4-59) defines the diagonal terms of the $[Q]$ matrix.

The individual terms of the $\{P\}$ vector must also be defined. Each of the 3N terms can be written as such:

$$P_n = -\frac{1}{R4_n}\left[P1_n + P2_n + P3_n\right] \tag{4-60}$$

in which

$$P1_n = R5_n \ddot{z}_{un}(t_i) + R6_n \dot{z}_{un}(t_i) + \omega_{un}^2 z_{un}(t_i) \tag{4-61}$$

$$P2_n = \sum_{k=1}^{3}\lambda_{unk} H_{ki} \tag{4-62}$$

$$P3_n = \sum_{k=1}^{3}\sum_{l=1}^{3}\left(\lambda_{unk} R3_k \alpha_{bkl} \Delta\ddot{d}_{gl}(t_{i+1})\right) - \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}(t_{i+1}) \tag{4-63}$$

Equation (4-57) can then be solved for the unknown $\{\Delta\ddot{z}_u\}$ vector:

$$\{\Delta\ddot{z}_u(t_{i+1})\} = [Q]^{-1}\{P\} \tag{4-64}$$

Equation (4-64) now represents a solution for the incremental accelerations of the superstructure floors. Recall that equation (4-34), the solution for the bearing accelerations, was in terms of the incremental modal accelerations of the superstructure. Now that these accelerations are known, the bearing level responses can be solved from equations (4-23), (4-28), and (4-34). However, as in Chapter 2, it is important to ensure that equilibrium is maintained throughout the solution procedure. Therefore, the bearing level response will be substituted into (4-47) to determine new superstructure modal accelerations, which are used to determine the next values for the bearing level response. This iterative process is described in detail below.

4.2 SUMMARY OF SOLUTION STEPS

As in Chapter 2, the solution procedure for the linear response of the base-isolated structure involves a number of steps. The steps listed here are extremely similar to those listed for the single-story structure; however, the multistory structure's response will be more mathematically demanding, as there are multiple levels for which to calculate the response quantities.

1. Assemble the mass and stiffness matrices, for the bearing level and for each level of the superstructure, as described in Appendix A.
2. Determine the modal matrices shown in equation (2-16) for the bearing level and the first floor by solving equations (2-14) and (2-15). The remaining superstructure modal matrices can be determined by solving the following equation:

$$\left[K_j\right]\left\{d_j\right\}=\omega_{jn}^2\left[M_j\right]\left\{d_j\right\} \qquad j=2,3,\ldots N \qquad (4\text{-}65)$$

3. Assemble the $\left[Q\right]$ matrix as described in equations (4-58) and (4-59). Assemble the $\left\{P\right\}$ vector as described in equations (4-60) through (4-63).

4. Solve equation (4-64) to determine the incremental modal acceleration values for the superstructure degrees of freedom. These values will act as the initial values for the time step.
5. To determine the initial values for the superstructure modal acceleration, velocity, and displacement at time t_{i+1}, substitute the values for superstructure incremental modal acceleration into equations (4-48), (4-49), and (4-50), respectively.
6. Substitute the values determined in steps 4 and 5 into equations (4-21) and (4-22) to determine the parameters required to calculate the bearing level modal response.
7. Substitute the parameters determined in step 6 into equations (4-23), (4-28), and (4-31) to determine initial values of the bearing level modal displacement, velocity, and acceleration, respectively.
8. Check equilibrium. Substitute the values for $\dot{z}_{un}\left(t_{i+1}\right)$, $z_{un}\left(t_{i+1}\right)$, and $\left\{\ddot{z}_b\left(t_{i+1}\right)\right\}$ into equation (4-48) to determine a second iteration value for $\ddot{z}_{un}\left(t_{i+1}\right)$. Use equation (4-48) to determine a second iteration value for the superstructure incremental modal accelerations.
9. Repeat steps 5-8 until the change in modal response between iterations is negligible. To determine the actual response of the structure, equations (4-43) and similar expressions involving velocity and displacement can be solved using the modal response. The values obtained from this calculation represent the actual response of the bearing level and the superstructure levels at time t_{i+1}.

5.0 MULTI-STORY NON-LINEAR ANALYSIS

As was the case with the single-story structure, nonlinearities often occur in the response of a structure to ground excitation. Since base-isolation seeks to restrict the superstructure to an elastic response, the only nonlinearity to be considered here is yielding and hardening of the bearings. For a more detailed explanation of the non-linearity, see Appendix C. The equation of motion of the bearings, taken from equation (4-8), is repeated here for convenience:

$$\begin{aligned}[M_t]\{\ddot{d}_b\}+[C_b]\{\dot{d}_b\}+[K_b]\{d_b\} &= -\sum_{i=1}^{N}\left([M_i]\{\ddot{d}_i\}\right) \\ &\quad -[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\operatorname{sgn}(\dot{d}_b)\}\end{aligned} \tag{5-1}$$

in which

$$[M_t]=[M_b]+\sum_{i=1}^{N}[M_i] \tag{5-2}$$

5.1 HILBER'S α METHOD

Equation (5-1) can be modified by the Hilber-α Method as seen in Chapter 3. This modification is as follows, evaluated at time t_{i+1}:

$$
\begin{aligned}
&[M_t]\{\ddot{d}_b(t_{i+1})\}+[C_b]\{\dot{d}_b(t_{i+1})\}+(1+\alpha)[K_b]\{d_b(t_{i+1})\} \\
&\quad -\alpha[K_b]\{d_b(t_i)\}=-\sum_{j=1}^{N}[M_j]\{\ddot{d}_j(t_{i+1})\}-[M_t]\{\ddot{d}_g(t_{i+1})\} \\
&\qquad -\mu\left(g+\ddot{d}_{gz}(t_{i+1})\right)[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b(t_i)\right)\right\}-\left\{\Delta R^{i+1}\right\}
\end{aligned}
\tag{5-3}
$$

The same expression, evaluated at time t_i, can be written as follows:

$$
\begin{aligned}
&[M_t]\{\ddot{d}_b(t_i)\}+[C_b]\{\dot{d}_b(t_i)\}+(1+\alpha)[K_b]\{d_b(t_i)\} \\
&\quad -\alpha[K_b]\{d_b(t_{i-1})\}=-\sum_{j=1}^{N}[M_j]\{\ddot{d}_j(t_i)\}-[M_t]\{\ddot{d}_g(t_i)\} \\
&\qquad -\mu\left(g+\ddot{d}_{gz}(t_i)\right)[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b(t_i)\right)\right\}-\left\{\Delta R^{i}\right\}
\end{aligned}
\tag{5-4}
$$

An incremental expression for the equation of motion can be written by subtracting equation (5-4) from equation (5-3). This new equation is written as follows:

$$
\begin{aligned}
&[M_t]\{\Delta\ddot{d}_b(t_{i+1})\}+[C_b]\{\Delta\dot{d}_b(t_{i+1})\}+(1+\alpha)[K_b]\{\Delta d_b(t_{i+1})\} \\
&\quad -\alpha[K_b]\{\Delta d_b(t_i)\}=-\sum_{j=1}^{N}[M_j]\{\Delta\ddot{d}_j(t_{i+1})\}-[M_t]\{\Delta\ddot{d}_g(t_{i+1})\} \\
&\qquad -\mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b(t_i)\right)\right\}-\{\Delta R\}
\end{aligned}
\tag{5-5}
$$

in which

$$\{\Delta R\}=\left\{\Delta R^{i+1}\right\}-\left\{\Delta R^{i}\right\} \tag{5-6}$$

$\left\{\Delta R^{i+1}\right\}\equiv$ residual forces at iteration i

$$\left\{\Delta\ddot{d}_{gz}(t_{i+1})\right\}=\left\{\ddot{d}_{gz}(t_{i+1})\right\}-\left\{\ddot{d}_{gz}(t_i)\right\} \tag{5-7}$$

5.2 NEWMARK'S β METHOD

To simplify the incremental equations of motion derived in Chapter 3, Newmark's β Method was implemented. This same method will be utilized here, but equations (3-8) and (3-9) must be rewritten in vector form as follows:

$$\left\{\dot{d}_b\left(t_{i+1}\right)\right\}=\left\{\dot{d}_b\left(t_i\right)\right\}+\left[(1-\gamma)\left\{\ddot{d}_b\left(t_i\right)\right\}+\gamma\left\{\ddot{d}_b\left(t_{i+1}\right)\right\}\right]\Delta t_i \tag{5-8}$$

$$\left\{d_b\left(t_{i+1}\right)\right\}=\left\{d_b\left(t_i\right)\right\}+\left\{\dot{d}_b\left(t_i\right)\right\}\Delta t_i+\left[\left(\frac{1}{2}-\beta\right)\left\{\ddot{d}_b\left(t_i\right)\right\}+\beta\left\{\ddot{d}_b\left(t_{i+1}\right)\right\}\right](\Delta t_i)^2 \tag{5-9}$$

These equations must then be converted into incremental form:

$$\begin{aligned}\left\{\Delta\dot{d}_b\left(t_{i+1}\right)\right\}&=\left[(1-\gamma)\left\{\ddot{d}_b\left(t_i\right)\right\}+\gamma\left\{\ddot{d}_b\left(t_{i+1}\right)\right\}\right]\Delta t_i\\&=\left\{\ddot{d}_b\left(t_i\right)\right\}\Delta t_i+\gamma\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}\Delta t_i\end{aligned} \tag{5-10}$$

$$\begin{aligned}\left\{\Delta d_b\left(t_{i+1}\right)\right\}&=\left\{\dot{d}_b\left(t_i\right)\right\}\Delta t_i+\left[\left(\frac{1}{2}-\beta\right)\left\{\ddot{d}_b\left(t_i\right)\right\}+\beta\left\{\ddot{d}_b\left(t_{i+1}\right)\right\}\right](\Delta t_i)^2\\&=\left\{\dot{d}_b\left(t_i\right)\right\}\Delta t_i+\frac{1}{2}\left\{\ddot{d}_b\left(t_i\right)\right\}(\Delta t)^2+\beta\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}(\Delta t_i)^2\end{aligned} \tag{5-11}$$

Equations (5-10) and (5-11) can then be substituted into equation (5-5), eliminating the unknown incremental velocity and displacement values from the equation. The simplified expression is now:

$$\begin{aligned}&[M_t]\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}+[C_b]\left\{\left\{\ddot{d}_b\left(t_i\right)\right\}+\gamma\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}\right\}\Delta t_i+\\&(1+\alpha)[K_b]\left\{\left\{\dot{d}_b\left(t_i\right)\right\}\Delta t_i+\frac{1}{2}\left\{\ddot{d}_b\left(t_i\right)\right\}(\Delta t_i)^2+\beta\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\}(\Delta t_i)^2\right\}=\\&\qquad\alpha[K_b]\left\{\Delta d_b\left(t_i\right)\right\}-\sum_{k=1}^{N}[M_k]\left\{\Delta\ddot{d}_k\left(t_{i+1}\right)\right\}-[M_t]\left\{\Delta\ddot{d}_g\left(t_{i+1}\right)\right\}\\&\qquad\qquad-\mu\Delta\ddot{d}_{gz}\left(t_{i+1}\right)[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b\left(t_i\right)\right)\right\}-\{\Delta R\}\end{aligned} \tag{5-12}$$

During each time step, the only true unknowns in this equation are the incremental accelerations of the bearing and superstructure levels. It is desirable to express this equation as a function of the superstructure accelerations, which will then be used to solve for the bearing accelerations. By rearranging the terms accordingly, the following expression can be obtained:

$$
\begin{aligned}
\left([M_t]+\gamma[C_b]\Delta t_i+\beta(1+\alpha)[K_b](\Delta t_i)^2\right)\left\{\Delta\ddot{d}_b(t_{i+1})\right\} &= -[C_b]\left\{\ddot{d}_b(t_i)\right\}\Delta t_i \\
&\quad -[K_b]\left((1+\alpha)\left(\left\{\dot{d}_b(t_i)\right\}\Delta t_i+\tfrac{1}{2}\left\{\ddot{d}_b(t_i)\right\}(\Delta t_i)^2\right)-\alpha\left\{\Delta d_b(t_i)\right\}\right) \\
&\quad -\sum_{k=1}^{N}[M_k]\left\{\Delta\ddot{d}_k(t_{i+1})\right\}-[M_t]\left\{\Delta\ddot{d}_g(t_{i+1})\right\} \\
&\quad -\mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b(t_i)\right)\right\}-\{\Delta R\}
\end{aligned}
\tag{5-13}
$$

Equation (5-13) can be further simplified as

$$
\left[\overline{K}(t_i)\right]\left\{\Delta\ddot{d}_b(t_{i+1})\right\} = -\left(\begin{aligned} &[C_b]\{D_1\}+[K_b]\{D_2\}+\sum_{k=1}^{N}[M_k]\left\{\Delta\ddot{d}_k(t_{i+1})\right\} \\ &\qquad +[M_t]\left\{\Delta\ddot{d}_g(t_{i+1})\right\} \\ &+\mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b(t_i)\right)\right\}+\{\Delta R\} \end{aligned}\right)
\tag{5-14}
$$

in which

$$
\left[\overline{K}(t_i)\right]=[M_t]+\gamma[C_b]\Delta t_i+\beta(1+\alpha)[K_b](\Delta t_i)^2
\tag{5-15}
$$

$$
\{D_1\}=\left\{\ddot{d}_b(t_i)\right\}\Delta t_i
\tag{5-16}
$$

$$
\{D_2\}=(1+\alpha)\left(\left\{\dot{d}_b(t_i)\right\}\Delta t_i+\tfrac{1}{2}\left\{\ddot{d}_b(t_i)\right\}(\Delta t_i)^2\right)-\alpha\left\{\Delta d_b(t_i)\right\}
\tag{5-17}
$$

Solving equation (5-14) for the unknown incremental accelerations at the bearing level:

$$
\left\{\Delta\ddot{d}_b(t_{i+1})\right\} = -\left[\overline{K}(t_i)\right]^{-1}\left(\begin{aligned} &[C_b]\{D_1\}+[K_b]\{D_2\}+\sum_{k=1}^{N}[M_k]\left\{\Delta\ddot{d}_k(t_{i+1})\right\} \\ &\qquad +[M_t]\left\{\Delta\ddot{d}_g(t_{i+1})\right\} \\ &+\mu\Delta\ddot{d}_{gz}(t_{i+1})[M_t]\left\{\operatorname{sgn}\left(\dot{d}_b(t_i)\right)\right\}+\{\Delta R\} \end{aligned}\right)
\tag{5-18}
$$

As mentioned previously, this equation is still a function of the superstructure accelerations, which are still unknown. Therefore, equation (5-18) must be solved simultaneously with another equation. The equation of motion of the superstructure is a second equation that depends upon both of the unknowns. Since the superstructure is assumed to maintain an elastic response, equation (4-45) applies here:

$$\{\ddot{z}_u\} + diag[2\xi_{un}\omega_{un}]\{\dot{z}_u\} + diag[\omega_{un}^2]\{z_u\} = [\lambda_u]\{\ddot{z}_b\} + [\alpha_u]\{\ddot{d}_g\} \tag{5-19}$$

in which

$$[\lambda_u] = -[\Phi_u]^T [M_{uc}][\Phi_b] \tag{5-20}$$

$$[\alpha_u] = -[\Phi_u]^T [M_{uc}] \tag{5-21}$$

Given equations (5-20) and (5-21), the following expression can be obtained:

$$[\lambda_u]\{\ddot{z}_b\} = -[\Phi_u]^T [M_{uc}][\Phi_b]\{\ddot{z}_b\} = [\alpha_u]\{\ddot{d}_b\} \tag{5-22}$$

Substituting equation (5-22) into equation (5-19) allows the following equation to be written:

$$\{\ddot{z}_u\} + diag[2\xi_{un}\omega_{un}]\{\dot{z}_u\} + diag[\omega_{un}^2]\{z_u\} = [\alpha_u]\{\ddot{d}_b\} + [\alpha_u]\{\ddot{d}_g\} \tag{5-23}$$

A single equation can then be written from the matrix expression of equation (5-23). Writing equation *n* from the matrix formulation evaluated at time t_{i+1}:

$$\begin{aligned} &\ddot{z}_{un}(t_{i+1}) + 2\xi_{un}\omega_{un}\dot{z}_{un}(t_{i+1}) + \omega_{un}^2 z_{un}(t_{i+1}) = \\ &\qquad \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{bk}(t_{i+1}) + \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}(t_{i+1}) \end{aligned} \tag{5-24}$$

Again the linear acceleration method will be used to interpolate the incremental acceleration values. For reference, equations (4-48), (4-49), and (4-50) are repeated here:

$$\ddot{z}_{un}(t_{i+1}) = \ddot{z}_{un}(t_i) + \Delta\ddot{z}_{un}(t_{i+1}) \tag{5-25}$$

$$\dot{z}_{un}(t_{i+1}) = \dot{z}_{un}(t_i) + \ddot{z}_{un}(t_i)\Delta t + \Delta\ddot{z}_{un}(t_{i+1})\frac{\Delta t}{2} \tag{5-26}$$

$$z_{un}(t_{i+1}) = z_{un}(t_i) + \dot{z}_{un}(t_i)\Delta t + \ddot{z}_{un}(t_i)\frac{\Delta t^2}{2} + \Delta\ddot{z}_{un}(t_{i+1})\frac{\Delta t^2}{6} \tag{5-27}$$

Substituting equations (5-25), (5-26), and (5-27) into equation (5-24) yields the following:

$$\begin{aligned} &R4_n\Delta\ddot{z}_{un}(t_{i+1}) + R5_n\ddot{z}_{un}(t_i) + R6_n\dot{z}_{un}(t_i) + \omega_{un}^2 z_{un}(t_i) = \\ &\qquad \sum_{k=1}^{3}\alpha_{unk}\left(\{\ddot{d}_{bk}(t_i)\} + \{\Delta\ddot{d}_{bk}(t_{i+1})\}\right) + \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}(t_{i+1}) \end{aligned} \tag{5-28}$$

This expression is an equation with two sets of unknowns. The incremental modal accelerations of both the superstructure and the bearings are unknown. Therefore, it is necessary to solve equations (5-28) and (5-18) simultaneously. However, first equation (5-18) must be written in a scalar form. The following definitions will be used for that purpose:

$$\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\} = \begin{Bmatrix} \Delta\ddot{d}_{b1}\left(t_{i+1}\right) \\ \Delta\ddot{d}_{b2}\left(t_{i+1}\right) \\ \Delta\ddot{d}_{b3}\left(t_{i+1}\right) \end{Bmatrix} = \begin{Bmatrix} \Delta\ddot{d}_{bx}\left(t_{i+1}\right) \\ \Delta\ddot{d}_{by}\left(t_{i+1}\right) \\ \Delta\ddot{d}_{b\theta}\left(t_{i+1}\right) \end{Bmatrix} \tag{5-29}$$

$$\{\delta_b\} = \left[\overline{K}\left(t_i\right)\right]^{-1}\left(\left[C_b\right]\{D_1\} + \left[K_b\right]\{D_2\} + \{\Delta R\}\right) = \begin{Bmatrix} \delta_{b1} \\ \delta_{b2} \\ \delta_{b3} \end{Bmatrix} \tag{5-30}$$

$$\left\{\delta_f\right\} = \mu\Delta\ddot{d}_{gz}\left(t_{i+1}\right)\left[\overline{K}\left(t_i\right)\right]^{-1}\left[M_t\right]\left\{\operatorname{sgn}\left(\dot{d}_b\left(t_i\right)\right)\right\} = \begin{Bmatrix} \delta_{f1} \\ \delta_{f2} \\ \delta_{f3} \end{Bmatrix} \tag{5-31}$$

$$\left[G_k\right] = \left[\overline{K}\left(t_i\right)\right]^{-1}\left[M_k\right] = \begin{bmatrix} G_{k11} & G_{k12} & G_{k13} \\ G_{k21} & G_{k22} & G_{k23} \\ G_{k31} & G_{k32} & G_{k33} \end{bmatrix} \qquad k = 1, 2, \ldots, N \tag{5-32}$$

$$\left[H\right] = \left[\overline{K}\left(t_i\right)\right]^{-1}\left[M_t\right] = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \tag{5-33}$$

Using these definitions, it is now possible to rewrite equation (5-18) as:

$$\left\{\Delta\ddot{d}_b\left(t_{i+1}\right)\right\} = -\left[\{\delta_b\} + \left\{\delta_f\right\} + \sum_{k=1}^{N}\left[G_k\right]\left\{\Delta\ddot{d}_k\left(t_{i+1}\right)\right\} + \left[H\right]\left\{\Delta\ddot{d}_g\left(t_{i+1}\right)\right\}\right] \tag{5-34}$$

However, equation (5-28) includes the superstructure accelerations as they are found in the overall superstructure acceleration vector $\left\{\ddot{d}_u\left(t_{i+1}\right)\right\}$, as opposed to the individual vectors

$\{\ddot{d}_k(t_{i+1})\}$. To write equation (5-34) in those same terms, it is necessary to realize that $\{\ddot{d}_k(t_{i+1})\}$ is a subset of $\{\ddot{d}_u(t_{i+1})\}$ and that relationship can be written as follows:

$$\ddot{d}_{kn}(t_{i+1}) = \ddot{d}_{u(3k-3+n)}(t_{i+1}) \qquad n = 1,2,3 \tag{5-35}$$

It will also be necessary to use the modal acceleration forms, so the following equalities will be necessary:

$$\{\ddot{d}_k(t_{i+1})\} = [\Phi_k]\{\ddot{z}_k(t_{i+1})\} \tag{5-36}$$

$$\ddot{d}_{kn}(t_{i+1}) = \sum_{m=1}^{3} \Phi_{knm}\ddot{z}_{km}(t_{i+1}) \tag{5-37}$$

From equations (5-35) and (5-37), the following expression can be obtained:

$$\ddot{d}_{u(3k-3+n)}(t_{i+1}) = \sum_{m=1}^{3N} \Phi_{u(3k-3+n)m}\ddot{z}_{um}(t_{i+1}) \tag{5-38}$$

Equation (5-37) is in a form which allows equation (5-18) to be broken down into a single scalar equation from its vector form. The resulting equation is shown below:

$$\Delta\ddot{d}_{bn}(t_{i+1}) = -\left[\delta_{bn} + \delta_{fn} + \sum_{k=1}^{N}\sum_{l=1}^{3} G_{knl}\Delta\ddot{d}_{kn}(t_{i+1}) + \sum_{k=1}^{3} H_{nk}\Delta\ddot{d}_{gk}(t_{i+1})\right] \tag{5-39}$$

By using equations (5-35) and (5-38), equation (5-39) can be rewritten as follows:

$$\Delta\ddot{d}_{bn}(t_{i+1}) = -\left[\begin{array}{r}\delta_{bn} + \delta_{fn} + \displaystyle\sum_{k=1}^{N}\sum_{l=1}^{3}\sum_{m=1}^{3N} G_{knl}\Phi_{u(3k-3+l)m}\Delta\ddot{z}_{um}(t_{i+1}) \\ + \displaystyle\sum_{k=1}^{3} H_{nk}\Delta\ddot{d}_{gk}(t_{i+1})\end{array}\right] \tag{5-40}$$

Substituting this result into equation (5-28) yields the following result:

$$R4_n \Delta\ddot{z}_{un}\left(t_{i+1}\right)+R5_n \ddot{z}_{un}\left(t_i\right)+R6_n \dot{z}_{un}\left(t_i\right)+\omega_{un}^2 z_{un}\left(t_i\right)= \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{bk}\left(t_i\right)-\sum_{k=1}^{3}\alpha_{unk}\left(\delta_{bk}+\delta_{fk}\right)- \sum_{j=1}^{3}\sum_{k=1}^{N}\sum_{l=1}^{3}\sum_{m=1}^{3N}\alpha_{unj}G_{kjl}\Phi_{u(3k-3+l)m}\Delta\ddot{z}_{um}\left(t_{i+1}\right)- \sum_{k=1}^{3}\sum_{l=1}^{3}\alpha_{unk}H_{kl}\Delta\ddot{d}_{gl}\left(t_{i+1}\right)+\sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right) \tag{5-41}$$

Grouping like terms in equation (5-41) yields:

$$R4_n \Delta\ddot{z}_{un}\left(t_{i+1}\right)+\sum_{j=1}^{3}\sum_{k=1}^{N}\sum_{l=1}^{3}\sum_{m=1}^{3N}\alpha_{unj}G_{kjl}\Phi_{u(3k-3+l)m}\Delta\ddot{z}_{um}\left(t_{i+1}\right)= -\left[\begin{array}{c} R5_n \ddot{z}_{un}\left(t_i\right)+R6_n \dot{z}_{un}\left(t_i\right)+\omega_{un}^2 z_{un}\left(t_i\right)- \\ \sum_{k=1}^{3}\alpha_{unk}\left(\ddot{d}_{bk}\left(t_i\right)+\ddot{d}_{gk}\left(t_{i+1}\right)-\delta_{bk}-\delta_{fk}\right)+ \\ \sum_{k=1}^{3}\sum_{l=1}^{3}\alpha_{unk}H_{kl}\Delta\ddot{d}_{gl}\left(t_{i+1}\right) \end{array}\right] \tag{5-42}$$

Equation (5-42) can then be expressed in a simplified into a matrix format:

$$\left[Q\right]\left\{\Delta\ddot{z}_u\left(t_{i+1}\right)\right\}=\left\{P\right\} \tag{5-43}$$

in which

$$Q_{nm}=\frac{1}{R4_n}\left[\sum_{j=1}^{3}\sum_{k=1}^{N}\sum_{l=1}^{3}\alpha_{unj}G_{kjl}\Phi_{u(3k-3+l)m}\right] \qquad n\neq m \tag{5-44}$$

$$Q_{nn}=1+\frac{1}{R4_n}\left[\sum_{j=1}^{3}\sum_{k=1}^{N}\sum_{l=1}^{3}\alpha_{unj}G_{kjl}\Phi_{u(3k-3+l)n}\right] \tag{5-45}$$

$$P_n=-\frac{1}{R4_n}\left[P1_n+P2_n+P3_n\right] \tag{5-46}$$

$$P1_n=R5_n \ddot{z}_{un}\left(t_i\right)+R6_n \dot{z}_{un}\left(t_i\right)+\omega_{un}^2 z_{un}\left(t_i\right) \tag{5-47}$$

$$P2_n=-\sum_{k=1}^{3}\alpha_{unk}\left(\ddot{d}_{bk}\left(t_i\right)+\ddot{d}_{gk}\left(t_{i+1}\right)-\delta_{bk}-\delta_{fk}\right) \tag{5-48}$$

$$P3_n = \sum_{k=1}^{3}\sum_{l=1}^{3}\alpha_{unk} H_{kl} \Delta\ddot{d}_{gl}\left(t_{i+1}\right) \tag{5-49}$$

Equation (5-43) can then be solved for the unknown incremental modal displacements as follows:

$$\left\{\Delta\ddot{z}_u\left(t_{i+1}\right)\right\} = \left[Q\right]^{-1}\left\{P\right\} \tag{5-50}$$

Equation (5-50) gives values for the superstructure incremental modal accelerations. From those values, the actual accelerations, velocities, and displacements of the superstructure can be derived from equation (5-36), shown only for acceleration.

Since the superstructure modal accelerations are now known, the bearing incremental accelerations can be determined from equation (5-40). These can then be used to determine the acceleration, velocity, and displacement of the bearing level at time t_{i+1} by using equation (5-7), (5-8), and (5-9) respectively. Again the most important values are the bearing acceleration and displacement, as discussed in Chapter 2.

5.3 SUMMARY OF SOLUTION STEPS

The solution procedure, as enumerated in the above text, can be condensed into a stepwise process as follows.

1. Select values for the three parameters used in Hilber's modification of Newmark's Method – α, β, and γ. Hilber suggests using the values of -0.1, 0.3025, and 0.6, respectively.
2. Assemble the mass matrices as described in Appendix A. Determine the stiffness matrix for the first floor from Appendix A.

3. To determine the stiffness of the bearing level, transfer the bearing level displacements to each individual bearing. Then the stiffness for each bearing must be determined from Appendix C. Those individual bearing stiffness values are then combined as shown in Appendix A.
4. Solve the generalized eigenvalue problem shown in equation (4-65) to determine the mode shapes of each floor of the superstructure.
5. Assemble the $[Q]$ matrix as shown in equations (5-54) and (5-55). Assemble the $\{P\}$ vector as shown in equations (5-46) through (5-49).
6. Solve equation (5-50) to determine the initial values of the incremental modal accelerations of the first floor. Using modal superposition, determine the incremental superstructure accelerations in the form of equation (5-36).
7. Substitute the values for $\{\Delta\ddot{z}_1(t_{i+1})\}$ into equation (5-40) to determine the incremental bearing level accelerations. Using these accelerations, determine the incremental bearing level velocity and displacement from equations (5-10) and (5-11), respectively.
8. Determine the displacement, velocity, and acceleration at time t_{i+1} from the previous values and the incremental values.
9. Substitute the values for bearing level displacement, velocity, and acceleration, along with the first floor acceleration, into equation (5-6) to determine the unknown residual force vector $\{\Delta R^{i+1}\}$.
10. From the bearing level displacements, determine the displacement of each individual bearing. From the bearing displacement, determine the force in that bearing. If a

bearing has yielded, its lateral force must be reduced to the yield value and the amount of the reduction must be added to the residual force vector $\{\Delta R^{i+1}\}$.

11. Determine $\{\Delta R\}$ from equation (3-6), which will then be used in the next time step.
12. Assemble the effective bearing stiffness matrix. Compare with the previous value for the time step. If the difference is neglible, proceed to the next time increment, beginning with step 4 of this procedure. Otherwise, return to step 5 and perform another iteration of calculations for the current time step.

6.0 CONCLUSION

A linear single story structure is the simplest possible base-isolated structure. Therefore, the derivation of an analysis procedure for this structure was presented first. The analysis method is more complex than the one-degree-of-freedom analysis, which disregards the effect of torsion. The more complex three-degree-of-freedom derivation shown in Chapter 2 will account for the torsion inherent in a non-symmetric structure. This torsion may increase or decrease the displacement and force maxima; therefore, the multi-dimensional analysis is more accurate than the one-degree-of-freedom method.

The bearings will not necessarily remain linear. Therefore, it is important to consider the effects of non-linear behavior on the dynamic response. A non-linear solution can be derived from the linear solution. In Chapter 3, the derivation of a non-linear single-story response incorporates inelastic bearing behavior through the use of Hilber's α method and the plasticity theory presented in Appendix C. It is important to note that due to the concept of base isolation, the superstructure is assumed to remain linear throughout the formulation.

Multi-story structures can also benefit greatly from base isolation. The linear solution presented in Chapter 2 is modified to apply to a multi-story structure in Chapter 4. In this derivation, both the bearings and the superstructure are assumed to remain linear throughout the dynamic response. The solution presented can then be applied to any multi-story base-isolated structure; however, it is counterproductive to apply base isolation to a structure with a very long period. Therefore, multi-story base isolation should be restricted to mid-height or shorter structures – for instance, up to eight or ten stories.

Bearing non-linearity is likely to occur in the dynamic response of a multi-story structure. Therefore, it is necessary to derive a solution for a multi-story structure that includes the methods presented in Chapter 3. The formulation for the multi-story structure presented in Chapter 4 was modified in a manner similar to that for the single story structure, incorporating Hilber's method and the plasticity theory into a comprehensive non-linear multi-story solution, presented in Chapter 5. As was the case in Chapter 3, the superstructure is assumed to remain linear throughout the response.

The four procedures presented in this thesis account for a wide range of structural response. Each formulation incorporates a torsional degree of freedom for each floor, which affects the one-dimensional response quantities. Additionally, each formulation allows for the use of friction-based bearings, which enhances the applicability of the solution. These methods will provide an accurate dynamic response for a wide variety of base-isolated structures, though further research is required to further enhance the analysis methods.

To evaluate the effectiveness of the analyses presented in this study, computer programs should be developed to perform the three-dimensional calculations. These results should be compared to one-dimensional results to determine the overall effect of the torsional degree-of-freedom. The contention of this study is that the effects are significant enough to require the use of the three-degree-of-freedom systems presented in the analyses in this thesis.

Another aspect of these analyses that can be improved through future work is seen in the work of Abe, et al (2004-a). It is apparent from the experimental results that the variation of an applied vertical load affects the response of the bearings; however, the model they present (2004-

b) does not incorporate the effects of vertical loading. Therefore, further research should be conducted to accurately model the effect of varying vertical loading on bearing properties, to account for the effects of vertical ground acceleration.

APPENDIX A

DERIVATION OF MASS AND STIFFNESS MATRICES

A.1 DETERMINATION OF MASS MATRIX

Given an arbitrary set of coordinates $OXYZ$ and floor centers of gravity G_i as shown in Figure 8, relationships can be developed between displacements along the arbitrary coordinate axis and a parallel axis through the center of gravity of each floor. The following definitions will be used in the derivation.

$G_i \equiv$ mass center of floor *i* (*b* for bearing floor, *1* for first floor)

$O_i \equiv$ origin of arbitrary coordinate axis

$u_i \equiv$ displacement of mass center G_i along $G_i X_i$

$v_i \equiv$ displacement of mass center G_i along $G_i Y_i$

$\theta_i \equiv$ rotational displacement of mass center G_i about $G_i Z_i$

$x_i \equiv$ displacement of floor *i* along $O_i X$

$y_i \equiv$ displacement of floor *i* along $O_i Y$

$e_i \equiv$ eccentricity between $G_i Y_i$ and $O_i Y$

$f_i \equiv$ eccentricity between $G_i X_i$ and $O_i X$

$e_b^1 \equiv$ eccentricity between $G_b Y_b$ and $G_1 Y_1$

$f_b^1 \equiv$ eccentricity between $G_b X_b$ and $G_1 X_1$

$m_i \equiv$ mass of floor *i*

$J_i \equiv$ mass moment of inertia of floor *i* with respect to its mass center

To derive a general formula for the mass matrix of a structure, a coordinate system is chosen arbitrarily at *O*. Therefore, transformations are required to express displacements with respect to this arbitrary axis as opposed to the floor mass center. The following equations represent a translation from the $O_1 XY$ coordinate system to the $G_1 X_1 Y_1$ coordinate system, assuming small rotations and slab rigidity:

$$u_1 = x_1 - f_1 \theta_1$$

$$v_1 = y_1 + e_1 \theta_1 \qquad \text{(A-1)}$$

$$\theta_1 = \theta_1$$

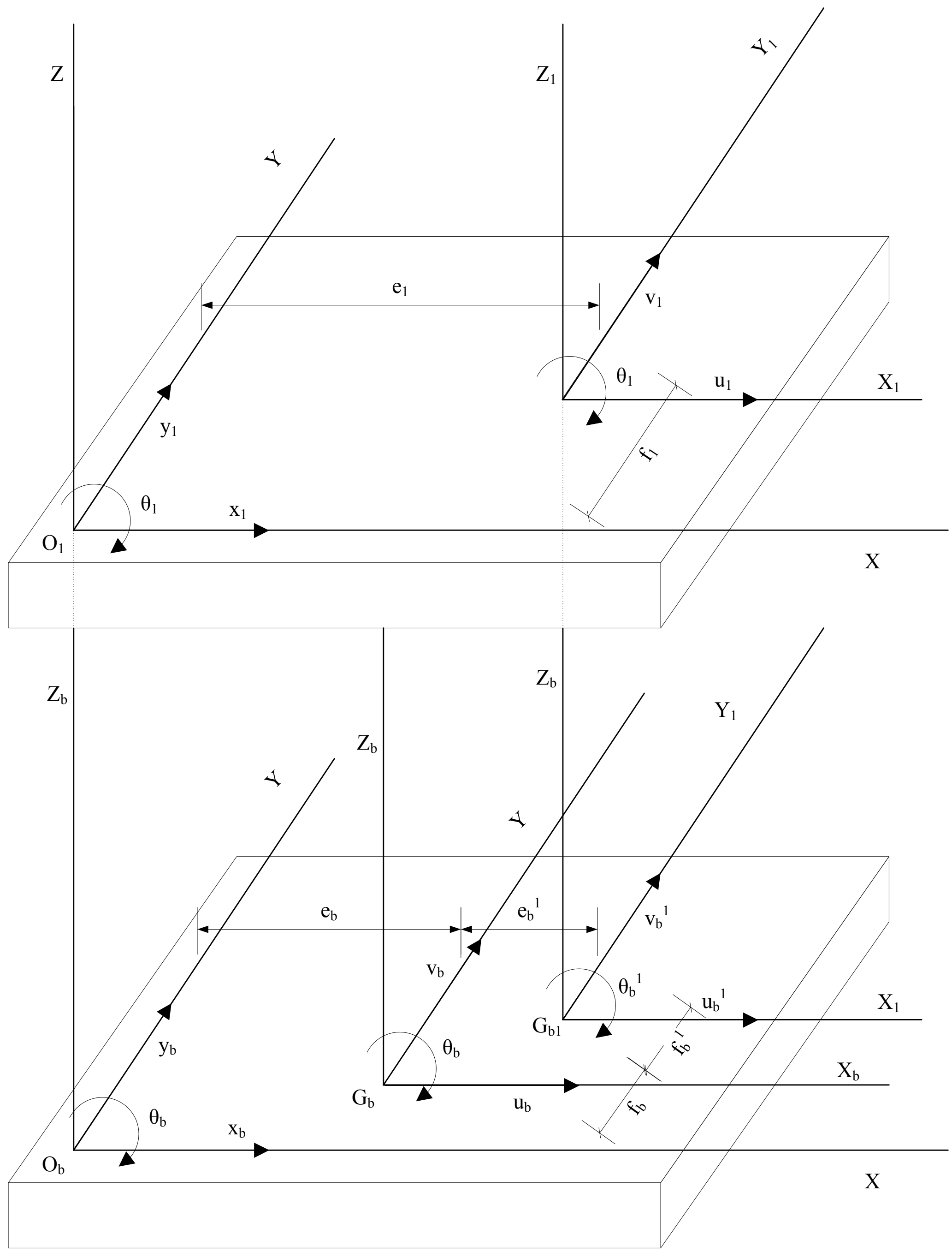

Figure 8 – Coordinate System

Similarly, the formulation for translating displacements from the $O_b XY$ coordinate system to the $G_b X_b Y_b$ coordinate system is as follows:

$$
\begin{aligned}
u_b &= x_b - f_b \theta_b \\
v_b &= y_b + e_b \theta_b \\
\theta_b &= \theta_b
\end{aligned}
\tag{A-2}
$$

Also, a relationship between the bearing floor and the first floor is desirable, since in this general formulation an allowance should be made for the floors to be non-concentric. The relationship between the $G_b X_b Y_b$ coordinate system and the $G_{b1} X_1 Y_1$ system is as shown in the following equations:

$$
\begin{aligned}
u_b^1 &= u_b - f_b^1 \theta_b \\
v_b^1 &= v_b + e_b^1 \theta_b \\
\theta_b^1 &= \theta_b
\end{aligned}
\tag{A-3}
$$

By substituting (A-2) into (A-3), the following relationship is developed, which represents a transformation from the $O_b XY$ system to the projection of the first floor axis on the bearing floor, the $G_{b1} X_1 Y_1$ system:

$$
\begin{aligned}
u_b^1 &= x_b - \left(f_b + f_b^1 \right) \theta_b \\
v_b^1 &= y_b + \left(e_b + e_b^1 \right) \theta_b \\
\theta_b^1 &= \theta_b
\end{aligned}
\tag{A-4}
$$

This expression for the relative first floor displacements will be used in the next step of the mass matrix derivation, in which the inertial forces are determined.

The inertial forces on any floor will act through the mass center of that floor. The first floor motions will be considered first. The motion of the first floor with respect to the fixed *OXY* coordinate system consists of three components: the motion of the ground, the motion of the bearings with respect to the ground, and the motion of the first floor with respect to the bearing floor. Writing these acceleration components into a series of equations, with ground rotation set to zero, yields the following expressions for the inertial forces acting upon the first floor:

$$\begin{aligned} Fx_{1G}^{I} &= m_1\left(\ddot{u}_1 + \ddot{u}_b^1 + \ddot{x}_g\right) \\ Fy_{1G}^{I} &= m_1\left(\ddot{v}_1 + \ddot{v}_b^1 + \ddot{y}_g\right) \\ F\theta_{1G}^{I} &= J_1\left(\ddot{\theta}_1 + \ddot{\theta}_b^1\right) \end{aligned} \qquad \text{(A-5)}$$

The next step is to substitute the derivatives of equations (A-1) and (A-4) into equation (A-5) to express all of the acceleration components in the *OXY* coordinate system. The constants e_1, f_1, e_b^1, and f_b^1 are unchanged in the derivatives. The inertial forces can now be written as

$$\begin{aligned} Fx_{1G}^{I} &= m_1\left(\left(\ddot{x}_1 - f_1\ddot{\theta}_1\right) + \left(\ddot{x}_b - \left(f_b + f_b^1\right)\ddot{\theta}_b\right) + \ddot{x}_g\right) \\ Fy_{1G}^{I} &= m_1\left(\left(\ddot{y}_1 + e_1\ddot{\theta}_1\right) + \left(\ddot{y}_b + \left(e_b + e_b^1\right)\ddot{\theta}_b\right) + \ddot{y}_g\right) \\ F\theta_{1G}^{I} &= J_1\left(\ddot{\theta}_1 + \ddot{\theta}_b\right) \end{aligned} \qquad \text{(A-6)}$$

By inspection of Figure 8, it can be seen that $f_1 = f_b + f_b^1$ and $e_1 = e_b + e_b^1$. Substituting these two relationships into equation (A-6) simplifies the formulations to

$$\begin{aligned} Fx_{1G}^{I} &= m_1\left(\left(\ddot{x}_1 - f_1\ddot{\theta}_1\right) + \left(\ddot{x}_b - f_1\ddot{\theta}_b\right) + \ddot{x}_g\right) \\ Fy_{1G}^{I} &= m_1\left(\left(\ddot{y}_1 + e_1\ddot{\theta}_1\right) + \left(\ddot{y}_b + e_1\ddot{\theta}_b\right) + \ddot{y}_g\right) \\ F\theta_{1G}^{I} &= J_1\left(\ddot{\theta}_1 + \ddot{\theta}_b\right) \end{aligned} \qquad \text{(A-7)}$$

These forces, though calculated using the *OXY* coordinates, act through the mass center of the floor. The superscripted *I* labels these forces as inertia. To formulate a mass matrix for any given structure, these forces must be transferred to the arbitrary *OXY* axes. Transferring these forces yields the following equations for the forces acting through the global origin *O*:

$$
\begin{aligned}
Fx_1^I &= Fx_{1G}^I \\
Fy_1^I &= Fy_{1G}^I \\
F\theta_1^I &= F\theta_{1G}^I - f_1 Fx_{1G}^I + e_1 Fy_{1G}^I
\end{aligned}
\tag{A-8}
$$

Since there are three equations of motion for each floor, it is convenient to express the system of equations in matrix form. This allows for more compact notation and greatly simplifies multistory calculations. Equation (A-8) for the first floor can be expressed as the following matrix equation:

$$\left\{F_1^I\right\} = \left[M_1\right]\left\{d_1\right\} + \left[M_1\right]\left\{d_b\right\} + \left[M_1\right]\left\{d_g\right\} \tag{A-9}$$

in which

$$\left\{F_1^I\right\} = \begin{Bmatrix} Fx_1^I \\ Fy_1^I \\ F\theta_1^I \end{Bmatrix} \qquad \left[M_1\right] = \begin{bmatrix} m_1 & 0 & -m_1 f_1 \\ 0 & m_1 & m_1 e_1 \\ -m_1 f_1 & m_1 e_1 & J_1 + m_1\left(f_1^2 + e_1^2\right) \end{bmatrix} \tag{A-10}$$

$$\left\{\ddot{d}_1\right\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\theta}_1 \end{Bmatrix} \qquad \left\{\ddot{d}_b\right\} = \begin{Bmatrix} \ddot{x}_b \\ \ddot{y}_b \\ \ddot{\theta}_b \end{Bmatrix} \qquad \left\{\ddot{d}_g\right\} = \begin{Bmatrix} \ddot{x}_g \\ \ddot{y}_g \\ \ddot{\theta}_g \end{Bmatrix} \tag{A-11}$$

The derivation of the inertial forces on the bearing floor is similar to that of the first floor. However, now the displacement is composed of only two elements, the motion of the ground and the motion of the bearing floor with respect to the ground. The equations for the inertial forces acting through the center of mass are:

$$Fx_{bG}^{1} = m_b\left(\ddot{u}_b + \ddot{x}_g\right)$$

$$Fy_{bG}^{1} = m_b\left(\ddot{v}_b + \ddot{x}_g\right) \qquad \text{(A-12)}$$

$$F\theta_{bG}^{1} = J_b\ddot{\theta}_b$$

Then, as was the case for the first floor derivation, these forces must be translated to the global *OXY* axis. The force transformation is as follows:

$$Fx_b^I = Fx_{bG}^I$$

$$Fy_b^I = Fy_{bG}^I \qquad \text{(A-13)}$$

$$F\theta_b^I = F\theta_{bG}^I - f_b Fx_{bG}^I + e_b Fy_{bG}^I$$

Expressing equation (A-13) in matrix form:

$$\left\{F_b^I\right\} = \left[M_b\right]\left\{\ddot{d}_b\right\} + \left[M_b\right]\left\{\ddot{d}_g\right\} \qquad \text{(A-14)}$$

in which

$$\left\{F_b^I\right\} = \begin{Bmatrix} Fx_b^I \\ Fy_b^I \\ F\theta_b^I \end{Bmatrix} \qquad \left[M_b\right] = \begin{bmatrix} m_b & 0 & -m_b f_b \\ 0 & m_b & m_b e_b \\ -m_b f_b & m_b e_b & J_b + m_b\left(f_b^2 + e_b^2\right) \end{bmatrix} \qquad \text{(A-15)}$$

Now inertial forces have been defined for both the first floor and the bearing floor of a single-story base-isolated structure. A multi-story structure would have a series of equations like those of the first floor, which would then require additional floor-to-floor relative displacements as in equation (A-3).

The mass matrices to be used in the calculation of structural response are given in equations (A-10) and (A-15), representing the first floor and the bearing floor, respectively.

A.2 DETERMINATION OF STIFFNESS MATRIX

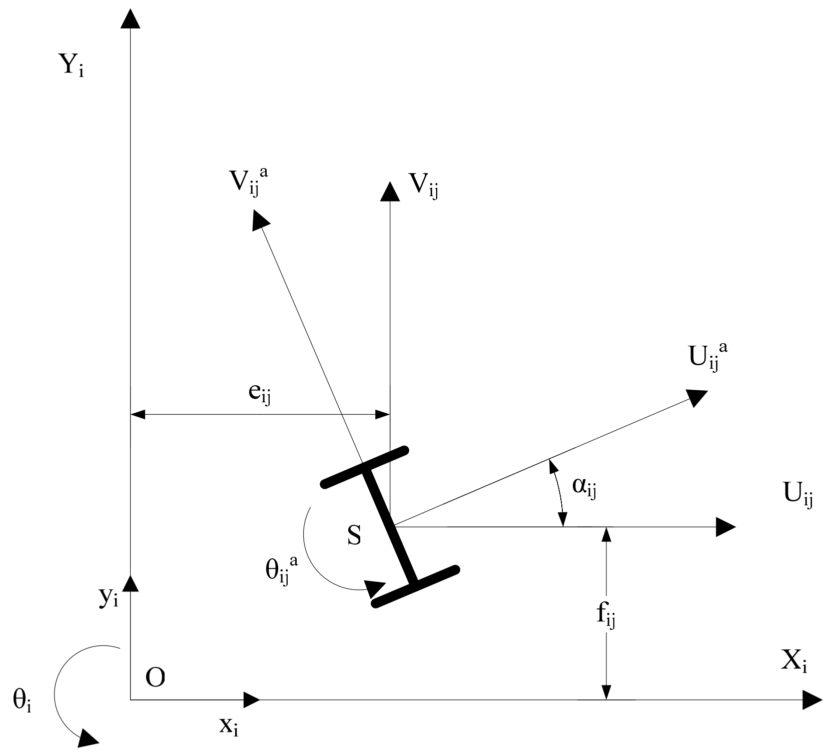

Figure 9 – Stiffness Element Coordinate System

Derivation of the stiffness matrix for a structure is typically more complex than that of the mass matrix. Whereas in the formulation of the mass matrix there was only one coordinate transformation required for each floor, the stiffness matrix demands a calculation for each element contributing to the stiffness of each floor. Figure 9 shows an individual element *j* on floor *i* of the structure. A general formulation is shown here, allowing any orientation of the stiffening elements.

First, displacements of an element in the local coordinates of that element, the $SU_{ij}^aV_{ij}^a$ axis, must be determined. These displacements can be related to the displacements of the element in the $SU_{ij}V_{ij}$ axis as follows:

$$u_{ij}^a = u_{ij}\cos\alpha_{ij} + v_{ij}\sin\alpha_{ij}$$

$$v_{ij}^a = -u_{ij}\sin\alpha_{ij} + v_{ij}\cos\alpha_{ij} \tag{A-16}$$

$$\theta_{ij}^a = \theta_{ij}$$

In matrix form, this set of equations becomes

$$\left\{\delta_{ij}^a\right\} = \left[T_{ij}\right]\left\{\delta_{ij}\right\} \tag{A-17}$$

Expanding the matrix equation:

$$\begin{Bmatrix} u_{ij}^a \\ v_{ij}^a \\ \theta_{ij}^a \end{Bmatrix} = \begin{bmatrix} \cos\alpha_{ij} & \sin\alpha_{ij} & 0 \\ -\sin\alpha_{ij} & \cos\alpha_{ij} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{ij} \\ v_{ij} \\ \theta_{ij} \end{Bmatrix} \tag{A-18}$$

The force-displacement relationship for the element in the $SU_{ij}^aV_{ij}^a$ coordinate system can be written as follows:

$$\left\{F_{ij}^a\right\} = \left[K_{ij}^a\right]\left\{\delta_{ij}^a\right\} \tag{A-19}$$

Expanding this matrix equation yields:

$$\begin{Bmatrix} F_{iju}^a \\ F_{ijv}^a \\ F_{ij\theta}^a \end{Bmatrix} = \begin{bmatrix} K_{ijxx}^a & K_{ijxy}^a & K_{ijx\theta}^a \\ K_{ijyx}^a & K_{ijyy}^a & K_{ijy\theta}^a \\ K_{ij\theta x}^a & K_{ij\theta y}^a & K_{ij\theta\theta}^a \end{bmatrix} \begin{Bmatrix} u_{ij}^a \\ v_{ij}^a \\ \theta_{ij}^a \end{Bmatrix} \tag{A-20}$$

However, the local displacements have already been determined in equations (A-17) and (A-18) as a function of the global displacements. Therefore, equation (A-19) can be rewritten as

$$\left\{F_{ij}^a\right\} = \left[K_{ij}^a\right]\ \left[T_{ij}\right]\left\{\delta_{ij}\right\} \tag{A-21}$$

The forces then must be transferred to the $SU_{ij}V_{ij}$ coordinate system. The relationships are very similar to those of the displacements. The transformation matrix used for the forces is the transpose of that used for the displacements; hence,

$$\{F_{ij}\} = [T_{ij}]^T \{F_{ij}^a\} \tag{A-22}$$

or, in expanded notation,

$$\begin{Bmatrix} F_{ijx} \\ F_{ijy} \\ F_{ij\theta} \end{Bmatrix} = \begin{bmatrix} \cos\alpha_{ij} & -\sin\alpha_{ij} & 0 \\ \sin\alpha_{ij} & \cos\alpha_{ij} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_{iju}^a \\ F_{ijv}^a \\ F_{ij\theta}^a \end{Bmatrix} \tag{A-23}$$

Now, by substituting equation (A-21) into equation (A-22), the global force-displacement relationship is determined in the $SU_{ij}V_{ij}$ coordinate system:

$$\{F_{ij}\} = [T_{ij}]^T [K_{ij}^a][T_{ij}]\{\delta_{ij}\} \tag{A-24}$$

As mentioned previously, this formulation is done for each member *j* on each floor *i*. Therefore, the contributions of each member must be summed to determine the total stiffness of each floor, as follows:

$$F_{ix} = \sum_{j=1}^{n} F_{ijx}$$

$$F_{iy} = \sum_{j=1}^{n} F_{ijy} \tag{A-25}$$

$$F_{i\theta} = \sum \left(- f_{ij} F_{ijx} + e_{ij} F_{ijy} + F_{ij\theta}\right)$$

Putting these equations into matrix form yields the following:

$$\{F_i\} = \sum_{j=1}^{n} [A_{ij}]^T \{F_{ij}\} \tag{A-26}$$

in which

$$\{F_i\} = \begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{i\theta} \end{Bmatrix} \quad [A_{ij}] = \begin{bmatrix} 1 & 0 & -f_{ij} \\ 0 & 1 & e_{ij} \\ 0 & 0 & 1 \end{bmatrix} \tag{A-27}$$

The forces $\{F_i\}$ now act through the origin O of the arbitrary OXY axis. However, the displacements were last written in terms of the $SU_{ij}V_{ij}$ coordinate system. These displacements can be transferred to the OXY axis as follows:

$$u_{ij} = x_{ij} - f_{ij}\theta_i$$

$$v_{ij} = y_{ij} + e_{ij}\theta_i \tag{A-28}$$

$$\theta_{ij} = \theta_i$$

Putting these equations into matrix form yields the following:

$$\{\delta_{ij}\} = [A_{ij}]\{d_i\} \tag{A-29}$$

Now both the forces and the displacements are formulated in the OXY coordinate system. Therefore, the force-displacement relationship can be written in that system as follows:

$$\{F_i\} = [K_i]\{d_i\} \tag{A-30}$$

in which

$$[K_i] = \sum_{j=1}^{n} [A_{ij}]^T [T_{ij}]^T [K_{ij}^a][T_{ij}][A_{ij}] \tag{A-31}$$

$$[K_i] = \begin{bmatrix} K_{ixx} & K_{ixy} & K_{ix\theta} \\ K_{iyx} & K_{iyy} & K_{iy\theta} \\ K_{i\theta x} & K_{i\theta y} & K_{i\theta\theta} \end{bmatrix} \tag{A-32}$$

Performing the matrix multiplication shown in equation (A-31) and simplifying the format of the equations with the following abbreviations

$$C = \cos\alpha_{ij} \qquad S = \sin\alpha_{ij} \tag{A-33}$$

yields the following values for the individual elements of the stiffness matrix $[K_i]$:

$$K_{ixx} = \sum_{j=1}^{n} \left(C^2 K_{ijuu}^a + S^2 K_{ijvv}^a - 2CSK_{ijuv}^a \right) \tag{A-34}$$

$$K_{iyy} = \sum_{j=1}^{n} \left(S^2 K^a_{ijuu} + C^2 K^a_{ijvv} + 2CSK^a_{ijuv} \right) \tag{A-35}$$

$$K_{ixy} = \sum_{j=1}^{n} \left(CS\left(K^a_{ijuu} - K^a_{ijvv}\right) + \left(C^2 - S^2\right) K^a_{ijuv} \right) \tag{A-36}$$

$$K_{ix\theta} = \sum_{j=1}^{n} \begin{pmatrix} -f_{ij}\left(C^2 K^a_{ijuu} + S^2 K^a_{ijvv}\right) + e_{ij} CS\left(K^a_{ijuu} - K^a_{ijvv}\right) \\ +\left(2 f_{ij} CS + e_{ij}\left(C^2 - S^2\right)\right) K^a_{ijuv} + CK^a_{iju\theta} - SK^a_{ijv\theta} \end{pmatrix} \tag{A-37}$$

$$K_{iy\theta} = \sum_{j=1}^{n} \begin{pmatrix} e_{ij}\left(S^2 K^a_{ijuu} + C^2 K^a_{ijvv}\right) - f_{ij} CS\left(K^a_{ijuu} - K^a_{ijvv}\right) \\ +\left(2e_{ij} CS - f_{ij}\left(C^2 - S^2\right)\right) K^a_{ijuv} + SK^a_{iju\theta} + CK^a_{ijv\theta} \end{pmatrix} \tag{A-38}$$

$$K_{i\theta\theta} = \sum_{j=1}^{n} \begin{pmatrix} \left(-f_{ij} C + e_{ij} S\right)^2 K^a_{ijuu} + \left(f_{ij} S + e_{ij} C\right)^2 K^a_{ijvv} \\ +2\left(\left(f_{ij} S + e_{ij} C\right)\left(-f_{ij} C + e_{ij} S\right)\right) K^a_{ijuv} \\ +2\left(-f_{ij} C + e_{ij} S\right) K^a_{iju\vartheta} + 2\left(f_{ij} S + e_{ij} C\right) K^a_{ijv\theta} + K^a_{ij\theta\theta} \end{pmatrix} \tag{A-39}$$

$$K_{iyx} = K_{ixy} \tag{A-40}$$

$$K_{i\theta x} = K_{ix\theta} \tag{A-41}$$

$$K_{i\theta y} = K_{iy\theta} \tag{A-42}$$

These formulae represent the contribution of member *j* to the overall stiffness of floor *i*. The results of equations (A-34) through (A-42), when combined into a single matrix, form the stiffness matrix for floor *i*.

A.3 DETERMINATION OF THE SHEAR CENTER LOCATION

The restoring force at floor *i*, equal to the stiffness matrix multiplied by the displacement vector, acts through the shear center of the floor. Therefore, the location of the shear center must be determined to properly translate the forces to the *OXY* axis to formulate the equation of motion for the structure. Figure 10 shows the shear center S_c of floor *i*. Note that the angle α is now considered clockwise positive, unlike the angle α_{ij} in the previous section.

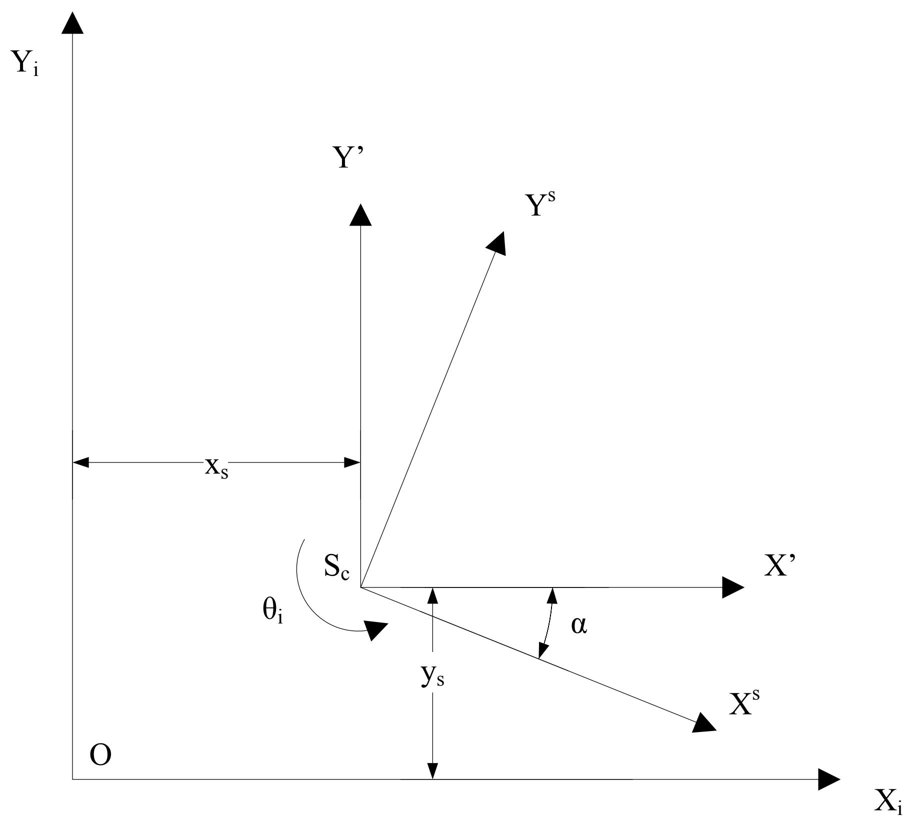

Figure 10 – Shear Center Coordinate System

To translate the shear forces from the shear center to the origin, first the forces will be transformed to the $S_c X'Y'$ coordinates, which are parallel to the global coordinates with an origin at the shear center. Then the forces will be translated to the origin of the global axes. The same procedure will be done with the displacements.

As shown in Figure 10, the following definitions for force vectors will be used to transfer the forces to the global axes:

$$\left\{F_i^s\right\} = \begin{Bmatrix} F_{ix}^s \\ F_{iy}^s \\ F_{i\theta}^s \end{Bmatrix} \qquad \left\{F_i'\right\} = \begin{Bmatrix} F_{ix}' \\ F_{iy}' \\ F_{i\theta}' \end{Bmatrix} \qquad \left\{F_i\right\} = \begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{i\theta} \end{Bmatrix} \tag{A-43}$$

To transform the forces from the primary shear axis, $S_c X^s Y^s$, to the $S_c X'Y'$ coordinate system, the following equations may be written:

$$\begin{aligned} F_{ix}' &= F_{ix}^s \cos\alpha + F_{iy}^s \sin\alpha \\ F_{iy}' &= -F_{ix}^s \sin\alpha + F_{iy}^s \cos\alpha \\ F_{i\theta}' &= F_{i\theta}^s \end{aligned} \tag{A-44}$$

The same equations can be written for the displacements, since the forces and displacements are assumed to be in the same directions. Equations (A-44) can then be rewritten in matrix form for both forces and displacements, recalling equations (A-43). These matrix equations are

$$\left\{F_i'\right\} = \left[T\right]\left\{F_i^s\right\} \qquad \left\{d_i'\right\} = \left[T\right]\left\{d_i^s\right\} \tag{A-45}$$

in which

$$\left[T\right] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A-46}$$

Then these forces must be translated from the shear center to the global origin. The equations to be used for this are similar to those presented in equations (A-25), though the notation is different. The translation is:

$$\begin{aligned} F_{ix} &= F_{ix}' \\ F_{iy} &= F_{iy}' \end{aligned} \tag{A-47}$$

$$F_{i\theta} = -y_s F'_{ix} + x_s F'_{iy} + F'_{i\theta}$$

Equations (A-47), again for both forces and displacements, can be put into matrix form as

$$\{F_i\} = [A']^T \{F'_i\} \qquad \{d_i\} = [A']^T \{d'_i\} \tag{A-48}$$

in which

$$[A'] = \begin{bmatrix} 1 & 0 & -y_s \\ 0 & 1 & x_s \\ 0 & 0 & 1 \end{bmatrix} \tag{A-49}$$

Now that the various force and displacement transformations have been derived, the force-displacement relationship must be developed. The starting point will be the simple force-displacement relationship for floor i, repeated from equation (A-30)…

$$\{F_i\} = [k_i]\{d_i\} \tag{A-50}$$

Now equations (A-48) are substituted into each side of the equation…

$$[A']^T \{F'_i\} = [k_i][A']^T \{d'_i\} \tag{A-51}$$

Premultiplying each side of the equation by $[A']$ and noting that, by definition, $[A'][A']^T = [I]$, the following result is obtained…

$$\{F'_i\} = [A'][k_i][A']^T \{d'_i\} \tag{A-52}$$

Substituting equations (A-45) into each side of the equation…

$$[T]\{F_i^s\} = [A'][k_i][A']^T [T]\{d_i^s\} \tag{A-51}$$

Premultiplying by $[T]^T$ to eliminate the $[T]$ term from the left side leaves only the force-displacement relationship for the shear center of floor i…

$$\{F_i^s\} = [k_i^s]\{d_i^s\} \tag{A-52}$$

in which

$$\left[k_i^s\right] = \left[T\right]^T \left[A'\right] \left[k_i\right] \left[A'\right]^T \left[T\right] \tag{A-53}$$

In a more expanded form, the matrix equation shown in equation (A-52) can be written as

$$\begin{Bmatrix} F_{ix}^s \\ F_{iy}^s \\ F_{i\theta}^s \end{Bmatrix} = \begin{bmatrix} k_{ixx}^s & k_{ixy}^s & k_{ix\theta}^s \\ k_{iyx}^s & k_{iyy}^s & k_{iy\theta}^s \\ k_{i\theta x}^s & k_{i\theta y}^s & k_{i\theta\theta}^s \end{bmatrix} \begin{Bmatrix} x_i^s \\ y_i^s \\ \theta_i^s \end{Bmatrix} \tag{A-54}$$

in which

$$k_{ixx}^s = C^2 k_{ixx} + S^2 k_{iyy} - 2CSk_{ixy} \tag{A-55}$$

$$k_{ixy}^s = CS\left(k_{ixx} - k_{iyy}\right) + \left(C^2 - S^2\right) k_{ixy} \tag{A-56}$$

$$k_{ix\theta}^s = Cy_s k_{ixx} + Sx_s k_{iyy} - \left(Cx_s + Sy_s\right) k_{ixy} + Ck_{ix\theta} - Sk_{iy\theta} \tag{A-57}$$

$$k_{iyy}^s = S^2 k_{ixx} + C^2 k_{iyy} + 2CSk_{ixy} \tag{A-58}$$

$$k_{iy\theta}^s = Sy_s k_{ixx} - Cx_s k_{iyy} + \left(Cy_s - Sx_s\right) k_{ixy} + Sk_{ix\theta} + Ck_{iy\theta} \tag{A-59}$$

$$k_{i\theta\theta}^s = y_s^2 k_{ixx} + x_s^2 k_{iyy} - 2x_s y_s k_{ixy} + 2y_s k_{ix\theta} - 2x_s k_{iy\theta} + k_{i\theta\theta} \tag{A-60}$$

$$k_{iyx}^s = k_{ixy}^s \tag{A-61}$$

$$k_{i\theta x}^s = k_{ix\theta}^s \tag{A-62}$$

$$C = \cos\alpha \qquad\qquad S = \sin\alpha \tag{A-63}$$

Because the force-displacement relationship in equation (A-54) is centered about the shear center of floor *i*, the stiffness matrix must be decoupled by definition. In other words, the off-diagonal terms (i.e. the terms other than k_{ixx}^s, k_{iyy}^s, or $k_{i\theta\theta}^s$) must equal zero. A series of mathematical operations are required to set the k_{ixy}^s term to zero. First, define an angle β such that

$$\sin 2\beta = \frac{2k_{ixy}}{\sqrt{\left(k_{iyy} - k_{ixx}\right)^2 + 4k_{ixy}^2}} = \frac{2k_{ixy}}{D} \tag{A-64}$$

in which

$$D = \sqrt{\left(k_{iyy} - k_{ixx}\right)^2 + \left(2k_{ixy}\right)^2} \tag{A-65}$$

From trigonometry, $\left(\cos 2\beta\right)^2 = 1 - \left(\sin 2\beta\right)^2$. Given this identity, equation (A-64) can be written as

$$\cos 2\beta = \frac{k_{iyy} - k_{ixx}}{\sqrt{\left(k_{iyy} - k_{ixx}\right)^2 + 4k_{ixy}^2}} = \frac{k_{iyy} - k_{ixx}}{D} \tag{A-66}$$

Multiplying equation (A-56) by $D/2$ gives the following equation:

$$\frac{2k_{ixy}^s}{D} = 2\cos\alpha\sin\alpha\frac{\left(k_{ixx} - k_{iyy}\right)}{D} + \left(\cos^2\alpha - \sin^2\alpha\right)\frac{2k_{ixy}}{D} \tag{A-67}$$

Substituting equations (A-64) and (A-66) into equation (A-67)…

$$\frac{2k_{ixy}^s}{D} = 2\cos\alpha\sin\alpha\left(-\cos 2\beta\right) + \left(\cos^2\alpha - \sin^2\alpha\right)\left(\sin 2\beta\right) \tag{A-68}$$

Recalling the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, equation (A-68) can be rewritten as

$$\frac{2k_{ixy}^s}{D} = -\sin 2\alpha\cos 2\beta + \cos 2\alpha\sin 2\beta \tag{A-69}$$

However, since $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$, equation (A-69) can be simplified to the following:

$$k_{ixy}^s = \frac{D}{2}\sin\left(2\beta - 2\alpha\right) \tag{A-70}$$

By inspection, k_{ixy}^s will only equal zero when $\beta = \alpha$.

In addition to setting the k_{ixy}^{s} term to zero, $k_{ix\theta}^{s}$ and $k_{iy\theta}^{s}$ must also equal zero. By solving equations (A-57) and (A-59) simultaneously in terms of x_s and y_s, the following expressions can be obtained:

$$x_s = \frac{k_{ixx}\,k_{iy\theta} - k_{ixy}\,k_{ix\theta}}{k_{ixx}\,k_{iyy} - k_{ixy}^{2}} \tag{A-71}$$

$$y_s = \frac{k_{ixy}\,k_{iy\theta} - k_{iyy}\,k_{ix\theta}}{k_{ixx}\,k_{iyy} - k_{ixy}^{2}} \tag{A-72}$$

Now the location of the shear center of floor *i* can be determined with respect to the global coordinate axes via equations (A-71) and (A-72). Also, the orientation *α* of the shear center can be determined via equations (A-64), (A-65), and (A-66).

Using equations (A-71) and (A-72), a simplified expression for $k_{i\theta\theta}^{s}$ can be written. The torsional stiffness equation, (A-60), becomes

$$k_{i\theta\theta}^{s} = k_{i\theta\theta} + y_s k_{ix\theta} - x_s k_{iy\theta} \tag{A-73}$$

Now, by using the general stiffness matrix from equation (A-32), the stiffness matrix for the shear center of floor *i* can be populated using equations (A-55), (A-58), and (A-73). The final matrix is

$$\begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{i\theta} \end{Bmatrix} = \begin{bmatrix} K_{ixx} & 0 & 0 \\ 0 & K_{iyy} & 0 \\ 0 & 0 & K_{i\theta\theta} \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ \theta_i \end{Bmatrix} \tag{A-74}$$

APPENDIX B

FRICTION

B.1 ADDITIONAL CONSIDERATIONS

The solution methods presented in Chapters 2-5 of this thesis allow for the inclusion of bearing friction in the structural response. If friction is to be considered in the solution, it will create non-linearities in the behavior which is not fully described in the previously presented solutions. Therefore, a detailed investigation of the frictional effects is undertaken here.

The frictional force is represented here by multiplying the total vertical acceleration (which is equal to the vertical ground acceleration plus gravitational acceleration g) times the mass matrix times the frictional constant μ. However, elementary physics introduces two separate values for μ, one for static friction and one for kinetic friction. The static coefficient μ_s represents the resistance to the onset of motion. The kinetic coefficient μ_k represents the resistance to continuing motion. Both of these coefficients will be required in the dynamic response of a structure. By definition, due to the reversal of direction of the motion, there are times when the velocity of the structure is reduced to zero. These conditions are defined as "non-sliding" phases in which the structure must overcome the static frictional force to return to motion.

To properly account for the non-linearities in the response due to friction, it is necessary to determine when the non-sliding phases occur. The method presented here uses the equations of motion to determine whether the frictional resistance to motion will overcome the dynamic forces on the structure. This formulation stems from the one-dimensional work of Mostaghel & Khodaverdian (1988).

B.2 APPLICATION TO A SINGLE-STORY STRUCTURE

By definition, a non-sliding phase is one in which the velocity of the bearing level is zero. Also, since the maximum static frictional force is assumed to be greater than the impelling forces, the acceleration of the bearings is also zero in non-sliding phases. Mathematically, a non-sliding phase can then be defined using equation (2-6), repeated here for convenience:

$$\begin{aligned}&[M_t]\{\ddot{d}_b\}+[C_b]\{\dot{d}_b\}+[K_b]\{d_b\}=\\&\quad-[M_1]\{\ddot{d}_1\}-[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\mathrm{sgn}(\dot{d}_b)\}\end{aligned} \qquad \text{(B-1)}$$

As mentioned, however, the accelerations and velocities of the bearing level, for each degree of freedom that is in a non-sliding phase, must be zero by definition. Therefore, equation (B-1) becomes:

$$[K_b]\{d_b\}=[M_1]\{\ddot{d}_1\}-[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\mathrm{sgn}(\dot{d}_b)\} \qquad \text{(B-2)}$$

Rearranging the terms to isolate the frictional term:

$$\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\mathrm{sgn}(\dot{d}_b)\}=-[M_1]\{\ddot{d}_1\}-[M_t]\{\ddot{d}_g\}-[K_b]\{d_b\} \qquad \text{(B-3)}$$

The $\{\mathrm{sgn}(\dot{d}_b)\}$ vector, as defined in Chapter 2, is composed of entirely positive and negative unit values. Therefore, by taking the absolute value of equation (B-3), the following result is obtained:

$$\left|\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{1\}\right| = \left|[M_1]\{\ddot{d}_1\}+[M_t]\{\ddot{d}_g\}+[K_b]\{d_b\}\right| \qquad \text{(B-4)}$$

in which $\{1\} \equiv \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, representing the absolute value of the $\{\mathrm{sgn}(\dot{d}_b)\}$ vector.

By definition, if the friction force is greater than the impelling forces for a degree of freedom, the structure is in a non-sliding phase for that degree of freedom. Also, the structure will generally start in a non-sliding phase. The condition for a non-sliding phase is as follows:

$$\left|\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{1\}\right| > \left|[M_1]\{\ddot{d}_1\}+[M_t]\{\ddot{d}_g\}+[K_b]\{d_b\}\right| \qquad \text{(B-5)}$$

Note that equation (B-5) is a matrix expression, representing three equations, one for each degree of freedom. Each equation must be evaluated to determine whether that particular degree of freedom will be in a sliding phase or a non-sliding phase. The three degrees of freedom, in order, are the x-direction, y-direction, and rotation.

Equation (B-5) can be used with the kinetic frictional coefficient following a sliding phase or with the static coefficient following a non-sliding phase. The structure will remain in a non-sliding phase until the impelling forces overcome the static frictional force. The condition to enter a sliding phase is as follows:

$$\left|\mu_s\left(g+\ddot{d}_{gz}\right)[M_t]\{1\}\right| < \left|[M_1]\{\ddot{d}_1\}+[M_t]\{\ddot{d}_g\}+[K_b]\{d_b\}\right| \qquad \text{(B-6)}$$

Again, this equation must be evaluated separately for each degree of freedom. If equation (B-6) is satisfied for one of those equations, that degree of freedom will enter a sliding phase.

The structure will then remain in a sliding phase until equation (B-5) is satisfied, using the coefficient of kinetic friction. If time period t_i has been established as a sliding phase, then the solutions presented in Chapter 2 and Chapter 3 can be applied to that time step. If time period t_i has been established as a non-sliding phase, a different solution method, presented below, is required to determine the superstructure response. The bearing response for a non-sliding phase is trivial, as the displacement is unchanged and both the velocity and acceleration vectors are zero vectors.

The solution procedure for the first floor in a non-sliding phase begins by evaluating equation (3-18) at time τ. Recall that equation (3-18) was derived from equation (2-58).

$$\ddot{z}_{1n}\left(t_{i+1}\right)+2\xi_{1n}\omega_{1n}\dot{z}_{1n}\left(t_{i+1}\right)+\omega_{1n}^{2}z_{1n}\left(t_{i+1}\right)= \sum_{k=1}^{3}\alpha_{1nk}\ddot{d}_{bk}\left(t_{i+1}\right)+\sum_{k=1}^{3}\alpha_{1nk}\ddot{d}_{gk}\left(t_{i+1}\right) \tag{B-7}$$

By the definition of a non-sliding phase, the acceleration and velocity of the bearing level is zero. Therefore, equation (B-7) can be written as:

$$\ddot{z}_{1n}\left(\tau\right)+2\xi_{1n}\omega_{1n}\dot{z}_{1n}\left(\tau\right)+\omega_{1n}^{2}z_{1n}\left(\tau\right)=\sum_{k=1}^{3}\alpha_{1nk}\ddot{d}_{gk}\left(\tau\right) \tag{B-8}$$

By the definition presented in equation (2-29), the following expression can be obtained:

$$\ddot{z}_{1n}\left(\tau\right)+2\xi_{1n}\omega_{1n}\dot{z}_{1n}\left(\tau\right)+\omega_{1n}^{2}z_{1n}\left(\tau\right)= \sum_{k=1}^{3}\alpha_{1nk}\ddot{d}_{gk}\left(t_{i}\right)+\left(\sum_{k=1}^{3}\alpha_{1nk}\Delta\ddot{d}_{gk}\left(t_{i+1}\right)\right)\frac{\tau}{\Delta t} \tag{B-9}$$

This can be written in simpler terms as follows:

$$\ddot{z}_{1n}\left(\tau\right)+2\xi_{1n}\omega_{1n}\dot{z}_{1n}\left(\tau\right)+\omega_{1n}^{2}z_{1n}\left(\tau\right)=A1_{ni}+B1_{ni+1}\frac{\tau}{\Delta t} \tag{B-10}$$

in which

$$A1_{ni} = \sum_{k=1}^{3} \alpha_{1nk} \ddot{d}_{gk}\left(t_i\right) \tag{B-11}$$

$$B1_{ni} = \sum_{k=1}^{3} \alpha_{1nk} \Delta\ddot{d}_{gk}\left(t_{i+1}\right) \tag{B-12}$$

The solution to equation (B-10) can be written as a combination of a complementary solution and a particular solution, as follows:

$$z_{1n}\left(\tau\right) = z_{1n}^{c}\left(\tau\right) + z_{1n}^{p}\left(\tau\right) \tag{B-13}$$

The complementary solution is of the same form as equation (2-33), though the solution is now for the first floor instead of the bearing level:

$$z_{1n}^{c} = e^{-\xi_{1n}\omega_{1n}\tau}\left(C1_n \sin\Omega_{1n}\tau + C2_n \cos\Omega_{1n}\tau\right) \tag{B-14}$$

in which

$$\Omega_{1n} = \omega_{1n}\sqrt{1-\xi_{1n}^{2}} \tag{B-15}$$

The particular solution is similar to that found in equation (2-35):

$$z_{1n}^{p} = C3_n + C4_n \frac{\tau}{\Delta t} \tag{B-16}$$

The constants in the particular solution can be determined as they were in Chapter 2. Those constant values are as follows:

$$C3_n = \frac{1}{\omega_{1n}^{2}}\left(A1_{ni} - \frac{2\xi_{1n}}{\omega_{1n}\,\Delta t} B1_{ni+1}\right) \tag{B-16}$$

$$C4_n = \frac{B1_{ni+1}}{\omega_{1n}^{2}} \tag{B-17}$$

Substituting equations (B-14) and (B-16) into equation (B-13) yields the following expression:

$$z_{1n}(\tau) = e^{-\xi_{1n}\,\omega_{1n}\,\tau}\left(C1_n \sin\Omega_{1n}\tau + C2_n \cos\Omega_{1n}\tau\right) + \frac{1}{\omega_{1n}^2}\left(A1_{ni} + \left(\tau - \frac{2\xi_{1n}}{\omega_{1n}}\right)\frac{B1_{ni+1}}{\Delta t}\right) \tag{B-18}$$

As in Chapter 2, the constants $C1_n$ and $C2_n$ will be solved by using the following identities:

$$z_{1n}(\tau = 0) = z_{1n}(t_i) \tag{B-19}$$

$$\dot{z}_{1n}(\tau = 0) = \dot{z}_{1n}(t_i) \tag{B-20}$$

Then the constant values can be determined as:

$$C1_n = \frac{1}{\Omega_{1n}}\left(\dot{z}_{1n}(t_i) + \xi_{1n}\,\omega_{1n}\,z_{1n}(t_i) - \frac{\xi_{1n}}{\omega_{1n}}A1_{ni} - \frac{\left(1 - 2\,\xi_{1n}^2\right)}{\omega_{1n}^2\,\Delta t}B1_{n\,i+1}\right) \tag{B-21}$$

$$C2_n = z_{1n}(t_i) - \frac{A1_{ni}}{\omega_{1n}^2} + \frac{2\,\xi_{1n}}{\omega_{1n}^3\,\Delta t}B1_{n\,i+1} \tag{B-22}$$

Now that the constants in the displacement function are known, it is a simple matter of derivation to determine the expressions for velocity and acceleration. By substituting equations (B-21) and (B-22) back into equation (B-18), the following expressions for displacement, velocity, and acceleration, respectively, can be written at time $\tau = \Delta t$:

$$z_{1n}(t_{i+1}) = D_{ni} + R1_n\,B1_{n\,i+1} + \frac{1}{\omega_{1n}^2}\left(A1_{ni} + \left(\Delta t - \frac{2\,\xi_{1n}}{\omega_{1n}}\right)\frac{B1_{n\,i+1}}{\Delta t}\right) \tag{B-23}$$

$$\dot{z}_{1n}(t_{i+1}) = \left(G_{ni} - \xi_{1n}\,\omega_{1n}\,D_{ni}\right) + \left(R2_n - \xi_{1n}\,\omega_{1n}\,R1_n\right)B1_{n\,i+1} + \frac{B1_{n\,i+1}}{\omega_{1n}^2\,\Delta t} \tag{B-24}$$

$$\ddot{z}_{1n}(t_{i+1}) = -H_{ni} - R3_n\,B1_{n\,i+1} \tag{B-25}$$

in which

$$D_{ni} = \frac{e^{-\xi_{1n}\,\omega_{1n}\,\Delta t}}{\Omega_{1n}}\left(E_{ni}\sin\Omega_{1n}\,\Delta t + F_{ni}\cos\Omega_{1n}\,\Delta t\right) \tag{B-26}$$

$$E_{ni} = \dot{z}_{1n}(t_i) + \xi_{1n}\,\omega_{1n}\,z_{1n}(t_i) - \frac{\xi_{1n}}{\omega_{1n}} A1_{ni} \tag{B-27}$$

$$F_{ni} = \Omega_{1n}\left(z_{1n}(t_i) - \frac{A1_{ni}}{\omega_{1n}^2}\right) \tag{B-28}$$

$$G_{ni} = e^{-\xi_{1n}\,\omega_{1n}\,\Delta t}\left(E_{ni}\cos\Omega_{1n}\,\Delta t - F_{ni}\sin\Omega_{1n}\,\Delta t\right) \tag{B-29}$$

$$H_{ni} = 2\,\xi_{1n}\,\omega_{1n}\,G_{ni} + \omega_{1n}^2\left(1 - 2\,\xi_{1n}^2\right)D_{ni} \tag{B-30}$$

$$R1_n = e^{-\xi_{1n}\,\omega_{1n}\,\Delta t}\left(-\frac{1-2\,\xi_{1n}^2}{\omega_{1n}^2\,\Omega_{1n}\,\Delta t}\sin\Omega_{1n}\,\Delta t + \frac{2\,\xi_{1n}}{\omega_{1n}^3\,\Delta t}\cos\Omega_{1n}\,\Delta t\right) \tag{B-31}$$

$$R2_n = \Omega_{1n} e^{-\xi_{1n}\,\omega_{1n}\,\Delta t}\left(-\frac{1-2\,\xi_{1n}^2}{\omega_{1n}^2\,\Omega_{1n}\,\Delta t}\cos\Omega_{1n}\,\Delta t - \frac{2\,\xi_{1n}}{\omega_{1n}^3\,\Delta t}\sin\Omega_{1n}\,\Delta t\right) \tag{B-32}$$

$$R3_n = 2\,\xi_{1n}\,\omega_{1n}\,R2_n + \omega_{1n}^2\left(1 - 2\,\xi_{1n}^2\right)R1_n \tag{B-33}$$

This completes the solution for the non-sliding phase. Note that this method only applies to the case in which all three degrees of freedom are in non-sliding phases. If one or two degrees of freedom are in non-sliding phases, then the method presented in Chapter 2 or Chapter 3 must be implemented, setting the appropriate velocity and acceleration values to zero.

As mentioned previously, the structure will remain in a non-sliding phase until the criterion shown in equation (B-6) is met, at which point the solution procedure presented in Chapter 2 or Chapter 3 can be used again.

B.3 APPLICATION TO A MULTI-STORY STRUCTURE

The process for applying the non-sliding condition to a multi-story structure is very similar to the process for the single-story structure presented above. A non-sliding phase can be defined beginning with equation (4-8), repeated here for convenience:

$$\begin{aligned}[M_t]\{\ddot{d}_b\}+[C_b]\{\dot{d}_b\}+[K_b]\{d_b\} &= -\sum_{i=1}^{N}\left([M_i]\{\ddot{d}_i\}\right)\\ &\quad -[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\operatorname{sgn}(\dot{d}_b)\}\end{aligned} \tag{B-34}$$

By definition of the non-sliding phase, however, the bearing level acceleration and velocity terms will be equal to zero for any degree of freedom which is in a non-sliding phase. Applying this definition to equation (B-34) yields the following:

$$[K_b]\{d_b\} = -\sum_{i=1}^{N}\left([M_i]\{\ddot{d}_i\}\right)-[M_t]\{\ddot{d}_g\}-\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\operatorname{sgn}(\dot{d}_b)\} \tag{B-35}$$

Note that, as was the case for the single-story formulation, each equation from the matrix expression must be evaluated separately. For simplicity, however, the matrix expression will be used throughout this appendix.

Isolating the frictional component of equation (B-35) yields the following result:

$$\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{\operatorname{sgn}(\dot{d}_b)\} = -\sum_{i=1}^{N}\left([M_i]\{\ddot{d}_i\}\right)-[M_t]\{\ddot{d}_g\}-[K_b]\{d_b\} \tag{B-36}$$

As in the single-story formulation, each component of the $\{\operatorname{sgn}(\dot{d}_b)\}$ vector is either a positive or negative one. Therefore, taking the absolute value of each side of equation (B-36),

$$\left|\mu\left(g+\ddot{d}_{gz}\right)[M_t]\{1\}\right| = \left|\sum_{i=1}^{N}\left([M_i]\{\ddot{d}_i\}\right)+[M_t]\{\ddot{d}_g\}+[K_b]\{d_b\}\right| \tag{B-37}$$

A non-sliding phase is defined by the frictional forces outweighing the impelling forces. Therefore, for a degree of freedom to enter a non-sliding phase, it must satisfy the following criterion:

$$\left|\mu\left(g+\ddot{d}_{gz}\right)\left[M_t\right]\{1\}\right| > \left|\sum_{i=1}^{N}\left(\left[M_i\right]\left\{\ddot{d}_i\right\}\right)+\left[M_t\right]\left\{\ddot{d}_g\right\}+\left[K_b\right]\left\{d_b\right\}\right| \qquad \text{(B-38)}$$

The coefficient of friction, μ, that is used in equation (B-38) depends upon the condition of motion. If the degree of freedom to be considered had been in a non-sliding phase, the coefficient of static friction should be used. If, however, the degree of freedom had been in a sliding phase, the coefficient of kinetic friction should be used. The criterion to enter a sliding phase is as follows:

$$\left|\mu\left(g+\ddot{d}_{gz}\right)\left[M_t\right]\{1\}\right| < \left|\sum_{i=1}^{N}\left(\left[M_i\right]\left\{\ddot{d}_i\right\}\right)+\left[M_t\right]\left\{\ddot{d}_g\right\}+\left[K_b\right]\left\{d_b\right\}\right| \qquad \text{(B-39)}$$

A degree of freedom that is in a sliding phase will remain that way until equation (B-38) is again satisfied. If a particular degree of freedom is in a sliding phase, the solution presented in Chapters 4 and 5 can be used. However, during the non-sliding phases, the alternative solution method presented below must be used. Since the bearing level is the only location for non-linearity, the solution for a non-sliding phase is by definition linear. The first step of the solution is to take the superstructure equation of motion presented in equation (5-24), repeated here for convenience:

$$\begin{aligned}\ddot{z}_{un}\left(t_{i+1}\right)+2\xi_{un}\omega_{un}\dot{z}_{un}\left(t_{i+1}\right)+\omega_{un}^{2}z_{un}\left(t_{i+1}\right)=\\ \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{bk}\left(t_{i+1}\right)+\sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right)\end{aligned} \qquad \text{(B-40)}$$

Note that this equation is another form of equation (4-47), from which the linear solution was derived.

Setting the bearing accelerations to zero yields the following result:

$$\ddot{z}_{un}\left(t_{i+1}\right)+2\xi_{un}\omega_{un}\dot{z}_{un}\left(t_{i+1}\right)+\omega_{un}^{2}z_{un}\left(t_{i+1}\right)=\sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right) \qquad \text{(B-41)}$$

Using the definition presented in equation (2-29), equation (B-41) can be expanded as follows:

$$\ddot{z}_{un}\left(t_{i+1}\right)+2\xi_{un}\omega_{un}\dot{z}_{un}\left(t_{i+1}\right)+\omega_{un}^{2}z_{un}\left(t_{i+1}\right)= \sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right)+\left(\sum_{k=1}^{3}\alpha_{unk}\Delta\ddot{d}_{gk}\left(t_{i+1}\right)\right)\frac{\tau}{\Delta t} \tag{B-42}$$

This expression can then be simplified as follows:

$$\ddot{z}_{un}\left(t_{i+1}\right)+2\xi_{un}\omega_{un}\dot{z}_{un}\left(t_{i+1}\right)+\omega_{un}^{2}z_{un}\left(t_{i+1}\right)=A1_{ni}+B1_{ni+1}\frac{\tau}{\Delta t} \tag{B-43}$$

in which

$$A1_{ni}=\sum_{k=1}^{3}\alpha_{unk}\ddot{d}_{gk}\left(t_{i+1}\right) \tag{B-44}$$

$$B1_{ni+1}=\sum_{k=1}^{3}\alpha_{unk}\Delta\ddot{d}_{gk}\left(t_{i+1}\right) \tag{B-45}$$

Now the solution to equation (B-43) can be written as a combination of a complementary solution and a particular solution, as shown here:

$$z_{un}\left(\tau\right)=z_{un}^{c}\left(\tau\right)+z_{un}^{p}\left(\tau\right) \tag{B-46}$$

As in the single story non-sliding formulation, the complementary solution is of the following form:

$$z_{un}^{c}=e^{-\xi_{1n}\,\omega_{un}\,\tau}\left(C1_{n}\sin\Omega_{un}\tau+C2_{n}\cos\Omega_{un}\tau\right) \tag{B-47}$$

in which

$$\Omega_{un}=\omega_{un}\sqrt{1-\xi_{un}^{2}} \tag{B-48}$$

The particular solution can be written as:

$$z_{un}^{p}=C3_{n}+C4_{n}\frac{\tau}{\Delta t} \tag{B-49}$$

in which

$$C3_{n}=\frac{1}{\omega_{un}^{2}}\left(A1_{ni}-\frac{2\xi_{un}}{\omega_{un}\,\Delta t}B1_{ni+1}\right) \tag{B-50}$$

$$C4_n = \frac{B1_{ni+1}}{\omega_{un}^2} \tag{B-51}$$

Combining the complementary and particular solutions, as shown in equation (B-46), yields the following equation:

$$\begin{aligned} z_{un}(\tau) = e^{-\xi_{un}\,\omega_{un}\,\tau}\left(C1_n \sin\Omega_{un}\tau + C2_n \cos\Omega_{un}\tau\right) \\ + \frac{1}{\omega_{un}^2}\left(A1_{ni} + \left(\tau - \frac{2\xi_{un}}{\omega_{un}}\right)\frac{B1_{ni+1}}{\Delta t}\right) \end{aligned} \tag{B-52}$$

As in the single story formulation, the constants $C1_n$ and $C2_n$ will be solved by using the following identities:

$$z_{un}(\tau = 0) = z_{un}(t_i) \tag{B-53}$$

$$\dot{z}_{un}(\tau = 0) = \dot{z}_{un}(t_i) \tag{B-54}$$

The values for the constants in equation (B-52) can now be defined as:

$$C1_n = \frac{1}{\Omega_{un}}\left(\dot{z}_{un}(t_i) + \xi_{un}\,\omega_{un}\,z_{un}(t_i) - \frac{\xi_{un}}{\omega_{un}}A1_{ni} - \frac{\left(1 - 2\,\xi_{un}^2\right)}{\omega_{un}^2\,\Delta t}B1_{n\,i+1}\right) \tag{B-55}$$

$$C2_n = z_{un}(t_i) - \frac{A1_{ni}}{\omega_{un}^2} + \frac{2\,\xi_{un}}{\omega_{un}^3\,\Delta t}B1_{n\,i+1} \tag{B-56}$$

Equation (B-52) now represents the modal displacement of the superstructure at time τ. Taking the time derivatives of equation (B-52) yields the modal velocity and acceleration at time τ. Evaluating these quantities at time $\tau = \Delta t$ grants the following expressions for modal displacement, velocity, and acceleration at time t_{i+1}:

$$z_{un}(t_{i+1}) = D_{ni} + R1_n\,B1_{n\,i+1} + \frac{1}{\omega_{un}^2}\left(A1_{ni} + \left(\Delta t - \frac{2\,\xi_{un}}{\omega_{un}}\right)\frac{B1_{n\,i+1}}{\Delta t}\right) \tag{B-57}$$

$$\dot{z}_{un}(t_{i+1}) = \left(G_{ni} - \xi_{un}\,\omega_{un}\,D_{ni}\right) + \left(R2_n - \xi_{un}\,\omega_{un}\,R1_n\right)B1_{n\,i+1} + \frac{B1_{n\,i+1}}{\omega_{un}^2\,\Delta t} \tag{B-58}$$

$$\ddot{z}_{un}\left(t_{i+1}\right) = -H_{ni} - R3_n\, B1_{n\,i+1} \tag{B-59}$$

in which

$$D_{ni} = \frac{e^{-\xi_{un}\,\omega_{un}\,\Delta t}}{\Omega_{un}}\left(E_{ni}\sin\Omega_{un}\,\Delta t + F_{ni}\cos\Omega_{un}\,\Delta t\right) \tag{B-60}$$

$$E_{ni} = \dot{z}_{un}\left(t_i\right) + \xi_{un}\,\omega_{un}\,z_{un}\left(t_i\right) - \frac{\xi_{un}}{\omega_{un}}A1_{ni} \tag{B-61}$$

$$F_{ni} = \Omega_{un}\left(z_{un}\left(t_i\right) - \frac{A1_{ni}}{\omega_{un}^2}\right) \tag{B-62}$$

$$G_{ni} = e^{-\xi_{un}\,\omega_{un}\,\Delta t}\left(E_{ni}\cos\Omega_{un}\,\Delta t - F_{ni}\sin\Omega_{un}\,\Delta t\right) \tag{B-63}$$

$$H_{ni} = 2\,\xi_{un}\,\omega_{un}\,G_{ni} + \omega_{un}^2\left(1 - 2\,\xi_{un}^2\right)D_{ni} \tag{B-64}$$

$$R1_n = e^{-\xi_{un}\,\omega_{un}\,\Delta t}\left(-\frac{1-2\,\xi_{un}^2}{\omega_{un}^2\,\Omega_{un}\,\Delta t}\sin\Omega_{un}\,\Delta t + \frac{2\,\xi_{un}}{\omega_{un}^3\,\Delta t}\cos\Omega_{un}\,\Delta t\right) \tag{B-65}$$

$$R2_n = \Omega_{un}e^{-\xi_{un}\,\omega_{un}\,\Delta t}\left(-\frac{1-2\,\xi_{un}^2}{\omega_{un}^2\,\Omega_{un}\,\Delta t}\cos\Omega_{un}\,\Delta t - \frac{2\,\xi_{un}}{\omega_{un}^3\,\Delta t}\sin\Omega_{un}\,\Delta t\right) \tag{B-66}$$

$$R3_n = 2\,\xi_{un}\,\omega_{un}\,R2_n + \omega_{un}^2\left(1 - 2\,\xi_{un}^2\right)R1_n \tag{B-67}$$

This completes the solution for a multi-story structure which is in a non-sliding phase. Note that the solution presented in this appendix only applies to the case in which all three degrees of freedom are in non-sliding phases. If only one or two degrees of freedom are in non-sliding phases, the method presented in Chapter 4 or Chapter 5 must be implemented with the appropriate velocities and accelerations set to zero.

If the structure is in a non-sliding phase, it will continue to act as a fixed-base structure until the condition shown in equation (B-39) is met. At that time step, the procedures presented in Chapter 4 and Chapter 5 apply once more.

APPENDIX C

PLASTICITY

As mentioned in Chapters 3 and 5, non-linearity arises in the response of the bearings to ground excitation. This non-linearity can be attributed to yielding and strain hardening. Yielding occurs when the shear forces on a bearing exceed its maximum resisting force. The bearing can no longer resist the forces applied to it; it deforms further without an increased load. Therefore, the behavior on the force-deformation plot is no longer linear. Strain hardening is a phenomenon in which a material achieves reserve strength after yielding. The material may attain a secondary force-deformation curve, which will not allow for linear analysis. It is necessary to accommodate any reserve strength in the response of the bearings, or the calculate response will be inaccurate.

C.1 HARDENING CRITERIA

Ziegler established a method of analysis of a hardening material, which will be used here to model the behavior of the bearings upon yielding. His model was a modification of Prager's model based upon kinematic hardening. Ziegler suggests that a hardening material can be described by the following three criteria:

a. An initial yielding condition, specifying the states of stress for which plastic flow first sets in

b. A flow rule, connecting the plastic strain increment with the stress and the stress increment

c. A hardening rule, specifying the modification of the yield condition in the course of plastic flow (Ziegler, 1959)

Ziegler's formulation for plasticity implements a stress-strain model to describe the behavior of the bearing and to determine yielding. This would be impractical for the analysis of the bearings, as stresses and strains are not necessary for other calculations in this study. Therefore, the formulation presented here is in terms of forces and displacements, which are analogous to the stresses and strains. Additionally, since plasticity depends greatly upon the load-displacement path, for the purposes of this study incremental forces and displacements will be considered.

As mentioned above, the hardening rule used for Ziegler's method is kinematic hardening. As demonstrated in Figure 11, this implies that plastic behavior changes the location, but not the orientation or size, of the yield function. Mathematically, this can be expressed as:

$$\{d\alpha\} = \{V - \alpha\} d\mu \qquad \text{(C-1)}$$

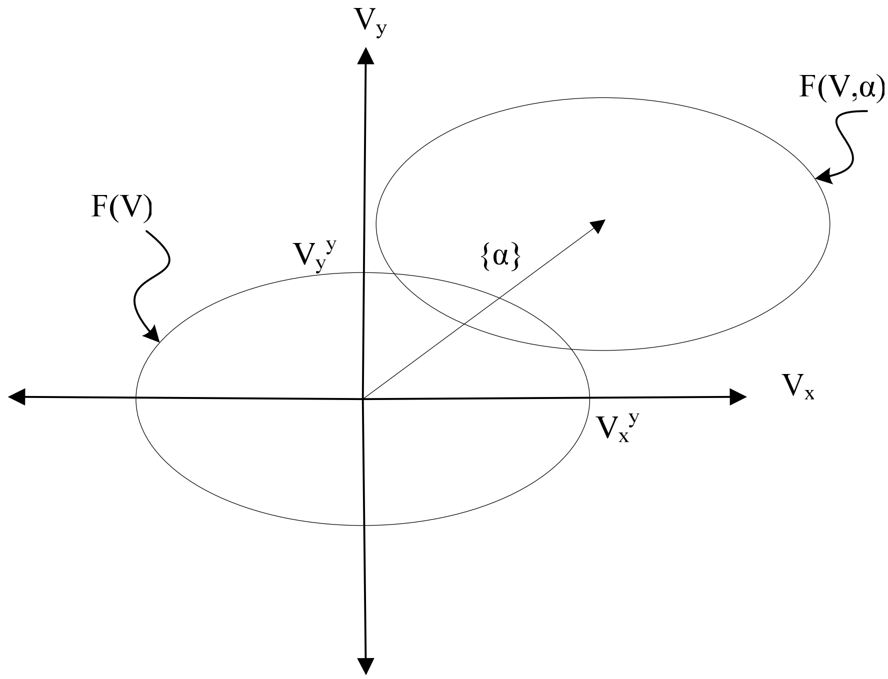

Figure 11 – Kinematic Hardening Schematic

Figure 11 shows the initial yield condition, assumed to be in the shape of an ellipse. This is similar to the von Mises yield function. The actual yield function shown here can be expressed mathematically as:

$$f(V,\alpha)=\left(\frac{V_x-\alpha_x}{V_x^y}\right)^2+\left(\frac{V_y-\alpha_y}{V_y^y}\right)^2-1 \qquad \text{(C-2)}$$

in which

$V_x^y \equiv$ yield force in the *x*-direction

$V_y^y \equiv$ yield force in the *y*-direction

In the derivation of the hardening rule, it will become necessary to define a vector normal to the yield function. This vector is defined and calculated as follows:

$$\left\{\frac{\partial f}{\partial V}\right\} = \{N\} = 2\begin{Bmatrix} \dfrac{V_x - \alpha_x}{\left(V_x^y\right)^2} \\ \dfrac{V_y - \alpha_y}{\left(V_y^y\right)^2} \end{Bmatrix} \tag{C-3}$$

in which

$V_x \equiv$ instantaneous shear force in the x-direction

$V_y \equiv$ instantaneous shear force in the y-direction

$\{\alpha\} = \begin{Bmatrix} \alpha_x \\ \alpha_y \end{Bmatrix} \equiv$ translation vector of the yield surface, see Figure 11

The final criterion for modeling hardening behavior is a flow rule. The flow rule is a relationship between plastic deformation and force increments, and is written as:

$$\{dU_p\} = \left\{\frac{\partial f}{dV}\right\} d\lambda = \{N\} d\lambda \qquad d\lambda > 0 \tag{C-4}$$

The goal of the formulation presented here is to express incremental force in terms of incremental displacement, to determine an effective stiffness which incorporates both elastic and inelastic displacements. A force-displacement relationship for elastic materials can be written, but plastic displacements can not be used in that formulation. To begin this plasticity formulation, a simple definition will be written. The total displacement at time t_{i+1} is equal to the displacement at the previous time step plus the incremental displacement:

$$\{U_{i+1}\} = \{U_i\} + \{dU_{i+1}\} \tag{C-5}$$

However, the incremental displacement is the sum of elastic and plastic incremental displacements, as shown in equation (C-6).

$$\{dU\} = \{dU_e\} + \{dU_p\} \tag{C-6}$$

Rearranging terms, the following expression can be written:

$$\{dU_e\} = \{dU\} - \{dU_p\} \tag{C-7}$$

By definition, the following incremental force-displacement relationship can be written for an elastic material:

$$\{dV\} = [K_e]\{dU_e\} \tag{C-8}$$

Now, by substituting equation (C-7) into the force-displacement relationship and rearranging terms, the following equation is obtained:

$$[K_e]\{dU\} - [K_e]\{dU_p\} - \{dV\} = \{0\} \tag{C-9}$$

The flow rule, equation (C-4), can be substituted here to replace the plastic displacements. Making that substitution leaves the following equation:

$$[K_e]\{dU\} - [K_e]\{N\}d\lambda - \{dV\} = \{0\} \tag{C-10}$$

Equation (C-10), derived from the elastic force-displacement relationship, is still well short of defining an elasto-plastic force-displacement function. To fully develop the model, it is necessary to write $d\lambda$ in terms of either the displacements or the forces. The following procedure can be used to solve for $d\lambda$, as suggested by Ziegler (1959).

First, it is necessary to make an assumption. The simplest assumption is that the vector $c\{dU_p\}$ is the projection of the translation vector $\{\alpha\}$ on the exterior normal of the yield surface. Therefore, the following must be true:

$$\{dV - cdU_p\}^T \{N\} = 0 \tag{C-11}$$

From equation (C-11), the following is evident:

$$\{dV\}^T \{N\} = c\{dU_p\}^T \{N\} \tag{C-12}$$

Transposing both sides of this equation leaves the following equality:

$$\{N\}^T\{dV\} = c\{N\}^T\{dU_p\} \tag{C-13}$$

Returning to equation (C-10), the next step to solve for $d\lambda$ is to premultiply each side of the equation by $\{N\}^T$, as follows:

$$\{N\}^T[K_e]\{dU\} - \{N\}^T[K_e]\{N\}d\lambda - \{N\}^T\{dV\} = \{0\} \tag{C-14}$$

Now equation (C-13) can be substituted into equation (C-14), yielding the following:

$$\{N\}^T[K_e]\{dU\} - \{N\}^T[K_e]\{N\}d\lambda - c\{N\}^T\{dU_p\} = \{0\} \tag{C-15}$$

Again recalling the flow rule, the plastic deformations can be replaced:

$$\{N\}^T[K_e]\{dU\} - \{N\}^T[K_e]\{N\}d\lambda - c\{N\}^T\{N\}d\lambda = \{0\} \tag{C-16}$$

Equation (C-16) can now be used to determine $d\lambda$ as a function of the total displacement. Rearranging the terms of equation (C-16) gives the following:

$$\{N\}^T[K_e]\{dU\} = \left(\{N\}^T[K_e]\{N\} + c\{N\}^T\{N\}\right)d\lambda \tag{C-17}$$

Solving for the parameter $d\lambda$ in terms of the total displacement vector:

$$d\lambda = \frac{\{N\}^T[K_e]}{\{N\}^T[K_e]\{N\} + c\{N\}^T\{N\}}\{dU\} \tag{C-18}$$

This definition can then be substituted back into equation (C-10). After rearranging the terms, the resulting equation is as follows:

$$\{dV\} = \left([K_e] - \frac{[K_e]\{N\}\{N\}^T[K_e]}{\{N\}^T[K_e]\{N\} + c\{N\}^T\{N\}}\right)\{dU\} \tag{C-19}$$

This is now an expression for force as a function of total increment of displacement, which was the goal of the plasticity formulation. Therefore, the term in the parentheses is the effective stiffness of the material given the combination of elastic and plastic displacements. The effective elasto-plastic stiffness is defined as:

$$\left[K_{ep}\right]=\left[K_e\right]-\frac{\left[K_e\right]\{N\}\{N\}^T\left[K_e\right]}{\{N\}^T\left[K_e\right]\{N\}+c\{N\}^T\{N\}} \tag{C-20}$$

Now that the effective stiffness has been determined, the plasticity theory itself can be examined to determine the quantities required for further plastic analysis. By definition of yielding, the forces must be such that the force function remains on the yield surface. Therefore, the change in forces must be tangential to the yield surface. This can be expressed mathematically as

$$\{N\}^T\{dV\}=0 \tag{C-21}$$

The same condition can be applied to the yield surface after it has been displaced due to plastic deformation. Graphically speaking, the change in forces must be tangential to the new yield surface. Therefore, the following expression can be written:

$$\{N\}^T\{dV-d\alpha\}=0 \tag{C-22}$$

Transposing this equation and substituting equation (C-1) for the $d\alpha$ term grants the following equation:

$$\{dV\}^T\{N\}-\{V-\alpha\}^T\{N\}d\mu=0 \tag{C-23}$$

Equation (C-23) can then be used to solve for the unknown $d\mu$, from Ziegler's hardening rule, in two steps:

$$\{dV\}^T\{N\}=\{V-\alpha\}^T\{N\}d\mu \tag{C-24}$$

$$d\mu=\frac{\{dV\}^T\{N\}}{\{V-\alpha\}^T\{N\}} \tag{C-25}$$

Therefore, the incremental change in location of the yield surface can now be solved using equation (C-1), since the unknown quantity $d\mu$ was determined in equation (C-25):

$$\{d\alpha\}=\{V-\alpha\}d\mu=\{V-\alpha\}\frac{\{dV\}^T\{N\}}{\{V-\alpha\}^T\{N\}} \tag{C-26}$$

C2. SOLUTION PROCEDURE

Now that the equations involved in the plastic analysis of the bearings have been derived, a solution will be calculated for use in this study by assuming an elastic stiffness matrix of the following form:

$$[K_e] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{C-27}$$

Note that the stiffness matrix is assumed to be de-coupled. This is for simplicity in the calculations. A more complex analysis including coupled stiffness matrices is beyond the scope of this study.

The following terms will be defined to allow for a more specific formulation of the effective stiffness:

$$s = 4\left(\frac{V_x - \alpha_x}{V_x^y}\right)^2 (k_x + c) + 4\left(\frac{V_y - \alpha_y}{V_y^y}\right)^2 (k_y + c) \tag{C-28}$$

$$\{\rho\} = \begin{Bmatrix} \rho_x \\ \\ \rho_y \end{Bmatrix} = \begin{Bmatrix} \dfrac{2(V_x - \alpha_x)}{(V_x^y)^2} \\ \dfrac{2(V_y - \alpha_y)}{(V_y^y)^2} \end{Bmatrix} \tag{C-29}$$

Multiplying out the terms to calculate $d\lambda$, as shown in equation (C-18), and substituting the appropriate values from equations (C-27), (C-28), and (C-29) yields the following:

$$d\lambda = \frac{1}{s}\begin{bmatrix} k_x\rho_x & k_y\rho_y \end{bmatrix}\begin{Bmatrix} dU_x \\ dU_y \end{Bmatrix} \tag{C-30}$$

Similarly, equation (C-25) can be used to determine $d\mu$ in terms of the identity shown in equation (C-29):

$$d\mu = \frac{\rho_x dV_x + \rho_y dV_y}{\rho_x (V_x - \alpha_x) + \rho_y (V_y - \alpha_y)} \tag{C-31}$$

The final step in the plastic analysis is to determine the change in the location of the yield surface. The relocation vector, as shown in equation (C-26), can be solved by substituting the result shown in equation (C-31) as follows:

$$\{d\alpha\} = \{V - \alpha\} d\mu = \{V - \alpha\} \frac{\rho_x dV_x + \rho_y dV_y}{\rho_x (V_x - \alpha_x) + \rho_y (V_y - \alpha_y)} \tag{C-32}$$

This shows the increment of the relocation vector. This increment is added to the current value of the vector to determine the value of $\{\alpha\}$ for the next time step.

Similarly, the incremental force can be written as follows:

$$\{dV\} = \left([K_e] - [K_e]\{N\}\left(\frac{1}{s}\right)\begin{bmatrix} k_x \rho_x & k_y \rho_y \end{bmatrix} \right)\{dU\} \tag{C-33}$$

This incremental force is then added to the current force to determine the force value for the next time step, and the process begins for the next time step.

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